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An aperiodic tiles machine ☆

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Abstract

The results we introduce in this work lead to get an algorithm which produces aperiodic sets of tiles using Voronoi diagrams. This algorithm runs in optimal worst-case time $O(n \log n)$. Since a wide range of new examples can be obtained, it could shed some new light on non-periodic tilings. These examples are locally isomorphic and exhibit the 5-fold symmetry which appears in Penrose tilings and quasicrystals. Moreover, we outline a similar construction using Delaunay triangulations and propose some related open problems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we relate one of the most versatile structures in Computational Geometry, namely Voronoi diagrams with aperiodic tilings, a field that, although considered by non-specialists merely as recreative mathematics, has important connections with crystallography, number theory, logic and theoretical computer science. In a first step, a *plane tiling* or simply a *tiling* can be considered as a countable family of closed sets which covers the plane without gaps or overlaps. These closed sets are called the *tiles* of the tiling. More explicitly, the union of tiles is the whole plane, and the interiors of them are pairwise disjoint. But that definition of tiling is too wide and gives way to some pathologic examples, thus some additional conditions must be imposed. In this way, the tiles we will consider in this paper are bounded polygons with a finite number of edges and, as further simplification, every tile is

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an isometric copy from a finite set of fixed tiles called *prototiles*. The best general reference here is [3] where a complete and accurate description of tilings, tiles and prototiles is given.

A tiling is *periodic* if there exists a translation that leaves the tiling invariant and it is non-periodic in any other case. A set of prototiles that always tiles non-periodically is called *aperiodic*. It was conjectured by Wang that there is no such a set of prototiles [11]. However, in 1966, Berger [1] found the first one with around 20,000 prototiles and since then, many other sets have been obtained.

We will prove in this paper that from any aperiodic set of prototiles, it is possible to construct infinitely many of those sets using Voronoi diagrams. As it is well-known, given a countable set of sites P in the Euclidean plane, the *Voronoi region* $V(p)$ associated with a site p consists of all the points at least as close to p as to any other site. The set of Voronoi regions associated with points in P constitutes a plane subdivision denoted by $\mathrm{Vor}(P)$ and called the *Voronoi diagram* generated by P (see [6] for details).

Since a Voronoi diagram records the information about proximity to a set of points, it is a powerful tool that can be applied in many fields such as archaeology, biology, architecture, motion planning or chemistry among others. One reason for its continuing success is that it verifies many topological and computational properties. In the literature, generally the set of sites is finite, but some authors have studied Voronoi diagrams for different classes of surfaces and this turns to be equivalent to study Voronoi diagrams for periodic, and therefore infinite, sets of sites. Thus, the method considered in [4] uses the fact that those surfaces are isometrically covered by the Euclidean plane, and then, finite sets of sites in the surface give way to periodic point sets in the plane. In that work, the authors prove that making use of the periodicity, Voronoi diagrams for such periodic set of sites in the plane can be computed using the same methods that work for finite set of sites in the plane. Similar ideas appear in some other works as [5] or [10] (see [6]), and their authors apply it to model some crystallographic processes and simulating Voronoi diagrams of a random point set. Nevertheless, we are not aware of any work dealing with infinite non-periodic set of sites. We will prove that a Voronoi diagram of an appropriate non-periodic site set is a non-periodic tiling of the plane, and, as we obtain those site sets by placing points in known prototiles, we thus obtain, in this way, a sort of machine for generating aperiodic sets of prototiles.

2. Aperiodic prototiles

As it has been pointed out in the Introduction, since Berger's first example, many other aperiodic sets of prototiles (or aperiodic prototiles for short) have appeared. Soon, the size of those examples decreased, and so, Robinson gave the first simple example. However, the most famous aperiodic prototiles are *kite* and *dart* described by Penrose [7] in 1974 (see Fig. 1 where τ means the golden ratio).

