Convergence to equilibrium for smectic-A liquid crystals in $3D$ domains without constraints for the viscosity*

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Abstract

In this paper, we focus on a smectic-A liquid crystal model in $3D$ domains, and obtain three main results: the proof of an adequate Lojasiewicz-Simon inequality by using an abstract result; the rigorous proof (via a Galerkin approach) of the existence of global in-time weak solutions that become strong (and unique) in long-time; and its convergence to equilibrium of the whole trajectory as time goes to infinity. Given any regular initial data, the existence of a unique global in-time regular solution (bounded up to infinite time) and the convergence to an equilibrium have been previously proved under the constraint of a sufficiently high level of viscosity. Here, all results are obtained without imposing said constraint.

Keywords: Liquid crystals, Navier-Stokes equations, Ginzburg-Landau potential, energy dissipation, convergence to equilibrium, Lojasiewicz-Simon's inequalities.

1 Introduction

We consider the following equations ([5]), which model a smectic-A liquid crystal confined in an open bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ within the time interval $(0, +\infty)$:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} - \lambda w \nabla \varphi + \nabla q = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

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$$\partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + \gamma w = 0, \tag{3}$$

$$\Delta^2\varphi - \nabla\cdot\boldsymbol{f}_\varepsilon(\nabla\varphi) - w = 0, \tag{4}$$

where

$$\boldsymbol{f}_\varepsilon(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F_\varepsilon(\boldsymbol{n}) = \frac{1}{\varepsilon^2}(|\boldsymbol{n}|^2-1)\boldsymbol{n}, \quad \forall\, \boldsymbol{n}\in\mathbb{R}^3$$

and $F_\varepsilon(\boldsymbol{n}) = \dfrac{1}{4\varepsilon^2}(|\boldsymbol{n}|^2-1)^2$ is the Ginzburg-Landau potential. Here, $\boldsymbol{u}:\Omega\times[0,+\infty)\mapsto\mathbb{R}^3$ is the flow velocity; $p:\Omega\times[0,+\infty)\mapsto\mathbb{R}$ describes a potential function (dependent of the fluid pressure); $\varphi:\Omega\times[0,+\infty)\mapsto\mathbb{R}$ is the layer variable, whose level sets represent the layer structure; and $w=\Delta^2\varphi-\nabla\cdot\boldsymbol{f}_\varepsilon(\nabla\varphi)$ is a variable related to the equilibrium equation with respect to the (smectic) elastic energy

$$E_e(\varphi) = \int_\Omega \left(\frac{1}{2}|\Delta\varphi|^2 + F_\varepsilon(\nabla\varphi)\right). \tag{5}$$

The constants $\nu>0$, $\lambda>0$, and $\gamma>0$ are some coefficients which depend on the viscosity, the elasticity and the time relaxation, respectively. The system (1)-(4) is completed with the (Dirichlet) boundary conditions

$$\boldsymbol{u}|_{\partial\Omega}=0, \quad \varphi|_{\partial\Omega}=\varphi_1, \quad \partial_{\rm n}\varphi|_{\partial\Omega}=\varphi_2, \tag{6}$$

where φ_1 and φ_2 are given time-independent functions, and the initial conditions

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \qquad \varphi(0)=\varphi_0 \quad \text{in } \Omega. \tag{7}$$

For compatibility, we assume $\boldsymbol{u}_0|_{\partial\Omega}=0$ with $\nabla\cdot\boldsymbol{u}_0=0$ and $\varphi_0|_{\partial\Omega}=\varphi_1$, $\partial_{\rm n}\varphi_0|_{\partial\Omega}=\varphi_2$.

The first mathematical results of problem (1)-(7) were obtained in [10]. For three-dimensional domains and time-independent boundary conditions, both the existence of global in-time weak solutions for the smectic-A problem (1)-(7) and pioneering research into its long-time behaviour are jointly studied in [10], and convergence of $\boldsymbol{u}(t)$ and $w(t)$ to zero as $t\to+\infty$ is attained, although the uniqueness of limit for the trajectories $\varphi(t)$ as $t\uparrow\infty$ is not assured. The regularity and time-periodicity of solutions of the problem (1)-(7) with time-dependent boundary conditions is studied in [3]. These results were previously studied for nematic liquid crystals in [9] and [1].

The convergence in infinite time of the whole trajectory was first solved in [14] for a nematic model with Dirichlet boundary conditions, thereby obtaining the convergence of the director vector $\boldsymbol{d}(t)$ (an average of preferential orientation of molecules) as $t\to+\infty$ towards an equilibrium of the elastic energy. In [15], a similar problem with stretching terms and periodic boundary conditions of $\boldsymbol{d}$ is treated. For these convergence results, suitable Lojasiewicz-Simon inequalities are used. In both cases above, in order to obtain a global in-time regular solution, a uniform in-time Gronwall theorem is used (see [13]), requiring either a sufficiently high viscosity coefficient or initial conditions sufficiently near to a global minimizer.

The long-time behaviour of a nematic liquid crystal model with time-dependent boundary conditions and external forces is studied in [6], while also imposing a high level of viscosity. For nematic models including stretching terms, in the recent paper [11], the authors show that any weak solution has a ω-limit set containing a single steady solution, thereby circumventing the use of the strong regularity (hence the viscosity constraint is rendered unnecessary).

Returning to the smectic-A problem (1)-(7), its long-time behaviour has already been studied in [12], where the imposition of both a high level of viscosity and periodic boundary conditions plays a main role. On the other hand, the convergence of the whole trajectory to equilibrium for a smectic-A model modified by penalization is given in [4], without imposing constraints for the viscosity.

