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Geometric tree graphs of points in convex position ☆

M.C. Hernando[a,*], F. Hurtado[b], A. Márquez[c], M. Mora[b], M. Noy[b]

[a] U. Politècnica de Catalunya, Dep. Matemàtica Aplicada I., Diagonal 647, 08028 Barcelona, Spain
[b] U. Politècnica de Catalunya, Dep. Matemàtica Aplicada II, Pau Gargallo 5, 08028 Barcelona, Spain
[c] U. de Sevilla, Dep. Matemática Aplicada I, Av. Reina Mercedes s/n, 41012 Sevilla, Spain

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Abstract

Given a set P of points in the plane, the geometric tree graph of P is defined as the graph $T(P)$ whose vertices are non-crossing spanning with straight edges trees of P, and where two trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges e and f. In this paper we concentrate on the geometric tree graph of a set of n points in convex position, denoted by G_n. We prove several results about G_n, among them the existence of Hamiltonian cycles and the fact that they have maximum connectivity. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

Given a connected graph G, the tree graph $T(G)$ is defined as the graph having as vertices the spanning trees of G, and edges joining two trees T_1, T_2 whenever $T_2 = T_1 - e + f$ for some edges e and f of G.

Tree graphs were introduced by Cummings [2] in connection with the study of electrical networks, showing that tree graphs are Hamiltonian. A simpler proof of the same fact was found later by Holzmann and Harary [6], and generalized to the base graph of a matroid. Liu [8] related the connectivity of $T(G)$ to the cyclomatic number of G. Later Liu showed that tree graphs have maximum connectivity, that is, connectivity equal to the minimum degree [9]. Additional results on tree graphs have been obtained recently [4].

Here we consider a geometric version of the problem. Given a set P of points in the plane, let $\mathscr{T}(P)$ be the set of non-crossing spanning trees of P (edges are straight line segments and do not cross). We define the *geometric tree graph* $T(P)$ as the graph having $\mathscr{T}(P)$ as vertex set and the same adjacencies as in combinatorial tree graph, that is, two non-crossing spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$. Geometric tree graphs have appeared previously in the work of Avis and Fukuda [1]

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* Corresponding author. E-mail: hernando@mal.upc.es.

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as a tool for enumerating spanning trees. They show that $T(P)$ is connected for any point set P in general position and has diameter bounded by $2n-4$ if $n=|P|$.

In this paper we concentrate on the combinatorial properties of the graphs $T(P)$ in the case where P is a point set in convex position. For any $n\geqslant 2$, we denote G_n the geometric tree graph of a set of n points in convex position.

In this paper we obtain a number of new results about the graphs G_n. In Section 2 we give definitions and preliminary results as the minimum and the maximum degree of G_n. In Sections 3 and 4 we determine the radius, and the group of automorphisms of G_n. We also show that the diameter of G_n is at least $3n/2-5$. In Section 6 we present a *tree of geometric trees*, a recursive construction of the graphs G_n in which a tree T in G_n gives rise to $\binom{d+2}{2}$ different trees in G_{n+1}, where d is the degree of the n-th vertex in T. This tool is then used to produce inductive proofs of two main results: G_n is a Hamiltonian graph for every $n\geqslant 3$, and G_n has connectivity equal to the minimum degree $2n-4$. We remark that this kind of construction has proved useful in solving similar problems for graphs of triangulations instead of tree graphs [7].

To determine the exact value of the diameter is the main open problem left in this paper. In the case of combinatorial tree graphs, the diameter is obviously bounded by $n-1$, because spanning trees satisfy the exchange property of the set of basis of a matroid. But this ceases to be true in the geometric case.

2. Definitions and preliminaries

2.1. Geometric tree graphs

Let $P=\{1,\ldots,n\}$ be a set of points in the plane, no three of them collinear. A *non-crossing spanning tree* for P is a spanning tree of P with edges given by straight line segments that do not cross. Let $\mathscr{T}(P)$ be the set of non-crossing spanning trees of P. The *geometric tree graph* $T(P)$ of the set of P has a vertex for every element of $\mathscr{T}(P)$ and two trees $T_1, T_2 \in \mathscr{T}(P)$ are *adjacent*, and we write $T_1 \sim T_2$, when there are edges $e\in T_1\backslash T_2$ and $f\in T_2\backslash T_1$ such that $T_2=T_1+f-e$.

An example is shown in Fig. 1

2.2. The graph G_n

Since any two sets of points, both in convex position, are equivalent with respect to their non-crossing spanning trees, all sets of n points in convex position have the same geometric tree graph, denoted simply by G_n (Fig. 2). So we are free to work with the set of P_n of vertices of a regular polygon. We assume, without loss of generality, that its vertices are labelled by integers 1 to n, sorted counterclockwise, and that 1 is the vertex with minimum x-coordinate. The arithmetic of the indices is done modulo n.

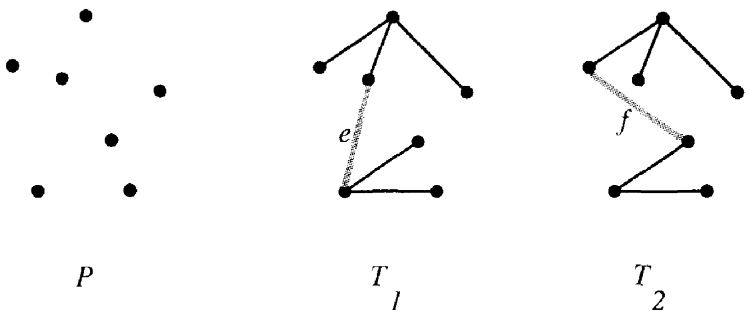

Fig. 1. Two trees adjacent in $\mathscr{T}(P)$.

