

# On the Computation of $A_\infty$ -Maps<sup>\*</sup>

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**Abstract.** Starting from a chain contraction (a special chain homotopy equivalence) connecting a differential graded algebra  $A$  with a differential graded module  $M$ , the so-called homological perturbation technique “tensor trick” [8] provides a family of maps,  $\{m_i\}_{i \geq 1}$ , describing an  $A_\infty$ -algebra structure on  $M$  derived from the one of algebra on  $A$ . In this paper, taking advantage of some annihilation properties of the component morphisms of the chain contraction, we obtain a simplified version of the existing formulas of the mentioned  $A_\infty$ -maps, reducing the computational cost of computing  $m_n$  from  $O(n!^2)$  to  $O(n!)$ .

**Keywords:**  $A_\infty$ -algebra, contraction, Basic Perturbation Lemma, transference, computation.

## 1 Introduction

At present,  $A_\infty$ -structures (or strong homotopy structures) find natural applications not only in Algebra, Topology and Geometry but also in Mathematical Physics, related to topics such as string theory, homological mirror symmetry or superpotentials [14,17,18]. Nevertheless, there are few methods for computing explicit  $A_\infty$ -structures, being the better known technique the *tensor trick* [8]. This tool is used in the context of Homological Perturbation Theory. Starting from a chain contraction  $c$  (a special chain homotopy equivalence, also called strong deformation retract) from a differential graded algebra  $A$  onto a differential graded module  $M$ , the tensor trick technique gives explicit formulas for computing a family of higher maps  $\{m_i\}_{i \geq 1}$  that provides an  $A_\infty$ -algebra structure on  $M$  (derived from the algebra structure on  $A$ ). However, the associated computational costs are extremely high (see [11,12,1]). In this paper, we are concerned about finding a more cost-effective formulation of the family of maps transferred to  $M$ . As it is shown in section 3, the use of annihilation properties of the component morphisms of the chain contraction allows to reformulate the  $A_\infty$ -maps on  $M$  (which depend on the mentioned component morphisms). Afterwards, in

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section 4 we carry out a theoretical study of the time and space invested in computing  $m_n$ , presenting the computational savings obtained, in comparison with the original formulas defined by the Basic Perturbation Lemma.

The results can be extended to the case of  $A$  being an  $A_\infty$ -algebra (then, another  $A_\infty$ -algebra structure is also induced on  $M$ ). We remark that such a transference can also be performed in the case of  $c$  being a general explicit chain homotopy equivalence.

Of course, all the results given in this paper can be easily translated into the context of coalgebras and  $A_\infty$ -coalgebras.

## 2 Notations and Preliminaries

We briefly recall here some basic definitions in Homological Algebra as well as the notations used throughout the paper. See [3] or [16] for further explanations.

Take a commutative unital ring  $A$ . Let  $(M, d)$  be a DG-module, that is, a  $A$ -module graded on the non-negative integers ( $M = \bigoplus_{n \geq 0} M_n$ ) and endowed with a differential  $d$  (of degree  $-1$ ). An element  $x \in M_n$  has degree  $n$ , what will be expressed by  $|x| = n$ . In the case that  $M_0 = A$ ,  $M$  is called *connected* and if, besides,  $M_1 = 0$ , then it is called *simply connected*. Given a connected DG-module,  $M$ , the *reduced* module  $\bar{M}$  is the one with  $\bar{M}_n = M_n$  for  $n > 1$  and  $\bar{M}_0 = 0$ .

We will denote the module  $M \otimes \cdots \otimes M$  by  $M^{\otimes n}$ , with  $M^{\otimes 0} = A$  and the morphism  $f \otimes \cdots \otimes f : M^{\otimes n} \rightarrow N^{\otimes n}$  by  $f^{\otimes n}$ . We adhere to Koszul convention for signs. More concretely, given  $f : M \rightarrow M'$ ,  $h : M' \rightarrow M''$ ,  $g : N \rightarrow N'$  and  $k : N' \rightarrow N''$  DG-module morphisms, then