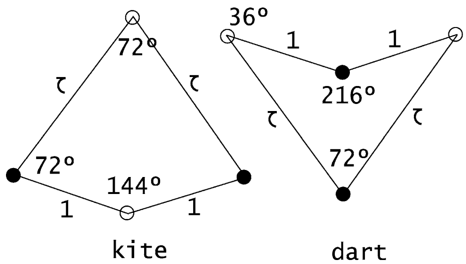

Fig. 1. The prototiles kite and dart.

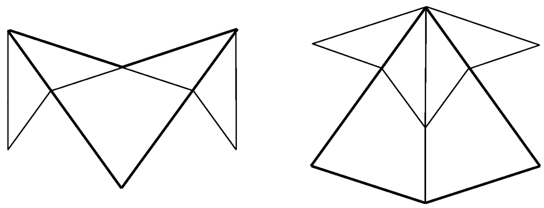

Fig. 2. Composing kite and dart.

A tiling by kites and darts will be called a *Penrose tiling*. Note that the vertices of kite and dart are colored. It means that if we put equal edges together then the colors of the vertices must also match. The colored vertices can be avoided if we replace the prototile edges by suitable Jordan curves but it is preferred to use them for simplicity. Generally speaking, a combinatorial rule like that is called a *matching rule* and it is usually defined in terms of labels in the vertices and edges of the prototiles.

Consider a finite set of points on Penrose's kite and dart. Now, if we tile the plane by those tiles we will get an infinite point set in the plane. Any infinite point set in the plane obtained in such a way will be called a *Penrose point set* and the Penrose tiling which generates it, is the *underlying tiling*. As it has been pointed out previously, we will study Voronoi diagrams on Penrose point sets. In fact, we could make a more general study, and most of the results here presented are valid in a wider context, but since Penrose tiling are the best known among aperiodic tiling, we consider it is worthy to show the conclusions in the way introduced here.

A *patch* of tiles in a given tiling is defined to be a finite number of tiles of the tiling such that their union is a topological disk. Given a tile in a given tiling its 1-*patch* is the set of its neighbors (the tiles that have non-empty intersection with it), its 2-*patch* is the set of tiles that are neighbors of some tile in its 1-patch and so on.

All aperiodic sets of prototiles discovered so far, depend on a surprising phenomenon discovered by Penrose [7]. Conway calls it *composition* or *inflation* and is a simple method to produce complicated tilings beginning with a patch or even a single tile. In the case of Penrose tilings, imagine that every dart is cut in half and then all short edges of the original pieces are glued together. The result is a new tiling by larger darts and kites (see Fig. 2). Detailed constructions are given in [3,8,9].

When this operation acting on a tiling or a patch of tiles is reversed, is called *decomposition* or *deflation*. The important point about the composition operation is its uniqueness which is closely related to the aperiodicity of prototiles.

Theorem 1 [3]. *If a tiling has a unique composition that leaves the tiling invariant then it is non-periodic. Hence, if there exists exactly one composition acting on any tiling of a set of prototiles then this set is aperiodic.*

It could seem that a Penrose tiling is an almost chaotic object but it presents a very regular local structure. For example in [3] it is shown that every tile vertex is surrounded by one of the seven patches of Fig. 3 called the *vertex neighborhoods*.

Another amazing local property of a Penrose tiling is that given a patch there exist some tiles which are forced by it. In Fig. 4, it is shown some forced tiles by five vertex neighborhoods. The other two neighborhoods called the ace and the sun do not force any tile.