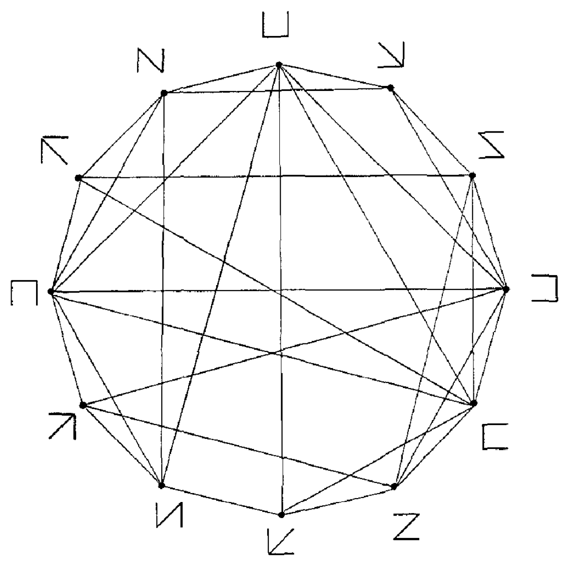

Fig. 2. The graph G_4.

Let us denote by $\mathscr{T}_n$ the set of all non-crossing spanning trees of P_n, that is, the vertex set of the graph G_n.

We summarize next what is known about the graphs G_n.

(i) G_n is connected and has diameter bounded above by $2n-4$ [1].

(ii) The number of vertices of G_n is $t_n = 1/(2n-1)\binom{3n-3}{n-1}$ [3,10], and every geometric tree graph of a set of n points has at least this number of vertices [5].

(iii) The chromatic number of G_n is in $\Theta(n^2)$ [4].

We finally remark a very useful property that will be used in the following sections.

Remark 2.1. Any tree $T \in \mathscr{T}_n$, $n \geqslant 3$, has at least two edges on the boundary of P_n, that is, two edges of the type $(i, i+1)$, and such that either the vertex i or the vertex $i+1$ is a leaf of T.

2.3. Maximum and minimum degree

The degrees of the vertices of G_n can be quite different. There are vertices with degree $\Theta(n)$ and vertices with degree $\Theta(n^3)$, as shown below.

There are some trees with a specially simple structure called *stars*. The star S_i is obtained by joining the vertex i to all the other vertices. Note that for $n = 2, 3$ all trees are stars. In order to obtain an adjacent tree of $\mathscr{T}_n$ from a star S_i we can only add an edge of the boundary of P_n that is not in S_i. There are $n-2$ edges of this kind. If $(k, k+1)$ is one of these edges, when it is added we must remove either the edge (i, k) or the edge $(i, k+1)$ of the cycle that appears in $S_i \cup (k, k+1)$. Then we conclude that the degree of a star in G_n is $2(n-2)$. Let $d_G(i)$ denote the degree of a vertex i in a graph G and $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree, respectively.

Proposition 2.2. $\delta(G_n) = 2n - 4$ *and only the stars have this degree.*

Other special trees are the *chains*. The chain C_i is obtained by taking all the edges in the boundary of P_n, except $(i, i+1)$.

Proposition 2.3. $\Delta(G_n) = \binom{n+1}{3} - n + 1$ *and only the chains have this degree.*

The proof of both results is by induction on n, removing a vertex of degree one incident to a boundary edge and considering the different cases that arise.

3. Center, radius and diameter

In this section we continue the study of properties of the graph G_n. We will denote by $d(T, T')$ the distance in G_n between two trees T and T' of $\mathscr{T}_n$, that is, the minimum number of edges we have to change from one of these trees in order to obtain the other one, so that at each exchange the resulting tree is non-crossing. The eccentricity $e(T)$ of $T \in \mathscr{T}_n$, is defined as the maximum distance between T and any other tree in $\mathscr{T}_n$. The radius of the graph G_n is the minimum of the eccentricities of the vertices of G_n, and the center of G_n is the set of all vertices that have eccentricity equal to the radius.

Remark 3.1. Let $T \in \mathscr{T}_n$ and let d_i be the degree of i in T, for $1 \leqslant i \leqslant n$. Then $d(T, S_i) = n - 1 - d_i$ (see [1]).

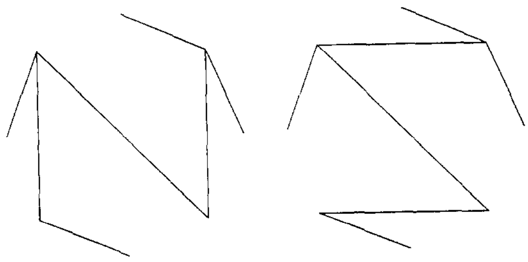

Fig. 3. $ch(T) = ch(T'), d_T(i) = d_{T'}(i) \forall i$.

Remark 3.2. Let $T \in \mathcal{T}_n$ and let $ch(T)$ be the number of edges of T in the boundary of P_n. Then

$$d(T, C_i) = n - ch(T) \quad \text{if } (i, i+1) \in T,$$
$$d(T, C_i) = n - 1 - ch(T) \quad \text{if } (i, i+1) \notin T.$$

The following result shows that the stars and the chains play a special role in the graph.

Theorem 3.3. *The radius of G_n is equal to $n-2$, and the center contains of the n stars $S_1, \ldots, S_n$ and the n chains $C_1, \ldots, C_n$.*

Proof. From Remark 3.1 it is obvious that the eccentricity of a star is equal to $n-2$, because any tree has at least one edge in common with any star. Since all trees $T \in \mathcal{T}_n$ have two edges on the boundary of P_n, the eccentricity of a chain is also $n-2$. It remains to show that if a tree T is neither a star nor a chain, then $e(T) \geqslant n-2$. It is sufficient to show that, for any of these trees T there is another T' that shares at most one edge with T, because then it is clear that $d(T, T') \geqslant n-2$. One removes a vertex of degree 1 incident to a boundary edge and the existence of T' is easily proved by induction. □

Remark 3.4. From Remarks 3.1 and 3.2 the distances from a tree to the stars and the chains are easily computed. One could think that these distances determine the tree. This is not so, moreover, as shown in Fig. 3, one can find two different trees with the same degree sequences and even the same edges on the boundary.

After having established the value of the radius, it is natural to ask about the diameter, that is, the maximum of the eccentricities of the vertices of G_n. An obvious upper bound for the diameter is twice the radius, i.e. $(2n-2)$, and a trivial lower bound is $n-1$. We give here a more precise lower bound.

