Solitary waves of a $\mathcal{PT}$-symmetric Nonlinear Dirac equation

Jesús Cuevas–Maraver, Panayotis G. Kevrekidis, Avadh Saxena, Fred Cooper, Avinash Khare, Andrew Comech and Carl M. Bender

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Abstract—In the present work, we consider a prototypical example of a $\mathcal{PT}$-symmetric Dirac model. We discuss the underlying linear limit of the model and identify the threshold of the $\mathcal{PT}$-phase transition in an analytical form. We then focus on the examination of the nonlinear model. We consider the continuation in the $\mathcal{PT}$-symmetric model of the solutions of the corresponding Hamiltonian model and find that the solutions can be continued robustly as stable ones all the way up to the $\mathcal{PT}$-transition threshold. In the latter, they degenerate into linear waves. We also examine the dynamics of the model. Given the stability of the waveforms in the $\mathcal{PT}$-exact phase we consider them as initial conditions for parameters outside of that phase. We find that both oscillatory dynamics and exponential growth may arise, depending on the size of the corresponding "quench". The former can be characterized by an interesting form of bi-frequency solutions that have been predicted on the basis of the $\mathrm{SU}(1,1)$ symmetry. Finally, we explore some special, analytically tractable, but not $\mathcal{PT}$-symmetric solutions in the massless limit of the model.

Index Terms—Nonlinear dynamical systems, nonlinear differential equations, bifurcation.

I. INTRODUCTION

The study of open systems bearing gain and loss (especially so in a balanced form) is a topic that has emerged over the past two decades as a significant theme of study [1]–[3]. While the realm of $\mathcal{PT}$-symmetry introduced by Bender and collaborators was originally intended as an alternative to the standard Hermitian quantum mechanics, its most canonical realizations (beyond the considerable mathematical analysis of the theme in its own right at the level of operators and spectral theory in mathematical physics) emerged elsewhere in physics. More specifically, in optical systems [4], [5] the

J. Cuevas–Maraver is with the Nonlinear Physics Group at Departamento de Física Aplicada I, Universidad de Sevilla. Escuela Politécnica Superior, C/ Virgen de África, 7, 41011-Sevilla, Spain and the Instituto de Matemáticas de la Universidad de Sevilla (IMUS). Edificio Celestino Mutis. Avda. Reina Mercedes s/n, 41012-Sevilla, Spain (email:jcuevas@us.es).

P.G. Kevrekidis is with the Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-4515, USA (email: kevrekid@math.umass.edu).

F. Cooper is with the Santa Fe Institute, Santa Fe, NM 87501, USA.

A. Khare is with the Department of Physics, Savitribai Phule Pune University, Pune 411007, India.

A. Comech is with the Department of Mathematics, Texas A&M University, College Station, TX 77843-3368 and the Institute for Information Transmission Problems, Moscow 127994, Russia.

C.M. Bender is with the Department of Physics, Washington University, St. Louis, MO 63130, USA.

P.G. Kevrekidis, A. Saxena and F. Cooper are with the Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA.

analogy of the paraxial approximation of Maxwell's equations and of the Schrödinger equation formed the basis on which the possibility of $\mathcal{PT}$-symmetric realizations initially in optical waveguide experiments was proposed and then experimentally implemented [6]. The success of this program motivated further additional initiatives in other directions of experimental interest, including, but not limited to, $\mathcal{PT}$-symmetric electronic circuits [7], [8], mechanical systems [9] and whispering-gallery microcavities [10].

Another theme of research that has been receiving increasing attention recently, both in the physics and in the mathematics community is that of the nonlinear Dirac equations (NLDEs). While such models were proposed in the context of high-energy physics over 50 years ago [11], [12], they have, arguably, been far less widespread than their non-relativistic counterpart [13], the nonlinear Schrödinger (NLS) equation [14], [15]. In recent years, however, there has been a surge of activity around NLDE models fueled to some extent by analytical solutions and computational issues arising in associated numerical simulations [16]–[19], as well as by the considerable progress achieved by rigorous techniques towards aspects of the spectral, orbital and asymptotic stability of solitary wave solutions of such models [20]–[25] and towards criteria for their spectral stability [26], [27]. Although our emphasis herein will be on the so-called Gross-Neveu model [28] (sometimes also referred to as the Soler model [29]), we also mention in passing that another main stream of activity in this direction has been towards the derivation of NLDEs in the context e.g. of bosonic evolution [30], [31] (or light propagation [32], [33]) in honeycomb optical lattices. In the latter context, contrary to what we will be focusing on below, there is no nonlinear interaction between the fields (of the two-component spinor).

Our aim in the present work is to connect these two budding areas of research, namely to propose a prototypical $\mathcal{PT}$-symmetric nonlinear Dirac equation model (PT-NLDE). Motivated by the considerable volume of activity, as well as analytical availability of solutions within the Hamiltonian limit, we will focus on the Gross-Neveu (or Soler) model. In the next section, we will present the mathematical formulation of the model. We will analyze its linear limit and discuss the existence of a $\mathcal{PT}$-symmetry breaking critical point, i.e., the point of a $\mathcal{PT}$-phase transition. Then, we will turn to the nonlinear variant of the model exploring the conditions for the existence of a standing wave solution, as well as discussing the linear (spectral) stability setup for such a solution. Finally,

we will briefly touch upon the fate of the conservation laws, such as the power (squared L^2 norm) and the energy. As a remarkable feature, we find that the energy remains *invariant* within the nonlinear PT-NLDE model, a feature that certainly distinguishes the model from its NLS counterpart. In Section III we will examine the numerical properties of the standing wave solutions. Remarkably, we will find that these standing wave solutions are stable *throughout* their interval of existence tending to a linear limit of vanishing amplitude as the linear threshold (i.e., the threshold of the underlying linear model) of the $\mathcal{PT}$-transition is approached. Since no instability is encountered within the exact $\mathcal{PT}$-phase, we consider the propagation beyond the relevant critical point (i.e., under a quench in the gain-loss parameter γ) finding (a) the possibility of oscillatory motion that we identify with a bi-frequency state and connect to the invariances of the model and (b) the possibility of exponential growth. Lastly, a special case of vanishing mass solutions is analytically identified and also numerically examined. The remarkable feature in this case is that these solutions do *not* respect the $\mathcal{PT}$-symmetry. In Section IV, we summarize our findings and present some interesting directions for future study.