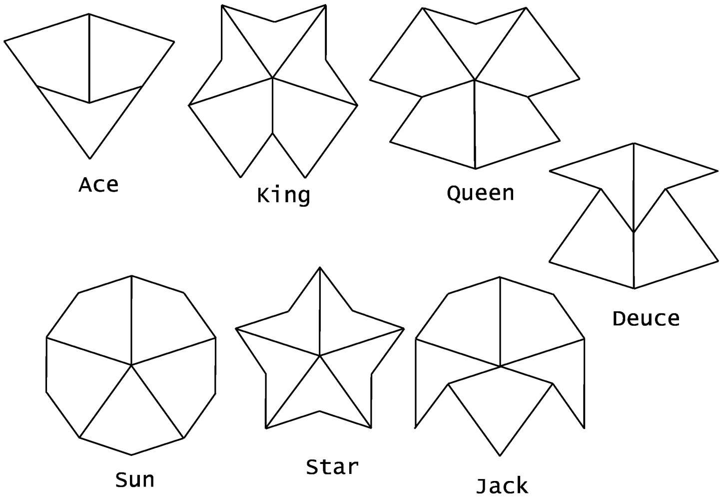

Fig. 3. The seven kinds of vertex neighborhoods.

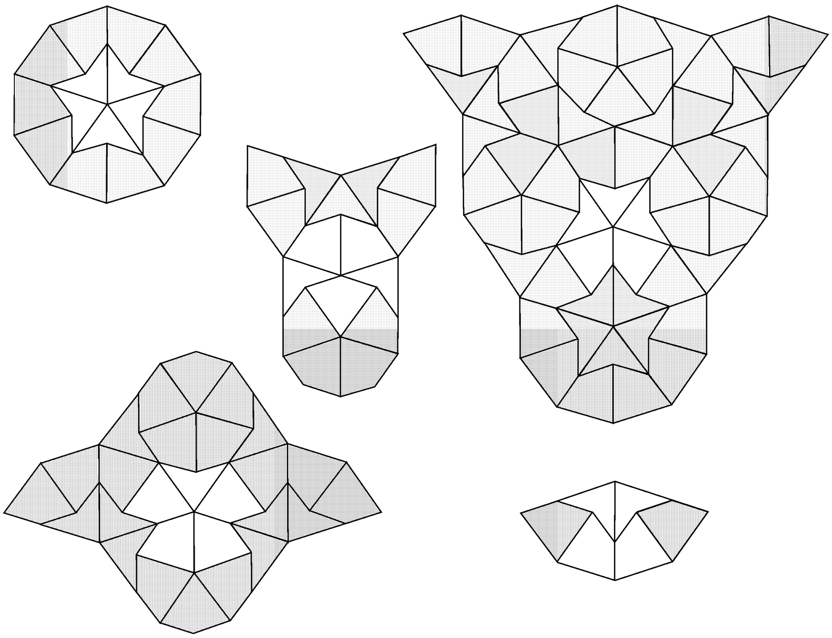

Fig. 4. Some forced tiles by vertex neighborhoods.

3. Voronoi diagrams as non-periodic tilings

We are now in position to show that the Voronoi diagram of a Penrose point set is a non-periodic plane tiling. This is established by the next two results.

Lemma 2. *The Voronoi diagram of a Penrose point set is a plane tiling.*

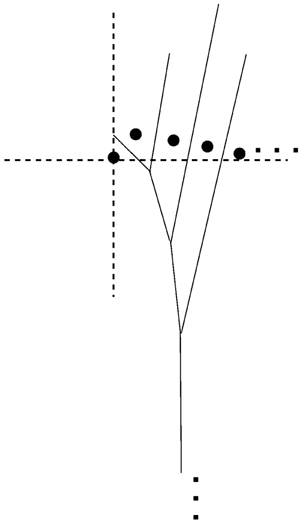

Fig. 5. A Voronoi region with an infinite number of edges.

Proof. We have to prove three conditions.

(1) The Voronoi regions must cover the plane without gaps or overlaps.
(2) All regions are bounded polygons with a finite number of edges.
(3) Every region is an isometric copy from a finite set of fixed tiles.

The first condition is immediate from the properties of Voronoi diagrams. Since it is well-known that a Voronoi diagram covers the plane without gaps and the interiors of two Voronoi regions are disjoint.

