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Reproductive and time periodic solutions for incompressible fluids

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Resumen

In this article, our aims is to review some of the results that are currently available concerning the existence, uniqueness and regularity of reproductive and time periodic solutions of the Navier-Stokes equations and some variants. By the way, we present some open problems.

1. Navier-Stokes equations

The modern theory of the Navier-Stokes equations began in the 1930s with Leray's pioneering work ([8]).

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) a bounded and regular enough domain filled by the fluid, and $[0, T]$ the time interval. We denote $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$.

In the case where the fluid is subject to the action of a body force $\boldsymbol{f}$, the Navier-Stokes equations can be written as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \text{div } \boldsymbol{u} = 0\,, \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x, t)$ is the velocity field evaluated at the point $\boldsymbol{x} \in \Omega$ and at time $t \in [0, T]$, $p = p(\boldsymbol{x}, t)$ is the pressure field and $\nu > 0$ is the coefficient of kinematical viscosity (which is taken constant). This system can be completed with several boundary conditions. For simplicity, we fix the following non-slip boundary conditions:

$$\boldsymbol{u}(\boldsymbol{x}, t) = 0, \;\; \boldsymbol{x} \in \partial\Omega, \;\; t > 0 \tag{2}$$

Finally, supplementary conditions in time must be considered. The more classical is the initial condition:

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega \tag{3}$$

Other possibility is to change this initial condition by the following time-periodic condition:

$$\boldsymbol{u}(0,\boldsymbol{x})=\boldsymbol{u}(T,\boldsymbol{x}),\quad \boldsymbol{x}\ \in\Omega. \tag{4}$$

Mathematical properties for system (1) have been deeply investigated over the years and are still the object of profound researches.

We introduce some space functions. Let $\mathcal{V}$ the vectorial space formed by all fields $\boldsymbol{v}\in C_0^\infty(\Omega)^d$ satisfying $\nabla\cdot\boldsymbol{v}=0$. We consider the Hilbert spaces $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) as the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). Furthermore, one has

$$\boldsymbol{H}=\{\boldsymbol{u}\in\boldsymbol{L}^2;\ \nabla\cdot\boldsymbol{u}=0,\ \boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\},\qquad \boldsymbol{V}=\{\boldsymbol{u}\in\boldsymbol{H}^1;\ \nabla\cdot\boldsymbol{u}=0,\ \boldsymbol{u}=0 \text{ on } \partial\Omega\} \tag{5}$$

We denote $L_0^2(\Omega)=\left\{p\in L^2(\Omega):\displaystyle\int_\Omega p\,dx=0\right\}$.

1.1. Main classical results for the initial-boundary problem

Definition 1 *Given $\boldsymbol{u}_0\in H$ and $\boldsymbol{f}\in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, it will said that $\boldsymbol{u}$ is a weak solution of the problem (1),(2), (3) in $(0,T)$, if $\boldsymbol{u}\in L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{H})$, and verifies (3) and the variational formulation*

$$\int_0^T\int_\Omega\Big\{-\boldsymbol{u}(t)\boldsymbol{v}'(t)+\nabla\boldsymbol{u}(t):\nabla\boldsymbol{u}(t)-(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{v}(t)\boldsymbol{u}(t)-\boldsymbol{f}(t)\boldsymbol{v}(t)\Big\}dxdt=0,$$

for all $\boldsymbol{v}\in C^1([0,T];\boldsymbol{H})\cap C([0,T];\boldsymbol{V})$, with compact support contained in $(0,T)$.

In addition, if $u_0\in\boldsymbol{V}$ and $f\in L^2(0,T;\boldsymbol{L}^2(\Omega))$ any weak solution will be a strong solution if

$$\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^2\cap\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{V}),\qquad \boldsymbol{u}_t\in L^2(0,T;\boldsymbol{H}),\qquad p\in L^2(0,T;H^1\cap L_0^2(\Omega))$$

and verifies the system (1) pointwise a.e. in $(0,T)\times\Omega$.

Remark: The previous definition can be extend to the case of final time $T=\infty$ changing the regularity $L^2(0,T)$ by $L^2_{\rm loc}(0,+\infty)$.

The following results are well known, see for instance [16].