II. MODEL EQUATIONS

The system of choice in the present context will be the Gross-Neveu model in its generalized $\mathcal{PT}$-symmetric (PT-NLDE) form:

$$i\partial_t U = \partial_x V - g(|U|^2 - |V|^2)^k U + mU + i\gamma V, \quad (1a)$$
$$i\partial_t V = -\partial_x U + g(|U|^2 - |V|^2)^k V - mV + i\gamma U. \quad (1b)$$

We will restrict our considerations to the one-dimensional setting, as is evident from the above. Eqs. (1) are $\mathcal{PT}$-symmetric because they are invariant under the transformation $\mathcal{P}$: $x \to -x$, $U \to U$, $V \to -V$ and $\mathcal{T}$: $t \to -t$, $i \to -i$, $U \to U$, and $V \to V$. This transformation assumes that $U(x,t)$ is spatially even and that $V(x,t)$ is spatially odd. Without loss of generality, we will set the mass $m = 1$, except when explicitly indicated otherwise, while we will also use $g = 1$ (the coefficient of the nonlinear term can be scaled out, hence, when it is present, we only care about its sign). The key addition in this model in comparison to the earlier proposal of [18] is the inclusion of the gain-loss term proportional to γ in the (implicit) form of the Dirac matrix γ_5 (cf. [1]) multipliying the spinor $(U, V)^T$. In our case of two-component spinors, the role of γ_5 is played by the Pauli matrix σ_1. We note in passing that in its linear form, the model can be converted under a suitable transformation (associated with the so-called $\mathcal{C}$-operator) to a Hamiltonian one with a reduced mass of $\sqrt{m^2 - \gamma^2}$ [1], [34]. Although the linear version of the model was proposed and analyzed in these works, to the best of our knowledge, there has not been any previous work in the realm of the nonlinear variant i.e., of the $\mathcal{PT}$-NLDE.

It is straightforward to see that in the linear case (of $g = 0$), plane waves $U = Ae^{i(\kappa x - \omega t)}$ and $V = iBe^{i(\kappa x - \omega t)}$ are solutions provided the dispersion relation $\omega = \pm\sqrt{m^2 + \kappa^2 - \gamma^2}$ is satisfied. Not only does the above formula have the characteristic Dirac form, but it also is consistent with the

equivalence of the linear $\mathcal{PT}$-Dirac equation with effective mass $\tilde{m} = \sqrt{m^2 - \gamma^2}$, as per the above discussion.

Having examined the linear states (plane waves) of the model, let us now turn to the nonlinear ones, and more specifically to standing waves. The relevant coherent structures will be of the form:

$$U(x,t) = \exp(-i\Lambda t)u(x), \qquad V(x,t) = \exp(-i\Lambda t)v(x). \quad (2)$$

Once such standing wave solutions are calculated (e.g., by fixed point methods as will be discussed in the next section), their linear stability is considered by means of a Bogoliubov-de Gennes linearized stability analysis. We note here, in passing, that unfortunately, contrary to what is the case for the Hamiltonian limit of the model with $\gamma = 0$, for which explicit solutions exist as:

$$u(x) = \sqrt{\frac{(m+\Lambda)\cosh^2(k\beta x)}{m+\Lambda\cosh(2k\beta x)}\left[\frac{(k+1)\beta^2}{g^2(m+\Lambda\cosh(2k\beta x))}\right]^{1/k}}, \quad (3a)$$

$$v(x) = \mathrm{sgn}(x)\sqrt{\frac{(m-\Lambda)\sinh^2(k\beta x)}{m+\Lambda\cosh(2k\beta x)}\left[\frac{(k+1)\beta^2}{g^2(m+\Lambda\cosh(2k\beta x))}\right]^{1/k}} \quad (3b)$$

(where $\beta = \sqrt{m^2 - \Lambda^2}$) in the present PT-NLDE we have been unable to identify such explicit solutions (with a notable exception for $m = 0$ discussed separately below). I.e, with the exception of the $m = 0$ solutions given below, we have not been able to identify additional analytical solitary wave solutions of the case with $\gamma \neq 0$. In the same vein, it does not appear to be straightforward to generalize the transformation of [1], [34] to the present nonlinear setting.

We now consider the linearization of the standing wave solutions in order to extract information about their spectral stability. More specifically, considering small perturbations [of order $\mathrm{O}(\delta)$, with $0 < \delta \ll 1$] of the standing wave solutions, we substitute the ansatz

$$U(x,t) = e^{-i\Lambda t}\left[u(x) + \delta(a_1(x)e^{i\omega t} + b_1(x)^* e^{-i\omega^* t})\right] \quad (4a)$$
$$V(x,t) = e^{-i\Lambda t}\left[v(x) + \delta(a_2(x)e^{i\omega t} + b_2(x)^* e^{-i\omega^* t})\right] \quad (4b)$$

into Eqs. (1), and then solve the ensuing [to $\mathrm{O}(\delta)$] eigenvalue problem $\omega(a_1, a_2, b_1^*, b_2^*)^T = \mathcal{M}(a_1, a_2, b_1^*, b_2^*)^T$ with $\mathcal{M}$ being

$$\mathcal{M} = \begin{pmatrix} L_1 & L_2 \\ -L_2^* & -L_1^* \end{pmatrix} - i\gamma \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad (5)$$

and

$$L_1 = \begin{pmatrix} g(|u|^2-|v|^2)^k - m + \Lambda & -\partial_x \\ \partial_x & -g(|u|^2-|v|^2)^k + m + \Lambda \end{pmatrix}$$
$$+ gk(|u|^2-|v|^2)^{k-1} \begin{pmatrix} |u|^2 & -uv^* \\ -u^*v & |v|^2 \end{pmatrix},$$

(6)

$$L_2 = gk(|u|^2-|v|^2)^{k-1} \begin{pmatrix} u^2 & -uv \\ -uv & v^2 \end{pmatrix},$$

(7)

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is the identity matrix. The potential existence of an eigenvalue with non-vanishing real part (i.e., an eigenfrequency ω with non-vanishing imaginary part) suggests the existence of a dynamical instability. If all the eigenvalues are imaginary (i.e., all eigenfrequencies ω are real), then the solution is spectrally (neutrally) stable.

