Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise[☆]

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Abstract

In this paper we investigate the long term behavior of a stochastic lattice dynamical system with a diffusive nearest neighbor interaction, a dissipative nonlinear reaction term, and a different multiplicative white noise at each node. We prove that this stochastic lattice equation generates a random dynamical system that possesses a global random attractor. In particular, we first establish an existence theorem for weak solutions to general random evolution equations, which is later applied to the specific stochastic lattice system to show that it has weak solutions and the solutions generate a random dynamical system. We then prove the existence of a random attractor of the underlying random dynamical system by constructing a random compact absorbing set and using an embedding theorem. The major novelty of this work is that we consider a different multiplicative white noise term at each different node, which significantly improves the previous results in the literature where the same multiplicative noise was considered at all the nodes. As a consequence, the techniques used in the existing literature are not applicable here and a new methodology has to be developed to study such systems.

Key words: stochastic lattice differential equations, random dynamical systems, random attractors, multiplicative noise
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1. Introduction

It is well known that stochastic lattice differential equations arise naturally in a wide variety of applications where the spatial structure has a discrete character and uncertainties or random influences, called noises, are taken into account. The lattice differential equations have been used to model systems such as cellular neural networks with applications to image processing, pattern recognition, and brain science (see [1] and references therein). They are also used to model the propagation of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites, in which case u_i represents the potential at the i-th active site (see e.g., [2, 3]). Lattice differential equations can also be found in chemical reaction theory (see e.g., [4, 5]). In the absence of noise, many works have been done on various aspects of solutions to deterministic lattice dynamical systems. We refer the readers to [6, 7] and references therein for traveling waves, and [8, 9] and references therein for the chaotic properties of solutions.

In this paper we will investigate the long term behavior for the following stochastic lattice differential equation with diffusive nearest neighbor interaction, a dissipative nonlinear reaction term and a different multiplicative white noise at each node:

$$du_i(t) = [\rho(u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i) + g_i]dt + \sigma_i u_i \circ dw_i(t), \quad i \in \mathbb{Z}, \quad (1.1)$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2 := \{(u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} u_i^2 < \infty\}$, $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$, ρ and σ_i are positive constants, f_i is a smooth function satisfying proper dissipative conditions, and w_i's are mutually independent Brownian motions. Here $\circ$ denotes the Stratonovich sense of the stochastic term.

The study of global random attractors was initiated by Ruelle [10]. The fundamental theory of global random attractors for stochastic partial differential equations was developed by Crauel and Flandoli [11], Flandoli and Schmalfuss [12], Imkeller and Schmalfuss [13], Schmalfuss [14] amongst others. Due to the unbounded fluctuations in the systems caused by the white noise, the concept of global pullback/random attractor was introduced to capture the essential dynamics with possibly extremely wide fluctuations. This is significantly different from the deterministic case.

The existence of a global attractor for the deterministic counterpart of (1.1) was established in [15]. For stochastic lattice dynamical systems with additive or multiplicative noise, the existence of global random attractors has been intensively analyzed in the recent literature (see e.g., Bates *et al.* [16], Caraballo *et al.* [17, 18], Caraballo and Lu [19], Han [20], Han *et al.* [21], amongst others). We emphasize that in the studies of stochastic lattice systems with multiplicative noise up to date, only a finite number of Wiener process is considered in each equation, being the same in all the equations, while the multiplicative noise considered here in Equation (1.1) is different at each node. This can be the result of an environmental effect on the whole domain of the system, either on each equation of the lattice (additive noise case) [16] or in some parameters of the model (multiplicative noise case). To be more precise, and focus on the multiplicative case, let us recall that in some previous lattice models analyzed in the existing literature, the term which is responsible for the dissipative character of the problem is sometimes split into two terms, one of which is linear. For that reason, the deterministic counterpart of our model (1.1) is given as (see, e.g. Caraballo and Lu [19])

$$du_i(t) = [\rho(u_{i-1} - 2u_i + u_{i+1}) - \lambda_i u_i - \tilde{f}_i(u_i) + g_i]dt, \quad i \in \mathbb{Z}.$$

Assuming now that the environmental effect produces a perturbation in the parameter λ_i in such a way that it becomes $\lambda_i - \sigma_i \dot{w}_i$ at each node $i \in \mathbb{Z}$, then our stochastically perturbed lattice becomes

$$du_i(t) = [\rho(u_{i-1} - 2u_i + u_{i+1}) - \lambda_i u_i - \tilde{f}_i(u_i) + g_i]dt + \sigma_i u_i \circ dw_i(t), \quad i \in \mathbb{Z},$$

which is precisely the model we are interested in, if we denote $f_i(u_i) = \lambda_i u_i + \tilde{f}_i(u_i)$.

In the present paper we will prove the existence of a global random attractor for the infinite dimensional random dynamical system generated by the stochastic lattice differential equation (1.1). An interesting feature of this structure is that, even though the spatial domain is unbounded and the solution operator is not smooth or compact, unlike parabolic type of partial differential equations on bounded domains, bounded sets of initial data converge in the pullback sense, under the forward flow to a random compact invariant set. It is worth mentioning again that the noise involved in the system is multiplicative, and more importantly, different at each node. We put emphasis again on this comment because, to the best of our knowledge, such systems have not been considered in the literature although they are fully justified by physical intuitions. More precisely, all the papers published on this topic to date consider at most a finite sum of noise in each node which means that the noise term is essentially the same at all the nodes.

In previous studies in the literature, the stochastic systems are first rewritten as a stochastic differential equation in the Hilbert space ℓ^2, then a suitable change of variable, involving usually an Ornstein-Uhlenbeck process, is performed in order to transform the stochastic equation into a random differential equation in ℓ^2. This has subsequently become a standard way of formalization (see e.g. [19, 18, 21]). However, due to the appearance of the infinitely many noise terms, this scheme cannot be applied to handle our problem and hence a new methodology is ought to be developed.

Unlike the previously described technique, we will perform a change of variable *first* to transform the stochastic lattice system (1.1) into a random lattice system. We then prove that the random dynamical system generated by the resulting random lattice system possesses a global random attractor. The existence of global random attractors for the original stochastic lattice system can be obtained once it can be shown that the random dynamical system generated by the transformed equation is conjugated with the original one in the same space of sequences ℓ^2.

After the transformation, we need to formulate the random lattice system as an abstract evolution equation in a Gelfand triple of Hilbert spaces formed by sequences. Instead of carrying out our analysis working only for our particular random lattice system, we will first develop an abstract theory for the existence of weak solutions[1] to general random differential equations defined in a Gelfand triple of Hilbert spaces. Then we will apply it to our lattice model as a particular example. For this reason the goal of this paper is two-fold:

1. proving a general theorem on the existence and uniqueness of weak solutions for random differential equations in Hilbert spaces;
2. proving, as a special application, that our stochastic lattice model (1.1) generates, after a suitable change of variable, a random dynamical system which possesses a global random attractor.

It is worth mentioning that the existence of the global random attractor for (1.1) is done by proving that the random dynamical system generated by (1.1) possesses a random compact absorbing set and taking benefit of the compact injections of the sequence of Hilbert spaces. In this way we avoid the calculation of uniform estimates on the tails of the solutions, as it has been done in all previous published works on this topic in order to prove either asymptotic compactness [19] or asymptotic nullness [21].

The rest of the paper is organized as follows. In Section 2, we introduce basic concepts concerning random dynamical systems and global random attractors. In Section 3, we perform a transformation which allows us to rewrite our lattice system as a random lattice system without white noise, which can be eventually written as an evolution equation in some appropriate Hilbert spaces. Section 4 is devoted to establishing the existence and uniqueness of weak solutions for general abstract random evolution equations, which will be used later to prove that (1.1) generates an infinite dimensional random dynamical system. The existence of the global random attractor for (1.1) is finally established in Section 5. Some closing remarks will be given in Section 6.

2. Preliminaries on random dynamical systems

We recall in this section some of the basic concepts related to random dynamical systems and the concept of random attractors (see [23], [24] and [12] for more details).

[1] The term "weak solution" throughout this paper refers to the weak solution in the PDE (partial differential equation) context, instead of the weak solution in the SDE (stochastic differential equation) context (see, e.g. [22]).

Let $(H, \|\cdot\|_H)$ be a separable Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

Definition 2.1. *$\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ is called a* ***metric dynamical system****, if*

(i) $\theta : \mathbb{R} \times \Omega \to \Omega$ *is* $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$*-measurable,*

(ii) θ_0 *is the identity on* Ω*,*

(iii) $\theta_{s+t} = \theta_t \circ \theta_s$ *for all* $s, t \in \mathbb{R}$*,*

(iv) $\theta_t\mathbb{P} = \mathbb{P}$ *for all* $t \in \mathbb{R}$*.*

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0), \mathbf{P})$, where

$$\mathcal{C}_0 = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

endowed with the compact open topology (see [23], Appendix A.2 and A.3), $\mathbf{P}$ is the corresponding Wiener measure on the Borel σ-algebra $\mathcal{B}(C_0)$. Let

$$\theta_t\omega(\cdot) = \omega\,(\cdot + t) - \omega\,(t)\,, \quad t \in \mathbb{R},$$

then $(\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0), \mathbf{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Definition 2.2. *A stochastic process $\varphi(t)$ is called a* ***continuous random dynamical system*** *over $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ if φ is $(\mathcal{B}[0,\infty)\times\mathcal{F}\times\mathcal{B}(H), \mathcal{B}(H))$-measurable, and for all $\omega \in \Omega$*

(i) the mapping $\varphi(t, \omega, \cdot) : H \to H$ *is continuous for* $(t, \omega) \in \mathbb{R}^+ \times \Omega$*,*
(ii) $\varphi(0, \omega, \cdot)$ *is the identity on* H*,*
(iii) $\varphi(s + t, \omega, \cdot) = \varphi(t, \theta_s\omega, \cdot) \circ \varphi(s, \omega, \cdot)$ *for all* $s, t \geq 0$ *(cocycle property).*

It is known that finite dimensional stochastic differential equations generate random dynamical systems (see Arnold [23] Chapter 1), but this is not necessarily true in general for infinite dimensional equations. However, for particular types of noise, as in our case, this can be achieved applying a suitable change of variable.

Definition 2.3. *A set-valued map $A : \Omega \to 2^H \setminus \emptyset$, $\omega \mapsto A(\omega)$, where $A(\omega)$ is closed for all $\omega \in \Omega$, is called a* ***random set*** *if for each $x \in H$ the map $\omega \mapsto dist(x, A(\omega))$ is measurable.*

Definition 2.4. *A random bounded set $B(\omega) \subset H$ is called* ***tempered*** *with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for $\omega \in \Omega$*

$$\lim_{t\to\pm\infty} \frac{\log^+ d(B(\theta_{-t}\omega))}{|t|} = 0\ ,$$

where $d(B) = \sup\limits_{x\in B} \|x\|_H$.