Let n be even, and let T_1 and T_2 be the following trees (see Fig. 4):

$$T_1 = \left\{(1,k) | 2 \leqslant k \leqslant \frac{n}{2}+1\right\} \cup \left\{\left(\frac{n}{2}+1, k\right) | \frac{n}{2}+2 \leqslant k \leqslant n\right\},$$
$$T_2 = \left\{\left(\frac{n}{2}+2, k\right) | 3 \leqslant k \leqslant \frac{n}{2}+1\right\} \cup \left\{(2,k) | \frac{n}{2}+2 \leqslant k \leqslant 1 \,(\mathrm{mod}\ n)\right\}.$$

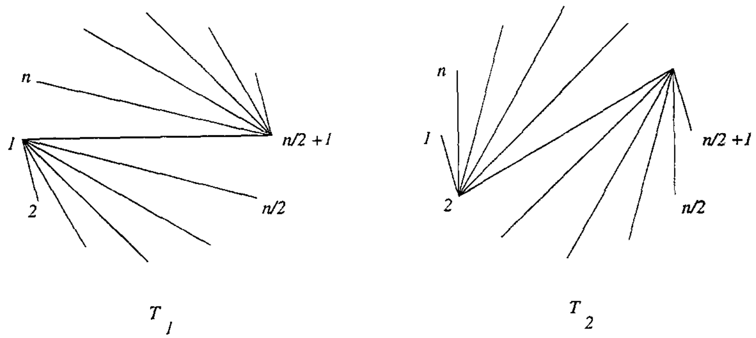

Fig. 4. Two trees with $d(T_1, T_2) = 3n/2 - 5$.

Theorem 3.5. *The diameter of G_n is at least $\lfloor 3n/2 \rfloor - 5$.*

Proof. We prove here the case when n is even. The case when n is odd is handled with a slight modification of the trees T_1 and T_2 defined above.

The edge $e = (2, n) \in T_2$ in an edge of T_2 that has the minimum number of crossings with the edges of T_1, and e intersects $n/2 - 1$ edges of T_1. We need at least $n/2 - 1$ changes before we can add the first edge of T_2. In the best case, we need a change for introducing each of the remaining edges of T_2 that do not appear in T_1. As T_1 and T_2 have in common two edges, we obtain

$$d(T_1, T_2) \geqslant \left(\frac{n}{2} - 1\right) + (n - 4) = \frac{3n}{2} - 5.$$

Finally, it is easy to find a path between T_1 and T_2 that has exactly this length. □

Taking into account this result, in order to maximize the eccentricity, we observe that after Remarks 3.1 and 3.2 we can conclude the following. If $T \in \mathscr{T}_n$ and $ch(T) \geqslant n/2$ or there is some vertex i such that $d_T(i) \geqslant n/2$, then $e(T) \leqslant 3n/2 - 2$. Then, it is natural to consider those trees that have few edges in the boundary of P_n and vertices with low degree. But we have obtained that if $T \in \mathscr{T}_n$ is such that $ch(T) = 2$, then $e(T) \leqslant 3n/2 - 2$.

4. Group of automorphisms

Let us denote by $\Gamma(G_n)$ the automorphism group of G_n. It is clear that any symmetry of the regular n-polygon will induce a corresponding automorphism on G_n. No more automorphisms are possible, as proved next.

Theorem 4.1. *The automorphism group $\Gamma(G_n)$ is isomorphic to the dihedral group D_n of the symmetries of a regular polygon with n sides.*

Proof. Let us consider the set of starts and chains

$$\mathscr{X} = \{S_1, \ldots, S_n, C_1, \ldots, C_n\}.$$

Moreover, it is straightforward to see that

$$d(S_i,S_j)=n-2,$$
$$d(C_i,C_j)=1,$$
$$d(C_i,S_j)=n-2 \quad \text{if } j=i \text{ or } j=i+1,$$
$$d(C_i,S_j)=n-3 \quad \text{otherwise.}$$

Now let γ be in $\Gamma(G_n)$. Since the S_i are the vertices of minimum degree and the C_i are those of maximum degree, we see that

$$\gamma(\{S_1,\dots,S_n\})=\{S_1,\dots,S_n\},$$
$$\gamma(\{C_1,\dots,C_n\})=\{C_1,\dots,C_n\}.$$

If $\gamma(S_1)=S_j$, as $d(S_1,C_1)=d(S_j,\gamma(C_1))$, then either $\gamma(C_1)=C_j$ or $\gamma(C_1)=C_{j-1}$. In the first case it follows that $\gamma(C_2)=C_{j+1}$, and in the second case that $\gamma(C_2)=C_{j-2}$. Proceeding in this way, we see that γ is either a rotation or a reflection of the index set $\{1,\dots,n\}$. This shows that the restriction of $\Gamma(G_n)$ to the set Z is equivalent to the dihedral group D_n.

That action on the set Z is crucial, as we conclude next by proving that $\gamma_{|\mathscr{Z}}=\mu_{|\mathscr{Z}}$ implies $\gamma=\mu$. Equivalently, we are going to show that if $\gamma_{|\mathscr{Z}}=1_{|\mathscr{Z}}$ then $\gamma=1$. Before proving this, assuming $\gamma_{|\mathscr{Z}}=1_{|\mathscr{Z}}$, we make two remarks.

(i) Let T be any tree and $d_i=d_T(i)$ the degree of T on the vertex i. From Remark 3.1 we know that $d(T,S_i)=n-1-d_i$. But, by hypothesis, γ is trivial on the stars, and an automorphism preserves distances, hence

$$d(\gamma(T),S_i)=d(\gamma(T),\gamma(S_i))=d(T,S_i)=n-1-d_i$$

and the vertex i has the same degree in T and $\gamma(T)$ for any i, $1\leqslant i\leqslant n$.

(ii) On the other hand, if $ch(T)$ is the number of edges that T has in the boundary, then

$$d(\gamma(T),C_i)=d(\gamma(T),\gamma(C_i))=d(T,C_i)=n-ch(T) \quad \text{if } (i,i+1)\in T,$$

$$d(\gamma(T),C_i)=d(\gamma(T),\gamma(C_i))=d(T,C_i)=n-1-ch(T) \quad \text{if } (i,i+1)\notin T,$$

hence T and $\gamma(T)$ have the same edges in the boundary.