Consequently, with respect to the above results, the main contribution that we will present in this paper is the identification of a unique critical point as the limit of the trajectory of $\varphi(t)$ as t approaches to infinity, for each global weak solution of the smectic-A model (1)-(7) that is strong over long periods, without imposing a high level of viscosity. Moreover, we consider of remarkable interest the following facts:

1. The proof of an adequate Lojasiewicz-Simon inequality by means of an abstract result given in [8] (see Theorem 4 below).

2. The rigorous proof, via a Galerkin approach, of the existence of weak solutions of the smectic-A problem (1)-(7), which are strong solutions in the case of long periods.

1.1 Notation

- In general, the notation will be abridged: $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, then $L^p(X)$ denotes the Banach space $L^p(0,T;X(\Omega))$. Moreover, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^3$.

- The L^p-norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, and the H^m-norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$). The inner product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The boundary $H^s(\partial\Omega)$-norm is denoted by $\|\cdot\|_{s;\partial\Omega}$.

- The space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^3$ satisfying $\nabla\cdot\boldsymbol{u} = 0$ is set as $\mathcal{V}$. The closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ and $\boldsymbol{H}^1$ are denoted as $\boldsymbol{H}$ and $\boldsymbol{V}$, which are Hilbert spaces for the norms $|\cdot|_2$

and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \qquad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

Note that if $\boldsymbol{u} \in \boldsymbol{H}$, since $\boldsymbol{u} \in \boldsymbol{L}^2$ and $\nabla\cdot\boldsymbol{u} \in \boldsymbol{L}^2$, therefore $\boldsymbol{u}\cdot\mathbf{n} = 0$ holds in $\boldsymbol{H}^{-1/2}(\partial\Omega)$.

- We will consider a sufficiently regular Ω in order to have the following equivalent norms:

$$\|\varphi\|_1 \approx |\nabla\varphi|_2 + \|\varphi|_{\partial\Omega}\|_{1/2;\partial\Omega} = |\nabla\varphi|_2 + \|\varphi_1\|_{1/2;\partial\Omega} \tag{8}$$

$$\|\varphi\|_2 \approx |\Delta\varphi|_2 + \|\varphi|_{\partial\Omega}\|_{3/2;\partial\Omega} = |\Delta\varphi|_2 + \|\varphi_1\|_{3/2;\partial\Omega} \tag{9}$$

$$\|\varphi\|_4 \approx |\Delta^2\varphi|_2 + \|\varphi_1\|_{7/2;\partial\Omega} + \|\varphi_2\|_{5/2;\partial\Omega} \tag{10}$$

- In the following, $C, K > 0$ will denote several constants, which depend only on the fixed data of the problem.

- For the sake of simplicity, henceforth we will consider $\nu, \lambda, \gamma = 1$.

2 Some preliminary results

2.1 Long-time behaviour

Assume the following starting point:

Let $E, \Phi \in L^1_{loc}(0, +\infty)$ be two positive functions with $E \in H^1(0,T)$ $\forall T > 0$, satisfying

$$E'(t) + \Phi(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty). \tag{11}$$

Therefore, E is a decreasing function with $E \in L^\infty(0, +\infty)$ and

$$\exists \lim_{t\to+\infty} E(t) = E_\infty \geq 0. \tag{12}$$

Moreover, by integrating (11), one has $\Phi \in L^1(0, +\infty)$.

The following result is proved in [2].

Lemma 1 *Let $\Phi \in L^1(0, +\infty)$ be a positive function such that $\Phi \in H^1(0,T)$ $\forall T > 0$, which satisfies*

$$\Phi'(t) \leq C_2(\Phi(t)^3 + 1). \tag{13}$$

Therefore, there exists a sufficiently large $T^* \geq 0$ such that $\Phi \in L^\infty(T^*, +\infty)$ and

$$\exists \lim_{t\to+\infty} \Phi(t) = 0.$$

We will extend this result for function sequences in order to uniformly bound them with respect to the index of sequence. Specifically,

Theorem 2 *Let Φ^m, E^m, be two positive function sequences, which satisfy (11) and (13) for some constant $C_2 > 0$ independent of m. Let $E(t) = \lim_{m\to+\infty} E^m(t)$ a.e. $t \in (0,+\infty)$. Therefore, for each $\varepsilon \in (0,1)$, there exists a sufficiently large time $T^* = T^*(\varepsilon) \geq 0$, independent of m, such that*

$$\|\Phi^m\|_{L^\infty(T^*,+\infty)} \leq \varepsilon.$$

Proof.

By construction, $E(t)$ is a decreasing positive function which satisfies (12) for a certain $E_\infty \geq 0$.

Let R^* and t be two times such that $R^* < t$. By integrating (11) in $[R^*, t]$ and taking the limit as $m \to +\infty$,

$$\int_{R^*}^t \Phi^m(s)\,ds \leq E^m(R^*) - E^m(t) \longrightarrow E(R^*) - E(t) \leq E(R^*) - E_\infty.$$

For each $\delta > 0$ given, we can choose a sufficiently large $R^* = R^*(\delta)$, such that $E(R^*) - E_\infty \leq \delta/2$. Therefore, there exists a sufficiently large number $m_0(\delta) \in \mathbb{N}$ such that

$$\int_{R^*}^t \Phi^m(s)\,ds \leq E(R^*) - E_\infty + \delta/2 \leq \delta, \quad \forall t \geq R^*, \quad \forall m \geq m_0(\delta).$$

Taking $t \to +\infty$, we have

$$\int_{R^*(\delta)}^{+\infty} \Phi^m(s)\,ds \leq \delta, \tag{14}$$

where $R^*(\delta)$ does not depend on m. Starting from (13) and (14), we are going to finish the proof of this theorem, using the lines provided in [2]. Indeed, from (14),

$$\frac{1}{\tau}\int_t^{t+\tau} \Phi^m(t)\,dt \leq \frac{\delta}{\tau}, \quad \forall \tau > 0, \quad \forall t \geq R^*(\delta). \tag{15}$$

Lemma 2.1 of [2] implies that, $\forall t \geq R^*(\delta)$ and $\forall \tau > 0$, there exist times $\bar{t} \in [t, t+\tau]$ such that:

$$\Phi^m(\bar{t}) \leq \frac{2\delta}{\tau}. \tag{16}$$

On the other hand, from (13), Lemma 2.2 of [2] implies that for any $\varepsilon < 1$, if $\Phi^m(t_0) \leq \varepsilon/3$, then $\Phi^m(t) \leq \varepsilon$ $\forall t \in [t_0, t_0 + S^*(\varepsilon)]$, where $S^*(\varepsilon) = \dfrac{\varepsilon}{3C_2}$ (that is independent of m).