Now consider a continuous random dynamical system φ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and let $\mathcal{D}$ denote the collection of random tempered sets in H.

Definition 2.5. *A random set $K \in \mathcal{D}$ is called an* ***absorbing set*** *in $\mathcal{D}$ if for $B \in \mathcal{D}$ and $\omega \in \Omega$ there exists $t_B(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \text{ for all } t \geq t_B(\omega).$$

Definition 2.6. *A random set $\mathcal{A}$ is called a* ***global random $\mathcal{D}$ attractor*** *(**pullback $\mathcal{D}$ attractor**) for φ if the following hold:*

(A1) $\mathcal{A} \in \mathcal{D}$, $\mathcal{A}(\omega)$ is compact set for $\omega \in \Omega$;
(A2) $\mathcal{A}$ is strictly invariant, i.e. for $\omega \in \Omega$ and all $t \geq 0$ it holds

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega);$$

(A3) $\mathcal{A}$ attracts all sets in $\mathcal{D}$, i.e., for all $B \in \mathcal{D}$ and a.e. $\omega \in \Omega$ it holds

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_H$ is the Hausdorff semi-metric (here $X \subset H, Y \subset H$).

The collection $\mathcal{D}$ is usually called the domain of attraction of $\mathcal{A}$.

Often it is enough to assume that the items (A1) - (A3) only hold for $\omega \in \tilde{\Omega}$, where $\tilde{\Omega} \in \mathcal{F}$ is of full $\mathbb{P}$ measure and $(\theta_t)_{t\in\mathbb{R}}$–invariant, i.e., $\theta_t\tilde{\Omega} = \tilde{\Omega}$ for all $t \in \mathbb{R}$.

The next proposition is an abstract result on the existence of global random attractor for a continuous random dynamical system. Although there are different versions of sufficient conditions ensuring the existence of global random attractors, the following one fits our purpose in this work, see [12].

Proposition 2.7. *Let $\varphi(t)$ be a continuous random dynamical system over (Ω, $\mathcal{F}$, $\mathbb{P}$, $(\theta_t)_{t\in\mathbb{R}}$). Suppose that $\varphi(t)$ has a random absorbing set $K \in \mathcal{D}$, $K(\omega)$ compact for $\omega \in \Omega$, then φ possesses a $\mathcal{D}$-random attractor $\mathcal{A}$ given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Moreover, $\mathcal{A}$ is unique in $\mathcal{D}$.

3. Mathematical preparation

Our goal is to study the global random attractor for the random dynamical system generated by (1.1). To this end we will first transform the stochastic lattice equation (1.1) containing white noise terms into a random lattice equation without white noise terms but with random coefficients, which can be written as a random evolution equation eventually. We will then investigate the random attractor for the resulting random evolution equation.

A finite version of system (1.1) as follows

$$\frac{du_i(t)}{dt} = \rho(u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i) + g_i + \sum_{j=1}^{N} \sigma_j u_i \circ \frac{dw_j(t)}{dt}, \quad i \in \mathbb{Z} \tag{3.1}$$

has been analyzed in [19], for a finite sum of noises in each node, which in principle is equivalent to the system with the same noise at all nodes. A similar version to (3.1) was later studied in [21] by using weighted spaces.

Note that in [19], [21] and any other existing work of stochastic lattice systems with single or a finite sum of noise at all nodes, the system can be reformulated as a stochastic equation in the (regular or weighted) space of infinite sequences, before the change of variable process transforming the stochastic equation into a random equation. However, due to the infinite number of noises in this problem, we have to perform the change of variables to (1.1) before formulating it as an evolution equation in appropriate spaces of infinite sequences. In doing so, we will obtain some operators which depend now on the random variable ω and which are not defined on the whole ℓ^2, their domains being a subspace of ℓ^2. As a consequence, the techniques used in [19], [21] and other existing works are not applicable to our problem in this work.

At this point, we will proceed with the transformation from a stochastic to a random system in a heuristic way without imposing any specific assumptions on the functions f_i, g_i in (1.1). Later we will investigate appropriate conditions on f_i and g_i to ensure the existence of a global random attractor.

Let $\mathcal{P} = (\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0), \mathbf{P})$ be a Wiener space. We consider the product space $(\Omega, \mathcal{F}, \mathbb{P}) := \Pi_i \mathcal{P}$ defined in the usual way. In particular, because $\mathcal{C}_0$ is a Polish space, $\mathcal{F}$ is generated by the product topology of $\mathcal{C}_0$ (see Kallenberg [25], Page 2). Note that Ω is a Fréchet–space where the convergence in this space is the component–wise convergence (point–wise convergence). Let us extend the Wiener–shift from $\mathbb{R} \times \mathcal{C}_0$ to $\mathbb{R} \times \Omega$ by

$$\theta_t \omega = (\cdots, \theta_t \omega_i, \cdots), \quad \Omega \ni \omega = (\omega_i)_{i \in \mathbb{Z}}.$$

Note that

$$\begin{aligned} t \mapsto \theta_t \omega_i &\quad \text{is continuous for any } \omega_i \in \mathcal{C}_0, \\ \omega_i \mapsto \theta_t \omega_i &\quad \text{is continuous for any } t \in \mathbb{R}, \end{aligned}$$

which gives the continuity (measurability) of the mappings $\theta_t \cdot$ on Ω with respect to the metric of the Fréchet–space and $\theta.\omega$ on $\mathbb{R}$. We obtain by Aliprantis and Border [26, Lemma 4.51, Page 153] the measurability of

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R} \times \Omega)) = (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F}).$$

The measures $\mathbf{P}$ obtained by the projections of $\mathbb{P}$ to $\mathcal{B}(\mathcal{C}_0)$ are still θ–ergodic.

For each $i \in \mathbb{Z}$ we consider

$$z_i(\theta_t\omega) = -\sigma_i \int_{-\infty}^{0} e^{\tau}\theta_t\omega_i(\tau)d\tau, \quad t \in \mathbb{R}, \quad \omega \in \Omega.$$

Note that each $z_i(\theta_t\omega)$ is an Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and solves the following Langevin equation (see, e.g., Caraballo et al. [27])

$$dz_i = -\, z_i\, dt + \sigma_i dw_i(t). \tag{3.2}$$

where $w_i(t)(\omega) = w_i(t, \omega) = \omega_i(t)$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$. Moreover, by a similar statement to that provided in [27] we have the following result.

Lemma 3.1. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\tilde{\Omega} \in \mathcal{F}$ of Ω of full measure such that for any $\omega \in \tilde{\Omega}$*

1. *For each $i \in \mathbb{Z}$,*
$$\lim_{t\to\pm\infty} \frac{|\omega_i(t)|}{t} = 0;$$
2. *For each $i \in \mathbb{Z}$ the random variable given by*
$$z_i(\omega) := -\sigma_i \int_{-\infty}^{0} e^{\tau}\omega_i(\tau)d\tau$$
is well defined, and the mapping
$$\begin{aligned}(t,\omega) \to z_i(\theta_t\omega) \quad &= \quad -\sigma_i \int_{-\infty}^{0} e^{\tau}\theta_t\omega_i(\tau)d\tau \\ &= \quad -\sigma_i \int_{-\infty}^{0} e^{\tau}\omega_i(t+\tau)d\tau + \sigma_i\omega_i(t)\end{aligned}$$
is a stationary solution of (3.2) with continuous trajectories;
3. *For each $i \in \mathbb{Z}$*
$$\begin{aligned}\lim_{t\to\pm\infty} \frac{|z_i(\theta_t\omega)|}{|t|} \quad &= \quad 0, \\ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z_i(\theta_\tau\omega)d\tau \quad &= \quad 0, \\ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z_i(\theta_\tau\omega)|d\tau \quad &= \quad \mathbb{E}|z_i| < \infty.\end{aligned}$$

Remark 3.2. *In the sequel we will consider θ defined as in (2) on $\tilde{\Omega}$ instead of Ω. This mapping possesses the same properties as the original one if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\tilde{\Omega}$ denoted also by $\mathcal{F}$.*

For each u_i ($i \in \mathbb{Z}$) satisfying (1.1), we perform the following change of variables

$$v_i(t) = e^{-z_i(\theta_t\omega)}u_i(t), \quad i \in \mathbb{Z}.$$

Then the equations for v_i $(i \in \mathbb{Z})$ read

$$\begin{aligned} \frac{dv_i}{dt} &= \rho\left(e^{z_{i-1}(\theta_t\omega)-z_i(\theta_t\omega)}v_{i-1} - 2v_i + e^{z_{i+1}(\theta_t\omega)-z_i(\theta_t\omega)}v_{i+1}\right) \\ &\quad + z_i(\theta_t\omega)v_i + g_i e^{-z_i(\theta_t\omega)} - e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)}v_i), \end{aligned} \tag{3.3}$$

with initial condition $v_i(0) = v_i^0 \in \mathbb{R}$ such that $v(0) := (v_i^0)_{i\in\mathbb{Z}} = \nu_0 \in \ell^2$.

Our goal is to justify that the lattice equations (3.3) can be reformulated as an abstract random evolution equation over an Gelfand evolution triplet $V \subset H \subset V'$, where H is a separable Hilbert space, V a separable Banach space with dense and compact embedding $V \subset H$, and V' denotes the dual space of V. For this purpose we will formulate an abstract random evolution equation in the next section, such that the conditions we impose for the lattice-system (3.3) can be fit to the coefficients of this abstract system. We will then show the existence of weak solutions to the evolution equation, and prove in addition that this solution generates a random dynamical system under certain assumptions.

Remark 3.3. *It is nontrivial to prove that the transformation from* (1.1) *to* (3.3) *is a conjugation for random dynamical systems. According to the existing results in the literature, to prove that the transformation generates conjugated random dynamical systems, we should verify assumptions such as in Caraballo and Lu [19, Lemma 2.3]. In particular, we need to check that the mapping* $T(\cdot,\cdot) : \Omega \times \ell^2 \to \ell^2$ *defined as*

$$T(\omega, u) = (e^{-z_i(\omega)}u_i)_{i\in\mathbb{Z}}$$

is a homeomorphism in ℓ^2 *for each fixed* $\omega \in \Omega$. *At this point, without proving the conjugation property, we will consider a new technique based on Gelfand triple, which allows us to prove the existence of random attractors for equation* (3.3).

4. Weak solutions to general random evolution equations

In this section, we will state and prove an existence theorem for weak solutions to general random evolution equations with particular type of operators and study the random dynamical systems generated by the weak solutions.

