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Some Properties of Complex Filiform Lie Algebras

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1. Introduction and Notations

The purpose of this paper is to study some properties of Filiform Lie Algebras (*FLA*) and to prove the following theorem: A *FLA*, of dimension n, is either derived from a Solvable Lie Algebra (*SLA*) of dimension $n+1$ or not derived from any *LA*.

From now on, we write $(A,B,C)=0$ to represent the Jacobi identity $[[A,B],C]+[[B,C],A]+[[C,A],B]=0$ and we denote by $\mathfrak{M}$ a complex *FLA* of dimension n [4], that is, a complex nilpotent *LA* admitting a base (1) $\{X_1,\dots,X_n\}$ such that

$$[X_1,X_2]=0\,,\ [X_1,X_j]=X_{j-1}\,,\ 3\leqslant j\leqslant n.$$

$\mathfrak{L}$ will denote a complex *SLA* of dimension $n+p$ such that (2) $\{X_1,\dots,X_n, U_1,\dots,U_p\}$ is a basis, where $\{X_1,\dots,X_n\}$ is the basis (1) above mentioned, and U_h $(h=1,\dots,p)$ are derivations of $\mathfrak{M}$ such that $(X_j,X_h,U_q)=0$, $1\leqslant j,h\leqslant n$, $1\leqslant q\leqslant p$.

First, we deduce some properties of *FLA*. Secondly, we study the conditions for the *FLA* $\mathfrak{M}$ to be derived from the *SLA* $\mathfrak{L}$, that is, for $[\mathfrak{L},\mathfrak{L}]=\mathfrak{M}$. We will use the following notation [3]:

$$[X_i,U_j]=\textstyle\sum_{h=1}^{n} a_{ij}^{h} X_h\,;\ [U_j,U_h]=\sum_{k=1}^{n} b_{jh}^{k} X_k\,;\ [X_i,X_j]=\sum_{l=1}^{n} c_{ij}^{l} X_l$$

Finally, i will denote the smallest natural number greater than 1 such that $[X_i,X_n]\neq 0$.

2. Some Properties of *FLA*

We distinguish two cases, depending on i:

CASE 1: Suppose that i does not exist, that is, $[X_j,X_n]=0$, $\forall j>1$.
In this case, we prove [3] then following:

THEOREM 2.1. *$[X_j,X_h]=0$ for all $j,h>1$.*

As a consequence, the only possible non-zero brackets are $[X_1,X_j]=-[X_j,X_1]=X_{j-1}$ $(j=3,...,n)$. Therefore, there exists only one *FLA* of this kind for each dimension.

CASE 2: Suppose that i exists and $[X_i,X_n]=c_{in}^{i-h}X_{i-h}+...+c_{in}^{2}X_2$. In this case we prove the following:

LEMMA 2.2. *$[X_j,X_h]=0$ for all $h>1$; $1<j<i$.*

THEOREM 2.3. *i is independent of the basis* (1).

As a consequence, the number h in the coefficient c_{in}^{i-h} is also independent of the basis (1).

THEOREM 2.4. *If $[X_i,X_n]=c_{in}^{i-h}X_{i-h}+...+c_{in}^{2}X_2$, then*

$$[X_i,X_{n-1}]=c_{in}^{i-h}X_{i-h-1}+...+c_{in}^{3}X_2.$$
$$[X_i,X_{n-2}]=c_{in}^{i-h}X_{i-h-2}+...+c_{in}^{4}X_2.$$
$$\dots$$
$$[X_i,X_{n-i+h+2}]=c_{in}^{i-h}X_2.$$

THEOREM 2.5. $c_{in}^{i-1}=...=c_{in}^{j-1}=...=c_{n-1,n}^{n-2}$.

