

NUMERICAL SOLUTION OF A LAPLACE EQUATION WITH DATA IN L^1 .

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Abstract. In this work, we address the numerical solution of the Laplace equation with data in L^1 by P_1 finite element schemes. Even if this is a simple problem, its analysis is difficult and requires new tools because finite element schemes are based on variational formulations which do not lend themselves to estimates in the L^1 norm.

The approach for analyzing this problem consists in applying some of the techniques that are used by Murat (cf. [5]) and Boccardo & Gallouet (cf. [2]) in constructing the renormalized solution of the problem. The key ingredient is the assumption that all the angles of the grid are acute; then the matrix of the system is an M matrix. Interestingly, with this sole assumption, we prove that u_h tends to u in measure in Ω .

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§1. Introduction

We address in this paper the numerical analysis of first degree finite element schemes approximating the solution of a Laplace equation with weak data, such as data in L^1 . Let Ω be a Lipschitz-continuous domain in two or three dimensions with a polygonal or polyhedral boundary $\partial\Omega$, cf. Grisvard [4]. Given an exterior force f in $L^1(\Omega)$, we want to find u such that

$$-\Delta u = f \text{ in } \Omega, \tag{1}$$

with the homogeneous boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{2}$$

This problem has a *unique renormalized solution* u and u belongs to $W_0^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$, where d denotes the dimension, cf. Boccardo & Gallouet [2], and Boccardo, Díaz, Giachetti & Murat [1].

Let $h > 0$ be a discretization parameter that will tend to zero and let \mathcal{T}_h be a family of triangulations of $\bar{\Omega}$. We discretize problem (1), (2) in the standard finite element space:

$$V_h = \{v_h \in C^0(\bar{\Omega}), \forall T \in \mathcal{T}_h, v_h|_T \in P_1, v_h|_{\partial\Omega} = 0\}, \tag{3}$$

and the discrete problem reads: Find $u_h \in V_h$ solution of

$$\forall v_h \in V_h, \int_{\Omega} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x}. \quad (4)$$

Note that this problem has a *unique solution*. Indeed, it is a square system of linear equations in finite dimension, and the integral in the right-hand side is well-defined because the functions of V_h belong to $L^\infty(\Omega)$ (and in fact to $W^{1,\infty}(\Omega)$). But the trouble is that the straightforward bound for this integral:

$$\left| \int_{\Omega} f v_h \, d\mathbf{x} \right| \leq \|f\|_{L^1(\Omega)} \|v_h\|_{L^\infty(\Omega)},$$

is useless as soon as the dimension is two, because the left-hand side of (4) is elliptic in $H^1(\Omega)$ and not in $L^\infty(\Omega)$. Therefore the methods of classical finite element analysis cannot be applied to (4).

We present one approach for analyzing (4). More precisely, it consists in applying some of the techniques that are used by Murat [5] and Boccardo & Gallouet [2] in constructing the renormalized solution of problem (1), (2). The key ingredient is the assumption that all the angles of the elements in \mathcal{T}_h are *acute*; then the matrix of the system (4) is an M matrix. Interestingly, with this sole assumption, we prove that u_h tends to u in measure in Ω . Recall that this condition also guarantees that the solution of the discrete problem (4) satisfies the maximum principle.

§2. The Laplace equation: solution by renormalization

Let us recall the definition of the truncation function $T_k : \mathbb{R} \mapsto [-k, k]$, for any real number $k > 0$:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ \text{sign}(s)k & \text{if } |s| > k. \end{cases} \quad (5)$$

For problem (1), (2), u is called a renormalized solution (cf. Di Perna & Lions [3]) if it satisfies:

$$u \in L^1(\Omega),$$

$$\forall k > 0, T_k(u) \in H_0^1(\Omega),$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{|u| \leq k} |\nabla u|^2 \, d\mathbf{x} = 0, \quad (6)$$

$$\forall \mu \in W_c^{1,\infty}(\mathbb{R}), \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

$$\int_{\Omega} \nabla u \cdot (\varphi \mu'(u) \nabla u) \, d\mathbf{x} + \int_{\Omega} \nabla u \cdot (\mu(u) \nabla \varphi) \, d\mathbf{x} = \int_{\Omega} f \mu(u) \varphi \, d\mathbf{x}. \quad (7)$$

Note that, as μ has a compact support, say its support is contained in $[-M, M]$ for a real number $M > 0$, then $\mu(u)(\mathbf{x}) \neq 0$ implies that $u(\mathbf{x}) = T_M(u)(\mathbf{x})$. And furthermore, since $\nabla T_M(u)(\mathbf{x}) = 0$ for all \mathbf{x} such that $|u(\mathbf{x})| > M$, we have

$$\mu'(u) \nabla u = \mu'(u) \nabla T_M(u).$$

Hence the product $\mu(u)\varphi$ belongs to $L_c^\infty(\Omega)$, $\nabla(\mu(u)\varphi)$ belongs to $L_c^2(\Omega)^d$ and all the integrals in (7) are well-defined.