Let H be a separable Hilbert space equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let V be a dense subspace of H with the inner product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V$, and assume that V is given a topological vector space structure for which the inclusion map is continuous. Then the triplet (V, H, V') forms a rigged Hilbert space where V' is the dual space of V equipped with the norm $\|\cdot\|_{V'}$. Let $\langle\cdot,\cdot\rangle$ denote the duality map between V and V'. Then the duality map is compatible with the inner product on H, in the sense that

$$\langle u, v\rangle = (u, v), \quad \forall u \in V \subset H,\ v \in H = H' \subset V'.$$

Let $(e_k)_{k\in\mathbb{N}} \in H$ be a complete orthonormal basis of H, and consider a sequence of finite dimensional linear subspaces $H_n \subset H_{n+1} \subset V \subset H$ given by

$$H_n = \operatorname{span}\{e_1, \cdots, e_n\}.$$

Define the projection $\pi_n : V \to H_n$ by

$$\pi_n \cdot = \sum_{j=1}^{n} (\cdot, e_j) e_j, \tag{4.1}$$

then $\pi_n : H \to H_n$ is an orthonormal projection. We assume that

$$\overline{\bigcup_n H_n}^{H} = H, \quad \overline{\bigcup_n H_n}^{V} = V,$$

where $\overline{\cdot}^H$ and $\overline{\cdot}^V$ denote the closure in the norm topology of H and V, respectively. π_n can be extended to V'.

We define a linear continuous operator $A : V \to V'$ such that:

$$\langle Av, v\rangle \geq \alpha \|v\|_V^2, \quad \|Av\|_{V'} \leq \alpha' \|v\|, \quad \forall v \in V. \tag{4.2}$$

We will investigate the following evolution system in a weak sense:

$$\frac{dv(t)}{dt} + Av(t) = F(\theta_t\omega, v(t)) + G(\theta_t\omega), \quad v(0) = \nu_0 \in H. \tag{4.3}$$

Definition 4.1. *An element $v \in L^2(0,T;V)$ with weak derivative $\frac{dv}{dt} \in L^2(0,T;V')$ is called a* **weak solution** *to* (4.3) *if for any $\xi \in V$ and any $\phi \in \mathcal{C}_0^\infty(0,T)$*

$$\begin{aligned} -\int_0^T (v(t), \xi)\phi'(t)dt &= -\int_0^T \langle Av(t), \xi\rangle \phi(t)dt \\ &\quad + \int_0^T \langle F(\theta_t\omega, v(t)) + G(\theta_t\omega), \xi\rangle \phi(t)dt. \end{aligned}$$

To prove the existence of a weak solution to (4.3), we impose the following standing assumptions for the mappings $F : \Omega \times V \to V'$ and $G : \Omega \to V'$:

(F1) $(\omega, t) \mapsto \langle F(\omega, v(t)), \xi\rangle$ is measurable for all $v \in L^2(0,T;V)$ and $\xi \in V$, and for every $\omega \in \Omega$, $\phi \in \mathcal{C}_0^\infty(0,T)$, $\xi \in \cup_{m\in\mathbb{N}} H^m$, and any sequence $v^{(n)}$ such that

$$v^{(n)} \to v \text{ strongly in } L^2(0,T;H),$$

we have

$$\lim_{n\to\infty} \int_0^T \langle F(\theta_t\omega, v^{(n)}(t)), \xi\rangle \phi(t)dt = \int_0^T \langle F(\theta_t\omega, v(t)), \xi\rangle \phi(t)dt.$$

(**F2**) Linear boundedness with respect to the V'-norm:

$$\begin{aligned}\|F(\omega,v)\|_{V'}^2 &\leq C_1(\omega)+C_2(\omega)\|v\|_H^2+C_3\|v\|_V^2\\ \langle F(\omega,v),v\rangle &\leq K_1(\omega)+K_2(\omega)\|v\|_H^2+\frac{\alpha}{2}\|v\|_V^2\end{aligned}$$

for any $v \in V$, where $t \mapsto C_j(\theta_t\omega)$, $t \mapsto K_j(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$, $j = 1, 2$, for all ω in a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of full measure, and $C_3 > 0$.

(**F3**) F is semi-Lipschitz continuous: there exists a positive random variable $M(\omega)$ such that

$$\langle -A(x-y)+F(\omega,x)-F(\omega,y),x-y\rangle \leq M(\omega)\|x-y\|_H^2 \quad \text{for any } x,\, y\in V,$$

where $t \mapsto M(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for all ω in a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of full measure.

(**F4**) G takes values in V', $\omega \mapsto \langle G(\omega), \xi\rangle$ is measurable and satisfies

$$t \mapsto \|G(\theta_t\omega)\|_{V'}^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$$

for all ω in a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of full measure.

The conditions we impose to our coefficients will ensure that $v \in L^2(0,T;V)$ and with a weak derivative in $L^2(0,T;V')$. We are ready to state and prove our main result of this section.

Theorem 4.2. *Let $A \in L(V,V')$ be the linear operator as defined in* (4.2)*, and assume that F and G satisfy assumptions* **(F1) - (F4)**. *Then*

(i) For any ω in a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of full measure and $\nu_0 \in H$, Eq. (4.3) *possesses a unique global solution $v(t)$ such that, for any $T > 0$, we have $v \in \mathcal{C}([0,T];H) \cap L^2(0,T;V)$ and its weak derivative $\frac{dv}{dt} \in L^2(0,T;V')$.*

(ii) The solution to (4.3) *generates a continuous random dynamical system φ.*

Proof. (i)-(a). We first prove the uniqueness. Suppose that $x(t), y(t) \in L^2(0,T;V)$, with weak derivatives $\frac{dx}{dt}, \frac{dy}{dt}$ in $L^2(0,T;V')$. By Proposition 23.23 in [28], x, y have a continuous version, i.e., $x(t)$, $y(t) \in \mathcal{C}([0,T];H)$. Hence we have by the positivity of A and assumption (F3) that

$$\begin{aligned}\|x(t)-y(t)\|_H^2 &= 2\int_0^t \langle \frac{d}{dt}x(\tau)-\frac{d}{dt}y(\tau),x(\tau)-y(\tau)\rangle d\tau\\ &= -2\int_0^t \langle A(x(\tau)-y(\tau)),x(\tau)-y(\tau)\rangle d\tau\\ &\quad +2\int_0^t \langle F(\theta_\tau\omega,x(\tau)\rangle - F(\theta_\tau\omega,y(\tau)),x(\tau)-y(\tau)\rangle d\tau\\ &\leq 2\int_0^t M(\theta_\tau\omega)\|x(\tau)-y(\tau)\|_H^2\, d\tau.\end{aligned}$$

In particular, the last integral exists by the continuity result. Then, thanks to the Gronwall lemma, we obtain the continuous dependence, and that $x(t) = y(t)$ for $t \in [0, T]$ since $x(0) = y(0)$.

(i)-(b) We now prove the compactness of Galerkin approximations. Recall that $\{e_j\}$ is the orthogonal basis of V and H_n is the subspace of H spanned by $e_1, \ldots, e_n$. Let $v^{(n)}$ satisfy the following ordinary differential equation

$$\begin{aligned} \left(\frac{dv^{(n)}(t)}{dt}, e_j\right) &= -\langle Av^{(n)}(t), e_j\rangle \\ &\quad +\langle F(\theta_t\omega, v^n(t)), e_j\rangle + \langle G(\theta_t\omega), e_j\rangle, \\ v^{(n)}(0) &= \pi_n\nu_0, \quad j = 1, \cdots, n, \end{aligned} \tag{4.4}$$

where π_n is defined as in (4.1). By the non-regularity of the coefficients we may consider this equation in integrated form. Then, by the conditions on the coefficients and taking the sum we obtain a unique local solution, $v^{(n)} \in \mathcal{C}([0, T(n, \omega)); H_n)$.

Replacing e_j in (4.4) by $(v^{(n)}(t), e_j)e_j$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v^{(n)}\|_H^2 &= -\langle Av^{(n)}, v^{(n)}\rangle + \langle F(\theta_t\omega, v^{(n)}(t)), v^{(n)}\rangle + \langle G(\theta_t\omega), v^{(n)}\rangle \\ &\le -\alpha\|v^{(n)}\|_V^2 + \langle F(\theta_t\omega, v^{(n)}(t)), v^{(n)}\rangle + \langle G(\theta_t\omega), v^{(n)}\rangle, \end{aligned}$$

and hence

$$\begin{aligned} \|v^{(n)}(t)\|_H^2 &\le \|\pi_n\nu_0\|^2 - 2\alpha\int_0^t \|v^{(n)}(\tau)\|_V^2 d\tau \\ &\quad +2\int_0^t \langle F(\theta_\tau\omega, v^{(n)}(\tau)), v^{(n)}(\tau)\rangle d\tau + 2\int_0^t \langle G(\theta_\tau\omega), v^{(n)}(\tau)\rangle d\tau \\ &\le \|\pi_n\nu_0\|^2 - 2\alpha\int_0^t \|v^{(n)}(\tau)\|_V^2 d\tau \\ &\quad +2\int_0^t \left(K_1(\theta_\tau\omega) + K_2(\theta_\tau\omega)\|v^{(n)}(\tau)\|_H^2 + \frac{\alpha}{2}\|v^{(n)}(\tau)\|_V^2\right) d\tau \\ &\quad +\frac{2}{\alpha}\int_0^t \|G(\theta_\tau\omega)\|_{V'}^2 d\tau + \frac{\alpha}{2}\int_0^t \|v^{(n)}(\tau)\|_V^2 d\tau \\ &= \|\pi_n\nu_0\|^2 + \frac{2}{\alpha}\int_0^t \|G(\theta_\tau\omega)\|_{V'}^2 d\tau + 2\int_0^t K_1(\theta_\tau\omega) d\tau \\ &\quad +2\int_0^t K_2(\theta_\tau\omega)\|v^{(n)}(\tau)\|_H^2 d\tau - \frac{\alpha}{2}\int_0^t \|v^{(n)}(\tau)\|_V^2 d\tau. \end{aligned} \tag{4.5}$$

Applying now the Gronwall lemma to (4.5) we obtain that $\{v^{(n)}\}_{n\in\mathbb{N}}$ is bounded in the space $\mathcal{C}([0, T(n, \omega)); H)$, and hence can be extended to any interval $[0, T]$. Then applying this boundedness and setting $t = T$ we have that $\{v^{(n)}\}_{n\in\mathbb{N}}$ is bounded in $L^2(0, T; V)$, and straightforwardly bounded in $\mathcal{C}([0, T]; H)\subset L^\infty(0, T; H)$.