$$(h \otimes k)(f \otimes g) = (-1)^{|k||f|}(hf \otimes kg).$$

On the other hand, if  $f : M^{\otimes i} \rightarrow M$  is a DG-module morphism and  $n$  is a non-negative integer, we will denote by  $f^{[n]} : M^{\otimes n} \rightarrow M^{\otimes n-i+1}$  the morphism

$$f^{[n]} = \sum_{j=0}^{n-i} 1^{\otimes j} \otimes f \otimes 1^{\otimes n-i-j}$$

and the morphism  $f^{[\ ]} : \bigoplus_{j \geq i} M^{\otimes j} \rightarrow \bigoplus_{k \geq 1} M^{\otimes k}$  will be the one such that  $f^{[\ ]}|_{M^{\otimes n}} = f^{[n]}$ .

We will denote by  $\uparrow$  and  $\downarrow$  the *suspension* and *desuspension* operators, which shift the degree by  $+1$  and  $-1$ , respectively. A given morphism of graded modules of degree  $k$ ,  $f : M \rightarrow N$ , induces another one between the suspended modules  $sf : sM \rightarrow sN$ , given by  $sf = (-1)^k \uparrow f \downarrow$ .

Given a DG-module  $(M, d)$ , the *tensor module* of  $M$ ,  $T(M)$ , is the DG-module

$$T(M) = \bigoplus_{n \geq 0} T^n(M) = \bigoplus_{n \geq 0} M^{\otimes n}$$

whose differential structure is provided by  $d_M^{[1]}$ . Every morphism of DG-modules  $f : M \rightarrow N$  induces another one  $T(f) : T(M) \rightarrow T(N)$ , such that  $T(f)|_{M^{\otimes n}} = f^{\otimes n}$ .

A *DG-algebra*,  $(A, d_A, \mu_A)$ , is a DG-module endowed with an associative product,  $\mu_A$ , compatible with the differential  $d_A$  and which has a unit  $\eta_A : \Lambda \rightarrow A$ , that is,  $\mu_A(\eta_A \otimes 1) = \mu_A(1 \otimes \eta_A) = 1$ . If there is no confusion, subscripts will be omitted. A *DG-coalgebra*  $(C, d_C, \Delta_C)$  is a DG-module provided with a compatible coproduct and counit  $\xi_C : C \rightarrow \Lambda$  (so,  $(\xi_C \otimes 1)\Delta_C = (1 \otimes \xi_C)\Delta_C = 1$ ).

In the case of the tensor module  $T(M)$ , a product,  $\mu$ , and a coproduct,  $\Delta$ , can be naturally defined on an element  $a_1 \otimes \cdots \otimes a_n \in T^n(M)$ , as follows:

$$\begin{aligned} \mu((a_1 \otimes \cdots \otimes a_n) \otimes (a_{n+1} \otimes \cdots \otimes a_{n+p})) &= a_1 \otimes \cdots \otimes a_{n+p}; \\ \Delta(a_1 \otimes \cdots \otimes a_n) &= \sum_{i=0}^n (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n). \end{aligned}$$

Therefore,  $T(M)$  acquires both structures of DG-algebra (denoted by  $T^a(M)$ ) and DG-coalgebra ( $T^c(M)$ ), though they are not compatible to each other (that is,  $(T(M), \mu, \Delta)$  is not a Hopf algebra).

We recall here two equivalent definitions of  $A_\infty$ -algebra (resp.  $A_\infty$ -coalgebra) [13,19].

- An  $A_\infty$ -algebra (respectively,  $A_\infty$ -coalgebra), is a DG-module  $(M, m_1)$  (resp.  $(M, \Delta_1)$ ) endowed with a family of maps

$$m_i : M^{\otimes i} \rightarrow M \quad (\text{resp.}, \Delta_i : M \rightarrow M^{\otimes i})$$

of degree  $i - 2$  such that, for  $n \geq 1$ ,

$$\sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0, \quad (1)$$

$$(\text{resp.}, \sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} (1^{\otimes i-n-k} \otimes \Delta_n \otimes 1^{\otimes k}) \Delta_{i-n+1} = 0). \quad (2)$$

- An  $A_\infty$ -algebra (resp.,  $A_\infty$ -coalgebra) is a graded module  $M$  endowed with a morphism of modules  $m : T(sM) \rightarrow M$  (resp.,  $\Delta : M \rightarrow T(s^{-1}M)$ ) such that the morphism  $d = -(\uparrow mT(\downarrow))^{[1]}$  (resp.,  $d = -(T(\downarrow)\Delta \uparrow)^{[1]}$ ) makes  $T^c(sM)$  (resp.,  $T^a(s^{-1}M)$ ) to be a DGA-coalgebra (resp., DGA-algebra).