Theorem 1.1 *For any $u_0\in\boldsymbol{H}$ and $\boldsymbol{f}\in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, the problem (1)-(2) has (at least) a weak solution. If $\Omega\subset\mathbb{R}^2$, one has uniqueness of weak solutions.*

Theorem 1.2 *For any $u_0\in\boldsymbol{V}$ and $\boldsymbol{f}\in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$, the problem (1)-(2) has a unique strong solution $(\boldsymbol{u},p)$ local in time, defined in $(0,T^\star)$ with $T^\star>0$ small enough. In fact, if a solution has the strong regularity, it coincides with any weak solution associated with the same data (this property is called weak/strong uniqueness). Moreover, this strong solution is global in time, defined in the whole time interval $(0,\infty)$ if either $\Omega\subset\mathbb{R}^2$ or $\Omega\subset\mathbb{R}^3$ and data $(\boldsymbol{u}_0,\boldsymbol{f})$ are small enough in their respective spaces $\boldsymbol{V}\times L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$.*

1.2. On the time-periodic weak solutions

Theorem 1.3 *[7] For any $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, there exists a weak solution of (1)-(2) and (4), (i.e. $\boldsymbol{u}$ has the so-called reproductive property: $\boldsymbol{u}(0,x) = \boldsymbol{u}(T,x)$).*

Notice that the time periodic extension, $\widetilde{\boldsymbol{u}}$, of any weak reproductive solution $\boldsymbol{u}$ to the whole time interval $(0,+\infty)$ is a periodic weak solution of (1)-(2) corresponding to the data, $\widetilde{\boldsymbol{f}}$, defined as the time periodic extension of $\boldsymbol{f}$.

Main ideas of the proof of Theorem 1.3

Let $\boldsymbol{u}^k$ the unique approximate solution of the Galerkin initial-boundary problem of Navier-Stokes in the finite-dimensional subspace $\boldsymbol{V}^k$, spanned by the first k elements of the "spectral" basis of $\boldsymbol{V}$ (orthogonal in $\boldsymbol{V}$ and orthonormal in $\boldsymbol{H}$), associated to a initial discrete data $\boldsymbol{u}_0^k \in \boldsymbol{V}^k$.

Since $\boldsymbol{V} \hookrightarrow \boldsymbol{H}$, there exists a Poincare constant $c_1 > 0$ such that $c_1\|\boldsymbol{u}^k\|_{L^2}^2 \le \|\nabla \boldsymbol{u}^k\|_{L^2}^2$, thus, from energy inequality, we have

$$\frac{d}{dt}\|\boldsymbol{u}^k\|_{L^2}^2 + c_1\|\boldsymbol{u}^k\|_{L^2}^2 \le \varepsilon\, \|\boldsymbol{f}\|_{H^{-1}}^2.$$

Therefore, multiplying by $e^{c_1 t}$ and integrating from 0 to T, we have

$$e^{c_1 T}\|\boldsymbol{u}^k(T)\|_{L^2}^2 \le \|\boldsymbol{u}^k(0)\|_{L^2}^2 + \int_0^T e^{c_1 t}\varepsilon\, \|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt. \tag{6}$$

Now, we define the operator $L^k : [0,T] \to \mathbb{R}^k$ as $L^k(t) = (c_1^k(t), ..., c_k^k(t))$, where $c_i^k(t),\ i = 1,\ldots,k$, are the coefficients of the expansion of $\boldsymbol{u}^k(t)$ in V^k.

Since we have choose the (orthonormal in L^2) spectral basis, $\|L^k(t)\|_{\mathbb{R}^k} = \|\boldsymbol{u}^k\|_{L^2}$.

We define the operator $\Phi^k : \mathbb{R}^k \to \mathbb{R}^k$ as follows: Given $L_0^k \in \mathbb{R}^k$, we define $\Phi^k(L_0^k) = L^k(T)$, where $L^k(t)$ are the coefficients of the Galerkin solution with initial value with coefficients L_0^k. It is easy to see that Φ^k is continuous and we want to prove that Φ^k has a fixed point.

For this, thanks to the Leray-Scahuder Theorem, it suffices to show that for all $\lambda \in [0,1]$, the possible solutions of the equation

$$L_0^k(\lambda) = \lambda \Phi^k(L_0^k(\lambda)), \tag{7}$$

are bounded independently of λ.