In addition, for any $0 \le s < t \le T$, we have

$$\begin{aligned}
\|v^{(n)}(t) - v^{(n)}(s)\|_{V'} &\le \int_s^t \|Av^{(n)}(\tau)\|_{V'} d\tau + \int_s^t \|F(\theta_\tau\omega, v^{(n)}(\tau))\|_{V'} d\tau + \int_s^t \|G(\theta_\tau\omega)\|_{V'} d\tau \\
&\le \alpha' \int_s^t \|v^{(n)}(\tau)\|_V d\tau + \int_s^t \|F(\theta_\tau\omega, v^{(n)}(\tau))\|_{V'} d\tau + \int_s^t \|G(\theta_\tau\omega)\|_{V'} d\tau.
\end{aligned}$$

Then by assumption (**F2**), there exists a constant $c > 0$ independent of n such that

$$\begin{aligned}
\|v^{(n)}(t) - v^{(n)}(s)\|_{V'} \quad \le \quad & c(t-s)^{\frac{1}{2}} \Bigg(\|v^{(n)}\|^2_{L^2(0,T;V)} + \int_0^T \|G(\theta_\tau\omega)\|^2_{V'} d\tau \\
& + \int_0^T C_1(\theta_\tau\omega) d\tau + \|v^{(n)}\|^2_{L^\infty(0,T;H)} \int_0^T C_2(\theta_\tau\omega) d\tau \Bigg)^{\frac{1}{2}},
\end{aligned}$$

which implies that $\{v^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{\frac{1}{2}}([0,T]; V')$. Collecting the above results we obtain that for any $v^{(n)}$ satisfying (4.4),

$$\sup_{n\in\mathbb{N}} \left\{ \|v^{(n)}\|_{L^\infty(0,T;H)} + \|v^{(n)}\|_{L^2(0,T;V)} + \|v^{(n)}\|_{\mathcal{C}^{\frac{1}{2}}([0,T];V')} \right\} < \infty.$$

According to the Dubinskii theorem of compact embedding [29, 30], $\{v^{(n)}\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;H) \cap \mathcal{C}([0,T]; V')$.

(i)-(c) We now prove that the cluster points of $\{v^{(n)}\}_{n\in\mathbb{N}}$ solve equation (4.3). First note that by the reflexivity and separability of V, any element from the set of cluster points is also in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Then by part (i)-(b), there exists v such that $\lim_{n\to\infty} v^{(n)} = v$ (by taking a subsequence if necessary), with respect to the strong topology of $L^2(0,T;H) \cap \mathcal{C}([0,T]; V')$ and the weak topology of $L^2(0,T;V)$ and the weak-star topology of $L^\infty(0,T;H)$.

Let $\xi \in \cup_{m\in\mathbb{N}} H^m$. It is clear that $\xi \in H^{\overline{m}}$ for some $\overline{m} \in \mathbb{N}$. Note that (4.4) implies for $n > \overline{m}$ that

$$\begin{aligned}
-\int_0^T (v^{(n)}(t), \xi)\phi'(t) dt \quad = \quad & -\int_0^T \langle Av^{(n)}(t), \xi\rangle \phi(t) dt - \int_0^T (v^{(n)}(t), \xi)_V \phi(t) dt \\
& + \int_0^T \langle F(\theta_t\omega, v^{(n)}(t)) + G(\theta_t\omega), \xi\rangle \phi(t) dt. \qquad (4.6)
\end{aligned}$$

We next show the convergence of each term in equation (4.6).

Since $v^{(n)}$ converges to v weakly in $L^2(0,T;V)$, we have that

$$\lim_{n\to\infty} \int_0^T (v^{(n)}(t), \xi)_V \phi(t) dt = \int_0^T (v(t), \xi)_V \phi(t) dt. \qquad (4.7)$$

Finally due to assumption (**F1**), we obtain

$$\lim_{n\to\infty} \int_0^T \langle F(\theta_t\omega, v^{(n)}(t)) + G(\theta_t\omega), \xi\rangle \phi(t) dt = \int_0^T \langle F(\theta_t\omega, v(t)) + G(\theta_t\omega), \xi\rangle \phi(t) dt. \qquad (4.8)$$

Collecting (4.7) to (4.8) and using that $\cup_{m\in\mathbb{N}} H^m$ is dense in V we obtain that v is a solution to (4.3). In fact, by the uniqueness proved in (i)-(a), the entire sequence $\{v^{(n)}\}$ converges to the solution v of (4.3).

In the next part of the proof we will show that v satisfies the initial condition. We can also obtain from estimates similar to the above that $\frac{dv}{dt} \in L^2(0,T;V')$, so that $v \in \mathcal{C}([0,T];H)$.

(ii) By the first part of the proof we have that the entire sequence $\{v^{(n)}\}_{n\in\mathbb{N}}$ converges to the solution v of (4.3) in the separable Banach space $\mathcal{C}([0,T];V')$. On the other hand, we know that by $v \in L^2(0,T;V)$, $\frac{dv}{dt} \in L^2(0,T;V')$ we have that $v \in C([0,T];H)$, so that $v(t) \in H$.

Denote by $v(t,\omega;\nu_0)$ and $v^{(n)}(t,\omega;\nu_0^{(n)})$ the solutions to (4.3) depending on their parameters. Since

$$\lim_{n\to\infty} v^{(n)}(\cdot,\omega;\nu_0^{(n)}) = v(\cdot,\omega,\nu_0) \quad \text{in } \mathcal{C}([0,T];V')$$

and the mapping $q_t : \mathcal{C}([0,T];V') \to V'$, $q_t u = u(t)$, is continuous, we have that $\omega \mapsto v(t,\omega;\nu_0)$ is $\mathcal{B}(V')$-measurable. In particular

$$\nu_0 = v(0,\omega,\nu_0) = \lim_{n\to\infty} q_0 v(\cdot,\omega,\nu_0) \in V'$$

so v satisfies the initial condition.

Moreover, by the separability of H, V' and the reflexivity of H we have that $H \cap \mathcal{B}(V') = \mathcal{B}(H)$ which implies the measurability of $\omega \mapsto v(t,\omega,\nu_0)$ w.r.t. $\mathcal{B}(H)$. Since $v(\cdot,\omega;\nu_0)$ is continuous we have the measurability of $(t,\omega) \mapsto H$ by Aliprantis and Border [26]. Similar to (i)-(a), we see that $H \ni \nu_0 \mapsto v(t,\omega;\nu_0) \in H$ is continuous for any (t,ω). Applying again [26] we see that v is measurable w.r.t. its arguments.

It then remains to verify the cocycle property, which follows easily from the uniqueness. In fact, any weak solution v to (4.3) on the interval $[\tau,\tau+s]$ can be shifted to $[0,s]$, still being a weak solution to (4.3) with initial condition $v(s) \in H$ if we replace the $\theta_t\omega$ term in each integral by $\theta_{t+\tau}\omega$. Let $v(\cdot,\omega;\nu_0)$ be a weak solution to (4.3) on $[0,\tau]$, $v^\tau(\cdot,\theta_\tau\omega;v(\tau))$ be a weak solution to (4.3) on $[0,s]$ and $\hat{v}$ be the concatenation of v and v^τ. Then we have for any $\xi \in V$ and $\phi \in \mathcal{C}_0^\infty(0,\tau+s)$,

$$\begin{aligned}
&\int_0^{\tau+s} (\hat{v}(t,\omega;\nu_0),\xi)\,\phi'(t)dt \\
=\ & \int_0^{\tau} (v(\cdot,\omega;\nu_0),\xi)\,\phi'(t)dt + \int_0^{s} (v^\tau(\cdot,\theta_\tau\omega;v(\tau)),\xi)\,\phi'(t)dt \\
=\ & -\int_0^{\tau+s} \langle A\hat{v}(t),\xi\rangle\phi(t)dt + \int_0^{\tau+s} \langle F(\theta_t\omega,\hat{v}(t)) + G(\theta_t\omega),\xi\rangle\phi(t)dt.
\end{aligned}$$

Indeed any $\phi \in \mathcal{C}_0^\infty(0,\tau+s)$ can be approximated in $L^2(0,\tau+s)$ by a sequence $\{\phi_n\}_{n\in\mathbb{N}}$, $\phi_n \in \mathcal{C}_0^\infty(0,\tau+s)$ and $\operatorname{supp}\phi_n \in (0,\tau)\cup(\tau,\tau+s)$.

This completes the proof that the solution to (4.3) generates a continuous random dynamical system. $\square$

In the next section we will show that the lattice equation (3.3) fits into the framework of the evolution equation (4.3), and prove that generates a random dynamical system which possesses a unique global random attractor. It is worth recalling that the lattice equation (3.3) is only a special case of (4.3), and that the existence result proved in Theorem 4.2 can be used to handle any system in the general format of (4.3).

5. Existence of global random attractors for the stochastic lattice dynamical system

We will now apply the general results proved in Section 4 to our special lattice equation (3.3). To this end, set

$$H = \ell^2 = \{\{u_i\}_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} u_i^2 = \|u\|_H^2 < \infty\},$$

with inner product

$$(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \text{for any } u, v \in H.$$

Denote by ϵ^i ($i \in \mathbb{Z}$) the element in H with value 1 at position i and 0 for all other components. Let $(\lambda_i)_{i\in\mathbb{Z}}$ be a sequence of positive numbers. In particular, we assume that

$$\begin{aligned} i \in \mathbb{Z}^+ &\mapsto \lambda_i \quad \text{is increasing,} \\ i \in \mathbb{Z}^- &\mapsto \lambda_i \quad \text{is decreasing,} \end{aligned}$$

and in addition that $\sum_{i\in\mathbb{Z}} \lambda_i^{-1+\kappa} < \infty$ for some positive $\kappa \in (0, 1)$, which ensures that $\sum_{i\in\mathbb{Z}} \lambda_i^{-1} < \infty$.

Define V to be

$$V = \{u \in H : \sum_{i\in\mathbb{Z}} \lambda_i u_i^2 := \|u\|_V^2 < \infty\},$$

where $\|\cdot\|_V$ is associated with the inner product given by

$$(u, v)_V := \sum_{i\in\mathbb{Z}} \lambda_i u_i v_i, \text{ for any } u, v \in V.$$

In addition, we define

$$V' = \{u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \lambda_i^{-1} u_i^2 := \|u\|_{V'}^2 < \infty\},$$

which is exactly the dual space of V. Then (V, H, V') forms a Gelfand triple.

To fit the lattice equation (3.3) in the framework of the random evolution equation (4.3), let us define the operator A_1 by

$$A_1 = 2\rho \mathrm{Id}_H,$$

where Id_H is the identity operator of H. Define the operator Γ by

$$(\Gamma(\omega)v)_i = -\rho \left(e^{z_{i-1}(\omega)-z_i(\omega)} v_{i-1} + e^{z_{i+1}(\omega)-z_i(\omega)} v_{i+1} \right), \quad v = (v_i)_{i\in\mathbb{Z}}. \quad (5.1)$$

Remark 5.1. *Notice that Γ does not generate a bounded linear operator from H to H. Hence the theory of Banach-space valued ordinary differential equations does not apply here. We will use the theory of weak solutions to evolution equations containing unbounded operators to analyze system* (3.3)*.*

The following lemma provides some properties of the linear operator Γ.