Finally, once the exact waveforms and their linear stability are identified, the corresponding full nonlinear dynamics of the scheme is monitored by means of the solution of Eqs. (1) in our numerical considerations of the next section. A natural, theoretically motivated aspect to consider in that regard is the (potential) preservation, by the numerical scheme, of the different conservation laws. To that effect, we examine the fate of the prototypical conservation laws (such as the power and the energy) in the context of our PT-NLDE model.

Based on the power density,

$$\rho(x) = |U(x,t)|^2 + |V(x,t)|^2,$$

(8)

we can define the charge (or power, as it is also referred e.g. in the context of optics)

$$Q = \int \rho(x)\mathrm{d}x.$$

(9)

From the dynamical equations (1) it is straightforward to show that the charge is not preserved. Instead, the following "moment equation" is satisfied:

$$\frac{dQ}{dt} = 4\gamma \int \mathrm{Re}(UV^*)\mathrm{d}x.$$

(10)

Note that in the case of a standing wave state, $dQ/dt = 0$ and charge is conserved.

Although the charge is not generally conserved, remarkably there is a conserved quantity in the form of the energy:

$$E = \int \Big[\mathrm{Re}(U^*\partial_x V - V^*\partial_x U) + m(|U|^2 - |V|^2)$$
$$- \frac{g}{k+1}[|U|^2 - |V|^2]^{k+1} \Big] \mathrm{d}x.$$

(11)

Notice that there is no γ dependence in this formula. That is, this is the same definition for the energy as in the $\gamma = 0$ limit. But, intriguingly, $dE/dt = 0$ even for $\gamma \neq 0$. This

is rather unusual in our experience in $\mathcal{PT}$-symmetric models and is effectively related to the special form of introducing $\mathcal{PT}$-symmetry through the γ_5 matrix. We note that in this form, it is not transparent (as it is e.g. in Schrödinger $\mathcal{PT}$-symmetric models [1]) which component corresponds to the gain and which one to the loss. Effectively, isolating the time-dependence and the γ-dependent term in the equations (i.e., $i\partial_t U = i\gamma V$ and $i\partial_t V = i\gamma U$ and momentarily ignoring the rest of the terms), it appears as if both components bear both gain and loss. Nevertheless, this γ-independent effective energy conservation that we will numerically corroborate below is certainly worth additional examination in order to determine its origin.

III. NUMERICAL RESULTS

A. Massive NLD equation with $k = 1$

In the numerical computations presented herein, we have utilized spectral collocation methods in order to approximate the spatial derivatives of Eq. (1). As discussed in [35], the Chebyshev collocation is, arguably, the most suitable method for approximating the relevant derivatives as it gives a better spectral accuracy. However, because of the non-Hermiticity of the system, the implementation of fixed-point methods, such as the Newton method used herein, requires a high amount of computer memory and more than ~ 500 grid points which, in turn, poses implementation challenges. Furthermore, the method has the drawback that the double humped solitons (which occur for $\Lambda \leq 1/3$ when $k = 1$), cannot be well resolved (i.e. the humps cannot be observed) because of the Chebyshev collocation including more points at the edge of the system in comparison to the center. Consequently, in what follows, a Fourier collocation scheme has been implemented. In suitable limit cases, we have checked that the results are similar for the different implementations and that no extra spurious eigenmodes arise in comparison to the Hermitian case. We note that hereafter we will focus on the case of $k = 1$ for our numerical implementation.

The first numerical result found by studying the standing wave solution is the $\mathcal{PT}$ transition point. We have checked that this transition takes place when $\gamma = \gamma_{\mathrm{PT}}$ with $\gamma_{\mathrm{PT}} = \sqrt{m^2 - \Lambda^2}$ (as we have taken $m = 1$, $\gamma_{\mathrm{PT}} = \sqrt{1 - \Lambda^2}$); notice that this is consonant with our analytical prediction from the previous section in the case of wavenumber $\kappa = 0$. Figure 1 shows the profile of a typical soliton with nonzero γ. Importantly, we have observed (see Fig. 2) that the non-Hermiticity does not introduce instabilities to our system. The relevant spectrum features a zero eigenvalue of algebraic multiplicity four and geometric multiplicity two. This is present in the spectrum of the linearized equation due to the U(1) symmetry and due to the translational symmetry, which are both preserved when $\gamma \neq 0$, hence both the algebraic and geometric multiplicity of this eigenvalue are preserved for all values of γ, as is the presence of two generalized eigenvectors. The spectrum also features the eigenfrequencies $\omega = \pm 2\Lambda$ [19] which can not leave the imaginary axis since their presence in the spectrum is due to the SU(1,1) invariance (see the more detailed discussion below), which is
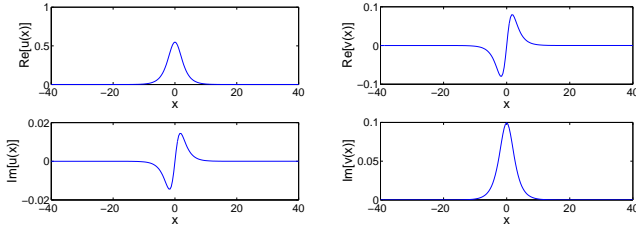
Fig. 1. Real and imaginary (top and bottom panels) part of the spinor components (left and right panels) for $\Lambda = 0.8$ and $\gamma = 0.3$.
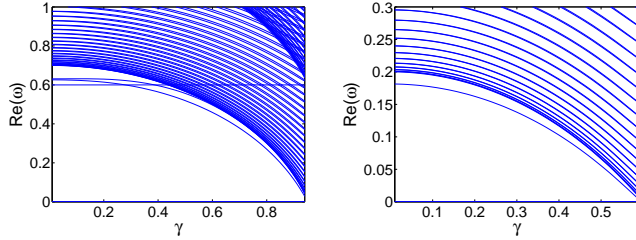