We prove now that if $\gamma_{|\mathscr{Z}}=1$ then $\gamma=1$, by induction on the number of vertices. Let U be any tree of $\mathscr{T}_n$, we want to show that $\gamma(U)=U$. Let i be a vertex such that $d_U(i)=1$ and $(i,i+1)\in U$ and let $\mathscr{T}(U)$ be the following set of trees in $\mathscr{T}_n$,

$$\mathscr{T}(U)=\{T\in\mathscr{T}_n \mid d_T(i)=1,(i,i+1)\in T\}.$$

We remark that after (i) and (ii) we can ensure that $\gamma(\mathscr{T}(U))\subseteq\mathscr{T}(U)$ and since γ is bijective, we have $\gamma(\mathscr{T}(U))=\mathscr{T}(U)$. In particular, it makes sense to consider the automorphism $\gamma_{|\mathscr{T}(U)}$, the restriction of γ to the set $\mathscr{T}(U)$. Besides, identifying P_{n-1} with $P_n\setminus\{i\}$, the subgraph induced on $\mathscr{T}(U)$ is isomorphic to G_{n-1}.

It is straightforward to verify that $\gamma_{|\mathscr{T}(U)}$ restricted to the set of stars and chains of G_{n-1} is the identity and applying induction we have that $\gamma_{|\mathscr{T}(U)}$ is the identity on G_{n-1}. We conclude that $\gamma(U)=U$ for all $U\in\mathscr{T}_n$. So, $\gamma=1$. $\square$

5. Tree of geometric trees

In this section we describe the main tool for proving the results that appear in the next two sections, on the Hamiltonicity and the connectivity of the graph G_n. This tool is a recursive construction of the graphs G_n in which a tree T of G_n gives rise to $\binom{d+2}{2}$ different trees of G_{n+1}. In this way we obtain an infinite tree, whose vertices are the trees in $\mathscr{T}_n$, for all n. This kind of construction has proved useful in solving similar problems for graphs of triangulations [7]. In this infinite tree, every $T \in \mathscr{T}_n$ has one father, belonging to $\mathscr{T}_{n-1}$, and some sons, belonging to $\mathscr{T}_{n+1}$.

If $T \in \mathscr{T}_n$ is such that $i_1 < i_2 < \cdots < i_d$ are the vertices adjacent to n in T, we construct its sons $S_{i,j}(T)$ as the trees of $\mathscr{T}_{n+1}$ defined as follows. We distinguish three kinds of sons:

Type 0: We add the edge $(n, n+1)$ to T and distribute between n and $n+1$ the edges (i_k, n) of T:

$$S_{0,0}(T) = T \cup \{(n, n+1)\}$$

and for k, $1 \leqslant k \leqslant d$,

$$S_{0,k}(T) = \{ (a,b) \mid a,b \neq n,\ (a,b) \in T\} \cup \{(n+1, i_p) \mid 1 \leqslant p \leqslant k\} \cup \\ \cup \{(n, i_p) \mid k \leqslant p \leqslant d\} \cup \{(n, n+1)\}.$$

Type 1: We split the edge (i_k, n) into the two edges (i_k, n) and $(i_k, n+1)$.

$$S_{1,k}(T) = \{ (a,b) \mid a,b \neq n, (a,b) \in T\} \cup \{(n+1, i_p) \mid 1 \leqslant p \leqslant k\} \cup \\ \cup \{(n, i_p) \mid k+1 \leqslant p \leqslant d\}$$

for $1 \leqslant k \leqslant d$.

Type $j, j \geqslant 2$: For every subset S of cardinality j of $\{i_1, \ldots, i_d\}$, $S = \{i_k, i_{k+1}, \ldots, i_{k+j-1}\}$, we build the chain $n+1, i_k, i_{k+1}, \ldots, i_{k+j-1}, n$:

$$S_{j,k}(T) = \{ (a,b) \mid a,b \neq n, (a,b) \in T\} \cup \{(n+1, i_p) \mid 1 \leqslant p \leqslant k\} \\ \cup \{(n, i_p) \mid k \leqslant p \leqslant d-j+2\} \\ \cup \{(i_k, i_{k+1}), (i_{k+1}, i_{k+2}), \ldots, (i_{k+j-2}, i_{k+j-1})\}.$$

In Fig. 5 we show all the sons of the star $S_4 \in \mathscr{T}_4$. The sons of type 0 are at the first floor, those of type 1 at the second one, and so on.

The number of sons of a tree $T \in \mathscr{T}_n$ depends on the degree of n in T. More exactly, if this degree is d, then the number of sons of T is

$$(d+1) + d + (d-1) + \cdots + 1 = \frac{(d+2)(d+1)}{2} = \binom{d+2}{2}.$$

We observe that, given a tree $T \in \mathscr{T}_{n+1}, T$ has a unique father. This father can be obtained by identifying n and $n+1$ and connecting n to all verties adjacent to n or

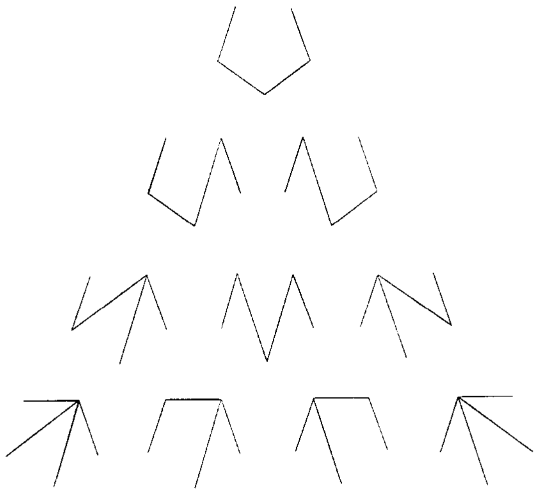

Fig. 5. Construction of the sons $S_{j,k}(S_4)$ of the star $S_4 \in \mathscr{T}_4$.

$n+1$ in T, and to all the vertices belonging to the path between n and $n+1$ in T. We see that this procedure generates all the trees of $\mathscr{T}_{n+1}$ from the trees of $\mathscr{T}_n$.