The *reduced bar construction* of a connected DG-algebra  $A$ ,  $\bar{B}(A)$ , is a DG-coalgebra whose module structure is given by

$$T(s\bar{A}) = \bigoplus_{n \geq 0} (s\bar{A} \otimes \overset{n \text{ times}}{\cdots} \otimes s\bar{A}).$$

The total differential  $d_B$  is given by the sum of the tensor differential,  $d_t$  (which is the natural one on the tensor product) and the *simplicial differential*,  $d_s$  (that depends on the product on  $A$ ):

$$d_t = - \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \uparrow d_A \downarrow \otimes 1^{\otimes n-i-1}; \quad d_s = \sum_{i=0}^{n-2} 1^{\otimes i} \otimes \uparrow \mu_A \downarrow^{\otimes 2} \otimes 1^{\otimes n-i-2}.$$

The coproduct  $\Delta_{\bar{B}} : \bar{B}(A) \rightarrow \bar{B}(A) \otimes \bar{B}(A)$  is the natural one on the tensor module.

In the context of homological perturbation theory, the main input data are *contractions* [4,9,15,7,10]: a contraction  $c : \{N, M, f, g, \phi\}$  from a DG-module  $N$  to a DG-module  $M$ , consists in a particular homotopy equivalence determined by two DG-module morphisms,  $f : N_{\star} \rightarrow M_{\star}$  and  $g : M_{\star} \rightarrow N_{\star}$  and a homotopy operator  $\phi : N_{\star} \rightarrow N_{\star+1}$  such that  $fg = 1_M$ , and  $\phi d_N + d_N \phi + gf = 1_N$ . Moreover, these data are also required to satisfy the annihilation properties:

$$f\phi = 0, \quad \phi g = 0, \quad \phi\phi = 0.$$

Given a DG-module contraction  $c : \{N, M, f, g, \phi\}$ , one can establish the following ones [7,8]:

- The *suspension contraction* of  $c$ ,  $sc$ , which consists of the suspended DG-modules and the induced morphisms:

$$sc : \{sN, sM, sf, sg, s\phi\},$$

being  $sf = \uparrow f \downarrow$ ,  $sg = \uparrow g \downarrow$  and  $s\phi = -\uparrow \phi \downarrow$ , which are briefly expressed by  $f, g$  and  $-\phi$ .

- The *tensor module contraction*,  $T(c)$ , between the tensor modules of  $M$  and  $N$ :

$$T(c) : \{T(N), T(M), T(f), T(g), T(\phi)\},$$

where

$$T(\phi)|_{T^n(N)} = \phi^{[\otimes n]} = \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes n-i-1}.$$

A morphism of graded modules  $f : N \rightarrow N$  is called *pointwise nilpotent* whenever for all  $x \in N$ ,  $x \neq 0$ , there exists a positive integer  $n$  such that  $f^n(x) = 0$ . A *perturbation of a DG-module*  $N$  consists in a morphism of graded modules  $\delta : N \rightarrow N$  of degree  $-1$ , such that  $(d_N + \delta)^2 = 0$ . A *perturbation datum* of the contraction  $c : \{N, M, f, g, \phi\}$  is a perturbation  $\delta$  of the DG-module  $N$  satisfying that the composition  $\phi\delta$  is pointwise nilpotent.

The main tool when dealing with contractions is the **Basic Perturbation Lemma** [2,5,15], which is an algorithm whose input is a contraction of DG-modules  $c : \{N, M, f, g, \phi\}$  and a perturbation datum  $\delta$  of  $c$  and whose output is a new contraction  $c_{\delta} : \{(N, d_N + \delta), (M, d_M + d_{\delta}), f_{\delta}, g_{\delta}, \phi_{\delta}\}$  defined by the formulas

$$d_{\delta} = f \delta \Sigma_c^{\delta} g; \quad f_{\delta} = f (1 - \delta \Sigma_c^{\delta} \phi); \quad g_{\delta} = \Sigma_c^{\delta} g; \quad \phi_{\delta} = \Sigma_c^{\delta} \phi;$$

where  $\Sigma_c^{\delta} = \sum_{i \geq 0} (-1)^i (\phi\delta)^i$ .