Fig. 2. Real part of the eigenfrequencies dependence for $\Lambda = 0.3$ (left) and $\Lambda = 0.8$ (right). Notice the existence on the left panel of a constant frequency at $2\Lambda = 0.6$. Notice also the approach of the frequencies towards 0, as per the discussed collision with the linear limit (eigenfunctions) of the problem.

also preserved for any γ; the relevant eigenfrequency, which persists under variations of γ, can be discerned in the left panel of Fig. 2. For the rest of the spectrum we note that according to [25], eigenfrequencies with nonzero imaginary part can only be born in the interval $\big(-1-|\Lambda|, 1+|\Lambda|\big)$. Here, however, all the eigenfrequencies remains inside this interval for all γ, as illustrated in Fig. 2, solely tending towards 0, as γ approaches $\gamma_{\rm PT}$.

Notice that the $\mathcal{PT}$ transition is caused by the nonlinear solutions colliding with (or degenerating into) linear modes. This fact can also be confirmed in the charge versus γ plot of Fig. 3, where the charge (norm) and energy tend to zero (while the width of the solution diverges) when the transition point is reached (actually, we have not been able to reach this point exactly, as the soliton width increases drastically when approaching this point). It should be noted here that this is a distinct phenomenology (again) in comparison to the NLS counterpart of the model. In the latter, typically at the $\mathcal{PT}$-phase transition a stable (center) and an unstable (saddle) solution collide and disappear in a saddle-center bifurcation. Here, a fundamentally different scenario arises through the degeneration of the nonlinear modes into linear ones. In the bottom panel of Fig. 3, we provide two-parameter diagrams of the relevant solutions as a function of the frequency Λ and the gain-loss parameter γ. The dependences strongly suggest a "combined" monoparametric dependence on $\gamma^2 + \Lambda^2$, although we have not been able to analytically identify solutions bearing this dependence, except in the limit of $m = 0$; see below. For this reason (the strong similarity of the dependence of monoparametric plots on Λ and γ), we do not show separately the dependence on Λ or fixed γ.

We now turn to the consideration of the dynamical evolution of solitons past the $\mathcal{PT}$ transition point. As indicated above, given their generic stability for $\gamma < \gamma_{\mathcal{PT}}$, we do not consider
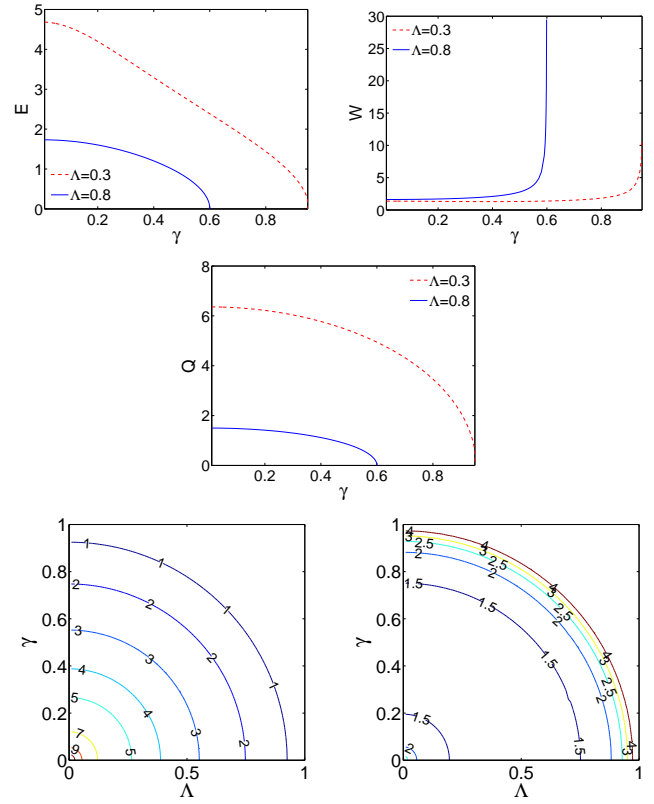

Fig. 3. The standing waves' energy E, width W and charge Q dependence for $\Lambda = 0.3$ and $\Lambda = 0.8$ are shown in the first three panels. The bottom panels present a two-parameter diagram of the dependence of the energy (bottom left) and width on the frequency Λ and the gain-loss parameter γ.

the latter case. In the case of $\gamma > \gamma_{\mathcal{PT}}$, we have firstly taken as initial condition the soliton for $\Lambda = 0.8$ and $\gamma = \gamma_0 = 0.59$ in the simulation with $\gamma = \gamma_s > \gamma_{\rm PT} = \sqrt{1 - \Lambda^2} = 0.6$. We observe that if γ_s is close enough to $\gamma_{\rm PT}$ (i.e., for a "shallow" quench), the density oscillates with a frequency that decreases with $\gamma_s - \gamma_{\rm PT}$ (see Fig. 4). Notice that the charge of the new soliton is always higher than the charge of the initial one (see the oscillations of Fig. 4) and that the maximum charge increases with γ_s. Interestingly, in all of these case examples we find that the (γ-independent) energy is very well conserved as is shown in Fig 5. When the maximum charge is above a threshold (this occurs for $\gamma_s \gtrsim 0.995$, i.e., for a deep quench), the frequency of the new soliton tends to zero and the solution starts to grow indefinitely as shown in Fig. 6. If a smaller value of γ_0 is taken, the same phenomenology persists, but the indefinite growth emerges for a smaller value of γ_s. In Fig. 5 we have confirmed that both the energy conservation law and the moment equation (10) for the power are satisfied in our dynamics. The same is true for the case of Fig. 6 where the charge grows exponentially (in the case shown in the figure, for which $\gamma_s = 1$, as $\sim \exp(0.088t)$; although the characteristic growth rate depends on γ_s). Here, the soliton does not collapse, as its shape and width are preserved during the growth. Again, this type of growth appears to be very different than, say, the collapse in the Hamiltonian NLS model [14]. In the latter, the width decreases and the amplitude increases, whereas here the entire solution grows without changing its spatial distribution.