We observe that any T has always the sons $S_{0,0}(T)$ and $S_{0,d}(T)$. These sons are a copy of T with a pending edge $(n, n+1)$, and will play an important role later. We denote them by $F(T)$ and $L(T)$, respectively. F and L stand for *first* and *last*, a name that will become clear later.

If $T \in \mathscr{T}_n$ is a son of $T_* \in \mathscr{T}_{n-1}$, we say that T_* is a father of T and we write $T_* = f(T)$. If T_1, T_2 have the same father, we say that T_1 and T_2 are *brothers*. The father $f(T)$ of T is easily obtained by reversing the process. The father is unique, hence we have an (infinite) tree as follows. Taking the unique vertex of G_2 as the root of this tree, at level $n-1$ we have all the trees of $\mathscr{T}_n$ that is, the vertices of G_n. In Fig. 6 we can see the first three levels of the tree.

The adjacencies in the graphs G_n are lifted up and down through the tree just constructed in a way we describe in the next two lemmas, which are immediate.

Lemma 5.1. *Let $T_1, T_2 \in \mathscr{T}_n$. The following properties hold:*
(a) *T_1, T_2 are adjacent if and only if $F(T_1), F(T_2)$ are adjacent and $L(T_1), L(T_2)$ are also adjacent.*
(b) *If T_1, T_2 are adjacent and $(i,n) \in T_1 \cap T_2$, and $i_1 < \cdots < i = i_r < \cdots < i_d$ are adjacent to n in T_1, and $j_1 < \cdots < j_r = i < \cdots < j_{d'}$ are adjacent to n in T_2, then $S_{j,i}(T_1), S_{j,i}(T_2)$ are adjacent for $j = 1, 2$.*
(c) *If T_1, T_2 are adjacent and have in common all the edges adjacent to n, then $S_{j,k}(T_1)$ and $S_{j,k}(T_2)$ are adjacent for all j, k.*

Lemma 5.2. *The sons of T induce a subgraph $\mathscr{S}_T$ in G_{n+1} that has the following properties:*

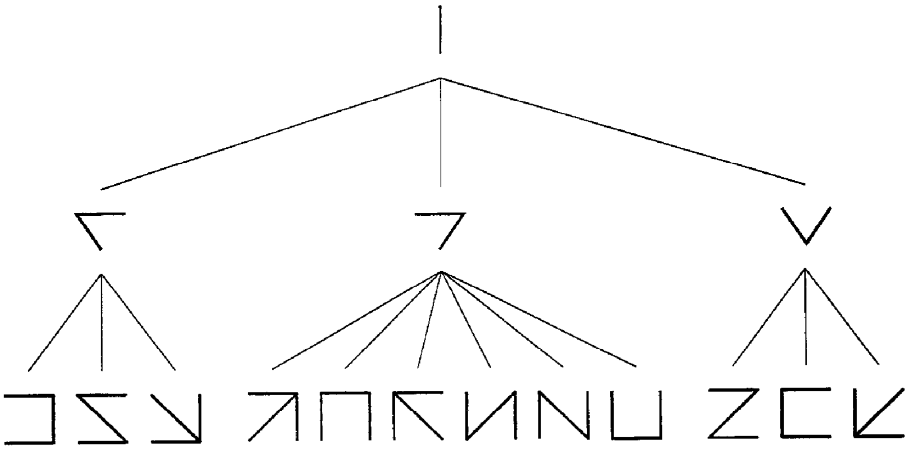

Fig. 6. First levels of a tree of geometric trees.

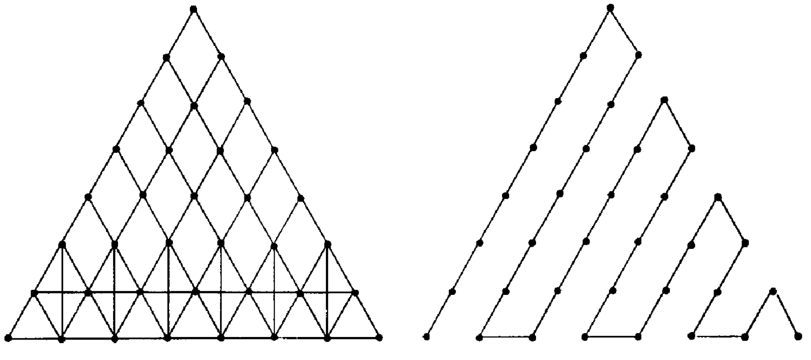

Fig. 7. The sons subgraph $\mathscr{S}_T$ and the brother-path having extremes $L(T)$ and $F(T)$.

(a) *$\mathscr{S}_T$ is 2-connected.*
(b) *$\mathscr{S}_T$ has a Hamiltonian path with extremes $F(T)$ and $L(T)$.*
(c) *The degree of the vertices of $\mathscr{S}_T$ is between* 2 *and* 6 (*in* $\mathscr{S}_T$).

In the rest of the paper we will refer to the subgraph of G_{n+1} induced by the set of sons of T and $\mathscr{S}_T$, and to the Hamiltonian path of Lemma 5.2(b) as a *brother-path* (from $F(T)$ to $L(T)$). Fig. 7 illustrates the last lemma. Each vertex of the figure represents a son of the tree T. Sons of type $S_{j,k}$ are at the $(j+1)$th-floor (bottom to top).

Because of Lemma 5.1 any substructure of G_n has an isomorphic copy in G_{n+1} via $F = \{F(T) = S_{0,0}(T) \mid T \in \mathscr{T}_n\}$ or via $L = \{L(T) = S_{0,d}(T) | T \in \mathscr{T}_n\}$. For this reason we can say that F and L are copies of G_n in G_{n+1}. We can obtain all the vertices of G_{n+1} from these two copies of G_n, joining the two copies $F(T), L(T)$ of each vertex T of G_n through the Hamiltonian path in $\mathscr{S}_T$ (see Lemma 5.2).