Lemma 5.2. *There exists a $(\theta_t)_{t\in\mathbb{R}}$- invariant set $\tilde{\Omega} \in \mathcal{F}$ of full measure such that the operator $\Gamma(\omega)$ defined in* (5.1) *can be considered as a linear operator $\Gamma(\omega) : H \to V'$ and satisfies*

$$\mathbb{E}\|\Gamma(\omega)\|^2_{L(H,V')} < \infty.$$

Moreover, the mapping $t \mapsto \|\Gamma(\theta_t\omega)\|_{L(H,V')} \in L^2_{\mathrm{loc}}(\mathbb{R})$ and is measurable. In addition $\omega \mapsto \Gamma(\omega)v$ is measurable for all $v \in V$ (or $v \in H$).

Proof. For any $v \in V$,

$$\begin{aligned}
\|\Gamma(\omega)v\|^2_{V'} &= \rho^2 \sum_{i\in\mathbb{Z}} \lambda_i^{-1} \left(e^{z_{i-1}-z_i} v_{i-1} + e^{z_{i+1}-z_i} v_{i+1} \right)^2 \\
&\le 2\rho^2 \sum_{i\in\mathbb{Z}} \lambda_i^{-1} \left(e^{2(z_{i-1}-z_i)} v_{i-1}^2 + e^{2(z_{i+1}-z_i)} v_{i+1}^2 \right) \\
&\le 2\rho^2 \|v\|_H^2 \cdot \left(\sup_{i\in\mathbb{Z}} \frac{e^{2(z_i-z_{i+1})}}{\lambda_i} + \sup_{i\in\mathbb{Z}} \frac{e^{2(z_i-z_{i-1})}}{\lambda_i} \right),
\end{aligned}$$

which implies that

$$\|\Gamma(\omega)\|_{L(H,V')} \le 2\rho^2 \cdot \left(\sup_{i\in\mathbb{Z}} \frac{e^{2(z_i-z_{i+1})}}{\lambda_i} + \sup_{i\in\mathbb{Z}} \frac{e^{2(z_i-z_{i-1})}}{\lambda_i} \right).$$

Observe that if z_i is $\mathcal{N}(0,\sigma^2)$ distributed then $2(z_i - z_{i-1})$ and $2(z_i - z_{i+1})$ are $\mathcal{N}(0, 8\sigma^2)$ distributed, and hence $\mathbb{E}[e^{2(z_i-z_{i\pm1})}] < \infty$. Also note that by $\sum_{i\in\mathbb{Z}} \lambda_i^{-1} < \infty$, we have

$$\mathbb{E} \sup_{i\in\mathbb{Z}} \frac{e^{2(z_i-z_{i\pm1})}}{\lambda_i} \le \sum_{i\in\mathbb{Z}} \mathbb{E} \frac{e^{2(z_i-z_{i\pm1})}}{\lambda_i} < \infty,$$

which implies that

$$\mathbb{E}\|\Gamma(\omega)\|^2_{L(H,V')} < \infty.$$

At the same time we obtain

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t \sup_{i\in\mathbb{Z}}\frac{e^{2(z_i(\theta_s\omega)-z_{i\pm1}(\theta_s\omega))}}{\lambda_i}ds=\mathbb{E}\sup_{i\in\mathbb{Z}}\frac{e^{2(z_i(\omega)-z_{i\pm1}(\omega))}}{\lambda_i}<\infty,$$

i.e., $t\mapsto\|\Gamma(\theta_t\omega)\|_{L(H,V')}\in L^2_{\rm loc}(\mathbb{R})$.

It remains to show that Γ is measurable. In fact, for any $v=(v_i)_{i\in\mathbb{Z}}\in H$, we have that $\omega\mapsto(\Gamma(\omega)v)_i\epsilon^i$ is measurable and in addition

$$\sum_{i\in\mathbb{Z}}\frac{((\Gamma(\omega)v)_i\epsilon^i)^2}{\lambda_i}<\infty.$$

This implies the measurability of Γ.

□

Let $A_2:V\to V'$ be defined by

$$(A_2u)_i=\lambda_iu_i,\quad i\in\mathbb{Z}.$$

Then equation (3.3) can be cast into the format of (4.3):

$$\frac{dv(t)}{dt}+Av(t)=F(\theta_t\omega,v(t))+G(\theta_t\omega),\tag{5.2}$$

where

$$A=A_1+A_2\quad,\qquad A_1=2\rho\mathrm{Id}_H,\quad(A_2v)=(\lambda_iv_i)_{i\in\mathbb{Z}},$$

$$F(\omega,v)=-\Gamma(\omega)v+\Big(\lambda_iv_i+z_i(\omega)v_i-e^{-z_i(\omega)}f_i(e^{z_i(\omega)}v_i)\Big)_{i\in\mathbb{N}},\tag{5.3}$$

$$G(\omega)=\Big(g_ie^{-z_i(\omega)}\Big)_{i\in\mathbb{N}}.\tag{5.4}$$

We will next verify that the operator F as defined in (5.3) satisfies assumptions **(F1) - (F3)** proposed in Section 4 with $\alpha=1$, and G as defined in (5.4) satisfies assumption **(F4)**. To this end, we make the following standing assumptions on function f_i and g_i:

(f0) $f_i:\mathbb{R}\to\mathbb{R}$ is continuous for each $i\in\mathbb{Z}$.

(f1) There exists $\beta=(\beta_i)_{i\in\mathbb{Z}}\in H$ such that

$$f_i(s)^2\le\lambda_is^2+\beta_i^2,\quad\forall s\in\mathbb{R}.$$

(f2) There exists $\gamma=(\gamma_i)_{i\in\mathbb{Z}}\in V$ such that

$$sf_i(s)\ge-\gamma_i^2+\frac{3\lambda_i}{4}s^2,\quad\forall s\in\mathbb{R}.$$

(f3) For any $\varepsilon>0$, there exists a constant $l>0$ such that

$$-(f_i(s_1)-f_i(s_2))(s_1-s_2)+2\varepsilon\lambda_i(s_1-s_2)^2\le l(s_1-s_2)^2,\quad\forall s_1,s_2\in\mathbb{R}.$$

(f4) $g = (g_i)_{i\in\mathbb{Z}} \in H$.

First, for any $v \in V$, we have

$$
\begin{aligned}
\|F(\omega, v)\|_{V'}^2 &= \sum_{i\in\mathbb{Z}} \frac{1}{\lambda_i}\Big[\rho(e^{z_{i-1}-z_i}v_{i-1} + e^{z_{i+1}-z_i}v_{i+1}) \\
&\qquad + \lambda_i v_i + z_i(\omega)v_i - e^{-z_i(\omega)} f_i(e^{z_i(\omega)}v_i)\Big]^2 \\
&\le 4\|\Gamma(\omega)v\|_{V'}^2 + 4\sum_{i\in\mathbb{Z}} \lambda_i v_i^2 + \sum_{i\in\mathbb{Z}} \frac{4}{\lambda_i} z_i^2(\omega) v_i^2 \\
&\quad + \sum_{i\in\mathbb{Z}} \frac{4}{\lambda_i} e^{-2z_i(\omega)} f_i^2(e^{z_i(\omega)}v_i),
\end{aligned}
$$

for which we have the following estimates of each term on the RHS:

$$
\|\Gamma(\omega)v\|_{V'}^2 \le \|\Gamma(\omega)\|_{L(H,V')}^2 \|v\|_H^2; \tag{5.5}
$$

$$
\sum_{i\in\mathbb{Z}} \frac{4}{\lambda_i} z_i^2(\omega) v_i^2 \le 4\|v\|_H^2 \cdot \sup_{i\in\mathbb{Z}} \frac{|z_i(\omega)|^2}{\lambda_i}. \tag{5.6}
$$

Due to assumption **(f1)** we have

$$
\begin{aligned}
\sum_{i\in\mathbb{Z}} \frac{4}{\lambda_i} e^{-2z_i(\omega)} f_i^2(e^{z_i(\omega)}v_i) &\le 4\sum_{i\in\mathbb{Z}} v_i^2 + \sum_{i\in\mathbb{Z}} \frac{4}{\lambda_i} e^{-2z_i(\omega)} \beta_i^2 \\
&\le 4\|v\|_H^2 + 4\sup_{i\in\mathbb{Z}} \frac{e^{-2z_i(\omega)}}{\lambda_i} \|\beta\|_H^2. \qquad (5.7)
\end{aligned}
$$

Let

$$
\begin{aligned}
C_1(\omega) &= 4\sup \frac{e^{-2z_i(\omega)}}{\lambda_i} \|\beta\|_H^2 \\
C_2(\omega) &= 4\|\Gamma(\omega)\|_{L(H,V')}^2 + 4\sup_{i\in\mathbb{Z}} \frac{|z_i(\omega)|^2}{\lambda_i} + 4.
\end{aligned}
$$

Since $z_i(\omega)$ and $2z_i(\omega)$ are Gauß random variables and the series of elements λ_i^{-1} is convergent, we have that $\sum_{i\in\mathbb{Z}} \frac{e^{\pm 2z_i(\omega)}}{\lambda_i} \in L^1(\Omega)$. Hence by the ergodic theorem and Lemma 5.2 we have $t \mapsto C_i(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$, $i = 1, 2$, on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure. Collecting (5.5) - (5.7) we obtain the first inequality in **(F2)**:

$$
\|F(\omega, v)\|_{V'}^2 \le C_1(\omega) + C_2(\omega)\|v\|_H^2 + 4\|v\|_V^2.
$$

Let $v \in L^2(0, T; V)$. Then by the continuity of f_i,

$$
(t, \omega) \mapsto e^{-z_i(\theta_t\omega)} f_i(e^{z_i(\theta_t\omega)} v_i(t)) \in \mathbb{R}
$$

is measurable. Hence

$$\sum_{i\in\mathbb{Z}} e^{-z_i(\theta_t\omega} f_i(e^{z_i(\theta_t\omega)}v_i(t,\omega))\xi_i, \quad \xi=(\xi_i)_{\in\mathbb{Z}}\in V,$$

is measurable as a sum of measurable mappings. Similarly, the other terms forming $\langle F(\theta_t\omega, v(t)),\xi\rangle$ are also measurable. Thus $(\omega,t)\mapsto\langle F(\theta_t\omega,v(t)),\xi\rangle$ is measurable for all $v\in L^2(0,T;V)$ and $\xi\in V$. Also, $\|F(\theta_t\omega,v)\|_{V'}$ is finite for almost all t and $t\mapsto F(\theta_t\omega,v(t))\in L^2(0,T,V')$ for all $\omega\in\Omega$.