Since $L_0^k(0) = 0$, it suffices to consider $\lambda \in (0,1]$. In this case, (7) is equivalent to $\Phi^k(L_0^k(\lambda)) = \frac{1}{\lambda}L_0^k(\lambda)$. Moreover, by the definition of Φ^k and (6), one obtains

$$e^{c_1 T}\|\frac{1}{\lambda}L_0^k(\lambda)\|_{\mathbb{R}^k}^2 \le \|L_0^k(\lambda)\|_{\mathbb{R}^k}^2 + 2c\int_0^T e^{c_1 T}\|f(t)\|_{H^{-1}}^2 dt,$$

which implies

$$\|L_0^k(\lambda)\|_{\mathbb{R}^k}^2 \le \frac{2c\int_0^T e^{c_1 T}\|f(t)\|_{H^{-1}}^2 dt}{e^{c_1 T} - 1} = M,$$

for each $\lambda \in (0,1]$. This bound is independent of $\lambda \in [0,1]$ and k. Consequently, Leray-Shauder Theorem implies the existence of at least one fixed point of Φ^k, that is the existence of reproductive Galerkin solution.

Thus, since previous estimates are independent of k, one has the same estimates for these reproductive Galerkin solutions.

Finally, the convergence of a subsequence to a reproductive solution of (1),(2), (4) hold.

1.3. Relation between weak periodic solutions and global solutions

Assume $\boldsymbol{f}\colon [0,+\infty) \to \boldsymbol{H}^{-1}(\Omega)$ and T-time periodic.

Navier-Stokes $2D$

One has (see Theorem 1.1) uniqueness of weak solution for the initial-boundary problem (associated to any initial data $\boldsymbol{u}_0$). Consequently, given a reproductive solution $\boldsymbol{u}$ associated to $\boldsymbol{u}(0) = \boldsymbol{u}(T) := \boldsymbol{u}_0$, then $\boldsymbol{u}$ is the (unique) solution of the initial-boundary problem associated to the initial data $\boldsymbol{u}_0$, which is defined for all time $t \in (0,\infty)$. Moreover, this solution is T-periodic, because in $(T,2T)$ must be equal to the reproductive solution defined as $\bar{\boldsymbol{u}}(t) = \boldsymbol{u}(t-T)$ (which verifies $\boldsymbol{u}(T) = \boldsymbol{u}(2T) = \boldsymbol{u}_0$) and so on.

Finally, using regularity of solution $\boldsymbol{u}$ for strictly positive times (see [4]), it is easy to prove that every periodic solution is regular.

Navier-Stokes $3D$

Since uniqueness of weak solution is not known, it is possible that the reproductive solution $\boldsymbol{u}$ and the global weak solution $\tilde{\boldsymbol{u}}$ associated to the initial data $\boldsymbol{u}_0 := \boldsymbol{u}(0) = \boldsymbol{u}(T)$ are different in $(0,T)$, although they coincide locally in time, near of the initial time $t = 0$.

1.4. Open problems

Navier-Stokes with large Reynolds number and a reaction term adding energy

Previous arguments of the proof of reproductive solutions are based on (exponential) decreasing of energy (thanks to dissipative terms). Naturally, the same argument, is applicable to models with energy strictly decreasing in finite time. But this is not always possible. For instance, we consider the following Navier-Stokes system with large Reynolds number and a reaction term adding energy:

$$\begin{aligned} &\partial_t \boldsymbol{u} - \varepsilon\Delta\boldsymbol{u} - \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = 0, \\ &\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \boldsymbol{u}_{|\Sigma} = 0. \end{aligned} \tag{8}$$

The energy inequality is

$$\partial_t \|\boldsymbol{u}\|^2_{L^2} + \varepsilon\|\nabla\boldsymbol{u}\|^2_{L^2} \le C(\|\boldsymbol{f}\|^2_{H^{-1}} + \|\boldsymbol{u}\|^2_{L^2}).$$

Assuming ε small enough such that $\|\boldsymbol{u}\|^2_{L^2} \not< \varepsilon\|\nabla\boldsymbol{u}\|^2_{L^2}$, the strictly decreasing in time of $\|\boldsymbol{u}\|^2_{L^2}$ is not clear. Consequently, the existence of time-periodic weak solutions of (8) remains as an open problem.

