Regular time-reproductive solutions for generalized Boussinesq model with Neumann boundary conditions for temperature

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1. Statement of the problem

$\Omega \subset \mathbb{R}^N$ regular bounded domain, $N = 2, 3$, $T > 0$.

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta) \nabla \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} - \alpha \boldsymbol{g} \theta + \nabla p = \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \theta - \nabla \cdot (k(\theta) \nabla \theta) + (\boldsymbol{u} \cdot \nabla) \theta = 0, \end{cases} \tag{1}$$

in $\Omega \times [0, \infty)$

- $\boldsymbol{u}(x,t) \in \mathbb{R}^N$ velocity of the fluid at point $x \in \Omega$ and time $t \in [0, T)$
- $p(x,t) \in \mathbb{R}$ (hydrostatic) pressure.
- $\theta(x,t) \in \mathbb{R}$ temperature.
- $\boldsymbol{g}(x,t) \in \mathbb{R}^N$ gravitational field
- $\boldsymbol{f}(x,t) \in \mathbb{R}^N$ resulting of external forces.
- $\alpha > 0$ constant associated to the coefficient of volume expansion.
- $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ kinematic viscosity,
- $k(\cdot) : \mathbb{R} \to \mathbb{R}$ thermal conductivity.

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in $\Omega \times [0, \infty)$

- Dirichlet-Neumann boundary conditions:

$$\boldsymbol{u} = 0, \qquad \partial_n \theta = 0 \qquad \text{on } [0, \infty) \times \partial\Omega,$$

- Time reproductive condition:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) \qquad \text{in } \Omega.$$

1. Statement of the problem

Previous results: Existence and uniqueness of initial value problem with Dirichlet's boundary conditions

S.A. Lorca, J.L. Boldrini, *Stationary solutions for generalized Boussinesq models*, J. Diferential Equations 124 (2), (1996).

Goal: To obtain strong estimates: H^2 for the velocity, H^3 for the temperature and consequently, existence of regular reproductive solution

Difficulties: Time reproductive condition

Nonlinear diffusion

Let us consider the following spaces

$$H^k_N = \left\{ \theta \in H^k;\ \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial\Omega\,,\ \int_\Omega \theta = 0 \right\}$$

where $k = 2, 3.$

$$|\Delta\theta|_2 \approx \|\theta\|_2 \text{ in } H^2_N$$

$$|\nabla\Delta\theta|_2 \approx \|\theta\|_3 \text{ in } H^3_N$$

Definition:

$(\boldsymbol{u}, p, \theta)$ is a regular solution in $(0,T)$, if

$$\boldsymbol{u} \in L^2(\boldsymbol{H}^2) \cap L^\infty(\boldsymbol{H}^1) \qquad \text{and} \qquad \partial_t \boldsymbol{u} \in L^2(\boldsymbol{L}^2),$$

$$p \in L^2(H^1),$$

$$\theta \in L^2(H^3_N) \cap L^\infty(H^2_N) \qquad \text{and} \qquad \partial_t \theta \in L^2(H^1_N),$$

satisfying (1) a.e. in $\Omega \times (0,T)$, boundary conditions and time reproductivity conditions in the sense of spaces $\boldsymbol{V}$ and H^2_N respectively.

Theorem

Let $T > 0$, $\Omega \subset \mathbb{R}^N$ *a bounded domain* ($N = 2$ *or* 3), $\partial\Omega \in C^{2,1}$.

- $\nu \in C^1(\mathbb{R})$, $\quad 0 < \nu_{min} \le \nu(s) \le \nu_{max}$, $\quad |\nu'(s)| \le \nu'_{max}$
- $k \in C^2(\mathbb{R})$, $\quad 0 < k_{min} \le k(s) \le k_{max}$, $\quad |k'(s)| \le k'_{max}$, $\quad |k''(s)| \le k''_{max}$.
- $\boldsymbol{f} \in L^2(\boldsymbol{L}^2)$, $\quad \boldsymbol{g} \in L^\infty(\boldsymbol{L}^2)$, $\quad \|\boldsymbol{f}\|_{L^2(L^2)} \le \delta$, $\quad \delta$ *small enough*

then there exists a regular (and small) reproductive solution of (1) *in* $(0, T)$. *Moreover, this solution also verifies* $\partial_t\theta(0) = \partial_t\theta(T)$.