Let us mention that the waveform with oscillating charge is fairly generic when the quench is not sufficiently deep to cause an exponential growth. Remarkably, such solitary waves with oscillating charge can be attained by performing an SU(1,1) transformation to a standing wave soliton (2) and have been predicted in [36]. This type of solution for a standing wave of frequency $\tilde{\Lambda}$ is, in fact, intrinsically connected to the invariance of the frequency $2\tilde{\Lambda}$ in the spectrum [19]. More specifically, these bi-frequency, oscillating charge coherent structures [which can be dubbed as SU(1,1) solitons] are of the form:

$$U(x,t)=\alpha_-\tilde{u}(x)\exp(-\mathrm{i}\tilde{\Lambda}t)-\mathrm{i}\alpha_+\tilde{v}^*(x)\exp(\mathrm{i}\tilde{\Lambda}t)\,,\quad (12)$$
$$V(x,t)=\alpha_-\tilde{v}(x)\exp(-\mathrm{i}\tilde{\Lambda}t)-\mathrm{i}\alpha_+\tilde{u}^*(x)\exp(\mathrm{i}\tilde{\Lambda}t)\,,\quad (13)$$

with

$$\alpha_\pm\in\mathbb{C},\qquad |\alpha_-|^2-|\alpha_+|^2=1.\quad (14)$$

In this case, $\{\tilde{u}(x),\tilde{v}(x)\}$ is the standing wave solution with frequency $\tilde{\Lambda}$. Consequently, the charge oscillates with a frequency $2\tilde{\Lambda}$ as long as $\gamma\neq 0$. There is an SU(1,1) family of solutions for each value of γ and $\tilde{\Lambda}$ which fulfills the same equations that the standing wave solutions (2) satisfy. As a result, when $\gamma_s\neq\gamma_0$, an SU(1,1) solution with $\gamma=\gamma_s$ is apparently dynamically manifested. Since these periodic SU(1,1) solutions and the standing wave solutions only exist for $\gamma<m\equiv 1$, it is natural to expect that there are no nontrivial fixed points for the dynamics for $\gamma>1$, hence giving rise to the observed growth dynamics.

We have confirmed that the dynamics observed, e.g., in Fig. 4 corresponds to SU(1,1) solutions. For instance, for the soliton with $\gamma_0=0.59$, $\Lambda=0.8$, when initializing it for $\gamma_s=0.9$, it spontaneously gives rise to an oscillatory state of the above form of Eq. (12) with $\tilde{\Lambda}=0.422$, $\alpha_-=1.0847$ and $\alpha_+=0.4201$. On the other hand, using a numerically exact (up to a prescribed tolerance) solution of our fixed point iteration scheme with a given frequency $\tilde{\Lambda}$ (for a desired γ), we can select values of α_- and α_+ and the exact form of Eq. (12) in order to construct, at will, such bi-frequency SU(1,1) solutions. An example of this form is shown in the panels of Fig. 7 (even for $\gamma=0$) for $\tilde{\Lambda}=0.5$, $\alpha_-=1.0500$ and $\alpha_+=0.3202$.

B. Massless NLD equation with $k=1$

Finally, we now turn to the analysis of the solutions in the special case of $m=0$. This case has the peculiarity that it features exact solutions available in analytical form even for $\gamma\neq 0$. Due to the functional form of those solutions, we had to use a Chebyshev collocation scheme with inhomogenous Dirichlet boundary conditions (to numerically identify their stability). As we will see below directly from their functional form, these solutions are not $\mathcal{PT}$-symmetric, hence the nonlinearity becomes responsible for the breaking of $\mathcal{PT}$-symmetry. This, in turn, implies that the linear stability eigenvalues do not come in quartets [but rather only in conjugate pairs i.e., in pairs with opposite Re(ω)]. Furthermore, although the solutions are stable for $\gamma=0$, they become immediately unstable, as soon as $\gamma\neq 0$.
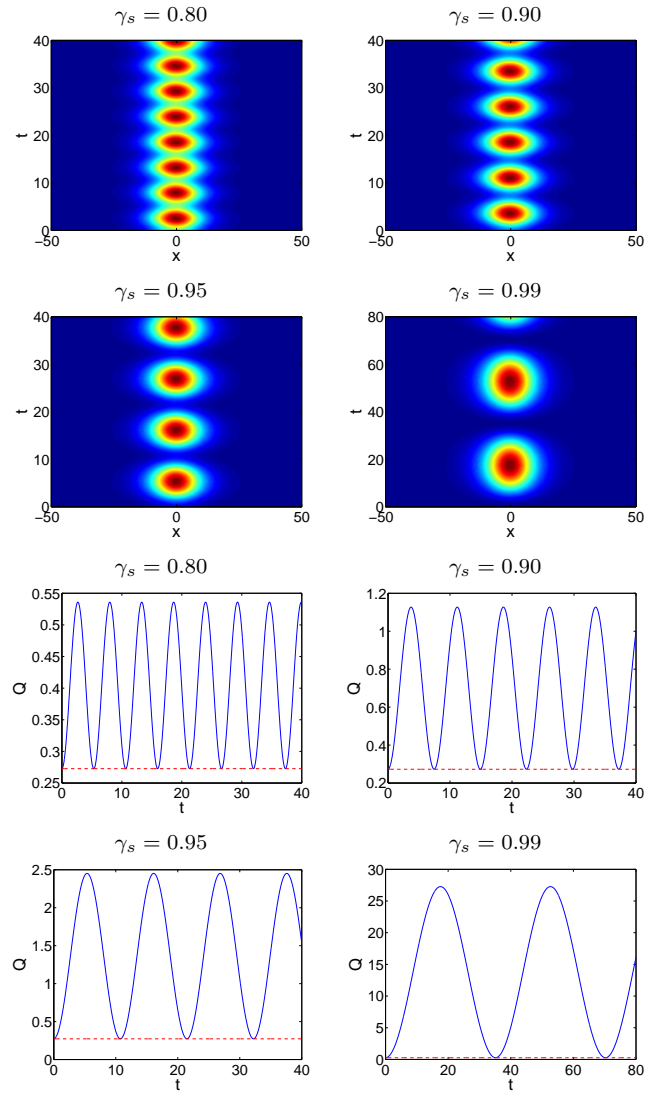