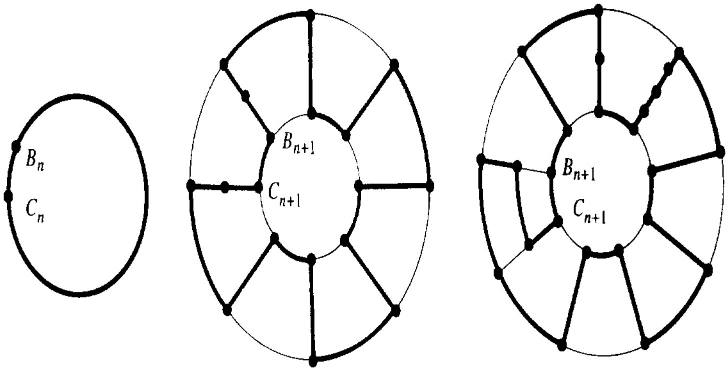

Fig. 8. Constructing a Hamiltonian cycle in G_{n+1} given a cycle in G_n.

6. Hamiltonicity and connectivity

As a first application of the tree introduced in the preceding section, we prove that the graph G_n is Hamiltonian by means of an inductive construction. We have to consider two special kinds of trees, C_n and B_n, defined as follows. C_n is the chain having all the edges of the boundary except $(1,n)$, and B_n is the tree having all its edges in common with C_n except the edge $(1,2)$ that is replaced by $(1,3)$ instead. It is clear that B_n and C_n are adjacent in G_n, and that the next properties are also satisfied.

Lemma 6.1. *The sons of C_n and B_n have the following properties*:
(a) *C_n has exactly three sons and they are connected through the path* $F(C_n) = S_{0,0}(C_n) \sim S_{1,1}(C_n) \sim S_{0,1}(C_n) = L(C_n)$.
(b) *B_n has exactly three sons and they are connected through the path* $F(B_n) = S_{0,0}(B_n) \sim S_{1,1}(B_n) \sim S_{0,1}(B_n) = L(B_n)$.
(c) $F(C_n) = C_{n+1}, F(B_n) = B_{n+1}$.
(d) $S_{i,j}(C_n) \sim S_{i,j}(B_n)$.

Theorem 6.2. *G_n is a Hamiltonian graph for all $n \geqslant 3$. Moreover, there is a Hamiltonian cycle in which C_n and B_n are adjacent.*

Proof. We proceed by induction on n. G_3 is K_3, and the basis of the induction is clear. Let us assume now that G_n has a Hamiltonian cycle C as in the statement. We obtain a copy of C in G_{n+1} via L, and a second and disjoint copy via F. For every tree T_n of G_n the vertices $L(T_n)$ and $F(T_n)$ are connected through the path formed by the sons of T_n, and all the vertices of G_{n+1} belong to some of these paths. By Lemma 6.1 we have $F(C_n) = C_{n+1}$ and $F(B_n) = B_{n+1}$. Taking into account these facts and Lemma 6.1 we construct a Hamiltonian cycle in G_{n+1} in the way depicted in Fig. 8. The case where G_n has an even number of vertices is shown in the middle of the figure, and the case where this number is odd is shown on the right. $\square$

As a second example of application of the tree introduced in Section 5, we compute the connectivity of the graph G_n.

Theorem 6.3. *The connectivity of the graph G_n is equal to $2n-4$.*

Proof. As the minimum degree is $2n-4$ we only have to prove that the graph remains connected when any $2n-5$ vertices are suppressed. This is clear for $n=3$. The case $n=4$ is easily proved by direct inspection. We assume now that the property holds for some $n-1\geqslant 3$ and proceed by induction: we will prove that G_n remains connected after the removal of any set W of $2n-5$ vertices. We distinguish three cases. Recall that F and L are isomorphic copies of G_{n-1} in G_n.

Case 1: $W\subset F$ or $W\subset L$. If $W\subset F$, we can construct a path between any two given nodes T and Y as follows: from T to $L(f(T))$, then from $L(f(T))$ to $L(f(Y))$, and finally to Y. The same proof applies when $W\subset L$.

Case 2: $|W\cap F|=2n-6$ or $|W\cap L|=2n-6$. If $|W\cap F|=2n-6$, there is only one vertex Z in W that is not in F. If $Z\neq L(f(T))$ and $Z\neq L(f(Y))$ then, because of the 2-connectivity of the subgraphs of sons, we can construct a path as in the preceding case. If $Z=L(f(T))$ or $Z=L(f(Y))$, it is easy to see that T has, at least, one adjacent vertex outside of $\mathscr{S}_f(T)$ and that it is not in F. The same proof applies when $|W\cap L|=2n-6$.

Case 3: $|W\cap L|\leqslant 2n-7$ and $|W\cap F|\leqslant 2n-7$. Because of the induction, the subgraphs $L-W$ and $F-W$ of G_n-W are connected. On the other hand, we know that the number of trees of G_{n-1} is $t_{n-1}>2n-7$, $(n\geqslant 4)$. Hence we can assure the existence of at least one complete brother-path in G_n-W going from F to L. In order to conclude the proof it is enough to prove the following claim:

Claim. *From any $T\in\mathscr{T}_n$ we can reach F or L in G_n-W.*

Proof. Let T be any vertex of G_n. If the brother path from T to $F(f(T))$ or from T to $L(f(T))$ is not broken, the statement holds. Suppose now that there is no path from T to $F(f(T))$ and from T to $L(f(T))$ in the subgraph $\mathscr{S}_{f(T)}$.

The main idea is to find neighbors of T in many different families, giving raise to many different ways of reaching F or L (see Fig. 9).

Let h be the distance between vertices $n-1$ and n in T and let $i_1<i_2<\cdots<i_d$ be the vertices adjacent to $n-1$ in $f(T)$. Let $ord(U)$ be the number of vertices of U, for any tree U.