Exterior domains

Assume Ω is an exterior domain where the Poincaré inequality is not true. Then, to show the existence of reproductive solutions one could use the "embedding domain technique" together with the Galerkin Method, obtaining reproductive solutions in a sequence of (bounded) truncated domains, see for instance [5, 14, 13]. However, since Poincaré imbedding is not applicable, it is not clear the control to the pass to the limit from truncated domains to the whole domain.

2. Reproductivity and maximum principle

Given $\boldsymbol{u} : Q \to \mathbb{R}^3$ such that $\nabla \cdot \boldsymbol{u} = 0$ in Q and $\boldsymbol{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we consider the (reproductive) diffusion-advection problem for the unknown $c : Q \to \mathbb{R}$ (a concentration):

$$\partial_t c - \Delta c + \boldsymbol{u} \cdot \nabla c = 0, \quad c_{|\Sigma} = c_\Sigma, \quad c(0) = c(T),$$

where $0 < \underline{c} \le c_\Sigma \le \overline{c}$ on Σ, for some constants $\underline{c}$ and $\overline{c}$. In particular,

$$\partial_t (c - \overline{c}) - \Delta(c - \overline{c}) + (\boldsymbol{u} \cdot \nabla)(c - \overline{c}) = 0 \quad \text{in } Q.$$

Multiplying by $(c - \overline{c})_+$ and integrating in Ω (notice that $(c - \overline{c})_+ = 0$ on Σ), one has

$$\frac{d}{dt} \int_\Omega \|(c - \overline{c})_+\|_{L^2}^2 + \int_\Omega \|\nabla (c - \overline{c})_+\|_{L^2}^2 \le 0.$$

Integrating in $t \in (0, T)$ and using the periodic condition $c(0) = c(T)$, one arrives at

$$\int_0^T \|\nabla (c - \overline{c})_+\|_{L^2}^2 = 0.$$

Hence $c \le \overline{c}$ in Q hold. Similarly $c \ge \underline{c}$ in Q hold.

Therefore, one has the following conclusion: The reproductive solution conserve the maximum principle.

In the following models, the maximum principle has an important role.

2.1. Generalized Boussinesq system, with diffusion depending on temperature

When the viscosity and heat conductivity are temperature dependent funtions in the Boussinesq system, one has the following system:

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta) \nabla \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \alpha \theta \mathbf{g} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \theta - \nabla \cdot (k(\theta) \nabla \theta) + (\boldsymbol{u} \cdot \nabla)\theta = 0, \end{cases} \tag{9}$$

where $\nu : \mathbb{R} \to \mathbb{R}^+$ and $k : \mathbb{R} \to \mathbb{R}^+$ are continuous functions (the kinematic viscosity and the thermal conductivity respectively).

The problem is to find a regular solution $\{\boldsymbol{u},\theta,p\}$ of (9) in $\Omega\times[0,T]$, together the following boundary Dirichlet data and time-periodic conditions:

$$\boldsymbol{u}=0,\quad \theta=0 \quad \text{on } [0,T)\times\partial\Omega, \qquad \text{and} \qquad \boldsymbol{u}(0)=\boldsymbol{u}(T),\quad \theta(0)=\theta(T) \quad \text{in } \Omega. \tag{10}$$

Thanks to the maximum principle, one has $\theta_{\text{mín}}\le\theta\le\theta_{\text{máx}}$ in Q. Then, there exists $\nu_{\text{mín}}>0$, $k_{\text{mín}}>0$, $\nu_{\text{máx}}>0$ and $k_{\text{máx}}>0$ such that

$$\nu_{\text{mín}}\le\nu(s)\le\nu_{\text{máx}} \quad \text{and} \quad k_{\text{mín}}\le k(s)\le k_{\text{máx}}, \quad \forall\, s\in[\theta_{\text{mín}},\theta_{\text{máx}}].$$

One can proves the existence of reproductive solution in the same way that in the Navier Stokes case, considering the equivalent problem that result changing ν by $\widetilde{\nu}$ and k by $\widetilde{k}$, where

$$\widetilde{\nu}(\theta)=\begin{cases}\nu(\theta_{\text{mín}}) & \text{if} \quad \theta<\theta_{\text{mín}}\\ \nu(\theta) & \text{if} \quad \theta_{\text{mín}}\le\theta\le\theta_{\text{máx}}\\ \nu(\theta_{\text{máx}}) & \text{if} \quad \theta>\theta_{\text{máx}}\end{cases} \qquad \widetilde{k}(\theta)=\begin{cases}k(\theta_{\text{mín}}) & \text{if} \quad \theta<\theta_{\text{mín}}\\ k(\theta) & \text{if} \quad \theta_{\text{mín}}\le\theta\le\theta_{\text{máx}}\\ k(\theta_{\text{máx}}) & \text{if} \quad \theta>\theta_{\text{máx}}\end{cases}$$

Indeed, it suffices to consider a Galerkin approximation for both variables, velocity and temperature, and to follow proof of theorem 2.3, changing u^k by (u^k,θ^k).