THEOREM 2.6. *If $c_{in}^{i-1}\neq 0$, then $i=4$.*

THEOREM 2.7. *If the coefficient $c_{4n}^{3}\neq 0$, then n is even.*

3. Conditions for a *FLA* to be Derived from another *SLA*

THEOREM 3.1. *A necessary condition for a FLA $\mathfrak{M}$ to be derived from the SLA $\mathfrak{L}$ is $a_{1j}^{1}\neq 0$ for some j $(j=1,2,...,p)$.*

To prove this theorem, we previously use the following: [3]

LEMMA 3.2. $a_{nh}^{1}=0$, $\forall h\geqslant 2$.

LEMMA 3.3. $a_{3h}^{3}=a_{2h}^{2}-a_{1h}^{1}$, $\forall h\geqslant 2$.

LEMMA 3.4. *If* $a^1_{1h}=0$, $\forall h=1,\dots,p$, *then* $b^1_{jh}=0$, $\forall\, 1\leqslant j,h\leqslant p$.

To study now if sufficiency also holds, we consider the following three cases:

CASE 1. Suppose that i does not exists, that is, $[X_j,X_h]=0$, $\forall j>1$. In this case, according to theorem 2.1, there only exists one *FLA* of this kind for each dimension n. This single algebra is always derived from a *SLA* of dimension $n+1$, one of it basis is $\{X_1,\dots,X_n,U\}$, with $[X_j,U]=\lambda_j X_j$, $\forall j$, and $\lambda_j=\lambda_2-(j-2)\lambda_1$, $3\leqslant j\leqslant n$ with $\lambda_1,\lambda_2\neq 0$.

CASE 2. Suppose that i exists and $c^{i-1}_{in}=0$, that is,

$$[X_i,X_n]=c^{i-q}_{in}X_{i-q}+\dots+c^3_{in}X_3+c^2_{in}X_2 \quad (1<q<i-2).$$

In this case, we prove the following

LEMMA 3.5. $a^{j-1}_{j-1,h}-a^j_{j,h}=a^1_{1,h}-a^n_{1,h}\,c^{j-1}_{jn}$ $(j\geqslant 3)$.

LEMMA 3.6. $a^n_{n,h}=q\,a^1_{1,h}$ $(q=1,2,\dots,i-2)$.

Consequently, if $a^1_{1,h}\neq 0$ for some h, then also $a^n_{n,h}\neq 0$. So, the *FLA* $\mathfrak{M}$, of dimension n, will be derived from an *SLA* $\mathfrak{L}$, of dimension $n+1$ having a basis $\{X_1,\dots,X_n,U_h\}$, where $\{X_1,\dots,X_n\}$ is the basis (1) of $\mathfrak{M}$.

CASE 3. Suppose that i exists and $c^{i-1}_{in}\neq 0$. In this case, $i=4$ (th. 2.6) and $c^3_{4n}=c^{j-1}_{jn}$ (th. 2.5). Taking lemma 3.5 into account we prove the following

LEMMA 3.7. $a^n_{n,h}=a^1_{1,h}-a^n_{1,h}\,c^3_{4n}$.

Then we distinguish:

CASE 3.1. If $a^1_{1,h}\neq 0$ and $a^n_{1,h}=0$ for some h, then $a^1_{1,h}=a^n_{n,h}\neq 0$. So $\mathfrak{M}$ will be derived from the *SLA* $\mathfrak{L}_h$, with $\{X_1,\dots,X_n,U_h\}$ as a basis. So sufficiency is also verified in this subcase.

CASE 3.2. If $a^1_{1,h}\neq 0$ and $a^n_{1,h}\neq 0$, then $a^1_{1,h}=a^n_{n,h}$. Therefore if $a^n_{n,h}\neq 0$ we obtain the same conclusion as in the case 3.1, but if $a^n_{n,h}=0$ then $\mathfrak{M}$ will not be derived from any *LA*, due to $X_n\notin\mathfrak{M}$ in this subcase, as we proved in [3].

So, sufficiency is not verified in this subcase of the case 3.2. only.

An immediate consequence of this th. 3.1 is the following

MAIN THEOREM 3.8. *A filiform Lie Algebra, of dimension n, is either derived from a SLA of dimension n+1 or not derived from any LA.*

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