In order to prove the second condition, first observe that given a Penrose point set P, every point is completely surrounded by other points, or more precisely, given a site p in P, there exist other sites in P such that p is in the convex hull of those other sites. This follows from the fact that in a Penrose tiling every kite (resp. dart) is surrounded by other forced kites (resp. darts). The reader can easily check this claim using the above results about vertex neighborhoods and forced tiles. Accordingly, every Voronoi region is contained in a patch of tiles so it is bounded and depends only on the position of those tiles. Thus, there exists a finite number of distinct Voronoi regions.

Therefore, we just have to make sure that a Voronoi region does not contain an infinite number of edges as in Fig. 5.

Suppose there exists a Penrose point p such that $V(p)$ contains infinitely many edges. Every edge e in $V(p)$ is defined by one Penrose point namely p_e such that p and p_e are equidistant from e. Since $V(p)$ is bounded, there exists a compact set which contains all the points p_e for all edges e in $V(p)$. Hence there exists an accumulation point for this set but this is a contradiction because a Penrose point set has no accumulation points. □

In order to prove the main theorem of this section, we will study the compositions which leave the Voronoi diagram of a Penrose point set invariant.

Theorem 3. *The Voronoi diagram of a Penrose point set is a non-periodic tiling.*

Proof. Let P be a Penrose point set. According to Theorem 1 we need to prove that there exists exactly one composition leaving $\mathrm{Vor}(P)$ invariant.

It is clear that we can define at least a composition in the following way: if $\mathcal{T}$ is the underlying tiling of P consider the usual composition of Penrose tilings. This operation also generates an inflated copy of P and consequently a composed copy of $\mathrm{Vor}(P)$.

On the other hand, suppose that there exists another composition which leaves $\mathrm{Vor}(P)$ invariant. From the composed Voronoi diagram we can reconstruct the inflate Penrose point set P and its underlying tiling which is an inflate copy of $\mathcal{T}$. But this is contrary to the fact that a Penrose tiling has a unique composition that leaves it invariant. □

This theorem may be summarized by saying that a new non-periodic tiling is constructed simply by choosing some points on Penrose's prototiles. We must point out that the set of prototiles does not depend on the underlying tiling used for generating the Penrose point set but on the position of the points lying on the original kites and darts.

4. Voronoi regions as aperiodic prototiles

In the above section we have proved that the Voronoi diagram of a Penrose set of sites is non-periodic. Particularly, we know that there are finitely many distinct Voronoi regions. Our next objective is to prove that if we consider the finite set of regions obtained in the previous section as a set of prototiles it is possible to give some matching rules which guarantee the aperiodicity of the prototiles so obtained. This is to say, any tiling constructed with those regions and following our rules will be a non-periodic tiling.

Firstly, we see in Fig. 6 that a Voronoi region could have a rhomb-shape, and a rhomb always admits a periodic tiling of the plane. Therefore, some matching rules are needed.

The labeling that we propose is a natural labeling in the following way. Since we start with a fixed Penrose tiling, we draw in each prototile the portions of the edges of that tiling that lie on it. Now we can match two prototiles that lead to a patch of a Penrose tiling (see Fig. 7).

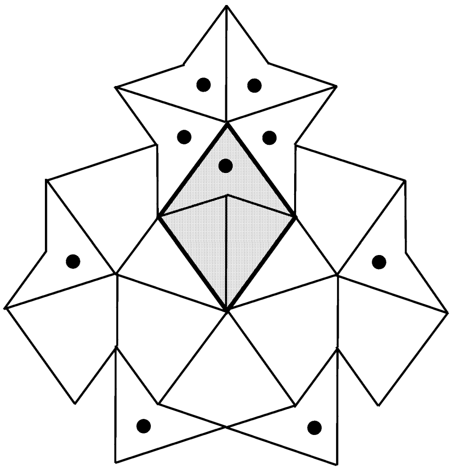

Fig. 6. An example that matching rules are needed.