2. The Galerkin initial-boundary problem

Let $\{\phi_i\}_{i\geq 1}$ and $\{\varphi_i\}_{i\geq 1}$ "special" basis of $\boldsymbol{V}$ and $\boldsymbol{H}_0^1(\Omega)$, respectively, formed by eigenfunctions of the Stokes and the Poisson problems following:

$$\left\{\begin{array}{rcll} -\Delta\phi_i & = & \lambda_i\phi_i & \text{in } \Omega \\ \phi_i & = & 0 & \text{on } \partial\Omega \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta\varphi_i & = & \mu_i\varphi_i & \text{in } \Omega \\ \partial_n\varphi_i & = & 0 & \text{on } \partial\Omega, \end{array}\right.$$

$\|\phi_i\|_1 = 1$, $\|\varphi_i\|_1 = 1$ for all i and $\int_\Omega \varphi_i = 0$.

$$\boldsymbol{u}_m(t) = \sum_{j=1}^{m} \xi_{i,m}(t)\phi_i \qquad \theta_m(t) = \sum_{j=1}^{m} \zeta_{i,m}(t)\varphi_i,$$

2. The Galerkin initial-boundary problem

For each $m \geq 1$, given $\boldsymbol{u}_{0m} \in \boldsymbol{V}^m$ and $\theta_{0m} \in W^m$, there exits a unique solution $(\boldsymbol{u}_m, \theta_m)$, with $\boldsymbol{u}_m : [0,T] \mapsto \boldsymbol{V}^m$ and $\theta_m : [0,T] \mapsto W^m$, verifying the following variational formulation a.e. in $t \in (0,T)$:

$$\begin{cases} (\partial_t \boldsymbol{u}_m(t), \mathbf{v}_m) + ((\boldsymbol{u}_m(t) \cdot \nabla)\boldsymbol{u}_m(t), \mathbf{v}_m) + (\nu(\theta_m(t))\nabla \boldsymbol{u}_m(t), \nabla \mathbf{v}_m) \\ \qquad -(\alpha\theta_m(t)\boldsymbol{g}, \mathbf{v}_m) - (\boldsymbol{f}, \mathbf{v}_m) = 0 \qquad \forall\, \mathbf{v}_m \in \boldsymbol{V}^m \\ (\partial_t \theta_m(t), e_m) + ((\boldsymbol{u}_m(t) \cdot \nabla)\theta_m(t), e_m) \\ \qquad +(k(\theta_m(t))\nabla\theta_m(t), \nabla e_m) = 0 \qquad \forall\, e_m \in W^m \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \quad \theta_m(0) = \theta_{0m}, \end{cases}$$

3. Differential inequalities in regular norms

$$(\boldsymbol{u}_m \text{system}, A\boldsymbol{u}_m) + (\boldsymbol{u}_m \text{system}, \partial_t \boldsymbol{u}_m)$$

$$\frac{d}{dt}\int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \nu_{min}\|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 \leq \delta\|\partial_t\theta_m\|_1^2$$

$$+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 + K(\|\boldsymbol{u}_m\|_1^6 + \|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + \|\boldsymbol{g}\|_{L^\infty(L^2)}^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2)$$

$$\text{for } \delta, \varepsilon > 0 \text{ small enough, and } K = K(\delta,\varepsilon) > 0$$

From viscosity term

3. Differential inequalities in regular norms

$$(\partial_t(\theta_m \mathrm{eq.}), \partial_t \theta_m) + (\theta_m \mathrm{eq.}, \Delta^2 \theta_m)$$

$$\frac{d}{dt}(\|\theta_m\|_2^2 + |\partial_t \theta_m|_2^2) + k_{min}(\|\theta_m\|_3^2 + \|\partial_t \theta_m\|_1^2)$$

$$\leq \delta |\partial_t \boldsymbol{u}_m|_2^2 + C_\delta(\|\theta_m\|_2^6 + \|\theta_m\|_2^4 |\partial_t \theta_m|_2^2 + \|\theta_m\|_2^2 \|\boldsymbol{u}_m\|_1^4)$$

for $\delta > 0$ small enough, and $C_\delta > 0$

3. Differential inequalities in regular norms

- $\Delta^2\theta_m \in W^m$ thanks to the election of spectral basis
- Integrating by parts in all terms, boundary terms vanish since

$$(\nabla\Delta\theta_m \cdot \boldsymbol{n})|_{\partial\Omega} = 0$$

One obtains:

$$-(\partial_t\nabla\theta_m, \nabla\Delta\theta_m) + (\nabla[\nabla\cdot(k(\theta_m)\nabla\theta_m)], \nabla\Delta\theta_m) - (\nabla(\boldsymbol{u}\cdot\nabla\theta_m), \nabla\Delta\theta_m) = 0.$$