2.2. Penalized Nematic liquid crystal model

We assume the following nematic liquid crystal model in $(0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^d$ for $d=2$ or 3 is an open bounded domain:

$$\begin{cases}\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\mu\Delta\boldsymbol{u}+\nabla p=-\lambda\nabla\cdot(\nabla\boldsymbol{d}^t\nabla\boldsymbol{d}), \quad \nabla\cdot\boldsymbol{u}=0,\\ \partial_t\boldsymbol{d}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}=\gamma(\Delta\boldsymbol{d}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})).\end{cases} \tag{11}$$

The positive constants ν, λ and γ, are the fluid viscosity, the elasticity constant and the relaxation time, respectively.

In this penalized model, the constraint $|\boldsymbol{d}|=1$ (where $|\cdot|$ is the punctual euclidean norm) is partially conserved to $|\boldsymbol{d}|\le1$ as consequence of the maximum principle for the Ginzburg-Landau equation considering the penalization function $\boldsymbol{f}_\varepsilon(\boldsymbol{d})=\varepsilon^{-2}(|\boldsymbol{d}|^2-1)\boldsymbol{d}$ where $\varepsilon>0$ is the penalization parameter.

The problem (11) is completed with the (Dirichlet) boundary conditions and the time-periodic conditions:

$$\boldsymbol{u}=0,\quad \boldsymbol{d}=\boldsymbol{h} \quad \text{on } \partial\Omega\times(0,T) \qquad \text{and} \qquad \boldsymbol{u}(0)=\boldsymbol{u}(T),\quad \boldsymbol{d}(0)=\boldsymbol{d}(T) \quad \text{in } \Omega. \tag{12}$$

In order to obtain the maximum principle for $|\boldsymbol{d}|^2$, we multiply the $\boldsymbol{d}$-system by $\boldsymbol{d}$ getting

$$\frac{1}{2}\partial_t|\boldsymbol{d}|^2+\frac{1}{2}\boldsymbol{u}\cdot\nabla|\boldsymbol{d}|^2-\gamma\Delta|\boldsymbol{d}|^2+\gamma|\nabla\boldsymbol{d}|^2+\gamma\boldsymbol{f}_\varepsilon(\boldsymbol{d})\cdot\boldsymbol{d}=0,$$

whence the following differential inequality holds for $c=|\boldsymbol{d}|^2$:

$$\partial_t c+\boldsymbol{u}\cdot\nabla c-2\gamma\Delta c+2\gamma\frac{1}{\epsilon^2}(c-1)c\le0.$$

Notice that, if $c = |\boldsymbol{d}|^2 \geq 1$ then $\frac{1}{\varepsilon^2}(c-1)c = \boldsymbol{f}_\varepsilon(\boldsymbol{d})\cdot\boldsymbol{d} \geq 0$. Therefore, assuming $|\boldsymbol{h}| \leq 1$, we can apply the maximum principle argument obtaining $|\boldsymbol{d}| \leq 1$.

This maximum principle is fundamental in order to obtain solution of the (11)-(12) problem because we can consider a equivalent problem changing $\boldsymbol{f}_\varepsilon$ by $\widetilde{\boldsymbol{f}}_\varepsilon$, the auxiliary function

$$\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) = \begin{cases} \boldsymbol{f}_\varepsilon(\boldsymbol{d}) & \text{if} \quad |\boldsymbol{d}| \leq 1, \\ 0 & \text{if} \quad |\boldsymbol{d}| > 1. \end{cases}$$

Indeed, if $(\boldsymbol{u}, p, \boldsymbol{d})$ is a solution of (11)-(12) with $\widetilde{\boldsymbol{f}}_\varepsilon$, in particular $|\boldsymbol{d}| \leq 1$ (because the maximum principle is also verified, since $\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d})\cdot\boldsymbol{d} \geq 0$ as $|\boldsymbol{d}| > 1$), then $(\boldsymbol{u}, p, \boldsymbol{d})$ is also a solution of (11)-(12) with $\boldsymbol{f}_\varepsilon$. The inverse statement is easy to verify.