The pointwise nilpotency of the composition  $\phi\delta$  guarantees that the sums are finite for each particular element.

### 3 Transferring $A_\infty$ -Algebras Via Homological Perturbation Theory

$A_\infty$ -algebras were first introduced by Stasheff in [20]. They are, roughly speaking, algebras which are associative “up to homotopy” (also called strongly homotopy associative algebras).

In the papers of Gugenheim, Stasheff and Lambe [6,9,8], they describe a technique called *tensor trick* by which, starting from a contraction between a DG-algebra  $A$  and a DG-module  $M$ , an  $A_\infty$ -algebra structure is induced on  $M$ . This transference also exists in the case that  $A$  is an  $A_\infty$ -algebra. Moreover, in the case that a general homotopy equivalence is established between  $A$  and  $M$ , it is also possible to derive a formulation for an  $A_\infty$ -algebra structure on  $M$ . We will mainly focus our efforts on obtaining computational improvements in the first case.

#### 3.1 Transference Via Contractions

Let us consider the contraction

$$c : \{A, M, f, g, \phi\},$$

where  $A$  is a connected DG-algebra and  $M$  a DG-module. The first step consists in tensoring, in order to obtain the underlying graded module of the bar construction of  $A$ ,

$$T(sc) : \{T^c(s\bar{A}), T^c(s\bar{M}), Tf, Tg, T(-\phi)\};$$

and then, considering the simplicial differential,  $d_s$ , which is a perturbation datum for this contraction, and using the Basic Perturbation Lemma, a new contraction is obtained,

$$\{\bar{B}(A), (T^c(s\bar{M}), \tilde{d}), \tilde{f}, \tilde{g}, \tilde{\phi}\},$$

where  $(T^c(s\bar{M}), \tilde{d})$  is called the *tilde bar construction* of  $M$  [20], denoted by  $\tilde{B}(M)$ . Then, the perturbed differential  $\tilde{d}$  induces a family of maps  $m_n : M^{\otimes n} \rightarrow M$  of degree  $n - 2$  that provides an  $A_\infty$ -algebra structure on  $M$ .

The transference of an  $A_\infty$ -algebra structure was also studied by Kadeishvili in [13] for the case  $M = H(A)$ . Using this technique, in the following theorem, an expression of a family of  $A_\infty$ -operations is given with regard to the component morphisms of the initial contraction. Although this formulation is implicitly derived from the mentioned papers [13] and [8], an explicit proof is given in [12].

**Theorem 1.** [13,8] *Let  $(A, d_A, \mu)$  and  $(M, d_M)$  be a connected DG-algebra and a DG-module, respectively and  $c : \{A, M, f, g, \phi\}$  a contraction between them. Then the DG-module  $M$  is provided with an  $A_\infty$ -algebra structure given by the operations*

$$\begin{aligned}
 m_1 &= -d_M \\
 m_n &= (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}, \quad n \geq 2 \quad (3)
 \end{aligned}$$

where

$$\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}.$$

As far as the computation of these formulas is concerned, we can take advantage of the annihilation properties of  $f$ ,  $g$  and  $\phi$  to deduce a more economical formulation for  $m_n$ .