3. Differential inequalities in regular norms

$$\frac{d}{dt}\int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \nu_{min}\|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 \leq \delta\|\partial_t\theta_m\|_1^2$$

$$+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 + K(\|\boldsymbol{u}_m\|_1^6 + \|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + \|\boldsymbol{g}\|_{L^\infty(L^2)}^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2)$$

$$+$$

$$\frac{d}{dt}(\|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2) + k_{min}(\|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2)$$

$$\leq \delta|\partial_t \boldsymbol{u}_m|_2^2 + C_\delta(\|\theta_m\|_2^6 + \|\theta_m\|_2^4|\partial_t\theta_m|_2^2 + \|\theta_m\|_2^2\|\boldsymbol{u}_m\|_1^4)$$

Adequate balance $\Rightarrow$ $||\boldsymbol{g}||_{L^\infty(L^2)}^2||\theta_m||_2^2$

4. Proof of theorem

We denote:

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m) + 1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t \theta_m|_2^2$$

$$\Psi_m(t) = \|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 + \|\theta_m\|_3^2 + \|\partial_t \theta_m\|_1^2$$

We obtain:

$$\begin{cases} \Phi_m' + C\Psi_m \leq \varepsilon \Psi_m \Phi_m^{1/2} + C_0(t) + D\Phi_m^3 \\ \Phi_m(0) = \Phi_{m0} \end{cases}$$

$$C_0(t) = C_0|\boldsymbol{f}|_2^2$$

4. Proof of theorem

First step: If $\Phi_m(0) \leq \delta$ and $\|\boldsymbol{f}\|_{L^2(L^2)} \leq \delta$, then $\Phi_m(t) < 2\delta \ \forall t \in [0,T]$.

Absurd argument: T^* such that $\Phi_m(T^*) = 2\delta \quad \Phi_m(s) < 2\delta \quad \forall s \in [0,T^*)$.

$$+$$

Poincaré inequality: $C_p > 0$ such that $\Phi_m(t) \leq C_p \Psi_m(t)$ (ε small enough)

$$\Rightarrow \quad \Phi'_m + C\Psi_m \leq \varepsilon \Psi_m \Phi_m^{1/2} + C_0(t) + D\Phi_m^3$$

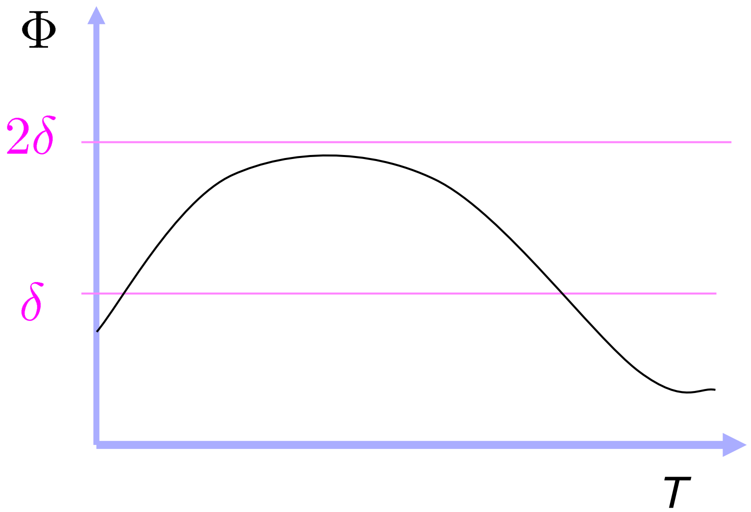
$$\Phi'_m + \tilde{C}\Phi_m \leq C_0(t) + D\Phi_m^3$$

Integrating in $[0,T^*]$: $\Phi_m(T^*) \leq \delta e^{-\bar{C}T^*} + \int_0^{T^*} C_0(t) < 2\delta. \qquad \longrightarrow\!\longleftarrow$

Second step: If $\Phi_m(0)$ and $\|\boldsymbol{f}\|_{L^2(L^2)}$ are small enough, then $\Phi_m(T) < \Phi_m(0)$.