In our particular case we can simply choose $\mathbb{R}^m$, $m\in\mathbb{N}$, as the finite-dimensional spaces H^m. The complete orthonormal basis of H generating H^m is given by

$$e_k=\begin{cases}\epsilon^{\frac{k}{2}}, \text{ if } k \text{ is even},\\ \epsilon^{-\frac{k-1}{2}}, \text{ if } k \text{ is odd}.\end{cases}$$

Let $\xi\in\cup_{m\in\mathbb{N}}H^m$ and let $(v^{(n)})_{n\in\mathbb{N}}$ be a sequence such that $v^{(n)}\to v$ strongly in $L^2(0,T;H)$ as $n\to\infty$. Then by the continuity of f_i we have that

$$e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)},v_i^{(n)}(t))\to e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)},v_i(t)),$$

for all i, ω and a.a. t. It is clear that $\xi\in H^{\overline{m}}$ for some $\overline{m}\in\mathbb{N}$. Consider, for example, that $\overline{m}$ is even. Then

$$\begin{aligned}&\left|\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)},v_i^{(n)}(t))\xi_i\right|^2\\ \leq\ & \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{-2z_i(\theta_t\omega)}f_i^2(e^{z_i(\theta_t\omega)},v_i^{(n)}(t))\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2\\ \leq\ & \left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\lambda_i\left(v_i^{(n)}(t)\right)^2+\sum_{i=1}^{\overline{m}}e^{-2z_i(\theta_t\omega)}\beta_i^2\right)\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2\\ \leq\ & C_{1,\overline{m}}\left(\left\|v^{(n)}(t)\right\|_H^2+1\right),\end{aligned}$$

for a.a. $t\in[0,T]$ and all n, where we have used condition **(f1)** and the fact that $t\mapsto z_i(\theta_t\omega)$ are continuous functions.

Since $v^{(m)}$ is convergent in $L^2(0,T;H)$, due to Lebesgue's theorem we deduce for every $\phi\in\mathcal{C}_0^\infty(0,T)$ that

$$\begin{aligned}&\lim_{m\to\infty}\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)},v_i^{(m)})\xi_i\right)\phi(t)dt\\ =\ & \int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{-z_i(\theta_t\omega)}f_i(e^{z_i(\theta_t\omega)},v_i)\xi_i\right)\phi(t)dt.\end{aligned}$$

Further, let us consider the term $\Gamma(\omega) v$. We have that

$$\begin{aligned} & \rho\left(e^{z_{i-1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i-1}^{(n)}(t) - e^{z_{i+1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i+1}^{(n)}(t)\right) \\ \longrightarrow \quad & \rho\left(e^{z_{i-1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i-1}(t) - e^{z_{i+1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i+1}(t)\right), \end{aligned}$$

for all i, ω and a.a. t. On the other hand,

$$\begin{aligned} & \left|\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \rho\left(e^{z_{i-1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i-1}^{(n)}(t) - e^{z_{i+1}(\theta_t\omega)-z_i(\theta_t\omega)} v_{i+1}^{(n)}(t)\right)\xi_i\right|^2 \\ \leq \quad & 2\rho^2\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{2(z_{i-1}(\theta_t\omega)-z_i(\theta_t\omega))}\left(v_{i-1}^{(n)}(t)\right)^2\right. \\ & \left.+\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} e^{2(z_{i+1}(\theta_t\omega)-z_i(\theta_t\omega))}\left(v_{i+1}^{(n)}(t)\right)^2\right)\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2 \\ \leq \quad & C_{2,\overline{m}}\left\|v^{(n)}(t)\right\|_H^2, \end{aligned}$$

for a.a. $t \in [0,T]$ and all n. Then as in the previous case we have

$$\lim_{n\to\infty}\int_0^T \langle\Gamma(\omega) v^{(n)},\xi\rangle\phi(t)\,dt = \int_0^T \langle\Gamma(\omega) v,\xi\rangle\phi(t)\,dt$$

for every $\phi \in \mathcal{C}_0^\infty(0,T)$.

In a similar way, we obtain that

$$\lim_{n\to\infty}\int_0^T \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} z_i(\theta_t\omega) v_i^{(n)}(t)\xi_i\phi(t)\,dt = \int_0^T \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} z_i(\theta_t\omega) v_i(t)\xi_i\phi(t)\,dt.$$

Finally, $v^{(n)} \to v$ in $L^2(0,T;H)$ implies that for any $\phi \in \mathcal{C}_0^\infty(0,T)$,

$$\begin{aligned} \lim_{n\to\infty}\int_0^T \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \lambda_i v_i^{(n)}(t)\xi_i\phi(t)\,dt &= \lim_{n\to\infty}\int_0^T \left(v^{(n)}(t),\xi\right)_V \phi(t)\,dt \\ &= \int_0^T (v(t),\xi)_V\,\phi(t)\,dt. \end{aligned}$$

Collecting all the terms of F we reach

$$\lim_{n\to\infty}\int_0^T \langle F(\theta_t\omega, v^{(n)}(t)),\xi\rangle\phi(t)dt = \int_0^T \langle F(\theta_t\omega, v(t)),\xi\rangle\phi(t)dt, \tag{5.8}$$

for any $\phi \in \mathcal{C}_0^\infty(0,T)$, $\xi \in \cup_{m\in\mathbb{N}} H^m$.

Hence, **(F1)** follows.

Next for any $v \in V$, we have

$$\begin{aligned}
-\langle \Gamma(\omega)v, v\rangle &\leq \|\Gamma(\omega)v\|_{V'}\|v\|_V \\
&\leq \|\Gamma(\omega)\|_{L(H,V')}\|v\|_H\|v\|_V \\
&\leq 2\|\Gamma(\omega)\|^2_{L(H,V')}\|v\|^2_H + \frac{1}{8}\|v\|^2_V; \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} z_i(\omega)v_i^2 &= \sum_{i\in\mathbb{Z}} \lambda_i^{-\frac{1}{2}} z_i(\omega)v_i\lambda_i^{\frac{1}{2}}v_i \\
&\leq 2\sup_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i}\|v\|^2_H + \frac{1}{8}\|v\|^2_V; \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
-\sum_{i\in\mathbb{Z}} e^{-z_i(\omega)} f_i(e^{z_i(\omega)}v_i)v_i &= -\sum_{i\in\mathbb{Z}} e^{-2z_i(\omega)} f_i(e^{z_i(\omega)}v_i)e^{z_i(\omega)}v_i \\
&\leq \sum_{i\in\mathbb{Z}} e^{-2z_i(\omega)}\left(\gamma_i^2 - \frac{3\lambda_i}{4}e^{2z_i(\omega)}v_i^2\right) \\
&\leq \|\gamma\|^2_V \sum_{i\in\mathbb{Z}}\frac{e^{-2z_i(\omega)}}{\lambda_i} - \frac{3}{4}\|v\|^2_V. \qquad (5.11)
\end{aligned}$$

Define

$$\begin{aligned}
K_1(\omega) &:= \|\gamma\|^2_V \sum_{i\in\mathbb{Z}}\frac{e^{-2z_i(\omega)}}{\lambda_i}, \\
K_2(\omega) &:= 2\left(\|\Gamma(\omega)\|^2_{L(H,V')} + \sup_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i}\right).
\end{aligned}$$

By Lemma 5.2 and the ergodic theorem again we have $t \mapsto K_j(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for $j = 1, 2$ on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure. Collecting (5.9)-(5.11) we obtain

$$\begin{aligned}
\langle F(\omega, v), v\rangle &= -\langle \Gamma(\omega)v, v\rangle + \sum_{i\in\mathbb{Z}}\left(\lambda_i v_i^2 - z_i(\omega)v_i^2 - (e^{-z_i(\omega)}f_i(e^{z_i(\omega)}v_i)v_i\right) \\
&\leq K_1(\omega) + K_2(\omega)\|v\|^2_H + \frac{1}{2}\|v\|^2_V, \quad \forall v \in V.
\end{aligned}$$

Hence, **(F2)** is proved. It remains to show that F satisfies **(F3)**. In fact, for any $x, y \in V$ we have

$$\begin{aligned}
-\langle \Gamma(\omega)(x-y), (x-y)\rangle &\leq \frac{1}{4\varepsilon}\|\Gamma(\omega)\|^2_{L(H,V')}\|x-y\|^2_H + \varepsilon\|x-y\|^2_V, \qquad (5.12) \\
\sum_{i\in\mathbb{Z}} z_i(\omega)(x_i - y_i)^2 &\leq \frac{1}{4\varepsilon}\sup_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i}\|x-y\|^2_H + \varepsilon\|x-y\|^2_V, \qquad (5.13)
\end{aligned}$$

and according to **(f3)** we have

$$
\begin{aligned}
&-\sum_{i\in\mathbb{Z}} e^{-z_i(\omega)}\left(f_i(e^{z_i(\omega)}x_i)-f_i(e^{z_i(\omega)}y_i)\right)(x_i-y_i)\\
&\quad\le \sum_{i\in\mathbb{Z}}[l-2\varepsilon\lambda_i](x_i-y_i)^2\\
&\quad\le l\|x-y\|_H^2-2\varepsilon\|x-y\|_V^2. \qquad (5.14)
\end{aligned}
$$

Define

$$
M(\omega):=l+\frac{1}{4\varepsilon}\left(\|\Gamma(\omega)\|^2_{L(H,V')}+\sup_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i}\right).
$$

Then $M(\omega)\in L^1(\Omega)$ and, by the ergodic theorem and Lemma 5.2, we have $t\mapsto M(\theta_t\omega)\in L^1_{\mathrm{loc}}(\mathbb{R})$ on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure. Collecting (5.12) - (5.14) we obtain

$$
\begin{aligned}
&\langle -A(x-y)+F(\omega,x)-F(\omega,y),x-y\rangle\\
\le\ & -\sum_{i\in\mathbb{Z}}\lambda_i(x_i-y_i)^2-\langle\Gamma(\omega)(x-y),(x-y)\rangle\\
&+\sum_{i\in\mathbb{Z}}\Big[\lambda_i(x_i-y_i)^2+z_i(\omega)(x_i-y_i)^2\\
&-e^{-z_i(\omega)}\left(f_i(e^{z_i(\omega)}x_i)-f_i(e^{z_i(\omega)}y_i)\right)(x_i-y_i)\Big]\\
\le\ & M(\omega)\|x-y\|_H^2,\quad \forall x,y\in V.
\end{aligned}
$$

Finally,

$$
\|G(\omega)\|^2_{V'}=\sum_{i\in\mathbb{Z}}\frac{1}{\lambda_i}g_i^2e^{-2z_i(\omega)}\le\|g\|_H^2\sup_{i\in\mathbb{Z}}\frac{e^{-2z_i(\omega)}}{\lambda_i}:=K_3(\omega)\in L^1(\Omega), \quad (5.15)
$$

i.e., G satisfies **(F4)**.

Therefore, from Theorem 4.2 we conclude immediately the following result.