By using (15) and (16) for $\delta = \dfrac{\varepsilon^2}{36C_2}$ and $\tau = \dfrac{S^*(\varepsilon)}{2}$, Theorem 2.3 of [2] gives

$$\Phi^m(t) \leq \varepsilon, \quad \forall\, t \geq R^*(\delta) + \frac{S^*(\varepsilon)}{2} = R^*(\delta) + \frac{\varepsilon}{6C_2} := T^*(\varepsilon). \tag{17}$$

Observe that bound (17) does not depend on m. Therefore, for each $\varepsilon < 1$, there exists a sufficiently large $T^* = T^*(\varepsilon)$ such that $\|\Phi^m\|_{L^\infty(T^*,+\infty)} \leq \varepsilon$. ■

2.2 Lojasiewicz-Simon inequality

It is standard procedure to use appropriate Lojasiewicz-Simon inequalities to study the convergence of trajectories in infinite time. It is not easy to find in the literature a demonstration of these types of inequalities associated to various Euler-Lagrange equations. Here, a particular Lojasiewicz-Simon inequality associated to the critical points of the elastic energy (5) is deduced, by using the abstract Theorem 4 presented below (Theorem 4.2 of [8]). Some extensions of this Lojasiewicz-Simon inequality are commented in the Remark 6 below.

We begin by recalling the following definitions:

Definition 3 *A bounded linear operator* $L : X_1 \mapsto X_2$ *between two Banach spaces* X_1 *and* X_2 *is called a Fredholm operator of index zero if* L *has a closed range* $R(L)$*, a finite dimensional kernel* $N(L)$ *and* $\dim \mathrm{N(L)} = \dim\,(\mathrm{X_2/R(L)}) < \infty$*. A* C^1 *map* $\mathcal{M} : U \subset X_1 \mapsto X_2$ *is called a Fredholm map of index zero if its Frèchet differential at each point are Fredholm operators of index zero.*

For instance, an invertible operator plus a compact operator is a Fredholm operator of index zero.

Theorem 4 *Assume the following hypotheses:*

- *Let* H *be a Hilbert space and* $A : D(A) \subset H \mapsto H$ *a linear self-adjoint and positive definite operator. In particular,* $H_A \equiv (D(A), \langle\cdot,\cdot\rangle_A)$ *is a Hilbert space endowed with the scalar product* $\langle u, v\rangle_A \equiv (Au, Av)_H$ *for all* $u, v \in D(A)$*.*
- *Let* X *and* $\widetilde{X}$ *be two Banach spaces such that the embeddings* $X \hookrightarrow H_A$ *and* $\widetilde{X} \hookrightarrow H$ *are continuous. Moreover,* $X \hookrightarrow \widetilde{X}$ *is also a continuous embedding.*
- *Let* $\mathcal{E} : X \mapsto \mathbb{R}$ *be a Fréchet-differentiable functional.*
- *Let* $\mathcal{M} = \mathcal{E}' : X \mapsto \widetilde{X}$ *be an analytic gradient map with the following properties:*
  - $\mathcal{M}$ *is a Fredholm map of index zero; i.e., for each* $u \in X$ *the bounded linear operator* $\mathcal{M}'(u) \in \mathcal{L}(X, \widetilde{X})$ *is a Fredholm operator of index zero.*
  - *For each fixed* $u \in X$*, the bounded linear symmetric operator* $\mathcal{M}'(u) : X \mapsto \widetilde{X}$ *has an extension* $\mathcal{M}_1(u) : H_A \mapsto H$*, which is a symmetric Fredholm operator of index zero.*
  - *The map* $\mathcal{R} : u \in X \mapsto \mathcal{M}_1(u)A^{-1} \in \mathcal{L}(H)$ *is continuous.*

Therefore, if $\bar{u} \in X$ *is a critical point of* $\mathcal{E}$*, i.e.* $\mathcal{E}'(\bar{u}) = 0$*, then positive constants* C*,* β_1 *and* $\sigma \in [1/2, 1)$ *exist such that*

$$|\mathcal{E}(u) - \mathcal{E}(\bar{u})|^\sigma \le C\,\|\mathcal{E}'(u)\|_H \qquad \forall u \in X \text{ with } \|u - \bar{u}\|_X < \beta_1.$$

This theorem is now going to be applied to the smectic-A model, by using strong norms.