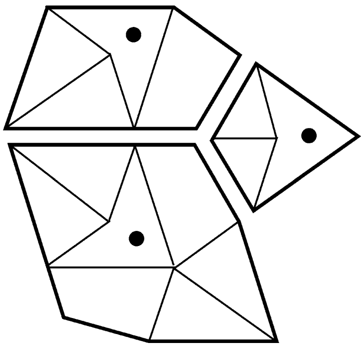

Fig. 7. How to match the prototiles.

Observe that two isometric Voronoi regions can admit more than one non-equivalent drawing. In this case we will think of it as two distinct prototiles. But, in any case, by the regular local structure of Penrose tilings, the number of prototiles so obtained is finite. For the sake of simplicity, from now on the set of prototiles obtained from regions of a Voronoi diagram of a Penrose set of sites will be called a *Voronoi–Penrose prototile set*. The compatibility of this procedure can be summarized in the following theorem.

Theorem 4. *Given a Voronoi diagram of a Penrose point set, there exists a set of matching rules for its Voronoi–Penrose prototiles such that any tiling following those rules induces a Penrose tiling. Reciprocally, if the two sets of prototiles induced by two Penrose tilings are the same, then the matching rules for those sets of prototiles are the same.*

Proof. We define the following matching rules:

(1) Whenever an unlabeled edge of a Voronoi–Penrose prototile exactly fits with an edge of the underlying tiling $\mathcal{T}$, we color its end-vertices as the underlying vertices in $\mathcal{T}$.
(2) For every prototile edge such that intersects an edge of $\mathcal{T}$, we choose a new color to label it and the rest of prototiles edges with the same type of intersection. Also, if an edge is not allowed to be adjacent to itself then we add an orientation for the edge and those of the same color.
(3) Finally, those edges which do not intersect with $\mathcal{T}$ remain unlabeled.

It is easy to see that two Voronoi regions share an edge when their portions of the underlying Penrose tiling match. So, given a tiling by the Voronoi–Penrose prototiles, a Penrose tiling is induced simply by considering these portions all together.

Reciprocally, the matching rules of a Voronoi–Penrose prototile $V(p)$ just depend on the tiles of the patch which contains the nearest neighbors of the site p. But it is well-known that every finite patch of a Penrose tiling appears in all Penrose tiling (see [3]) so, the matching rules of $V(p)$ remain invariant. □

Obviously, Theorem 4 leads to many consequences. The first one is that our Voronoi regions constitute an aperiodic set of prototiles. Namely,

Corollary 5. *Given a set of Voronoi–Penrose prototiles, there exists a set of matching rules which enforces them to be aperiodic.*

Hence, we have added a new method to the two other known so far in order to produce aperiodic tiles (see [2]).

Another important consequence of Theorem 4 is that given any set of prototiles obtained by our procedure, there exists an uncountable number of plane tilings with that set of prototiles.

5. Properties of the Voronoi diagram of Penrose point sets

In the previous section we have introduced our “machine” for generating aperiodic sets of prototiles. In this point we are going to investigate some of their properties. A first remarkable fact is that we can obtain an aperiodic set as big as we want with this method.

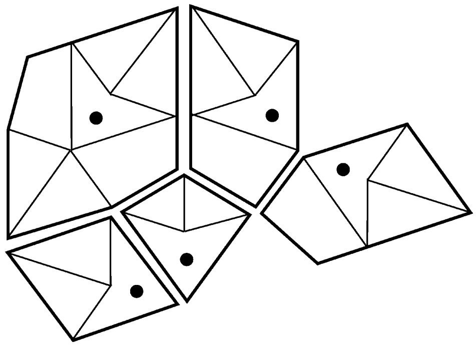

Fig. 8. A set of five Voronoi–Penrose prototiles.