**Theorem 2.** *Any composition of the kind  $\phi^{[\otimes s]} \mu^{(s)}$  ( $s = 2, \dots, n-1$ ) in the formula (3), which is given by*

$$\left( \sum_{j=0}^{s-1} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes s-j-1} \right) \circ \left( \sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes s-i-1} \right),$$

can be reduced to the following sum

$$\sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes s-i-1}. \quad (4)$$

Moreover, given a composition of the kind

$$(\phi^{[\otimes s-1]} \mu^{(s-1)}) \circ (\phi^{[\otimes s]} \mu^{(s)}) \quad s = 3, \dots, n-2,$$

for every index  $i$  in the sum (4) of  $\phi^{[\otimes s]} \mu^{(s)}$ , the formula of  $\phi^{[\otimes s-1]} \mu^{(s-1)}$  in such a composition can be reduced to

$$\sum_{j=i-1, j \geq 0}^{s-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes 1^{\otimes s-j-2}. \quad (5)$$

In other words, the whole composition  $(\phi^{[\otimes 2]} \mu^{(2)}) \circ \dots \circ (\phi^{[\otimes n-1]} \mu^{(n-1)})$  in the formula of  $m_n$  can be expressed by

$$\sum_{i_{n-1}=0}^{n-2} \left( \sum_{i_{n-2}=i_{n-1}-1}^{n-3} \left( \dots \left( \sum_{i_2=i_3-1}^1 (\phi \mu)^{(2, i_2)} \right) \dots \right) (\phi \mu)^{(n-2, i_{n-2})} \right) (\phi \mu)^{(n-1, i_{n-1})},$$

where  $(\phi \mu)^{(k, j)} = (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes 1^{\otimes k-j-1}$  and each addend exists whenever the corresponding index  $i_k \geq 0$ .

*Proof.* Let us prove the formula 4 of  $\phi^{[\otimes s]} \mu^{(s)}$  for any  $s = n-1, n-2, \dots, 2$ , by induction over the number  $k = n-s$  of factors of the type  $\phi^{[\otimes *]} \mu^{(*)}$  that are composed, following the scheme

$$m_n = (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \underbrace{\phi^{[\otimes n-2]} \mu^{(n-2)}}_{k=2} \underbrace{\phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}}_{k=1} \quad (6)$$

At the same time, we will prove the major reduction of terms given by (5) for  $s = n - 2, \dots, 2$ .

–  $k = 1$  The composition of morphisms  $\phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}$  can be written as

$$\left( \sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2} \right) \circ \left( \sum_{i=0}^{n-2} (-1)^{i+1} g^{\otimes i} \otimes \mu_A g^{\otimes 2} \otimes g^{\otimes n-i-2} \right).$$

Now, using the facts that  $fg = 1$  and  $\phi g = 0$ , it is simple to see that the only non null elements are those where  $\phi$  is applied over  $\mu_A$ , so the original formula of  $\phi^{[\otimes n-1]} \mu^{(n-1)}$  is simplified to

$$\sum_{i=0}^{n-2} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes n-i-2}.$$

–  $k = 2$  In this case, taking into account the formula obtained for  $k = 1$ ,

$$\phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n} = \sum_{i=0}^{n-2} (-1)^{i+1} g^{\otimes i} \otimes \phi \mu_A g^{\otimes 2} \otimes g^{\otimes n-i-2} \quad (7)$$

and that  $\phi^{[\otimes n-2]} \mu^{(n-2)}$  is the composition

$$\left( \sum_{j=0}^{n-3} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-3} \right) \circ \left( \sum_{i=0}^{n-3} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes n-i-3} \right),$$

we can use the annihilation properties  $\phi g = 0$  and  $\phi^2 = 0$ , to conclude that the factor  $\phi$  in  $\phi^{[\otimes n-2]}$  has to be applied over  $\mu_A$  and hence,

$$\phi^{[\otimes n-2]} \mu^{(n-2)} = \sum_{j=0}^{n-3} (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes (gf)^{\otimes n-j-3}. \quad (8)$$

Now, considering the composition of the sum (7) with (8), one can observe that, since  $f\phi = 0$ , for each index  $i$  in the sum (7), the only addends of (8) that have to be considered for the composition are those  $j \geq i - 1$ . On the other hand,  $fg = 1$  is also satisfied, so

$$\phi^{[\otimes n-2]} \mu^{(n-2)} = \sum_{j=i-1}^{n-3} (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes 1^{\otimes n-j-3}.$$