Fig. 4. Top two rows: Dynamics for $\Lambda=0.8$ using as initial condition the soliton with $\gamma=0.59$, but evolving Eq. (1) for $\gamma=\gamma_s$, shown in the respective panels. Each panel shows the contour plot of the space-time evolution of the soliton density. Bottom two rows: same as the top but with each panel showing the time evolution of the soliton charge. The dashed line corresponds to the charge of the initial condition.

There are two main exact solutions for a given value of γ:

$$u_1(x)=\frac{1}{2\sqrt{-g\Lambda}}\left[(-\Lambda+\rho\tanh(\rho x))-\mathrm{i}\gamma\right],\quad (15a)$$

$$v_1(x)=\frac{1}{2\sqrt{-g\Lambda}}\left[(-\Lambda-\rho\tanh(\rho x))-\mathrm{i}\gamma\right],\quad (15b)$$

and

$$u_2(x)=\frac{1}{2\sqrt{-g\Lambda}}\left[(\rho+\Lambda\tanh(\rho x))+\mathrm{i}\gamma\tanh(\rho x)\right],\quad (16a)$$

$$v_2(x)=\frac{1}{2\sqrt{-g\Lambda}}\left[(-\rho+\Lambda\tanh(\rho x))+\mathrm{i}\gamma\tanh(\rho x)\right],\quad (16b)$$

with $\rho=\sqrt{\gamma^2+\Lambda^2}$. This pair of solutions can be transformed to another pair by virtue of the transformation $\{u,v\}\rightarrow\{v^*,-u^*\}$, as Eq. (1) is invariant under transformations
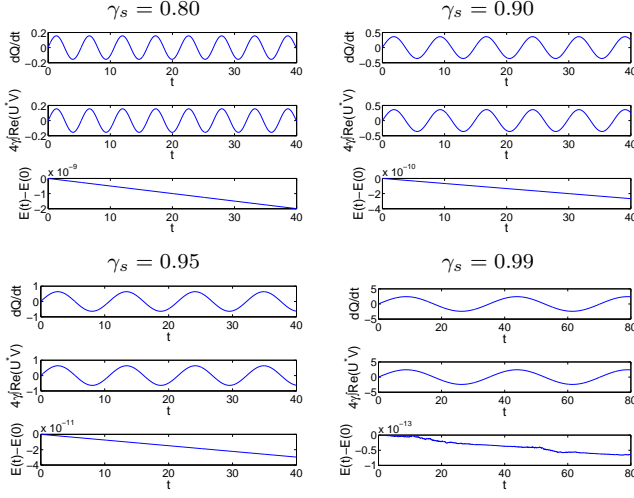
Fig. 5. Dynamics for $\Lambda = 0.8$ using as initial condition the soliton with $\gamma_0 = 0.59$. Each panel shows the time evolution of the time derivative of the soliton charge and compares it with the right hand side of Eq. (10); the bottom panels for each example of γ illustrate the time dependence of the energy fluctuations.
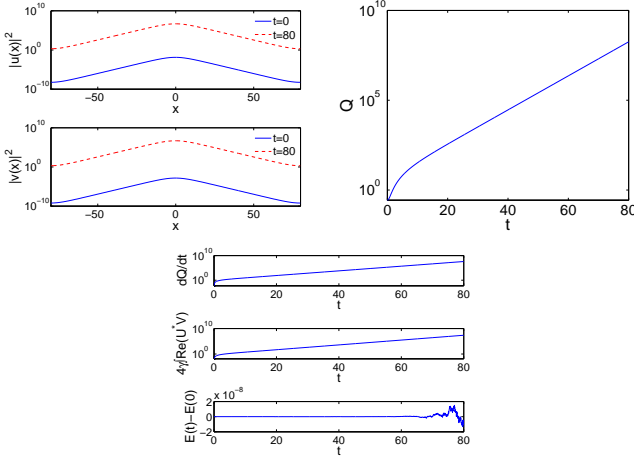