Theorem 6. *For all $n \in \boldsymbol{N}$, $n \geqslant 5$, there exists a set of at least, n Voronoi–Penrose prototiles.*

Proof. In Fig. 8 we can observe that the set of prototiles has five elements. Obviously, adding suitable new points we get new regions. Thus, we only need to add enough points to get the result. □

By definition, every tiling vertex belongs to a finite number of tiles. Since there exists a finite number of different Voronoi regions of a Penrose point set, as it was stated in Lemma 2, the next result holds.

Proposition 7. *There exists a finite number of vertex neighborhoods in every Voronoi diagram of a Penrose point set.*

Given a Voronoi diagram of a Penrose point set $\mathrm{Vor}(P)$, some of its properties are consequences from those of the underlying Penrose tiling as the next result

Proposition 8. *Every patch of tiles in a Voronoi diagram of a Penrose point set is congruent with infinitely many patches in every tiling by the same prototiles.*

6. Algorithmic considerations

Now we seek an algorithm that taking as its input the points placed in Penrose's prototiles gives as its output their associate Voronoi–Penrose prototiles. The existence of that algorithm is based on the following lemma.

Lemma 9. *All Voronoi neighbors of a site p in a Penrose point set lie on the 3-patch of the dart or kite where p lies on.*

Proof. As it can be checked in Fig. 3, every dart belongs to a king, a queen or a star and a kite is always in a jack or a deuce. First suppose that p lies in a dart which is part of a king. The 3-patch of this dart contains some other sites such that p is in their convex hull, this can be checked in Fig. 4. Then, the Voronoi neighbors of p are in the convex hull or closer than some of those sites, so they are contained in the 3-patch of the dart where p lies on. More elaborate but similar arguments can be applied to the other cases.

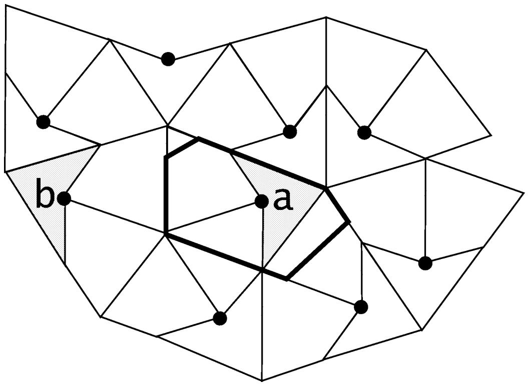

Fig. 9. b is not in the 2-patch of the tile containing a, but a and b are Voronoi neighbors.

Moreover, the above result is not true for 2-patches as Fig. 9 shows. □

As a corollary of Lemma 9 we get the algorithm we are looking for.

Theorem 10. *It is possible to construct an optimal* $O(n \log n)$ *algorithm that taking as its input n points placed in Penrose's prototiles gives as its output their associate Voronoi–Penrose prototiles* (*with their matching rules*).

Proof. It is enough to consider all the eleven 3-patches for a fixed dart and for a fixed kite and to compute the Voronoi regions that arise from sites in those two tiles. It is an immediate consequence of Lemma 9 that this procedure constructs all Voronoi–Penrose prototiles and their matching rules. □

7. Delaunay triangulations

Once we have created a machine to generate aperiodic tiles using Voronoi diagrams, the obvious subsequent step is to consider Delaunay triangulations. This new point of view is very attractive because it gives rise not only to convex regions as in the case of Voronoi regions but to triangular tiles as well.

A first difficulty that must be solved is that any Voronoi diagram of a Penrose point set has vertices with a degree greater than 3, hence its dual is not a triangulation but what is called by some authors a *pretriangulation*, with some regions that are convex k-polygons with $k \geqslant 4$. Partitioning those regions by $k-2$ non-intersecting line segments and joining the vertices we get a triangulation that is known as a *Delaunay triangulation* of the original point set.

The aperiodicity of Delaunay triangulations will be a consequence of the next result.