The results in this section are valid in two and three dimensions. Let us make precise the family of triangulations \mathcal{T}_h : it is made of triangles in two dimensions or tetrahedra in three dimensions, with diameter bounded by h , and such that any two elements are either disjoint or share a vertex, a whole side or a whole face. We number the interior vertices of \mathcal{T}_h from 1 to N and the boundary vertices from $N + 1$ to P , and denote them by \mathbf{a}_i , $1 \leq i \leq P$. As usually, we denote by h_T the diameter of the element T and by ρ_T the diameter of the ball inscribed in T . On this family of triangulations, we define the space V_h by (3):

$$V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}), \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_1, v_h|_{\partial\Omega} = 0\},$$

the discrete problem by (4): Find $u_h \in V_h$ solution of

$$\forall v_h \in V_h, \int_{\Omega} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x},$$

and the interpolation operator $\Pi_h : \mathcal{C}^0(\overline{\Omega}) \mapsto V_h$ by

$$\Pi_h(v) \in V_h, \Pi_h(v)(\mathbf{a}_i) = v(\mathbf{a}_i), \quad 1 \leq i \leq N. \quad (8)$$

It satisfies in particular:

$$\|\Pi_h(v)\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)}. \quad (9)$$

In this section, we shall use the following assumption:

Hypothesis (H) The inner angles (dihedral angles in three dimensions) of all elements T are not larger than $\frac{\pi}{2}$.

This implies, in particular that the family of triangulations is regular in the sense of Ciarlet: there exists a constant σ , independent of h , such that

$$\forall T \in \mathcal{T}_h, \sigma_T := \frac{h_T}{\rho_T} \leq \sigma. \quad (10)$$

In practice, this assumption can be restrictive, because it complicates the construction of the triangulation. But it has the great advantage that the matrix of the system (4) is *diagonally dominant*. This matrix \mathcal{M} has entries:

$$(\mathcal{M})_{i,j} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{x},$$

where φ_i denote the basis function of V_h that takes the value 1 at \mathbf{a}_i and zero at all other nodes. Clearly, \mathcal{M} is symmetric, positive definite. The fact that it is diagonally dominant follows from the following well-known result.

Lemma 1. *Under Hypothesis (H), we have in dimension $d = 2$ or 3 :*

$$\forall i \neq j, (\mathcal{M})_{i,j} \leq 0. \quad (11)$$

Proposition 2. *Under Hypothesis (H), \mathcal{M} is diagonally dominant. It is also an M matrix.*

Then we have the following fundamental theorem.

Theorem 3. *Under Hypothesis (H), the solution u_h of (4) satisfies for all real numbers $k > 0$:*

$$|\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2 \leq k \|f\|_{L^1(\Omega)}. \quad (12)$$

Proof. Let u_h be the solution of (4) and let us choose $v_h = \Pi_h(T_k(u_h))$. Observe that $T_k(u_h)$ vanishes on $\partial\Omega$ because $u_h = 0$ on $\partial\Omega$. Therefore $\Pi_h(T_k(u_h))$ belongs to V_h and we have

$$\int_{\Omega} \nabla u_h \cdot \nabla \Pi_h(T_k(u_h)) \, d\mathbf{x} = \int_{\Omega} f \Pi_h(T_k(u_h)) \, d\mathbf{x}. \quad (13)$$

By definition, $|T_k(u_h)| \leq k$ in Ω and (9) yields:

$$\|\Pi_h(T_k(u_h))\|_{L^\infty(\Omega)} \leq k. \quad (14)$$

This gives the upper bound for the right-hand side of (13):

$$\left| \int_{\Omega} f \Pi_h(T_k(u_h)) \, d\mathbf{x} \right| \leq k \|f\|_{L^1(\Omega)}. \quad (15)$$