Case $h=1$: In this case T contains the edge $(n-1,n)$ and T is a son of type $0, T=\mathscr{S}_{0,k}(f(T))$ for some k. Besides, the father of T is obtained by contracting the edge $(n-1,n)$. If $k=0$ or $k=d$, T is already in F or L. For $1\leqslant k<d$ consider the trees T_{n-1},T_n with root $n-1$ and n, respectively, obtained by deleting the edge $(n-1,n)$ of T. The trees of the form $T+e-f$, where e and f are both edges of T_{n-1} or both edges of T_n, give rise to adjacent trees in different families and that are not in $\mathscr{S}_{f(T)}$. Hence, by Proposition 2.2, the number of families different from $\mathscr{S}_{f(T)}$

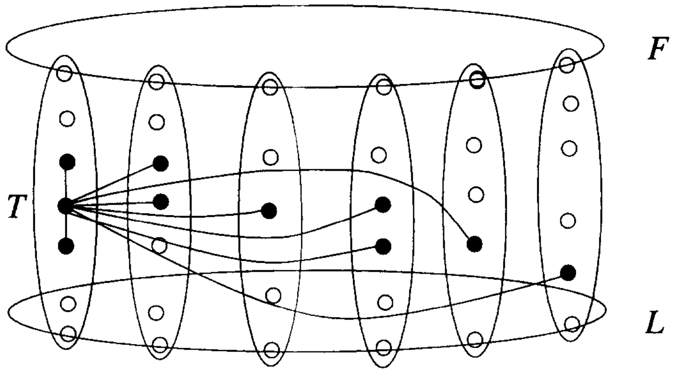

Fig. 9. Neighbors of T distributed in different families.

containing vertices adjacent to T is at least

$$\begin{aligned}2(ord(T_{n-1})-2)+2(ord(T_n)-2) &= 2(ord(T_{n-1})+2(ord(T_n)-4)\\ &= 2(n-4)=2n-8.\end{aligned}$$

Consider now the two trees $T+(i_k,i_{k+1})-(i_k,n)$ and $T+(i_k,i_{k+1})-(i_{k+1},n-1)$. These trees are not brothers and do not belong to the preceding families. Therefore, in order to disconnect T from L and F it is necessary to remove at least $2n-6$ vertices which are not in $\mathscr{S}_{f(T)}$, plus two in $\mathscr{S}_{f(T)}$, i.e. $2n-4$ vertices.

Case $h \geqslant 2$: Consider the path $\mathscr{P}$ of length h from n to $n-1$ in T, $n=j_0 \sim j_1 \sim \cdots \sim j_h = n-1$. For each vertex j_l, $l>0$, of $\mathscr{P}$ let $T_{j_l,b}$ be the tree obtained as the subgraph generated by the set of vertices $i \leqslant j_l$ such that the path from i to j_l in T does not contain the edge (j_{l-1},j_l). Analogously, for each vertex j_l, $l<h$, we define the tree $T_{j_l,f}$ as the subgraph generated by the set of vertices $i>j_l$ such that the path from i to j_l in T does not contain the edge (j_l,j_{l+1}) (see Fig. 10).

We distinguish three *classes* of edges in $\mathscr{P}$. An edge (j_l,j_{l+1}) is of class 2 if both trees $T_{j_l,f}$ and $T_{j_l+1,b}$ have order $\geqslant 2$; it is of class 1 if exactly one of them has order $\geqslant 2$, and of class 0 if both trees have order 1. Denote by r and s the number of edges of $\mathscr{P}$ of class 2 and 1, respectively. Now the number of trees of order $\geqslant 2$ defined in this way is $2r+s$.

Consider first the trees adjacent to T obtained by making a single interchange in one of the trees $T_{j_l,b}$ or $T_{j_l,f}$. The new trees belong to different families and are not in $\mathscr{S}_{f(T)}$. Then, again by Proposition 2.2, the number of distinct families containing neighbors of T is at least,

$$\begin{aligned}&\sum_{ord(T_{j_l,b})\geqslant 2} 2(ord(T_{j_l,b})-2)+\sum_{ord(T_{j_l,f})\geqslant 2} 2(ord(T_{j_l,f})-2)\\ &=2\left(\sum_{ord(T_{j_l,b})\geqslant 2} ord(T_{j_l,b})+\sum_{ord(T_{j_l,f})\geqslant 2} ord(T_{j_l,f})\right),\end{aligned}$$

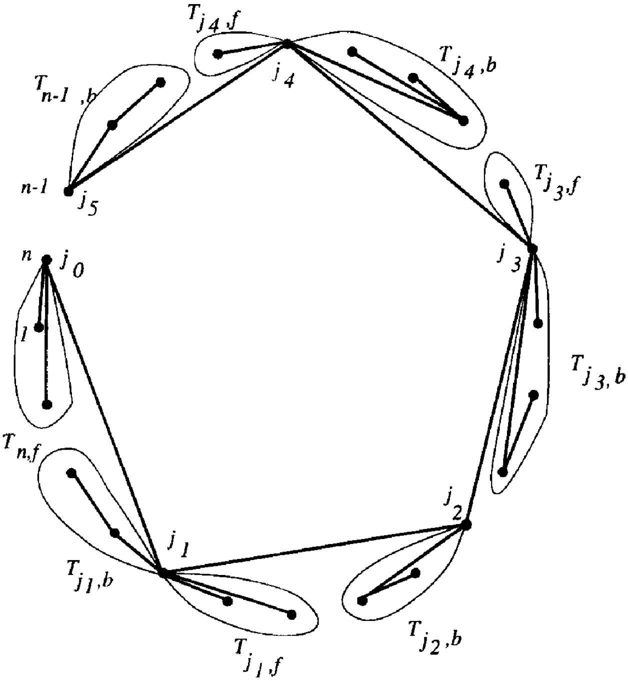

Fig. 10. The path $\mathscr{P}$.

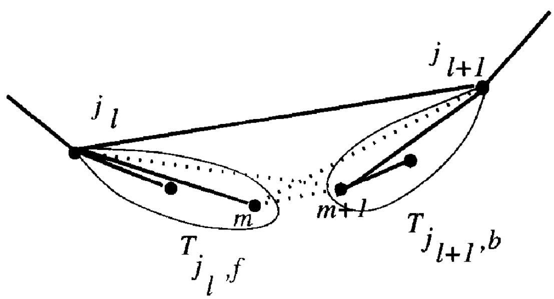

Fig. 11. Constructing special trees adjacent to T.

$$-4(|\{T_{j_l,b}|ord(T_{j_l,b})\geqslant 2\}|+|\{T_{j_l,f}|ord(T_{j_l,f})\geqslant 2\}|)$$
$$=2(n-(h+1)+2r+s)-4(2r+s)=2n-2-(4r+2s+2h). \quad (1)$$

Consider now an edge (j_l,j_{l+1}) of class 2. There exists a vertex m, $j_l<m<m+1<j_{l+1}$ such that $m\in T_{j_l,f}$ and $m+1\in T_{j_{l+1},b}$ (see Fig. 11). The addition of the edge $(m,m+1)$ creates a cycle C in T. Let e be an edge of $C\cap T_{j_l,f}$ and f an edge of $C\cap T_{j_{l+1},b}$. The trees $T+(m,m+1)-e, T+(m,m+1)-f, T+(m,j_{l+1})-e, T+(j_l,m+1)-f$ are not in $\mathscr{S}_{f(T)}$ and belong to new families.