–  $\boxed{k = m}$  Finally, let us assume that the proposition is true for  $\phi^{[\otimes n-k]} \mu^{(n-k)}$  for all  $k = 1, \dots, m-1$ . Now, considering, on one hand,  $\phi^{[\otimes n-m]} \mu^{(n-m)}$ ,

$$\left( \sum_{j=0}^{n-m-1} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-m-1} \right) \left( \sum_{i=0}^{n-m-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes n-i-m-1} \right)$$

and that, on the other hand, the composition of morphisms

$$\phi^{[\otimes n-m+1]} \mu^{(n-m+1)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}$$

by induction hypothesis, is a sum of elements that are tensor product of factors of the type  $\phi(\text{something})$  or  $g$ , using again the annihilation properties, it follows that

$$\phi^{[\otimes n-m]} \mu^{(n-m)} = \sum_{j=0}^{n-m-1} (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes (gf)^{\otimes n-j-m-1}.$$

Since, by induction hypothesis,

$$\phi^{[\otimes n-m+1]} \mu^{(n-m+1)} = \sum_{i=0}^{n-m} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes n-m-i},$$

taking into account that  $fg = 1$  and the fact that  $f\phi = 0$ , again we can reduce the number of terms of  $\phi^{[\otimes n-m]} \mu^{(n-m)}$  to

$$\sum_{j=i-1}^{n-m-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes n-m-i-1},$$

where  $i$  is the index corresponding to the term of the preceding sum that is being composed with  $\phi^{[\otimes n-m]} \mu^{(n-m)}$ .

We can generalize the results showed above to the case that the “big” DG-module of a given contraction is an  $A_\infty$ -algebra. The stability of the  $A_\infty$ -structures with respect to the contractions follows from the paper [8]. In fact, it is possible to extract the next theorem as an implicit consequence of the results there.

**Theorem 3.** *Given  $c : \{A, M, f, g, \phi\}$  a contraction, where  $(A, m_1, m_2, \dots)$  is a connected  $A_\infty$ -algebra and  $M$  is a DG-module, then  $M$  inherits an  $A_\infty$ -algebra structure.*

*Proof.* The proof follows the same scheme as in theorem 1 (and for that reason, we will only sketch it slightly), making use of the tensor trick and the Basic Perturbation Lemma, with the difference that, now, the perturbation datum for the contraction



$$T(sc) : \{T^c(s\bar{A}), T^c(s\bar{M}), T(f), T(g)T(-\phi)\}$$

is the one induced by the  $A_\infty$ -maps

$$d_m|_{(s\bar{A})^{\otimes n}} = - \sum_{k=2}^n \sum_{i=0}^{n-k} 1^{\otimes i} \otimes \uparrow m_k \downarrow^{\otimes k} \otimes 1^{\otimes n-k-i}.$$

Since the family of maps  $\{m_i\}_{i \geq 1}$  defines an  $A_\infty$ -algebra structure on  $A$ ,  $d_{\bar{B}} = d_t + d_m$  is a differential on  $T^c(s\bar{A})$  (in fact,  $(T^c(s\bar{A}), d_{\bar{B}})$  is the tilde bar construction of  $A$ ). On the other hand, the pointwise nilpotency of  $T(-\phi)d_m$  follows because  $d_m$  reduces the simplicial dimension, while  $T(-\phi)$  keeps it the same.

Thanks to the Basic Perturbation Lemma, a new differential is obtained on  $T^c(s\bar{M})$ ,  $\tilde{d}$ , given by the formula:

$$\tilde{d} = d_t + T(f)d_m \sum_{i \geq 0} (-1)^i (T(-\phi)d_m)^i T(g).$$

This way,  $\tilde{d}$  induces a family of maps  $\{m_i^M\}_{i \geq 1}$  on  $M$ , where  $m_n^M$ , up to sign, can be expressed by

$$f m_n g^{\otimes n} + \sum_{l=1}^{n-2} \sum_{2 \leq k_1 < \dots < k_l \leq n-1} \pm f m_{k_1} (\phi^{[\otimes k_1]} m_{k_2-k_1+1}^{(k_1)} \dots (\phi^{[\otimes k_l]} m_{n-k_l+1}^{(k_l)})) g^{\otimes n}$$

where  $m_{n-k+1}^{(k)} : A^{\otimes n} \rightarrow A^{\otimes k}$  is given by

$$m_{n-k+1}^{(k)} = \sum_{i=0}^{n-k+1} 1^{\otimes i} \otimes m_{n-k+1} \otimes 1^{\otimes k-i-1}.$$

Notice that, since  $m_i$  is a map of degree  $i-2$ ,  $m_n^M$  has degree  $n-2$ .