Similarly, integrating in $[0, T]$,

$$\Phi_m(T) \le \Phi_m(0)e^{-\bar{C}T} + \underbrace{\int_0^T C_0(s)}_{<<}.$$



Third step: Existence of approximate reproductive solution.

Given $(\boldsymbol{u}_{m0}, \theta_{m0}) \in V^m \times W^m$,

$$\begin{array}{rcl} L^m : [0,T] & \mapsto & \mathbb{R}^m \times \mathbb{R}^m \\ t & \mapsto & (\xi_{1m}(t), ..., \xi_{mm}(t), \zeta_{1m}(t), ..., \zeta_{mm}(t)) \end{array}$$

$(\xi_{1m}(t), ..., \xi_{mm}(t))$, $(\zeta_{1m}(t), ..., \zeta_{mm}(t))$ coefficients of $\boldsymbol{u}_m(t)$ and $\theta_m(t)$

Given $L_0^m = L^m(0)$

$$\bar{B} = \{(\xi_{1m}, ..., \xi_{mm}, \zeta_{1m}, ..., \zeta_{mm}) := L_0^m : \Phi_m(0) \leq \delta\}.$$

$$\begin{array}{rcl} \mathcal{R}^m : \bar{B} \subset \mathbb{R}^m \times \mathbb{R}^m & \mapsto & \mathbb{R}^m \times \mathbb{R}^m \\ L_0^m & \mapsto & \mathcal{R}^m(L_0^m) = L^m(T) \end{array}$$

Brouwer Theorem.

Four step: Pass to the limit in reproductive approximate solutions

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m) + 1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t \theta_m|_2^2 \leq 2\delta$$

(independent of m) for small data.

$$(\boldsymbol{u}_m) \text{ uniformly bounded in } L^\infty(H^1) \cap L^2(H^2),$$
$$(\theta_m) \text{ uniformly bounded in } L^\infty(H^2_N) \cap L^2(H^3_N),$$
$$(\partial_t \boldsymbol{u}_m) \text{ uniformly bounded in } L^2(L^2),$$
$$(\partial_t \theta_m) \text{ uniformly bounded in } L^\infty(L^2) \cap L^2(H^1).$$

and

$$(\boldsymbol{u}_m) \text{ is relatively compact in } L^2(H^1)$$
$$(\theta_m) \text{ is relatively compact in } L^2(H^2).$$

Sufficient to pass to the limit in equations.

4. Proof of theorem

θ_m is relatively compact in $C([0,T];H^1)$

$$\begin{array}{l}\theta_m(T) \longrightarrow \theta(T) \text{ in } H^1(\Omega) \\ \| \\ \theta_m(0) \longrightarrow \theta(0) \text{ in } H^1(\Omega)\end{array}$$

$\theta_m(T)$ and $\theta_m(0)$ are bounded in $H^2(\Omega)$

$$\Rightarrow \quad \theta(T) = \theta(0) \text{ in } H^2(\Omega)$$

$\partial_{tt}\theta_m$ is uniformly bounded in $L^2((H^1)')$

$\partial_t\theta_m$ is uniformly bounded in $L^\infty(L^2)$

$$\Rightarrow \partial_t\theta_m \text{ is relatively compact in } C([0,T];(H^1)')$$

$$\begin{array}{l}\partial_t\theta_m(T) \longrightarrow \partial_t\theta(T) \text{ in } (H^1)'(\Omega) \\ \| \\ \partial_t\theta_m(0) \longrightarrow \partial_t\theta(0) \text{ in } (H^1)'(\Omega)\end{array}$$

$$\Rightarrow \partial_t\theta(0) = \partial_t\theta(T)$$

5. Some comments and open problems

- Dirichlet boundary condition is imposed for the temperature θ, the boundary terms do not vanish in the integration by parts.

- The uniqueness remains open. Higher regularity for the velocity is necessary. H^3-regularity for $\boldsymbol{u}$ and Dirichlet condition ?