Theorem 5.3. *There exists a θ–invariant set $\tilde{\Omega}$ of full measure such that for any $\nu_0\in H$ and any $T>0$ equation (5.2) has a unique weak solution $v=(v_i)_{i\in\mathbb{Z}}\in\mathcal{C}([0,T];H)\cap L^2(0,T;V)$ on $[0,T]$, with initial condition $v(0)=\nu_0$, and a weak derivative in $L^2(0,T;V')$.*

The following lemma provides some qualitative and quantitative properties of the random variables K_1 and K_3 appearing above.

Lemma 5.4. *The random variables $K_1(\omega)$ and $K_3(\omega)$ are tempered.*

Proof. To obtain the temperedness of K_1 we have to show that

$$
\mathbb{E}\sup_{t\in[0,1]}\log^+K_1(\theta_t\omega)\le\mathbb{E}\sup_{t\in[0,1]}K_1(\theta_t\omega)<\infty.
$$

Noting that

$$z_i(\theta_t\omega) = z_i(\omega) - \int_0^t z_i(\theta_s\omega)ds + \sigma_i\omega_i(t),$$

we consider

$$\begin{aligned}\mathbb{E}e^{\sup_{t\in[0,1]}|z_i(\theta_t\omega)|} \leq &\left(\mathbb{E}e^{4|z_i(\omega)|}\right)^{\frac{1}{4}}\left(\mathbb{E}e^{\int_0^1 4|z_i(\theta_s\omega)|ds}\right)^{\frac{1}{4}} \\ &\cdot\left(\mathbb{E}e^{4\sigma_i\sup_{t\in[0,1]}\omega_i(t)}\right)^{\frac{1}{4}}\left(\mathbb{E}e^{4\sigma_i\sup_{t\in[0,1]}(-\omega_i(t))}\right)^{\frac{1}{4}}. \qquad (5.16)\end{aligned}$$

First, since each z_i is a Gauß random variable,

$$\mathbb{E}e^{4|z_i(\omega)|} < \infty. \qquad (5.17)$$

Next, due to the Jensen inequality, the Fubini Theorem, and the fact that each z_i is a Gauß random variable, we have

$$\mathbb{E}e^{\int_0^1 4|z_i(\theta_s\omega)|ds} \leq \mathbb{E}\int_0^1 e^{4|z_i(\theta_s\omega)|}ds < \infty. \qquad (5.18)$$

Finally, by the properties of $\omega_i(t)$ we have

$$\mathbb{E}e^{4\sigma_i\sup_{t\in[0,1]}\pm\omega_i(t)} \leq \int_0^\infty e^{4\sigma_i s}\sqrt{\frac{2}{\pi}}e^{-\frac{s^2}{2}}ds < \infty. \qquad (5.19)$$

Thanks to (5.16) - (5.19) and the facts that $(\lambda_i^{-1})_{i\in\mathbb{Z}} \in \ell^2$ and $(z_i)_{i\in\mathbb{Z}}$ is an iid–sequence we deduce the temperedness of K_1 and K_3. $\square$

Further, we shall obtain some useful estimates for the solutions v to the lattice random differential equation (5.2).

Lemma 5.5. *For any $\omega\in\Omega$, there exist functions $J_1(K,T,\omega)$ and $J_2(K,T,\omega)$, which are bounded for any (K,T) in bounded sets, such that*

$$\|v\|^2_{\mathcal{C}([0,T],H)} \leq J_1(\|\nu_0\|_H, T, \omega), \quad \int_0^T \|v(t)\|_V^2 dt \leq J_2(\|\nu_0\|_H, T, \omega).$$

Proof. First notice that $\langle A_1 v, v\rangle = 2\rho\|v\|_H^2$ and $\langle A_2 v, v\rangle = \|v\|_V^2$. Then from the estimate of $\langle F(\omega,v),v\rangle$ and (5.15) we obtain

$$\begin{aligned}&\frac{1}{2}\|v(t)\|_H^2 + 2\rho\int_0^t\|v(s)\|_H^2 ds + \int_0^t \|v(s)\|_V^2 \\ &\leq \frac{1}{2}\|\nu_0\|_H^2 + \int_0^t (K_1(\theta_s\omega) + K_3(\theta_s\omega))ds \\ &\quad + \int_0^t K_2(\theta_s\omega)\|v(s)\|_H^2 ds + \frac{1}{2}\int_0^t\|v(s)\|_V^2,\end{aligned}$$

and consequently the so-called energy inequality:

$$
\begin{aligned}
&\|v(t)\|_H^2 + 4\rho\int_0^t \|v(s)\|_H^2 ds + \int_0^t \|v(s)\|_V^2 \\
&\le \|\nu_0\|_H^2 + 2\int_0^t (K_1(\theta_s\omega) + K_3(\theta_s\omega))ds + 2\int_0^t K_2(\theta_s\omega)\|v(s)\|_H^2 ds
\end{aligned}
\tag{5.20}
$$

Define

$$
Q(\omega) := -\lambda_0 - 4\rho + 2K_2(\omega).
$$

Then for any $\omega \in \Omega$ the mapping $t \mapsto Q(\theta_t\omega)$ is locally integrable. Moreover, by using the Gronwall Lemma we obtain

$$
\|v(t)\|_H^2 \le e^{\int_0^t Q(\theta_r\omega)dr}\|\nu_0\|_H^2 + 2\int_0^t e^{\int_s^t Q(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds. \tag{5.21}
$$

It follows immediately that $\|v\|^2_{\mathcal{C}([0,T],H)} \le J_1(\|\nu_0\|_H, T, \omega)$, where $J_1(\|\nu_0\|_H, T, \omega)$ is defined by

$$
J_1(\|\nu_0\|_H, T, \omega) := e^{\int_0^T |Q(\theta_r\omega)|dr}\|\nu_0\|_H^2 + 2\int_0^T e^{\int_s^T |Q(\theta_r\omega)|dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.
$$

By inequality (5.20), we have that $\int_0^T \|v(t)\|_V^2 dt \le J_2(\|\nu_0\|_H, T, \omega)$, where

$$
\begin{aligned}
J_2(\|\nu_0\|_H, T, \omega) \quad := \quad & \|\nu_0\|_H^2 + 2\int_0^T K_2(\theta_s\omega)J_1(\|\nu_0\|_H, T, \omega)ds \\
& + 2\int_0^T (K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.
\end{aligned}
$$

□

Up to here we have shown a global existence and uniqueness result for (5.2). Denote by $v(t,\omega,\nu_0)$ the solution to (5.2) at time $t \ge 0$ starting in $\nu_0 \in H$ for a noise–path $\omega \in \Omega$.

Theorem 5.6. *(i) Let $\omega \in \Omega$ and assume that* **(f1) - (f4)** *hold. Then the solution to system* (5.2) *generates a continuous random dynamical system $\varphi(t,\omega,\nu_0)$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space H:*

$$
\varphi(t,\omega,\nu_0) = v(t,\omega,\nu_0), \quad \nu_0 \in H,\ t \ge 0,\ \omega \in \Omega.
$$

(ii) Suppose that $\mathbb{E}Q < 0$. Then the random dynamical system φ has a closed positively invariant tempered pullback absorbing set $\mathcal{K}(\omega)$ in H given by the ball in H, $\mathbb{B}_H(0, R(\omega))$, centered at 0 with radius

$$
R(\omega) := 2\left(\int_{-\infty}^0 e^{\int_s^0 Q(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds\right)^{\frac{1}{2}}.
$$

Proof. (i) This follows directly from Theorem 4.2.

(ii) According to (5.21) we have

$$\|\varphi(t,\omega,\nu_0)\|_H^2 \le e^{\int_0^t Q(\theta_r\omega)dr}\|\nu_0\|_H^2 + 2\int_0^t e^{\int_s^t Q(\theta_r\omega)dr}(K_1(\theta_s\omega)+K_3(\theta_s\omega))ds.$$

Replacing ω by $\theta_{-t}\omega$ in φ we obtain

$$\begin{aligned}\|\varphi(t,\theta_{-t}\omega,\nu_0)\|_H^2 &\le e^{\int_0^t Q(\theta_{r-t}\omega)dr}\|\nu_0\|_H^2\\ &\quad +2\int_0^t e^{\int_s^t Q(\theta_{r-t}\omega)dr}(K_1(\theta_{s-t}\omega)+K_3(\theta_{s-t}\omega))ds\\ &= e^{\int_{-t}^0 Q(\theta_r\omega)dr}\|\nu_0\|_H^2\\ &\quad +2\int_{-t}^0 e^{\int_s^0 Q(\theta_r\omega)dr}(K_1(\theta_s\omega)+K_3(\theta_s\omega))ds.\end{aligned}$$

First notice that since $\mathbb{E}Q<0$, then for any $\nu_0\in B(\theta_{-t}\omega)$,

$$\lim_{t\to\infty} e^{\int_{-t}^0 Q(\theta_r\omega)dr}\|\nu_0\|_H^2 \le \lim_{t\to\infty} e^{\int_{-t}^0 Q(\theta_r\omega)dr}\mathrm{d}(B(\theta_{-t}\omega))^2 = 0.$$

In addition by the temperedness of K_1 and K_3 we have

$$\int_{-\infty}^0 e^{\int_s^0 Q(\theta_r\omega)dr}(K_1(\theta_s\omega)+K_3(\theta_s\omega))ds<\infty.$$

Letting

$$R^2(\omega)=4\int_{-\infty}^0 e^{\int_s^0 Q(\theta_r\omega)dr}(K_1(\theta_s\omega)+K_3(\theta_s\omega))ds,$$

gives that the ball $\mathcal{K}(\omega):=\overline{\mathbb{B}_H(0,R(\omega))}$ is a pullback absorbing set in H. The temperedness of $R(\omega)$ follows immediately from $\mathbb{E}Q<0$ and the fact that K_1 and K_3 are tempered. It is easy to show that $K(\omega)$ is positively invariant. □

Consider some $\kappa>0$ satisfying

$$\sum_{i\in\mathbb{Z}}\lambda_i^{\kappa-1}<\infty,$$

see the beginning of this section. We now introduce the space

$$V_\kappa=\{u=(u_i)_{i\in\mathbb{Z}}:\sum_{i\in\mathbb{Z}}\lambda_i^\kappa u_i^2:=\|u\|_{V_\kappa}^2<\infty\},$$

equipped with the inner product

$$(u,v)_{V_\kappa}=\sum_{i\in\mathbb{Z}}\lambda_i^\kappa u_iv_i,\quad u=(u_i)_{i\in\mathbb{Z}},\qquad v=(v_i)_{i\in\mathbb{Z}}.$$

Then we have the compact imbedding $V_\kappa\subset H$ (see [31] page 94). It then remains to show the compactness of the random dynamical system $\varphi(t,\omega,\nu_0)$, which will be presented in the following lemma.