Lemma 5 (Strong Lojasiewicz-Simon inequality for smectic-A problems) *Let $\mathcal{S}$ be the following set of equilibrium points related to the elastic energy $E_e(\varphi) = \int_\Omega \left(\frac{1}{2}|\Delta\varphi|^2 + F_\varepsilon(\nabla\varphi)\right)$:*

$$\mathcal{S} = \{\varphi \in H^4(\Omega) : \ \Delta^2\varphi - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla\varphi) = 0 \ a.e \ in \ Q \ , \ \varphi|_{\partial\Omega} = \varphi_1, \ \partial_{\rm n}\varphi|_{\partial\Omega} = \varphi_2\}.$$

If $\overline{\varphi} \in \mathcal{S}$, there are three positive constants C, β, and $\theta \in (0, 1/2)$ which depend on $\overline{\varphi}$, such that for all $\varphi \in H^4$ with $\varphi|_{\partial\Omega} = \varphi_1$, $\partial_{\rm n}\varphi|_{\partial\Omega} = \varphi_2$ and $\|\varphi - \overline{\varphi}\|_3 \le \beta$, then

$$|E_e(\varphi) - E_e(\overline{\varphi})|^{1-\theta} \le C\,|w|_2 \tag{18}$$

where $w = w(\varphi) := \Delta^2\varphi - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla\varphi)$.

Proof. The proof is divided into two steps.

Step 1 *(Application of Theorem 4): $\exists\,\beta_1, C > 0$ such that if $\|\varphi - \overline{\varphi}\|_4 \le \beta_1$, then (18) holds.*

Let $\phi \in H^4(\Omega)$ be the "lifting" function defined as the (strong) solution of the problem:

$$\Delta^2\phi = 0 \text{ in } \Omega, \quad \phi|_{\partial\Omega} = \varphi_1, \quad \partial_{\rm n}\phi|_{\partial\Omega} = \varphi_2. \tag{19}$$

Theorem 4 is going to be applied for the following spaces and operators:

$$H \equiv \widetilde{X} = L^2(\Omega), \qquad X \equiv H_A = H_0^2(\Omega) \cap H^4(\Omega),$$
$$A = \Delta^2 : \xi \in X \mapsto A\xi = \Delta^2\xi \in H \text{ and } \langle \xi, \psi \rangle_A = (\Delta^2\xi, \Delta^2\psi)_{L^2} \quad \forall, \xi, \psi \in D(A),$$
$$\mathcal{E} : \xi \in X \mapsto \mathcal{E}(\xi) = E_e(\xi + \phi) = \int_\Omega \left(\frac{1}{2}|\Delta(\xi + \phi)|^2 + F_\varepsilon(\nabla(\xi + \phi))\right) \in \mathbb{R},$$
$$\mathcal{M} = \mathcal{E}' : \xi \in X \mapsto H, \text{ such that } \mathcal{M}(\xi) = \Delta^2\xi - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla(\xi + \phi)),$$

and $\mathcal{M}_1(\xi) = \mathcal{M}'(\xi)$, where for each $\xi \in X$,

$$\mathcal{M}'(\xi) : \psi \in X \mapsto \mathcal{M}'(\xi)(\psi) = \Delta^2\psi - \nabla \cdot ((\boldsymbol{f}_\varepsilon)'(\nabla(\xi + \phi))\nabla\psi) \in H.$$

Indeed, $\mathcal{M}'(\xi)$ is a Fredholm operator of index zero, because $\mathcal{M}'(\xi)$ is the sum of the invertible operator A and the compact operator $\psi \in X \to -\nabla \cdot ((\boldsymbol{f}_\varepsilon)'(\nabla(\xi + \phi))\nabla\psi) \in H$.

Moreover, the map $\mathcal{R} : \xi \in X \mapsto \mathcal{M}'(\xi)A^{-1} \in \mathcal{L}(H)$ is well-posed because $A^{-1} \in \mathcal{L}(H; X)$ and $\mathcal{M}'(\xi) \in \mathcal{L}(X; H)$. It remains to be proved that $\mathcal{R}$ is (sequentially) continuous. Let $\xi_n \to \xi$ in X as $n \to \infty$. Therefore,

$$\|\mathcal{R}(\xi_n) - \mathcal{R}(\xi)\|_{\mathcal{L}(H)} = \|\mathcal{M}'(\xi_n)A^{-1} - \mathcal{M}'(\xi)A^{-1}\|_{\mathcal{L}(H)} \le \|\mathcal{M}'(\xi_n) - \mathcal{M}'(\xi)\|_{\mathcal{L}(X;H)}\|A^{-1}\|_{\mathcal{L}(H;X)}$$

and

$$\|\mathcal{M}'(\xi_n)-\mathcal{M}'(\xi)\|_{\mathcal{L}(X;H)} = \sup_{\psi\in X\setminus\{0\}} \frac{\|\mathcal{M}'(\xi_n)(\psi)-\mathcal{M}'(\xi)(\psi)\|_H}{\|\psi\|_X}$$

$$= \sup_{\psi\in X\setminus\{0\}} \frac{|\nabla\cdot\Big(((\boldsymbol{f}_\varepsilon)'(\nabla(\xi+\phi))-(\boldsymbol{f}_\varepsilon)'(\nabla(\xi_n+\phi)))\nabla\psi\Big)|_2}{\|\psi\|_4}$$

$$\leq \sup_{\psi\in X\setminus\{0\}} \frac{\|\,((\boldsymbol{f}_\varepsilon)'(\nabla(\xi+\phi))-(\boldsymbol{f}_\varepsilon)'(\nabla(\xi_n+\phi)))\nabla\psi)\,\|_1}{\|\psi\|_4}$$

$$\leq C\,\|(\boldsymbol{f}_\varepsilon)'(\nabla(\xi+\phi))-(\boldsymbol{f}_\varepsilon)'(\nabla(\xi_n+\phi))\|_1$$

By taking into account that $\|(\boldsymbol{f}_\varepsilon)'(\nabla(\xi+\phi))-(\boldsymbol{f}_\varepsilon)'(\nabla(\xi_n+\phi)))\|_{H^1}\to 0$ as $n\to\infty$ if $\xi_n\to\xi$ in H^4, then the continuity of the operator $\mathcal{R}$ has been proved.