If we examine the formula above in low dimensions, we obtain, up to sign:

$$\begin{aligned} m_2^M &= \pm f m_2 g^{\otimes 2}; \\ m_3^M &= \pm f m_3 g^{\otimes 3} \pm f m_2 \phi^{[\otimes 2]} m_2^{(2)} g^{\otimes 3}; \\ m_4^M &= \pm f m_4 g^{\otimes 4} \pm f m_2 \phi^{[\otimes 2]} m_3^{(2)} g^{\otimes 4} \pm f m_3 \phi^{[\otimes 3]} m_2^{(3)} g^{\otimes 4} \\ &\quad \pm f m_2 \phi^{[\otimes 2]} m_2^{(2)} \phi^{[\otimes 3]} m_2^{(3)} g^{\otimes 4}. \end{aligned}$$

Notice that only the last addend of each map is the one induced in the case of  $A$  being an algebra, instead of the  $2^{n-2}$  addends generated in these cases (the number of subsets of a set of  $n-2$  elements). At each addend of each  $A_\infty$ -map, one can obtain a reduction in number of terms, of the same nature than the one showed in theorem 2.

**Theorem 4.** *Any composition of the kind  $\phi^{[\otimes s]} m_r^{(s)}$  in the formula of  $m_n^M$ , which is given by*

$$\left( \sum_{j=0}^{s-1} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes s-j-1} \right) \circ \left( \sum_{i=0}^r 1^{\otimes i} \otimes m_r \otimes 1^{\otimes s-i-1} \right),$$

can be reduced to the following sum

$$\sum_{i=0}^r 1^{\otimes i} \otimes \phi m_r \otimes 1^{\otimes s-i-1}.$$

*Proof.* This proof is completely dual to the one of theorem 2, so it is left to the reader.

### 3.2 Transference Via Homotopy Equivalences

In [10], a general chain homotopy equivalence  $e$  between two DG-modules  $M$  and  $M'$  is considered as a pair of chain contractions  $\{\hat{M}, M, f, g, \phi\}$  and  $\{\hat{M}, M', f', g', \phi'\}$ , where  $\hat{M}$  is a “big” DG-module obtained from  $e$ . Our interest here is to compute the  $A_\infty$ -algebra structure on  $M'$  derived from that of  $M$ . Having at hand the mentioned characterization of chain homotopy equivalence and the results of the previous subsection, our task is then reduced to determine the transferring of  $A_\infty$ -structures via chain contractions in the sense from-small-to-big. In a more formal way, our main problem here is the transference of the  $A_\infty$ -algebra structure from a “small” DG-module  $N$  to a “big” DG-module  $M$  via the chain contraction  $c : \{M, N, f, g, \phi\}$ . The following propositions are straightforward and, in particular, allow to design an algorithmic method for transferring  $A_\infty$ -structures via chain homotopy equivalences:

**Proposition 1.** *Let  $c : \{M, N, f, g, \phi\}$  be a chain contraction and let  $(N, \mu)$  be a DG-algebra with product  $\mu$ . Then,  $M$  has a structure of DG-algebra, provided by the product  $\mu_M = g\mu(f \otimes f)$ .*

**Proposition 2.** *Let  $c : \{M, N, f, g, \phi\}$  be a chain contraction and let  $(N, \mu)$  be an  $A_\infty$ -algebra with higher maps  $(n_1, n_2, n_3, \dots)$ . Then, the DG-module  $M$  inherits a structure of  $A_\infty$ -algebra, given by the maps  $(g n_1 f, g n_2 f^{\otimes 2}, g n_3 f^{\otimes 3}, \dots)$ .*

## 4 Computational Advantages: Theoretical Study

In this section we are concerned about the theoretical study of the time and space invested in computing the maps of an  $A_\infty$ -algebra structure induced by a contraction  $c : \{A, M, f, g, \phi\}$ . We will focus on the case of  $A$  being an algebra. In particular, we will make a comparison between the original formulas defined by the Basic Perturbation Lemma and the reduced formulas obtained in the previous section.