Fig. 6. Dynamics for $\Lambda = 0.8$ and $\gamma_s = 1$ when the soliton with $\gamma_0 = 0.59$ is used as initial condition. The left panel shows both components of the spinors at different times. The right panel shows the total charge as a function of time. The lower panel compares the time derivative of the charge and the right hand side of Eq. (10) and shows the time dependence of the energy fluctuations. One can clearly observe the (spatially independent) exponential growth of the waveform.
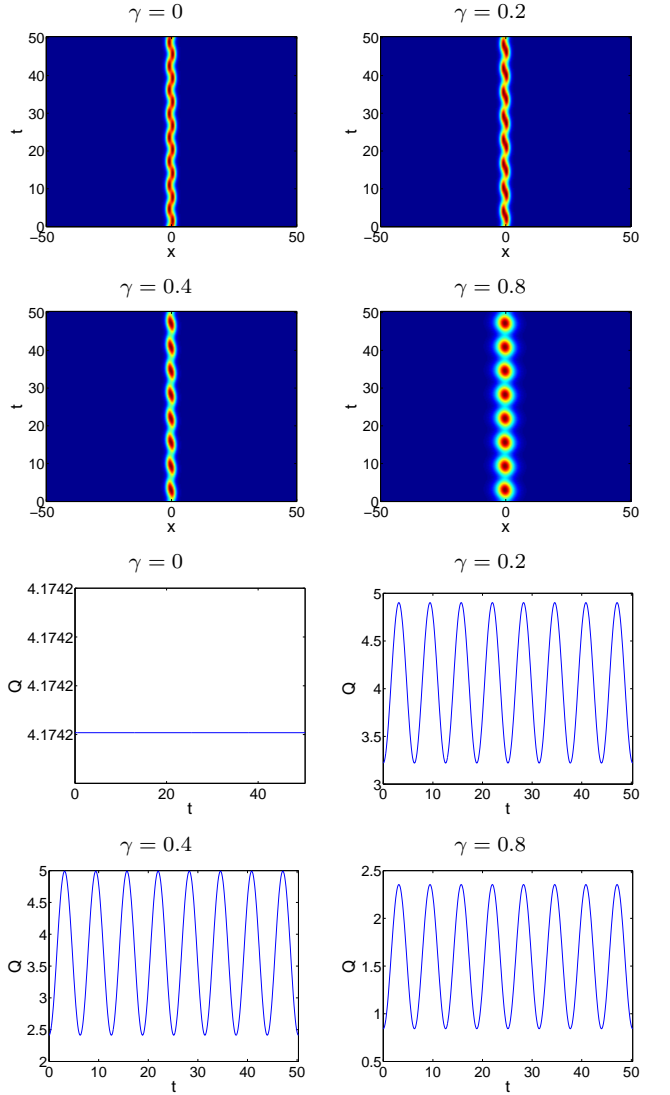
Fig. 7. Dynamics for SU(1,1) solitons with $\tilde{\Lambda} = 0.5$, $\alpha_- = 1.0500$ and $\alpha_+ = 0.3202$. Each panel shows the time evolution of the soliton density. The bottom four panels show the temporal evolution of the charge for each of the examples in the top panels.

$\{U, V\} \to \{V^*, -U^*\}$ when $m = 0$. This transformation introduces the change $\omega \to \omega^*$ into the stability eigenfrequencies spectrum. It is worth noticing that both solutions (15) and (16) correspond to the same density, i.e.

$$|u_1(x)|^2 + |v_1(x)|^2 = |u_2(x)|^2 + |v_2(x)|^2 \qquad (17)$$

Figure 8 shows the spectral plane for both solutions of the massless equation for $\gamma = 0.01$ and $\Lambda = -0.9$. It can be observed that the stability eigenfrequencies for the first solution are the complex conjugates of the ones of the second solution. Nevertheless, in neither case does the spectrum present a four-fold symmetry, given the nonlinearity-induced breaking of $\mathcal{PT}$-symmetry.
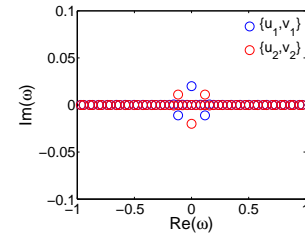

Fig. 8. Spectral plane of the exact analytical solutions of Eqs. (15)-(16) for $m = 0$, $\Lambda = -0.9$ and $\gamma = 0.01$. The spectra of the two solutions are conjugate, but both reflect their non-$\mathcal{PT}$-symmetric nature in the absence of eigenfrequency quartets within the spectral plane.
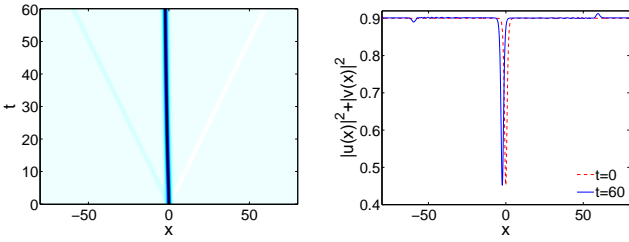
Fig. 9. Dynamics of the soliton (15) for $m = 0$, $\Lambda = -0.9$ and $\gamma = 0.01$. Left panel shows the density plot (darker color means deeper amplitude) whereas right panel compares the initial density with that of an intermediate time.

We have performed simulations in order to observe the effect of unstable modes for both $\{u_1, v_1\}$ and $\{u_2, v_2\}$ solutions. Those simulations have been performed using a Chebyshev collocation method with Neumann boundary conditions using a 2nd-3rd order Runge-Kutta integrator supplemented by a trapezoidal rule and the backward differentiation formula of order 2 (TR-BDF2 algorithm) [37]. We have used as initial condition the standing wave perturbed along its unstable eigenmode corresponding to the imaginary eigenfrequency direction. As a result of the simulation (see Fig. 9), we can observe that the soliton dip starts to slowly move along the system with constant speed and preserving its shape whereas a counter-propagating pair of small amplitude waves is emitted. This behaviour is found for both $\{u_1, v_1\}$ and $\{u_2, v_2\}$ solutions. If the standing waves were perturbed following the direction of the eigenmode corresponding to the complex eigenvalue pair, the density dip would remain at rest and radiation in form of a staggered perturbation is emitted; in fact, the eigenmode is spurious because its spatial profile possesses a zigzag tail which is not compatible with solutions in the continuum limit, hence we do not consider it further here.