In order to apply Theorem 4, the boundary conditions must be lifted by using the function ϕ given in (19). In fact, function $\overline{\xi}=\overline{\varphi}-\phi$ (recall that $\overline{\varphi}\in\mathcal{S}$) satisfies $\overline{\xi}|_{\partial\Omega}=0$ and $\partial_{\rm n}\overline{\xi}|_{\partial\Omega}=0$ and represents a critical point of $\mathcal{E}(\xi)$. Let $\varphi\in H^4(\Omega)$ with $\varphi|_{\partial\Omega}=\varphi_1$, $\partial_{\rm n}\varphi|_{\partial\Omega}=\varphi_2$ and $\|\varphi-\overline{\varphi}\|_4\leq\beta_1$ ($\beta_1>0$ given in Theorem 4). If we define $\xi=\varphi-\phi\in X$, then $\|\xi-\overline{\xi}\|_4\leq\beta_1$ and, owing to Theorem 4:

$$|E_e(\varphi)-E_e(\overline{\varphi})|^{1-\theta}=|\mathcal{E}(\xi)-\mathcal{E}(\overline{\xi})|^{1-\theta}\leq C\,\|\mathcal{E}'(\xi)\|_H$$
$$=C\,|\Delta^2\xi-\nabla\cdot\boldsymbol{f}_\varepsilon(\nabla(\xi+\phi))|_2=C\,|w(\varphi)|_2.$$

Hence (18) holds.

Step 2: *(Relaxing the local approximation $\|\varphi-\overline{\varphi}\|_4\leq\beta$ by $\|\varphi-\overline{\varphi}\|_3\leq\beta$) There exits $\beta>0$ and $C>0$ such that if $\varphi\in H^4(\Omega)$ and $\|\varphi-\overline{\varphi}\|_3\leq\beta$, then (18) holds.*

In this step, a similar argument is followed to that in Lemma 4.4 of [12]. Since $\varphi-\overline{\varphi}=\xi-\overline{\xi}$, this is reduced to the homogeneous functions $\xi,\overline{\xi}$. From (10), there exists $M>0$ such that

$$\|\xi-\overline{\xi}\|_4\leq M|\Delta^2(\xi-\overline{\xi})|_2$$

and by using Sovolev's embeddings and $\|\xi\|_3\leq\|\,\overline{\xi}\|_3+\beta\leq C$, we obtain

$$|\nabla\cdot(\boldsymbol{f}_\varepsilon(\nabla(\xi+\phi))-\boldsymbol{f}_\varepsilon(\nabla(\overline{\xi}+\phi)))|_2\leq C(\beta)\,\|\xi-\overline{\xi}\|_3,$$

$$|\mathcal{E}(\xi)-\mathcal{E}(\overline{\xi})|^{1-\theta}\leq C(\beta)\,\|\xi-\overline{\xi}\|_2^{1-\theta}\leq C(\beta)\,\|\xi-\overline{\xi}\|_3^{1-\theta}$$

where $C(\beta)$ depends on β (and $\|\overline{\xi}\|_3$). In particular, since $\|\xi-\overline{\xi}\|_3<\beta$, then

$$|\nabla\cdot(\boldsymbol{f}_\varepsilon(\nabla(\xi+\phi))-\boldsymbol{f}_\varepsilon(\nabla(\overline{\xi}+\phi)))|_2+|\mathcal{E}(\xi)-\mathcal{E}(\overline{\xi})|^{1-\theta}<C(\beta)(\beta+\beta^{1-\theta}).$$

Therefore, there exists a (sufficiently small) $\beta\in(0,1]$ independent of ξ, such that

$$C(\beta)(\beta+\beta^{1-\theta})<\frac{\beta_1}{2M}.$$

For any $\xi \in H^4(\Omega)$ satisfying $\|\xi - \overline{\xi}\|_3 < \beta$ (that is, for any $\varphi \in H^4(\Omega)$ satisfying $\|\varphi - \overline{\varphi}\|_3 < \beta$), there are only two possibilities: either $\|\xi - \overline{\xi}\|_4 < \beta_1$ and then (18) holds by using Step 1; or $\|\xi - \overline{\xi}\|_4 > \beta_1$. In this latter case,

$$\begin{aligned}|w(\varphi)|_2 &= |\Delta^2(\xi - \overline{\xi}) - \nabla \cdot (\boldsymbol{f}_\varepsilon(\nabla(\xi + \phi)) - \boldsymbol{f}_\varepsilon(\nabla(\overline{\xi} + \phi)))|_2 \\ &\geq \frac{1}{M}\|\xi - \overline{\xi}\|_4 - |\nabla \cdot (\boldsymbol{f}_\varepsilon(\nabla(\xi + \phi)) - \boldsymbol{f}_\varepsilon(\nabla(\overline{\xi} + \phi)))|_2 \\ &> \frac{\beta_1}{M} - \frac{\beta_1}{2M} = \frac{\beta_1}{2M} > |\mathcal{E}(\xi) - \mathcal{E}(\overline{\xi})|^{1-\theta} = |E_e(\xi) - E_e(\overline{\xi})|^{1-\theta},\end{aligned}$$

and hence (18) holds. ■

Remark 6 *The Lojasiewicz-Simon inequality given in Lemma 5 has been formulated in a "strong sense". However, other versions are also possible. For example, Theorem 2.1 of [7] for homogeneous Dirichlet conditions and the comments given in [14] for the non-homogeneous Dirichlet case show a "weak" version where, if* $\|\varphi - \overline{\varphi}\|_1 \leq \beta$, *then* $|E_e(\varphi) - E_e(\overline{\varphi})|^{1-\theta} \leq C\|w\|_{-2}$ *holds. Futhermore, an "intermediate" version has been applied in [12] for periodic boundary conditions, where* $|E_e(\varphi) - E_e(\overline{\varphi})|^{1-\theta} \leq C\|w\|_{-1}$ *if* $\|\varphi - \overline{\varphi}\|_2 \leq \beta$.