If the edge (j_l, j_{l+1}) is of class 1, suppose that the tree $T_{j_l,f}$ has order $\geqslant 2$ (if the tree with order $\geqslant 2$ is $T_{j_{l+1},b}$ we proceed analogously). The addition of the edge $(j_{l+1}-1, j_{l+1})$ creates a cycle C in T. Consider the tree $T+(j_{l+1}-1, j_{l+1})-e$, where e is an edge of $C \cap T_{j_l,f}$. If the edge (j_l, j_{l+1}) is not incident with $n-1$ or n, consider the tree $T+(j_{l+1}-1, j_{l+1})-(j_l, j_{l+1})$. These trees do not belong to $\mathscr{S}_{f(T)}$ and are in new families.

Therefore, we can assure the existence of neighbors of T in at least

$$2n-2-(4r+2s+2h)+4r+2s-2=2n-2h-4 \tag{2}$$

different families distinct from $\mathscr{S}_{f(T)}$.

The proof now proceeds according to the different values of h.

Subcase $h \geqslant 4$: For each l, $1 \leqslant l \leqslant h-1$, the trees $T+(n-1,n)-(j_l, j_{l+1})$ and $T+(n-1, j_l)-(j_l, j_{l+1})$ are brothers but are neither in $\mathscr{S}_{f(T)}$ nor in any of the families considered before. We have found $2n-2h-4+2(h-1)=2n-6$ trees adjacent to T and such that at most two are in the same family. In order to disconnect T from F and L it is necessary to remove at least two vertices of $\mathscr{S}_{f(T)}$ and the $2n-6$ vertices considered before, i.e., a minimum of $2n-4$ vertices.

Subcase $h=3$: By (2) we have already $2n-10$ trees in different families and not in $\mathscr{S}_{f(T)}$. The trees $T+(n-1,n)-(n,j_1)$ and $T+(j_2,n)-(j_1,n)$, and the trees $T+(n-1,n)-(j_2,n-1)$ and $T+(j_1,n-1)-(j_2,n-1)$ are two pairs of brothers in distinct families and belong to new families. Then, disconnecting T from F and L is possible only after the removal of at least $2n-10+4$ vertices that are not in $\mathscr{S}_{f(T)}$ plus two of $\mathscr{S}_{f(T)}$, that is, after the removal of $2n-4$ vertices.

Subcase $h=2$: This means that T is a son of type 1. By 2 we have at least $2n-8$ neighbors of T that are not in $\mathscr{S}_{f(T)}$ and in different families. We already know that the trees of $\mathscr{S}_{f(T)}$ that are sons of type 0 or type j, $j \geqslant 2$, are connected to F or to L in $G_n - W$.

Let $d=d_{f(T)}(n-1)$ be the degree of $n-1$ in $f(T)$. If $d \geqslant 2$, taking into account the adjacencies of T in $\mathscr{S}_{f(T)}$, it is necessary to remove at least four vertices of $\mathscr{S}_{f(T)}$ in order to disconnect T from F and L (see Fig. 7). Therefore, it is necessary to remove at least $2n-8+4=2n-4$ vertices to disconnect T from F and L.

If $d=1$, we observe that $|\mathscr{S}_{f(T)}|=3$, so we have to remove the two remaining vertices of its family to disconnect T from F and L. Moreover, in this case, $r=0$ and $0 \leqslant s \leqslant 2$ (recall that r and s are the number of edges in the path $\mathscr{P}$ of class 2 and 1, respectively). We cannot have $s=0$ because we are assuming $n \geqslant 4$.

If $s=1$, by Eq. (1), we have at least $2n-2-(2+4)=2n-8$ neighbors of T in different families. Suppose that the edge of class 1 is (j_1,n) (if the edge of class 1 is $(j_1,n-1)$ we proceed analogously). We consider the trees $T+(1,n)-(j_1,n)$, $T+(1,n)-e$, where e is an edge of the tree $T_{j_1,b}$ and of the cycle formed when adding edge $(1,n)$ to T. These trees may be brothers but are neither in the families of the $2n-8$ trees considered before nor in $\mathscr{S}_{f(T)}$. That is, we have to remove at least $2n-8+2+2=2n-4$ vertices to disconnect T from F and L.

Finally, if $s=2$, by 1, we have at least $2n-2-(4+4)=2n-10$ trees in different families. Consider now the trees $T+(1,n)-(j_1,n)$, $T+(1,n)-e$, where e is in $T_{j_1,b}$ and in the cycle formed when adding edge $(1,n)$ to T, and $T+(n-2,n-1)-(j_1,n-1)$, $T+(n-2,n-1)-f$, where f is in $T_{j_1,f}$ and in the cycle formed when adding edge $(1,n)$ to T. In this way, we obtain four trees that are not in $\mathscr{S}_{f(T)}$, and no more than two are in the same family. This implies that we have to remove, besides the two vertices of $\mathscr{S}_{f(T)}, 2n-10+4=2n-6$ vertices to disconnect T from F and L, that is $2n-6+2=2n-4$ vertices.

This finishes the proof of the claim and the proof of the theorem. □

7. Conclusions and open problems

We have obtained many basic properties of the graphs G_n. We consider that to determine exactly the diameter is the main open problem left in this paper. An efficient algorithm for finding shortest paths in G_n would also be interesting.

On the other hand, when the set P of points is not in convex position, it would be interesting to relate the position of the points to the properties of $T(P)$, as well as to try to characterize the graphs which are geometric tree graphs for some P.

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