Now, we must derive a lower bound for the left-hand side of (13). We split it into:

$$\int_{\Omega} \nabla u_h \cdot \nabla \Pi_h(T_k(u_h)) \, d\mathbf{x} = \int_{\Omega} \nabla(u_h - \Pi_h(T_k(u_h))) \cdot \nabla \Pi_h(T_k(u_h)) \, d\mathbf{x} + |\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2, \quad (16)$$

and we must prove that

$$\int_{\Omega} \nabla(u_h - \Pi_h(T_k(u_h))) \cdot \nabla \Pi_h(T_k(u_h)) \, d\mathbf{x} \geq 0.$$

With the vector notation $\mathbf{u}_h = (u_h(\mathbf{a}_i))_{i=1}^N$ and $\mathbf{T}_k(u_h) = (T_k(u_h(\mathbf{a}_i)))_{i=1}^N$, and using the symmetry of \mathcal{M} , we can write:

$$\begin{aligned} \int_{\Omega} \nabla(u_h - \Pi_h(T_k(u_h))) \cdot \nabla \Pi_h(T_k(u_h)) \, d\mathbf{x} &= (\mathcal{M}(\mathbf{u}_h - \mathbf{T}_k(u_h)), \mathbf{T}_k(u_h)) = \\ &= (\mathbf{u}_h - \mathbf{T}_k(u_h), \mathcal{M}\mathbf{T}_k(u_h)), \end{aligned} \quad (17)$$

where (\cdot, \cdot) denotes the Euclidean scalar product. Therefore, it remains to establish that

$$(\mathbf{u}_h - \mathbf{T}_k(u_h), \mathcal{M}\mathbf{T}_k(u_h)) \geq 0. \quad (18)$$

This result follows from the discussion splits according to the value of $(\mathbf{u}_h)_i$ with respect to k . By substituting (18) into (16) and the result into (13), we derive

$$|\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2 \leq \int_{\Omega} f \Pi_h(T_k(u_h)) \, d\mathbf{x}, \quad (19)$$

and (12) follows by applying (15) to (19). \square

For $k \in \mathbb{N}$, the estimate (12), uniform in h , allows us to extract a subsequence (that we still denote by h) such that

$$\forall k \in \mathbb{N}, \lim_{h \rightarrow 0} \Pi_h(T_k(u_h)) = v_k \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \quad (20)$$

The strong convergence implies convergence in measure and convergence of a subsequence (again denoted by h) almost everywhere in Ω . Furthermore, (14) gives

$$|v_k(\mathbf{x})| \leq k \text{ a.e. in } \Omega, \quad (21)$$

thus showing that v_k is the truncation by T_k of some function. We propose to construct a function u such that

$$v_k = T_k(u), \quad \lim_{h \rightarrow 0} u_h = u \text{ in measure in } \Omega,$$

and u is the renormalized solution of (1), (2). For this, we require several technical lemmas.

Lemma 4. *Let v_k be the limit function of (20). Then, we have for any real number s such that $0 < s \leq k$:*

$$\lim_{h \rightarrow 0} T_s(\Pi_h(T_k(u_h))) = T_s(v_k) \text{ a.e. in } \Omega.$$

Proof. This convergence follows readily from the convergence almost everywhere of $\Pi_h(T_k(u_h))$ to v_k and the continuity of the truncation operator T_s with respect to this convergence. \square

Lemma 5. *For any real number s and integer k such that $0 < s < k$, let $A_h(k, s)$ be the set*

$$A_h(k, s) = \cup \{T \in \mathcal{T}_h; \exists \mathbf{x} \in T, \exists \mathbf{y} \in T, |u_h(\mathbf{x})| \geq k, |u_h(\mathbf{y})| \leq s\}. \quad (22)$$

Then

$$|A_h(k, s)| \leq \frac{k}{(k-s)^2} h^2 \|f\|_{L^1(\Omega)}. \quad (23)$$

Proof. Let $T \subset A_h(k, s)$; from the definition (22), we know that there exist two points \mathbf{x} and \mathbf{y} in T , such that

$$u_h(\mathbf{x}) \geq k \text{ or } u_h(\mathbf{x}) \leq -k \quad \text{and} \quad -s \leq u_h(\mathbf{y}) \leq s.$$

Then the fact that u_h belongs to \mathbb{P}_1 in T implies that T has two vertices, say \mathbf{a}_i and \mathbf{a}_j such that

$$u_h(\mathbf{a}_i) \geq k \text{ or } u_h(\mathbf{a}_i) \leq -k \quad \text{and} \quad -s \leq u_h(\mathbf{a}_j) \leq s.$$