3 The Smectic Model

Definition 7 *A pair* $(\boldsymbol{u}, \varphi)$ *is said to be a global weak solution of* (1)-(7) *in* $(0, +\infty)$ *if*

$$\begin{gathered}\boldsymbol{u} \in L^\infty(0, +\infty; \boldsymbol{L}^2(\Omega)) \cap L^2(0, +\infty; \boldsymbol{V}), \quad w \in L^2(0, +\infty; L^2(\Omega)), \\ \varphi \in L^\infty(0, +\infty; H^2(\Omega)),\end{gathered} \tag{20}$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } Q, \quad \boldsymbol{u}|_\Sigma = 0, \quad \varphi|_\Sigma = \varphi_1, \quad \partial_{\mathrm{n}}\varphi|_\Sigma = \varphi_2,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{ in } \Omega,$$

and it satisfies the variational formulation:

$$\langle \partial_t \boldsymbol{u}, \bar{\boldsymbol{u}} \rangle + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \bar{\boldsymbol{u}}) + (\nabla \boldsymbol{u}, \nabla \bar{\boldsymbol{u}}) - (w \nabla \varphi, \bar{\boldsymbol{u}}) = 0 \quad \forall\, \bar{\boldsymbol{u}} \in \boldsymbol{V}, \tag{21}$$

$$\langle \partial_t \varphi, \bar{w} \rangle + (\boldsymbol{u} \cdot \nabla \varphi, \bar{w}) + (w, \bar{w}) = 0, \quad \forall\, \bar{w} \in L^2 \tag{22}$$

$$(\Delta \varphi, \Delta \bar{\varphi}) - (\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi), \bar{\varphi}) - (w, \bar{\varphi}) = 0, \quad \forall\, \bar{\varphi} \in H^2. \tag{23}$$

Moreover, from the weak regularity of (φ, w) given in (20), (23) and (10), it can be deduced that $\varphi \in L^2_{loc}(0, +\infty; H^4)$ whenever $\varphi_1 \in H^{7/2}(\partial\Omega)$ and $\varphi_2 \in H^{5/2}(\partial\Omega)$, i.e. $\varphi \in L^2(0, T; H^4)$ for all $T > 0$.

Definition 8 *A weak solution* $(\boldsymbol{u},\varphi)$ *is said to be a strong solution of* (1)-(7) *in* $(0,+\infty)$ *if*

$$\begin{gathered}\boldsymbol{u}\in L^\infty(0,+\infty;\boldsymbol{H}^1(\Omega))\cap L^2_{loc}(0,+\infty;\boldsymbol{H}^2(\Omega)),\quad \partial_t\boldsymbol{u}\in L^2_{loc}(0,+\infty;\boldsymbol{L}^2(\Omega)),\\ \partial_t\varphi\in L^\infty(0,+\infty;L^2(\Omega))\cap L^2_{loc}(0,+\infty;H^2(\Omega)),\end{gathered}\tag{24}$$

and it satisfies the fully differential system (1)-(3) *point-wise in* $(0,+\infty)\times\Omega$.

Moreover, for regular domains, one has

$$\varphi\in L^\infty(0,+\infty;H^4)\cap L^2_{loc}(0,+\infty;H^6),\qquad w\in L^\infty(0,+\infty;L^2)\cap L^2_{loc}(0,+\infty;H^2)$$

whenever $\varphi_1\in H^{11/2}(\partial\Omega)$ and $\varphi_2\in H^{9/2}(\partial\Omega)$.

3.1 Energy Equality and Weak Estimates

If $(\boldsymbol{u},\varphi,w)$ is a regular enough solution of (1)-(4), (6), (7), then by taking $\bar{\boldsymbol{u}}=\boldsymbol{u}$, $\bar{w}=\boldsymbol{w}$ and $\bar\varphi=\partial_t\varphi$ as a test function in (21), (22) and (23) respectively, one has

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|_2^2+|\nabla\boldsymbol{u}|_2^2-(w\,\nabla\varphi,\boldsymbol{u})=0,$$

$$(\partial_t\varphi,w)+(\boldsymbol{u}\cdot\nabla\varphi,w)+|w|_2^2=0,$$

$$\frac{d}{dt}\left(\frac{1}{2}|\Delta\varphi|_2^2+\int_\Omega F_\varepsilon(\nabla\varphi)\right)-(w,\partial_t\varphi)=0.$$

Through adding these three equalities, the term $(w,\partial_t\varphi)$ is cancelled and the nonlinear convective term $(\boldsymbol{u}\cdot\nabla\varphi,w)$ plus the elastic term $-(w\,\nabla\varphi,\boldsymbol{u})$ also vanish, thereby yielding at the following *energy equality*:

$$\frac{d}{dt}E(\boldsymbol{u}(t),\varphi(t))+|\nabla\boldsymbol{u}|_2^2+|w|_2^2=0.\tag{25}$$

This energy equality illustrates the dissipative character of the model with respect to the total free energy $E(\boldsymbol{u},\varphi)=E_k(\boldsymbol{u})+E_e(\varphi)$, where $E_k(\boldsymbol{u})=\frac{1}{2}\int_\Omega|\boldsymbol{u}|^2$ is the kinetic energy and $E_e(\varphi)$ is the elastic energy defined in (5). Moreover, assuming the initial estimate $|\boldsymbol{u}_0|_2^2\le C$ and $\|\varphi_0\|_2^2\le C$, the following uniform bounds at the infinite time interval $(0,+\infty)$ hold:

$$\boldsymbol{u}\text{ in }L^\infty(0,+\infty;\boldsymbol{H})\cap L^2(0,+\infty;\boldsymbol{V}),\quad w\text{ in }L^2(0,+\infty;L^2),\quad \varphi\text{ in }L^\infty(0,+\infty;H^2).\tag{26}$$

In particular, from the bound of w in $L^2(0,+\infty;L^2)$ and (10), one has the finite time bound

$$\varphi\text{ in }L^2(0,T;H^4),\quad\forall T>0.$$

For instance, weak solutions furnished by a limit of Galerkin approximate solutions which satisfy the corresponding energy inequality (by replacing the equality $=0$ with the inequality ≤ 0 in (25)) can be obtained, which suffices to rigorously prove all previous estimates.