Lemma 11. *A Penrose point set is non-periodic.*

Proof. Several proofs of this fact can be provided. Suppose that the Penrose point set is created by placing $n > 0$ sites in each kite and $m > 0$ sites in each dart (if $n = 0$ or $m = 0$, a similar proof can be given). It is known that the ratio of kites to darts tends to τ (the golden ratio) in any sequence of patches increasing in size (see [3]). Thus if we split our Penrose point set into two subsets K and D in the obvious way, and we denote by k_l (respectively d_l) to the cardinal of the set of points in K (respectively D) whose distance to a fixed origin is at most l, we obtain that k_l/d_l tends to $\tau(n/m)$ as l tends to infinity. And this is impossible in a periodic tiling (that ratio must be rational in a periodic tiling). □

As the Penrose point set is the vertex set of its Delaunay triangulation, we get immediately the aperiodicity of this last tiling.

Theorem 12. *The Delaunay triangulation of a Penrose point set is a non-periodic tiling.*

Proof. The non-periodicity is an immediate consequence of Lemma 11. We omit here the rest of the proof that runs parallel to those of Lemma 2 and Theorem 3. □

As in the case of Voronoi regions, we can draw the original darts and kites on the Delaunay triangles in order to obtain matching rules. The prototiles so obtained will be called *Delaunay–Penrose prototiles*.

Lemma 13. *Given a Penrose triangulation of a Penrose point set, there exists a set of matching rules for its triangles such that any tiling following those rules induces a Penrose dart and kite tiling.*

Therefore, we can prove that Delaunay triangles are as well suitable aperiodic prototiles.

Corollary 14. *Given a Delaunay triangulation of a Penrose point set, there exists a set of matching rules for its triangles which enforces them to be aperiodic.*

Of course, it is possible to mimic all properties of Section 5 for the case of Delaunay triangulations, but in order to avoid tedious repetitions we omit them here.

However a version of Theorem 6 is more interesting in the case of Delaunay triangulations than for Voronoi diagrams. In order to obtain that result, we consider the *Penrose rhombs* that are another aperiodic set of prototiles (see Fig. 10). If we split each rhomb as Fig. 11 shows we obtain a set of four triangles that obviously constitute an aperiodic set of prototiles that will be called P_3'.

Theorem 15. *Any tiling by P_3' is a Delaunay triangulation.*

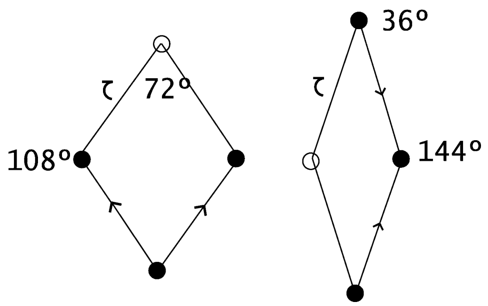

Fig. 10. An new aperiodic set of prototiles: the Penrose rhombs.

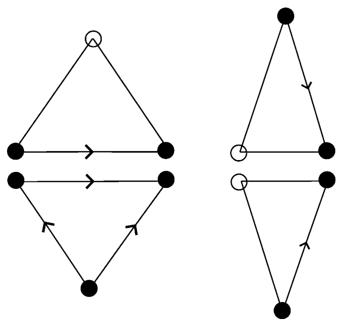

Fig. 11. Delaunay triangles which are a subdivision of Penrose rhombs.