Regarding the original formulas of the  $A_\infty$ -algebra maps, we must say that experimental results can be obtained with [1], a software developed in order to perform low dimension computations. This software is based on the initial formulation for the map  $m_n : M^{\otimes n} \rightarrow M$  given in theorem 1:

$$m_n = (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}, \quad n \geq 2.$$

We will take  $n \geq 3$ , since no improvement is obtained in the case  $n = 2$ .

As for complexity in space, let us consider the number of addends generated in the sum above. Taking into account that

$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes k-i-1} \quad \text{and} \quad \mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1},$$

the result of applying  $m_n$  to an element  $x_1 \otimes x_2 \otimes \dots \otimes x_n$  has  $(n-1)!$  addends.

Concerning complexity in time, let us assume that each component morphism of the initial contraction,  $f$ ,  $g$  and  $\phi$ , consumes a unit of time when applied (that is, each one of these morphisms is considered a basic operation); we will also make this assumption for the composition  $gf$  which is applied in different terms of the morphisms  $\phi^{[\otimes k]}$ .

Notice that applying  $g^{\otimes n}$  is  $O(n)$  in time.

On the other hand, the number of operations of each addend of the form  $1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes k-i-1}$  is  $k-i$  and the one of each addend  $1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}$  is 1. That is, the number of basic operations can be expressed by

$$n + 2(n-1)!^2 + (n-1)! \sum_{k_i \in \{1, 2, \dots, i\}} (k_2 + 1 + k_3 + 1 + \dots + k_{n-1} + 1),$$

where  $n$  comes from  $g^{\otimes n}$ ,  $2(n-1)!$  from the two operations  $f\mu$  at the end of each addend and the big sum corresponds to the operations on the composition

$$\phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)}.$$

Notice that the sum is multiplied by  $(n-1)!$  because of all the possibilities for taking an addend  $1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}$  of each  $\mu^{(k)}$ . The sum above can be expressed by

$$n + n(n-1)!^2 + \frac{(n+3)(n-2)}{4} (n-1)!^2.$$

Therefore, the complexity of the algorithm becomes  $O(n!^2)$  in time.

Now, taking into account the first reduction of terms in the sums involved in  $m_n$  (theorem 2), any composition of morphisms of the form  $\phi^{[\otimes s]} \mu^{(s)}$ , which had  $s^2$  addends, is reduced to a sum with  $s$  terms. So, the total number of addends is now  $(n-1)!$ .

As for the number of operations, now it is  $O(n)$  for each addend. Moreover, the number of operations is, exactly,

$$n + (n-1)!(2n-2),$$

and hence  $O(n!)$  in time.

Finally, considering that the upla of indexes  $(i_2, i_3, \dots, i_{n-1})$  for the sums must be taken so that  $i_k \geq i_{k+1} - 1$ , we eliminate (for  $n \geq 4$ )

$$S_n = \sum_{k=1}^{n-3} \sum_{i=1}^k i \cdot k! = \sum_{k=1}^{n-3} \frac{k(k+1)!}{2}$$

addends, so the number of addends becomes  $(n - 1)! - S_n$ . Now, taking into account that  $(n - 1)!$  can be expressed by

$$(n - 1)! = 2 + \sum_{k=1}^{n-3} (k + 1)! + \sum_{k=1}^{n-3} k(k + 1)!,$$

it is easy to see that

$$\frac{(n - 1)!}{2} < (n - 1)! - S_n < (n - 1)!,$$

so the algorithm is still  $O((n - 1)!)$  in space. However, the final number of addends,  $(n - 1)! - S_n$ , is much "closer" to  $\frac{(n-1)!}{2}$  than to  $(n - 1)!$ , as it is shown in the following comparative table.

$n$	5	10	50	100	1000
$((n - 1)! - S_n)/(n - 1)!$	0,708333	0,563704	0,510421	0,505103	0,500050

Summing up, the order of complexity in time and space of the original formula versus the new one is presented in the following table.

	original formula		new formula	
	time	space	time	space
$m_n$	$O(n!^2)$	$O((n - 1)!^2)$	$O(n!)$	$O((n - 1)!)$

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