3.2 Strong Estimates

From (23) and (10), we have for each $t \in (0, +\infty)$:

$$\|\varphi(t)\|_4 \leq C(\|\varphi_1\|_{7/2;\partial\Omega} + \|\varphi_2\|_{5/2;\partial\Omega} + |w(t)|_2 + |\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla\varphi(t))|_2). \tag{27}$$

By using weak estimates $\|\varphi(t)\|_2 \leq C$ and

$$|\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla\varphi(t))|_2 \leq C|\nabla_n \boldsymbol{f}_\varepsilon(\nabla\varphi(t))|_3 |D^2\varphi(t)|_6 \leq C\|\varphi(t)\|_3, \tag{28}$$

we obtain

$$\|\varphi(t)\|_3 \leq C\|\varphi(t)\|_2^{1/2}\|\varphi(t)\|_4^{1/2} \leq C(1 + |w(t)|_2^{1/2} + \|\varphi(t)\|_3^{1/2}).$$

Hence

$$\|\varphi(t)\|_3 \leq C(1 + |w(t)|_2^{1/2}). \tag{29}$$

On the other hand, from (3), it follows that

$$|w(t)|_2 \leq C(|\partial_t\varphi(t)|_2 + |\boldsymbol{u}(t)|_3|\nabla\varphi(t)|_6) \leq C(|\partial_t\varphi(t)|_2 + \|\boldsymbol{u}(t)\|_1^{1/2}). \tag{30}$$

Hence, from (29) and (30)

$$\|\varphi(t)\|_3 \leq C(1 + |\partial_t\varphi(t)|_2^{1/2} + \|\boldsymbol{u}(t)\|_1^{1/4}). \tag{31}$$

By means of taking $-A\boldsymbol{u} + \partial_t\boldsymbol{u}$ as a test function in the $\boldsymbol{u}$-system (1) (A being the Stokes operator), and by applying Hölder and Young's inequalities and the interpolation inequality

$$\|\varphi\|_{W^{1,\infty}} \leq C\|\varphi\|_2^{1/2}\|\varphi\|_3^{1/2},$$

we attain:

$$\begin{aligned}
\frac{d}{dt}|\nabla\boldsymbol{u}|_2^2 + |A\boldsymbol{u}|_2^2 + |\partial_t\boldsymbol{u}|_2^2 &\leq C\left(|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|_2 + |(\nabla\varphi)w|_2\right)\left(|A\boldsymbol{u}|_2 + |\partial_t\boldsymbol{u}|_2\right) \\
&\leq C\left(|\boldsymbol{u}|_6|\nabla\boldsymbol{u}|_3 + |\nabla\varphi|_\infty|w|_2\right)\left(\|\boldsymbol{u}\|_2 + |\partial_t\boldsymbol{u}|_2\right) \\
&\leq C\left(\|\boldsymbol{u}\|_1^{3/2}\|\boldsymbol{u}\|_2^{3/2} + \|\boldsymbol{u}\|_1^{3/2}\|\boldsymbol{u}\|_2^{1/2}|\partial_t\boldsymbol{u}|_2 + \|\varphi\|_2^{1/2}\|\varphi\|_3^{1/2}|w|_2(\|\boldsymbol{u}\|_2 + |\partial_t\boldsymbol{u}|_2)\right) \\
&\leq \frac{1}{2}\|\boldsymbol{u}\|_2^2 + \frac{1}{2}|\partial_t\boldsymbol{u}|_2^2 + C\left(\|\boldsymbol{u}\|_1^6 + \|\varphi\|_3|w|_2^2\right).
\end{aligned}$$

Therefore, by using (30) and (31), we obtain

$$\frac{d}{dt}\|\boldsymbol{u}\|_1^2 + \frac{1}{2}\|\boldsymbol{u}\|_2^2 + \frac{1}{2}|\partial_t\boldsymbol{u}|_2^2 \leq C\left(\|\boldsymbol{u}\|_1^6 + (1 + |\partial_t\varphi|_2^{1/2} + \|\boldsymbol{u}\|_1^{1/4})(|\partial_t\varphi|_2^2 + \|\boldsymbol{u}\|_1)\right). \tag{32}$$

On the other hand, by deriving the w-equation (3) and φ-equation (4) with respect to t, taking $\partial_t\varphi$ as a test function in both these derivations, adding, and taking into account that

