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Complex Nilpotent Lie Algebras of Dimension 7 which are not Derived from Others

FRANCISCO J. ECHARTE[1], JOSÉ R. GÓMEZ[2] AND JUAN NÚÑEZ[1]

[1] *Dpto. de Algebra, Computación, Geometría y Topología, Facultad de Matemáticas, Univ. de Sevilla, 41012-Sevilla, Spain*
[2] *Dpto. de Matemática Aplicada I, Facultad de Informática y Estadística, Univ. de Sevilla, 41012-Sevilla, Spain*

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INTRODUCTION AND NOTATIONS

There does not exist, at present, any classification of complex Nilpotent Lie Algebras of dimension greater than 7. Goze and Ancochea, by the introduction of a new invariant which they call the "characteristic sequence", which corresponds to the maximal dimensions of Jordan blocks of a certain nilpotent matrix, obtained the classification of complex nilpotent Lie Algebras of dimension 7 [2], and later, by the same way, they also obtained the classification of complex filiform Lie Algebras of dimension 8 [1]. Filiform Lie algebras, as it is known, are a subclass of nilpotent Lie algebras.

However, the importance of these authors' work is not only to have obtained these classifications, but, above all, to have devised techniques which can be applied in any dimension. So, Echarte and Gómez Martín [4] and recently Boza, Echarte and Núñez [3] classified by this way, filiform Lie algebras of dimension 9 and 10, respectively.

Therefore, as Goze and Ancochea totally solved the problem of the classification of Filiform Lie algebras of greater dimension, although, of course, this requires hard and complicated calculations which are not easy without the use of a computer, the main problem which is now considered is the search for new results, rather than the classification itself, such as the attainment of new invariants or the description of the irreducible components of varieties of Lie algebras.

In this sense, we present in this paper the list of the complex nilpotent Lie algebras of dimension 7 which are not derived from other Lie algebras. Since a complex filiform Lie algebra is a characteristically nilpotent Lie algebra (that is, all its derivations are nilpotent), if and only if that algebra is not a derived algebra [6], the knowlwdge of complex nilpotent Lie algebras which are not derived could be useful to obtain next the classification of characteristically nilpotent Lie algebras, which does not exist, at present, for any dimension. To obtain this result, we develop a procedure which permits to determine if a complex nilpotent Lie algebra is a derived Lie algebra.

Let **L** be a solvable Lie algebra. The complex nilpotent Lie algebra **M** is said to be a derived algebra from **L** if $[L,L] \equiv M$, where $[L,L]$ represents any linear combination of all brackets among fields of a basis of **L** . It is known that a Lie algebra is solvable if and only if its derived algebra is nilpotent. However, the correspondence $L \mapsto [L,L]$ is not a bijection.

From now on, we denote by B_M a basis $\{e_1, \dots, e_n\}$ of **M** and by B_L the corresponding basis $= \{e_1, \dots, e_n, u_1, \dots, u_p\}$ of **L** . If we suppose that **M** is a derived algebra from **L** , then we have $[e_i, u_j] = \sum_{h=1}^{n} a_{i,j}^h e_h$ Hence, by using the Jacobi identity $[[e_i,e_j],u_h] + [[e_j,u_h],e_i] + [[u_h,e_i],e_j] = 0$, which we will denote by $(e_i, e_j, u_h) = 0$, we can obtain the relations between the coefficients $a_{i,j}^h$.

1. Main Theorem

Theorem 1.1. *Between the* 120 *complex nilpotent Lie algebras of dimension* 7 *of Goze and Ancochea's classification, there are only* 9 *algebras and one uniparametric family of algebras which are not derived from other Lie algebras. These algebras, defined by one of their basis and the non-zero brackets, are the following:*

g_1
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 3) \\ [e_4,e_7] &= e_2 \\ [e_5,e_7] &= e_3 \\ [e_6,e_7] &= e_2+e_4 \end{aligned}$$

g_2
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 3) \\ [e_4,e_7] &= e_2 \\ [e_5,e_6] &= -e_2 \\ [e_5,e_7] &= e_2 \\ [e_6,e_7] &= e_3 \end{aligned}$$

g_3
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 3) \\ [e_5,e_7] &= e_2 \\ [e_6,e_7] &= e_2+e_3 \end{aligned}$$

g_4
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 4) \\ [e_2,e_6] &= e_3 \\ [e_2,e_7] &= e_3+e_4 \\ [e_5,e_7] &= \alpha e_3 \\ [e_6,e_7] &= e_2+\alpha e_4 \end{aligned}$$