IV. Conclusions and Future Challenges

In the present work, we introduced a $\mathcal{PT}$-symmetric extension of the Gross-Neveu (or Soler) model as a prototypical example of a $\mathcal{PT}$-symmetric nonlinear Dirac equation (PT-NLDE). We have found that the model possesses a number of remarkable and previously unexplored (some even in the standard Hamiltonian case) characteristics. The $\mathcal{PT}$-symmetric variant used here involved the Dirac matrix γ_5 (whose role in the case of spinors with two components was played by the Pauli matrix σ_1) that, as we argued, introduced an *unusual* form of a $\mathcal{PT}$-symmetric setting where for our two-component model, it was not transparent that one field bears gain and the other loss, but both, in principle, carry both gain and loss. Perhaps this special nature of the problem is responsible for the remarkable feature that the Hamiltonian form of the energy (which is γ-independent) is still conserved in the presence of the gain-loss parameter γ (while the power is not). On the other hand, the nonlinear solutions identified here presented an unusual $\mathcal{PT}$ phase transition (for the case of unit mass) whereby they degenerated into the linear limit of the problem. The solutions (which presented an implicit mono-parametric dependence on $\gamma^2 + \Lambda^2$) were found to be stable everywhere within the regime of exact $\mathcal{PT}$-symmetry hence we attempted to consider dynamics beyond this interval. There, we found the prototypical formation of time-periodic solutions whose bi-frequency character was attributed to the SU(1,1) symmetry of the equation. Importantly such periodic solutions exist even in the Hamiltonian limit of the model. Beyond a critical value of γ, the solutions were found to feature an unusual (spatially independent) growth. Finally, exact analytical waveforms could only be identified in the special limit of the massless problem with $m = 0$. These solutions had an unexpected characteristic as well in that the nonlinearity enabled their breaking of the $\mathcal{PT}$-symmetry for all values of (Λ, γ) for which they were found to exist.

Clearly, this is an unusual and exceptionally rich class of models at the interface of the emerging theme of $\mathcal{PT}$-symmetric media and the mathematically highly nontrivial realm of the nonlinear Dirac equations. Understanding in more depth any one of the above features [the energy independence on γ, the exponential, spatially independent growth, the spectrum of the massless solutions or that of the SU(1,1) solutions] would already enable significant advances in this nascent field of $\mathcal{PT}$-NLDE models. Generalizations including variants of the model [such as the integrable Thirring model, or the physically motivated self-interacting model (i.e., only featuring respectively nonlinear terms of the form $|U|^2 U$ and $|V|^2 V$ in Eq. (1))] would be particularly worthwhile to consider, as would extensions to other settings such as two-dimensional ones. Such extensions are presently under consideration and relevant results will be reported in future publications.

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Jesús Cuevas–Maraver received the B.Sc. in Physics from the University of Sevilla, Seville, Spain, in 1999. Subsequently, he become member of the Nonlinear Physics Group at the same university and in 2000, he joined the Applied Physics Department, receiving the PhD in Physics in 2003 for his work in intrinsic localized modes in nonlinear lattices. In May 2009 he become *Profesor Titular* (Associate Professor) at the same department and from 2014 is member of the Mathematics Institute of the University of Seville (IMUS). His main research interest comprises the study of localized waves in nonlinear media, ranging from photonic systems to biomolecules or Bose-Einstein condensates.

Panayotis G. Kevrekidis earned his B.Sc. from the University of Athens (1996, Physics). He joined the Physics Department of Rutgers earning his M.S. (1998), M.Phil. and Ph.D. (2000) in Physics. He then assumed a post-doctoral post split between the Program in Applied and Computational Mathematics of Princeton University and the Theoretical Division and the Center for NonLinear Studies of the Los Alamos National Laboratory in 2000-2001. From 09/2001, he joined the Department of Mathematics and Statistics of the University of Massachusetts (UMass), Amherst, as an Assistant Professor. He became an Associate Professor at UMass in 2005 and a Full Professor in 2010.

Avadh Saxena is Group Leader of the Condensed Matter and Complex Systems group at the Theoretical Division at Los Alamos National Laboratory, New Mexico, USA. He is also an affiliate of the Center for Nonlinear Studies at Los Alamos. He received his Ph. D. in Physics from Temple University, Philadelphia in 1986 and after a postdoc at the Pennsylvania State University joined Los Alamos in 1990. His expertise is in condensed matter physics, phase transitions in functional materials a variety of nonlinear phenomena.

Fred Cooper completed his studies in Physics from City College New York in 1964, and received his Ph. D. in Physics from Harvard University in 1968. He was an Instructor at Cornell University from 1968-1970, an Assistant Professor of Physics at Belfer Graduate School of Science from 1970-1975 abad a Staff Member at Los Alamos National Labs (LANL) in the Theory division from 1975-2012, and was Group Leader of the Elementary Particle Physics group T-8 from 1995-2001. He was Program Director for Theoretical Physics at the National Science Foundation from 2002-2009 He currently is an external Professor at the Santa Fe Institute and a guest scientist at LANL. His expertise is in Elementary Particle Physics, the theory of Bose Einstein Condensates, Nonlinear dynamics, and Pattern Formation in Chemical reactions. He is a member of the APS.

Avinash Khare got his Masters in Physics from Holkar college Indore, India and his Ph.D. in Theoretical High Energy Physics from Saha Institute of Nuclear Physics, Kolkata, India in 1971. After a postdoctoral position at University of Tokyo, he moved to Institute of Physics, Bhubaneswar where he retired as Senior Professor in 2010. Since then he worked as Raja Ramanna Fellow at Indian Institute of Science Education and Research, Pune and in June 2015 he has moved to Physics Department, Savitribai Phule Pune University, Pune, India as INSA Senior Scientist.

Andrew Comech completed his M.S. studies in Theoretical Physics at Moscow Institute for Physics and Technology in 1993, and received his Ph.D. in Mathematics from Columbia University in 1997. He is currently an Associate Professor at Mathematics Department at Texas A&M University, working on stability properties of solitary wave solutions.

Carl M. Bender got his undergraduate degree in physics from Cornell University and his PhD in theoretical physics from Harvard University in 1969. After a postdoctoral position at the Institute for Advanced Study (Princeton) and Assistant and Associate Professorships at MIT, he moved to Washington University in St. Louis, where he is the Wilfred R. and Ann Lee Konneker Distinguished Professor of Physics.