Hence

$$T_k(u_h)(\mathbf{a}_i) = k \text{ or } T_k(u_h)(\mathbf{a}_i) = -k,$$

and

$$-s \leq T_k(u_h)(\mathbf{a}_j) = u_h(\mathbf{a}_j) \leq s.$$

In both cases, since the gradient of $\Pi_h(T_k(u_h))$ is a constant vector in T , this means that

$$\|\nabla \Pi_h(T_k(u_h))\| \geq \frac{k-s}{h_T} \geq \frac{k-s}{h},$$

where $\|\cdot\|$ denotes the Euclidean vector norm. Therefore

$$|\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2 \geq \int_{A_h(k,s)} \|\nabla \Pi_h(T_k(u_h))\|^2 d\mathbf{x} \geq |A_h(k,s)| \frac{(k-s)^2}{h^2},$$

and (23) follows from this lower bound and (12). \square

Lemma 5 states that the measure of $A_h(k, s)$ tends to zero as h tends to zero. The next lemma shows that the measure of the set

$$B = \{\mathbf{x} \in \Omega; T_s(\Pi_h(T_k(u_h)))(\mathbf{x}) \neq T_s(u_h)(\mathbf{x})\} \quad (24)$$

also tends to zero with h .

Lemma 6. *For each real number s and integer k with $0 < s < k$, the set B is contained in $A_h(k, s)$.*

Proof. The statement is equivalent to the implication: if $\mathbf{x} \notin A_h(k, s)$, then $\mathbf{x} \notin B$. Thus, suppose that $\mathbf{x} \notin A_h(k, s)$; the argument depends upon the position of u_h with respect to that of k and s .

- If $|u_h(\mathbf{x})| \geq k$, then all points \mathbf{y} in the same element T satisfy $|u_h(\mathbf{y})| > s$, i.e. either $u_h(\mathbf{y}) > s$ or $u_h(\mathbf{y}) < -s$. Hence we have for all points in T :

$$|T_s(u_h)| = s, \quad |T_k(u_h)| > s, \quad |\Pi_h(T_k(u_h))| > s,$$

and thus

$$|T_s(\Pi_h(T_k(u_h)))| = s.$$

As all the quantities (inside the absolute values) have the same sign, we see that $\mathbf{x} \notin B$.

- If $|u_h(\mathbf{x})| \leq s$, then all points \mathbf{y} in the same element T satisfy $|u_h(\mathbf{y})| < k$. Hence we have for all points in T :

$$T_k(u_h) = u_h, \quad \Pi_h(T_k(u_h)) = u_h \text{ and thus } T_s(\Pi_h(T_k(u_h))) = T_s(u_h),$$

this implies that $\mathbf{x} \notin B$.

- If $s < |u_h(\mathbf{x})| < k$ and if there exists $\mathbf{y} \in T$ such that $|u_h(\mathbf{y})| \leq s$, then for all points $\mathbf{z} \in T$, we must have $|u_h(\mathbf{z})| < k$, and we revert to the previous case. If there exists $\mathbf{y} \in T$ such that $|u_h(\mathbf{y})| \geq k$, then we revert to the first case. Therefore, the only possibility that remains is that $s < |u_h(\mathbf{y})| < k$ for all points $\mathbf{y} \in T$. Then we have for all points in T :

$$T_k(u_h) = u_h, \text{ and thus } \Pi_h(T_k(u_h)) = u_h,$$

this implies that $\mathbf{x} \notin B$. □

Proposition 7. *For each real number s and integer k with $0 < s < k$, we have*

$$\lim_{h \rightarrow 0} T_s(u_h) = T_s(v_k) \text{ in measure in } \Omega. \quad (25)$$

Proof. Let $0 < s < k$; for any $\varepsilon > 0$, we define the set

$$C_\varepsilon = \{\mathbf{x} \in \Omega; |T_s(u_h) - T_s(v_k)| > \varepsilon\}.$$

We are going to prove that $|C_\varepsilon|$ tends to zero with h . For this, we observe that, in view of Lemma 6,

$$|C_\varepsilon| \leq |C_\varepsilon \cap A_h(k, s)| + |C_\varepsilon \cap A_h^c(k, s)| \leq |A_h(k, s)| + |C_\varepsilon \cap B^c|. \quad (26)$$

But, if $\mathbf{x} \notin B$, then

$$T_s(\Pi_h(T_k(u_h))) = T_s(u_h).$$

As a consequence,

$$|C_\varepsilon \cap B^c| \leq |\{\mathbf{x} \in \Omega; |T_s(\Pi_h(T_k(u_h))) - T_s(v_k)| > \varepsilon\}|.$$