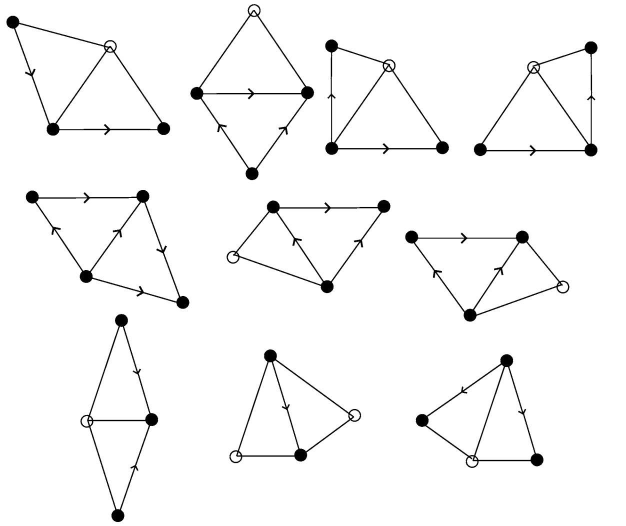

Fig. 12. If any edge is flipped then a smaller angle appears.

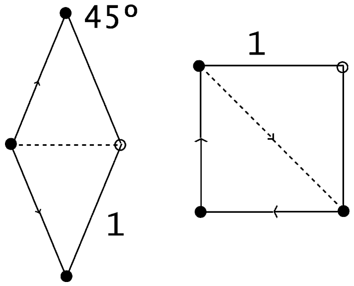

Fig. 13. Delaunay triangles which are a subdivision of Ammann rhombs.

Proof. It is well known that a triangulation of a set of points that satisfies the local max-min criterion is a Delaunay triangulation of that set (see [6]). Thus, it suffices to check that giving a tiling by P_3', if an edge is flipped then a smaller angle than the original one is created. This can easily be done since there exists only ten possible distinct edges as Fig. 12 shows. □

Note that in Theorem 15 the Penrose rhombs cannot be substituted by Penrose darts and kites. As a corollary of Theorem 15 we can obtain a version of Theorem 6 for Delaunay–Penrose prototiles.

Corollary 16. *For all $n \in \boldsymbol{N}$, $n \geqslant 4$ there exists a set of at least, n Delaunay–Penrose prototiles.*

In a similar way, a subdivision of the aperiodic set of Ammann rhombs [3] (see Fig. 13) can be obtained as a Delaunay triangulation of a Penrose point set.

8. Conclusions and open problems

We have provided a new method for generating infinite collections of aperiodic prototiles based on considering Voronoi diagrams of aperiodic infinite point sets, generated by tiling the plane with Penrose's darts and kites, where previously some sites have been placed. Some of the properties of those tilings have been considered. From an algorithmic point of view, it is possible to construct one of those collections with n prototiles in optimal time $O(n \log n)$.

In fact, we have seen in a first step that the Voronoi diagram of Penrose point sets are non-periodic and that with some appropriate matching rules, the prototiles so obtained constitute an aperiodic set of prototiles. Regarding those matching rules, it can be derived a new kind of condition. As each prototile emerges from a Voronoi region, each tile has an associate site then we can think of covering the plane with a set of Voronoi–Penrose tiles in such a way that the edges of the tiling constitute the Voronoi

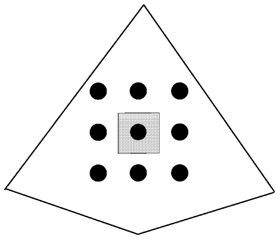

Fig. 14. To be a Voronoi diagram is not a matching rule.

diagram of the associate sites. Unfortunately, this method can lead to periodic tilings as Fig. 14 shows (one of the prototiles is a square with its site just in the center).

However, it is not clear for us what happens when only one site is selected in each dart and kite. There are many other open questions that are worth to consider, but we mention here just three of them. The first one is that we have not obtained as a Voronoi diagram any of the most-known collections of aperiodic prototiles (among the convex ones), but possibly some subdivision of those prototiles can be obtained as in Theorem 15 for Delaunay triangulations.

We have presented a Voronoi–Penrose set of prototiles of size 5, but we do not know if 5 is the minimum size of such kind of collections. Finally, it could be interesting to find an aperiodic set $\mathcal{S}$ of prototiles such that placing points on each element of $\mathcal{S}$ and following the process described in this paper the same set is obtained.

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