g_5
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 4) \\ [e_2,e_6] &= e_3 \\ [e_2,e_7] &= e_4 \\ [e_5,e_7] &= e_3 \\ [e_6,e_7] &= e_2+e_4 \end{aligned}$$

g_6
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 4) \\ [e_2,e_7] &= e_3 \\ [e_5,e_6] &= e_3 \\ [e_5,e_7] &= e_4 \\ [e_6,e_7] &= e_2+e_5 \end{aligned}$$

g_7
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 4) \\ [e_2,e_6] &= e_3 \\ [e_2,e_7] &= e_4 \\ [e_5,e_6] &= e_3 \\ [e_5,e_7] &= e_3+e_4 \\ [e_6,e_7] &= e_4+e_5 \end{aligned}$$

g_8
$$\begin{aligned} [e_1,e_i] &= e_{i-1} \quad (i \geq 4) \\ [e_2,e_6] &= e_3 \\ [e_2,e_7] &= e_3+e_4 \\ [e_6,e_7] &= e_2 \end{aligned}$$

g_9 $[e_1,e_i]= e_{i-1}$ $(i\geq 4)$
$[e_2,e_5]= e_3$
$[e_2,e_6]= e_4$
$[e_2,e_7]= e_3+e_5$

g_{10} $[e_1,e_i]= e_{i-1}$ $(i\geq 4)$
$[e_2,e_6]= e_3$
$[e_2,e_7]= e_3+e_4$

Proof. (This is a sketch of the proof). We consider g_1. If g_1 were derived from **L** and $B_{g1}=\{e_1,\ldots,e_7\}$ is a basis of g_1 and $B_{L1}=\{e_1,\ldots,e_7,u_1,\ldots,u_p\}$ is a basis of **L**, then it will be ied $[e_i,u_j]=\sum_{h=1}^{n}a_{i,j}^{h}e_h$

Now, we use a set of iterative calculations based on all possible Jacobi identities of the kind $(e_i,e_j,u_k)=0$ with $1\leq i,j\leq 7$ and $k=1,\ldots,p$, from which we can determine whether the coefficients $a_{i,i}$ vanish or not. This in turn determines whether the Filiform Lie algebra g_1 is derived from the solvable Lie algebra **L**.

By this procedure, we obtain in this case (by U we denote any u_k), that

1. $[e_1,U]=a_{1,2}e_2+a_{1,3}e_3+a_{1,4}e_4+a_{1,5}e_5+a_{1,6}e_6$
2. $[e_2,U]=0,\quad [e_3,U]=a_{3,2}e_2,\quad [e_4,U]=a_{4,2}e_2+a_{3,2}e_3$
3. $[e_5,U]=a_{5,2}e_2+a_{4,2}e_3+a_{3,2}e_4,\quad [e_6,U]=a_{6,2}e_2+a_{5,2}e_3+a_{4,2}e_4+a_{3,2}e_5$
4. $[e_7,U]=a_{7,2}e_2+(a_{6,2}-a_{1,4}-a_{1,6})e_3+(a_{5,2}-a_{1,5})e_4+(a_{4,2}-a_{1,6})e_5+a_{3,2}e_6$

Therefore, we can see that the field e_1 is not either in the brackets $[e_i,e_j]$ or in the brackets $[e_i,u_k]$. Moreover, if we state $[u_i,u_j]=\sum_{h=1}^{7}c_{i,j}^{h}e_h$ with $1\leq i,j\leq p$, we can deduce, from $(e_3,u_i,u_j)=0$, that $c_{i,j}^{1}=0$. So, the field e_1 is never in the brackets $[u_i,u_j]$ and consequently, the complex nilpotent Lie algebra g_1 is not derived from any other Lie algebra. By applying this method to the rest of the algebras above mentioned we obtain the same result with respect to the existence of $a_{i,i}$ with $1\leq i\leq 7$.

Finally, by using the same method, it is easy to prove that if we consider any of the nilpotent Lie algebras of dimension 7 of Goze and Ancochea's classification different from $g_1,\ldots,g_{10}$, we obtain that the coefficients $a_{i,i}$ with $1\leq i\leq 7$ appear in all the expressions $[e_j,U]$ with $1\leq j\leq 7$. So, we deduce that all of these algebras (different from g_k $(1\leq k\leq 10)$) are derived from others. It finishes the proof. ∎

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