Now thanks to Lemma 4,

$$\lim_{h \rightarrow 0} T_s(\Pi_h(T_k(u_h))) = T_s(v_k) \text{ in measure in } \Omega.$$

This amounts to say that for any $\varepsilon > 0$, for any real number s and integer k such that $0 < s < k$,

$$\lim_{h \rightarrow 0} |C_\varepsilon \cap B^c| = 0.$$

This, together with (23), and (26) imply that $\lim_{h \rightarrow 0} |C_\varepsilon| = 0$, and (25) follows from the fact that this limit holds for any ε . \square

The uniqueness of the limit and the fact that Proposition 7 holds for any pair s and k , with $0 < s < k$, give the next corollary.

Corollary 8. *For any real number s and integers k and ℓ with $0 < s < k$ and $0 < s < \ell$, we have*

$$T_s(v_k) = T_s(v_\ell). \quad (27)$$

Corollary 9. *For any integers k and ℓ with $1 \leq \ell < k$, we have*

$$v_\ell = T_\ell(v_k) \text{ a.e. in } \Omega; \quad (28)$$

in particular, for almost all $\mathbf{x} \in \Omega$ such that $|v_k(\mathbf{x})| \leq k - 1$,

$$v_k(\mathbf{x}) = v_{k-1}(\mathbf{x}). \quad (29)$$

Proof. Applying (27) with $1 \leq \ell < k$ and $s < \ell$, we find

$$\forall s < \ell, T_s(v_k) = T_s(v_\ell). \quad (30)$$

But v_k and v_ℓ belong to $H^1(\Omega) \cap L^\infty(\Omega)$ and on this space, the truncation operator T_s is continuous with respect to s . Therefore, passing to the limit as s tends to ℓ in (30), we obtain (28), thus proving also (29). \square

This corollary suggests to define u by

$$u = \sum_{k=1}^{\infty} v_k \chi_{k-1 \leq |v_k| < k}, \quad (31)$$

i.e. for all \mathbf{x} such that $k - 1 \leq |v_k(\mathbf{x})| < k$, $u(\mathbf{x}) = v_k(\mathbf{x})$. Thus (29) implies that

$$v_k(\mathbf{x}) = T_k(u(\mathbf{x})), \quad (32)$$

and for almost every \mathbf{x} such that $|u(\mathbf{x})| \leq s < k$, we have

$$u(\mathbf{x}) = T_s(u(\mathbf{x})) = T_s(v_k(\mathbf{x})).$$

Then the uniqueness of the limit and Proposition 7 show that for each real number $s > 0$,

$$\lim_{h \rightarrow 0} T_s(u_h(\mathbf{x})) = u(\mathbf{x}) \text{ in measure in the set } \{\mathbf{x}; |u(\mathbf{x})| \leq s\}.$$

As $s > 0$ is arbitrary, this yields the following theorem:

Theorem 10. *The function u defined by (31) is the limit of u_h :*

$$\lim_{h \rightarrow 0} u_h(\mathbf{x}) = u(\mathbf{x}) \text{ in measure in } \Omega. \quad (33)$$

In order to prove that u defined by (31) is a renormalized solution of (1), (2), we approximate (1), (2) by regularizing f . For any $\varepsilon > 0$, let $f^\varepsilon \in L^2(\Omega)$ be an arbitrary approximation of f such that

$$\lim_{\varepsilon \rightarrow 0} f^\varepsilon = f \text{ in } L^1(\Omega), \quad (34)$$

and let $u^\varepsilon \in H^1(\Omega)$ be the unique solution of

$$-\Delta u^\varepsilon = f^\varepsilon \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega. \quad (35)$$

It is proven in Murat [5] that the renormalized solution of (1), (2) is the limit a.e. in Ω of the solution u^ε of (35). Hence, if we show that u is the limit of u^ε , the uniqueness of the limit will prove that u is the renormalized solution of (1), (2).