Now, the key is that $|\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d})| \leq \frac{1}{\varepsilon^2}$ $\forall \boldsymbol{d} \in \mathbb{R}^3$. Then, existence of weak reproductive solution of this model can be proved (see [1]). The main steps of the proof are to prove existence and uniqueness of solution for a Galerkin initial-boundary problem, to obtain the reproductivity of approximate solution with the argument of Theorem 1.3 and to pass to the limit.

Remark: An interesting open problem in this context is the asymptotic behavior as $\varepsilon \to 0$ of the reproductive solutions of this liquid crystal model (11)-(12). For the initial-boundary problem, this asymptotic behavior is studied in [3]

3. Regularity of periodic solutions via regularity of reproductive solutions

We consider the time-periodic boundary problem associated to 3D Navier Stokes model with data $\boldsymbol{f}$.

Let $\boldsymbol{u}$ be a reproductive solution in $[0, T]$ (given in Theorem 1.3). The problem is to obtain regularity for this solution. A possible argument is to prove that there exists at least one time $t_\star \in [0, T]$ such that $\|\boldsymbol{u}(t_\star)\|_{H^1}$ is small enough. In fact, we can find that $t_\star$ exists, integrating in $(0, T)$ the energy inequality

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|^2_{L^2} + \nu\|\nabla\boldsymbol{u}(t_\star)\|^2_{L^2} \leq \frac{1}{\nu}\|\boldsymbol{f}\|^2_{H^{-1}}$$

and applying the reproductive condition $u(0) = u(T)$, arriving at

$$\nu\int_0^T \|\nabla\boldsymbol{u}(t)\|^2_{L^2} \leq \frac{1}{\nu}\int_0^T ||\boldsymbol{f}(t)||^2_{H^{-1}}.$$

Assuming external forces $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm, hence $\displaystyle\int_0^T \|\nabla\boldsymbol{u}(t)\|^2_{L^2} \leq \varepsilon T$. From integral mean value theorem, there exists $t_\star \in [0, T]$ such that $\|\nabla\boldsymbol{u}(t_\star)\|^2_{L^2} \leq \varepsilon$.

On the other hand, let $\overline{\boldsymbol{u}}$ be the unique regular strong solution (see Theorem 1.2) with initial data $\boldsymbol{u}(t_\star)$ and the same force $\boldsymbol{f}$. Moreover, following the proof of this type of global in time results with small data (see for instance ([16]), one has $\|\nabla\bar{\boldsymbol{u}}(t)\|^2_{L^2} \leq 2\,\varepsilon$ for each $t \geq t_\star$ (here $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm is necessary).

By uniqueness of weak-strong solution (Theorem 1.2), one has $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}$ in $[t_\star, T]$ and therefore $\boldsymbol{u}$ is regular in $[t_\star, T]$. In particular, $\|\nabla \boldsymbol{u}(T)\|^2_{L^2} = \|\nabla \overline{\boldsymbol{u}}(T)\|^2_{L^2} \leq 2\,\varepsilon$. Therefore $\|\nabla \boldsymbol{u}(0)\|^2_{L^2} \leq 2\,\varepsilon$, hence $\boldsymbol{u}$ is a strong solution in $[0,T]$. Finally, in $[T, 2\,T]$, $\boldsymbol{u}(t-T) \equiv \overline{\boldsymbol{u}}(t)$ and so on. The precedent argument is used, for instance in [11].

Therefore, we arrive at the following conclusion: The periodic extension of a reproductive solution $\boldsymbol{u}$ is a regular solution in $[0, +\infty)$ for small enough external forces $\boldsymbol{f}$.

Previous argument is based on to obtain $\displaystyle\int_0^T \|\boldsymbol{u}(t)\|^2_{H^1} dt$ small enough, only assuming force $\boldsymbol{f}$ small enough (in particular, $\|\boldsymbol{u}(t_\star)\|^2_{H^1}$ is small for some $t_\star \in [0,T]$). But, there are some fluids models, where this is not always possible to obtain.

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