Lemma 5.7. *There exists a full θ-invariant set $\tilde{\Omega}$ of Ω such that for any $\omega \in \tilde{\Omega}$ there exists a function $J_3(K,\omega)$ which is bounded for $K \geq 0$ in a bounded set such that*

$$\|\varphi(1,\omega,\nu_0)\|^2_{V_\kappa} \leq J_3(\|\nu_0\|_H,\omega).$$

Proof. Consider the Galerkin-approximations $v^{(n)}$ as defined in (4.4) and notice that

$$\begin{aligned}\frac{d}{dt}\left(t\|v^{(n)}(t)\|^2_{V_\kappa}\right) &= \|v^{(n)}(t)\|^2_{V_\kappa} + t\frac{d}{dt}\|v^{(n)}(t)\|^2_{V_\kappa} \\ &= \|v^{(n)}(t)\|^2_{V_\kappa} + 2t\left(\frac{d}{dt}v^{(n)}(t), v^{(n)}(t)\right)_{V_\kappa}.\end{aligned} \tag{5.22}$$

Integrating (5.22) over the interval $[0,1]$ gives

$$\|v^n(1)\|_{V_\kappa} = \int_0^1 \|v^{(n)}(t)\|^2_{V_\kappa} dt + \int_0^1 2t\left(\frac{d}{dt}v^{(n)}(t), v^{(n)}(t)\right)_{V_\kappa} dt. \tag{5.23}$$

The first term on the right hand side of (5.23) satisfies

$$\int_0^1 \|v^{(n)}(t)\|^2_{V_\kappa} dt \leq \lambda_0^{\kappa-1}\int_0^1 \|v^{(n)}(t)\|^2_V dt \leq \lambda_0^{\kappa-1} J_2(\|\nu_0\|_H, 1, \omega).$$

To estimate the second term on the right hand side of (5.23), we first estimate step by step

$$\left(\frac{d}{dt}v^{(n)}(t), v^{(n)}(t)\right)_{V_\kappa} = \left(-A_1 v^{(n)} - A_2 v^{(n)} + F(v^{(n)}) + G, v^{(n)}\right)_{V_\kappa}. \tag{5.24}$$

It is straightforward to obtain

$$\left(-A_1 v^{(n)}, v^{(n)}\right)_{V_\kappa} = -2\rho\|v^{(n)}\|^2_{V_\kappa}; \tag{5.25}$$

$$\left(-A_2 v^{(n)}, v^{(n)}\right)_{V_\kappa} = -\|v^{(n)}\|^2_{V_{1+\kappa}}. \tag{5.26}$$

By definition (5.3) we have

$$\begin{aligned}\left(F(v^{(n)}), v^{(n)}\right)_{V_\kappa} &= \left(-\Gamma(\omega)v^{(n)}, v^{(n)}\right)_{V_\kappa} \\ &\quad + \sum_{i\in\mathbb{Z}} \lambda_i^\kappa\left(\lambda_i(v_i^{(n)})^2 + z_i(\omega)(v_i^{(n)})^2 - e^{-z_i} f_i(e^{z_i} v_i^{(n)})v_i^{(n)}\right)\end{aligned} \tag{5.27}$$

The terms on the right-hand side of (5.27) have the following estimates:

$$
\begin{aligned}
\left(-\Gamma(\omega)v^{(n)}, v^{(n)}\right)_{V_\kappa} &= \rho\sum_{i\in\mathbb{Z}}\lambda_i^\kappa(e^{z_{i-1}(\omega)-z_i(\omega)}v_{i-1}^{(n)} + e^{z_{i+1}(\omega)-z_i(\omega)}v_{i+1}^{(n)})v_i^{(n)} \\
&\leq 2\rho^2\sum_{i\in\mathbb{Z}}\frac{\left(e^{z_{i-1}(\omega)-z_i(\omega)}v_{i-1}^{(n)} + e^{z_{i+1}(\omega)-z_i(\omega)}v_{i+1}^{(n)}\right)^2}{\lambda_i^{1-\kappa}} \\
&\quad +\frac{1}{4}\sum_{i\in\mathbb{Z}}\lambda_i^{1+\kappa}(v_i^{(n)})^2 \\
&\leq K_4(\omega)\|v^{(n)}\|_H^2 + \frac{1}{4}\|v^{(n)}\|_{V_{1+\kappa}}^2, \qquad (5.28)
\end{aligned}
$$

where

$$
K_4(\omega) := 4\rho^2\left(\sup_{i\in\mathbb{Z}}\left(\frac{e^{2z_{i-1}(\omega)-2z_i(\omega)}}{\lambda_i^{1-\kappa}}\right) + \sup_{i\in\mathbb{Z}}\left(\frac{e^{2z_{i+1}(\omega)-2z_i(\omega)}}{\lambda_i^{1-\kappa}}\right)\right) \in L^1(\Omega);
$$

$$
\sum_{i\in\mathbb{Z}}\lambda_i^{\kappa+1}(v_i^{(n)})^2 = \|v^{(n)}\|_{V_{1+\kappa}}^2; \qquad (5.29)
$$

$$
\begin{aligned}
\sum_{i\in\mathbb{Z}}\lambda_i^\kappa z_i(\omega)(v_i^{(n)})^2 &\leq \sum_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i^{1-\kappa}}(v_i^{(n)})^2 + \frac{1}{4}\sum_{i\in\mathbb{Z}}\lambda_i^{1+\kappa}(v_i^{(n)})^2 \\
&\leq \sup_{i\in\mathbb{Z}}\frac{|z_i(\omega)|^2}{\lambda_i^{1-\kappa}}\|v^{(n)}\|_H^2 + \frac{1}{4}\|v^{(n)}\|_{V_{1+\kappa}}^2; \quad (5.30) \\
-\sum_{i\in\mathbb{Z}}\lambda_i^\kappa e^{-z_i}f_i(e^{z_i}v_i^{(n)})v_i^{(n)} &= -\sum_{i\in\mathbb{Z}}\lambda_i^\kappa e^{-2z_i}f_i(e^{z_i}v_i^{(n)})e^{z_i}v_i^{(n)} \\
&\leq \sum_{i\in\mathbb{Z}}\lambda_i^\kappa e^{-2z_i}\left(\gamma_i^2 - \frac{3\lambda_i}{4}e^{2z_i}(v_i^{(n)})^2\right) \\
&\leq \sup_{i\in\mathbb{Z}}\frac{e^{-2z_i(\omega)}}{\lambda_i^{1-\kappa}}\|\gamma\|_V^2 - \frac{3}{4}\|v^{(n)}\|_{V_{1+\kappa}}^2. \qquad (5.31)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
(G, v^{(n)})_{V_\kappa} &\leq \sum_{i\in\mathbb{Z}}\frac{g_i^2e^{-2z_i}}{\lambda_i^{1-\kappa}} + \frac{1}{4}\sum_{i\in\mathbb{Z}}\lambda_i^{1+\kappa}(v_i^{(n)})^2 \\
&\leq \sup_{i\in\mathbb{Z}}\frac{e^{-2z_i(\omega)}}{\lambda_i^{1-\kappa}}\|g\|_H^2 + \frac{1}{4}\|v^{(n)}\|_{V_{1+\kappa}}^2. \qquad (5.32)
\end{aligned}
$$

Collecting (5.24) through (5.32) and simplifying the sum we obtain

$$
\begin{aligned}
\left(\frac{d}{dt}v^{(n)}(t), v^{(n)}(t)\right)_{V_\kappa} &\leq -2\rho\|v^{(n)}\|_{V_\kappa}^2 + K_6(\theta_t\omega) \\
&\quad + \left(K_4(\theta_t\omega) + K_5(\theta_t\omega)\right)\|v^{(n)}\|_H^2 \\
&\leq \left(K_4(\theta_t\omega) + K_5(\theta_t\omega)\right)J_1(\|\nu_0\|, 1, \omega) + K_6(\theta_t\omega),
\end{aligned}
$$

where

$$\begin{aligned} K_5(\omega) &:= \sup_{i\in\mathbb{Z}} \frac{|z_i(\omega)|^2}{\lambda_i^{1-\kappa}}; \\ K_6(\omega) &:= \sup_{i\in\mathbb{Z}} \frac{e^{-2z_i(\omega)}}{\lambda_i^{1-\kappa}}(\|\gamma\|_V^2 + \|g\|_H^2). \end{aligned}$$

Again due to the fact the $\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa-1} < \infty$ and the Gauß character of z_i we have that $\mathbb{E}K_j < \infty$ for $j = 4, 5, 6$. Then, by (5.23) we obtain that there exists a constant $J_3(\|\nu_0\|, \omega)$ such that

$$\|v^{(n)}(1)\|_{V_\kappa} \leq J_3(\|\nu_0\|_H, \omega). \tag{5.33}$$

Noticing that the right hand side of (5.33) is independent of n, we have

$$\|v(1)\|_{V_\kappa} \leq J_3(\|\nu_0\|_H, \omega),$$

which completes the proof. □

Finally we are ready to state the main result of this work in the following theorem.

Theorem 5.8. *Let φ be the continuous random dynamical system generated by (5.2). Then φ has a unique random attractor with respect to the system $\mathcal{D}$ given by the tempered random sets.*

Proof. We know from Theorem 5.6 that φ is a continuous random dynamical system having a positively–invariant absorbing set $\mathcal{K} \in \mathcal{D}$. Define

$$B(\omega) := \overline{\varphi(1, \theta_{-1}\omega, \mathcal{K}(\theta_{-1}\omega))}^H \subset \mathcal{K}(\omega)$$

where the latter inclusion follows by the positive invariance of $\mathcal{K}$. This inclusion ensures that $B \in \mathcal{D}$. By Lemma 5.7 the set B is compact and straightforwardly B is absorbing. The existence of a random global attractor follows directly from Proposition 2.7. □

6. Closing Remarks

This work is motivated by realizing the physical limitation of considering exactly the same multiplicative noise at each node in a lattice system under random influences. In particular, when the randomness comes from an environmental noise that affects the whole system, but differently at each node, it is more realistic to consider a different noise (but with similar structure) at different node. Driven by this motivation, we studied in this work a stochastic lattice dynamical system with different multiplicative noise at each different node. This has never been done in the literature, mainly due to its technical difficulty.

One major difficulty is that due to the infinite structure of the noise, we have to perform a change of variables by using iid-sequence of one-dimensional Ornstein-Uhlenbeck processes, before writing the system in an operator formulation. This requires an extended definition of the traditional Wiener-shift. The resulting random system involves an operator that does not necessarily map the space ℓ^2 into itself. Therefore we have to formulate the system as an abstract random evolution equation, instead of an ordinary operator-differential equation. This triggers the study of existence and uniqueness for weak solutions to an random evolution equation over a Gelfand evolution triplet. Note that this result can be applied to handle more general systems than the lattice system considered in this work.

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