Theorem 11. *We have:*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u \text{ a.e. in } \Omega. \quad (36)$$

Proof. We discretize (35) in V_h by: Find $u_h^\varepsilon \in V_h$ satisfying:

$$\forall v_h \in V_h, \quad \int_{\Omega} \nabla u_h^\varepsilon \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f^\varepsilon v_h \, d\mathbf{x}. \quad (37)$$

This problem has a unique solution and for each $\varepsilon > 0$,

$$\lim_{h \rightarrow 0} u_h^\varepsilon = u^\varepsilon \text{ strongly in } H_0^1(\Omega). \quad (38)$$

Then subtracting (37) from (4), we obtain

$$\forall v_h \in V_h, \quad \int_{\Omega} \nabla(u_h - u_h^\varepsilon) \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} (f - f^\varepsilon) v_h \, d\mathbf{x}, \quad (39)$$

and applying (12) to (39), we derive in particular for any integer $k \geq 1$:

$$|\Pi_h(T_k(u_h - u_h^\varepsilon))|_{H^1(\Omega)}^2 \leq k \|f - f^\varepsilon\|_{L^1(\Omega)}. \quad (40)$$

Hence this and (34) imply that for each integer $k \geq 1$ and real number $h > 0$,

$$\lim_{\varepsilon \rightarrow 0} \Pi_h(T_k(u_h - u_h^\varepsilon)) = 0 \text{ strongly in } H_0^1(\Omega), \quad (41)$$

and this convergence is uniform with respect to h .

Besides this, applying (20) and (32) to (40), there exists a measurable function w^ε such that

$$\lim_{h \rightarrow 0} \Pi_h(T_k(u_h - u_h^\varepsilon)) = T_k(w^\varepsilon) \text{ weakly in } H_0^1(\Omega);$$

similarly, applying (33) to (40), we have

$$\lim_{h \rightarrow 0} (u_h - u_h^\varepsilon) = w^\varepsilon \text{ in measure in } \Omega.$$

Then applying (33) to u_h and (38), we obtain

$$w^\varepsilon = u - u^\varepsilon \text{ a.e. in } \Omega.$$

It remains to prove that $\lim_{\varepsilon \rightarrow 0} w^\varepsilon = 0$. The lower semi-continuity of the norm for the weak convergence and (40) yield:

$$|T_k(w^\varepsilon)|_{H^1(\Omega)}^2 \leq k \|f - f^\varepsilon\|_{L^1(\Omega)}.$$

Hence, for any integer $k \geq 1$

$$\lim_{\varepsilon \rightarrow 0} |T_k(w^\varepsilon)|_{H^1(\Omega)} = 0,$$

thus implying the desired limit. \square

Corollary 12. *The function u defined by (31) is the renormalized solution of (1), (2) and it belongs to $W_0^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$, where d is the dimension.*

The next theorem collects the main results of this section.

Theorem 13. *If Hypothesis (H) holds, the sequence of solutions u_h of (4) satisfies for all integers $k \geq 1$:*

$$\lim_{h \rightarrow 0} \Pi_h(T_k(u_h)) = T_k(u) \text{ strongly in } H_0^1(\Omega),$$

and

$$\lim_{h \rightarrow 0} u_h(\mathbf{x}) = u(\mathbf{x}) \text{ in measure in } \Omega,$$

where $u \in W_0^{1,q}(\Omega)$ for any $q < \infty$ in one dimension, $q < 2$ in two dimensions and $q < 3/2$ in three dimensions, is the renormalized solution of (1), (2).

Proof. It suffices to establish the strong convergence of $\Pi_h(T_k(u_h))$. From (19), we have

$$\limsup_{h \rightarrow 0} |\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2 \leq \int_{\Omega} f T_k(u) d\mathbf{x}.$$

But, since on one hand, $u \in W_0^{1,q}(\Omega)$ and on the other hand, $T_k(u) \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and $\nabla T_k(u) = \mathbf{0}$ in regions with positive measure where $|u(\mathbf{x})| \geq k$ we can multiply both sides of (1) by $T_k(u)$, integrate over Ω and apply Green's formula:

$$\int_{\Omega} f T_k(u) d\mathbf{x} = \int_{\Omega} \nabla u \cdot \nabla T_k(u) d\mathbf{x}.$$

But

$$\int_{\Omega} \nabla u \cdot \nabla T_k(u) \, d\mathbf{x} = |T_k(u)|_{H^1(\Omega)}^2,$$

therefore

$$\limsup_{h \rightarrow 0} |\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2 \leq |T_k(u)|_{H^1(\Omega)}^2.$$

On the other hand, the lower semi-continuity of the norm for the weak convergence gives

$$|T_k(u)|_{H^1(\Omega)}^2 \leq \liminf_{h \rightarrow 0} |\Pi_h(T_k(u_h))|_{H^1(\Omega)}^2.$$

Hence

$$\lim_{h \rightarrow 0} |\Pi_h(T_k(u_h))|_{H^1(\Omega)} = |T_k(u)|_{H^1(\Omega)},$$

whence the strong convergence. □

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