$(\boldsymbol{u}\cdot\nabla\partial_t\varphi,\partial_t\varphi)=0$ and also the term $(\partial_t w,\partial_t\varphi)$ is cancelled, we then have:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\partial_t\varphi|_2^2+|\Delta\partial_t\varphi|_2^2=-(\partial_t\boldsymbol{u}\cdot\nabla\varphi,\partial_t\varphi)+(\partial_t(\nabla\cdot\boldsymbol{f}_\varepsilon(\nabla\varphi)),\partial_t\varphi)\\
&\le|\partial_t\boldsymbol{u}|_2|\nabla\varphi|_6|\partial_t\varphi|_3+\Big(|\nabla_n\boldsymbol{f}_\varepsilon(\nabla\varphi)|_3|\nabla^2\partial_t\varphi|_2+|\nabla_n^2\boldsymbol{f}_\varepsilon(\nabla\varphi)|_6|\nabla^2\varphi|_2|\partial_t\nabla\varphi|_6\Big)|\partial_t\varphi|_6\\
&\le C(|\partial_t\boldsymbol{u}|_2|\partial_t\varphi|_2^{1/2}\|\partial_t\varphi\|_1^{1/2}+\|\partial_t\varphi\|_2\|\partial_t\varphi\|_1+\|\partial_t\varphi\|_2^{3/2}|\partial_t\varphi|_2^{1/2})\\
&\le\frac{1}{8}|\partial_t\boldsymbol{u}|_2^2+\frac{1}{2}\|\partial_t\varphi\|_2^2+C|\partial_t\varphi|_2^2,
\end{aligned}
\tag{33}
$$

where (28) and $\|\partial_t\varphi\|_2=|\Delta\partial_t\varphi|_2$ have been applied (because $\partial_t\varphi|_{\partial\Omega}=0$). Therefore, from (33)

$$
\frac{d}{dt}|\partial_t\varphi|_2^2+\|\partial_t\varphi\|_2^2\le\frac{1}{4}|\partial_t\boldsymbol{u}|_2^2+C|\partial_t\varphi|_2^2. \tag{34}
$$

From the addition of (32) and (34), it follows that:

$$
\begin{aligned}
&\frac{d}{dt}(\|\boldsymbol{u}\|_1^2+|\partial_t\varphi|_2^2)+\frac{1}{2}\|\boldsymbol{u}\|_2^2+\frac{1}{4}|\partial_t\boldsymbol{u}|_2^2+\|\partial_t\varphi\|_2^2\\
&\qquad\le C\left(\|\boldsymbol{u}\|_1^6+(1+|\partial_t\varphi|_2^{1/2}+\|\boldsymbol{u}\|_1^{1/4})(|\partial_t\varphi|_2^2+\|\boldsymbol{u}\|_1)\right).
\end{aligned}
\tag{35}
$$

By denoting

$$
\Phi(t):=\|\boldsymbol{u}\|_1^2+|\partial_t\varphi|_2^2,\qquad \Psi(t):=\frac{1}{2}\|\boldsymbol{u}\|_2^2+\frac{1}{4}|\partial_t\boldsymbol{u}|_2^2+\|\partial_t\varphi\|_2^2,
$$

then (35) can be rewritten as

$$
\Phi'+\Psi\le C(\Phi^3+\Phi+\Phi^{1/2}+\Phi^{5/4}+\Phi^{3/4}+\Phi^{9/8})\le C(\Phi^3+1). \tag{36}
$$

Observe that $\Phi\in L^1(0,+\infty)$ since $|\partial_t\varphi|_2\in L^2(0,+\infty)$. Indeed, from the w-equation (3):

$$
|\partial_t\varphi|_2\le C\Big(|w|_2+\|\boldsymbol{u}\|_1\|\nabla\varphi\|_1\Big)\le C\Big(|w|_2+\|\boldsymbol{u}\|_1\Big),
$$

and $|w|_2+\|\boldsymbol{u}\|_1\in L^2(0,+\infty)$.

Therefore, the entire hypothesis of Theorem 2 holds, then there exists a sufficiently large $T^*_{reg}\ge 0$ such that the following (regular) estimates hold in $(T^*_{reg},+\infty)$:

$$
\boldsymbol{u}\in L^\infty(T^*_{reg},+\infty;\boldsymbol{H}^1),\qquad \partial_t\varphi\in L^\infty(T^*_{reg},+\infty;L^2).
$$

By integrating (36) in $[0,t]$ for all $t>0$, the following local (regular) estimates in $(T^*_{reg},+\infty)$ are obtained:

$$
\boldsymbol{u}\in L^2_{loc}(T^*_{reg},+\infty;\boldsymbol{H}^2),\quad \partial_t\boldsymbol{u}\in L^2_{loc}(T^*_{reg},+\infty;\boldsymbol{L}^2),\quad \partial_t\varphi\in L^2_{loc}(T^*_{reg},+\infty;H^2).
$$

By using the w-equation (3), one has, for each $t\in(0,+\infty)$:

$$
|w(t)|_2\le C(|\partial_t\varphi(t)|_2+\|\boldsymbol{u}(t)\|_1), \tag{37}
$$

hence

$$w \in L^\infty(T^*_{reg}, +\infty; L^2)$$

and from (29),

$$\varphi \in L^\infty(T^*_{reg}, +\infty; H^3).$$

Futhermore, from (3), we have

$$\|w(t)\|_2 \le C(\|\partial_t \varphi(t)\|_2 + \|\boldsymbol{u}(t)\|_2 \|\varphi(t)\|_3),$$

hence

$$w \in L^2_{loc}(T^*_{reg}, +\infty; H^2).$$

Observe that, through combining (3) and (4), $\varphi(t)$ is the solution of the bilaplacian problem

$$\begin{cases} \Delta^2 \varphi = \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi) - w \quad \text{in } \Omega, \\ \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_{\rm n} \varphi|_{\partial\Omega} = \varphi_2 \quad \text{on } \partial\Omega. \end{cases}$$

By means of using the H^4 and H^6 regularity of this problem and bounding the right-hand-side terms, and from the weak regularity and the strong regularity of φ and w previously proved, we have

$$\varphi \in L^\infty(T^*_{reg}, +\infty; H^4) \cap L^2_{loc}(T^*_{reg}, +\infty; H^6).$$

3.3 Existence of global weak solutions with long-time strong regularity

The existence of solutions of (1)-(7) can be justified by the Galerkin Method [3]. Given some fixed regular basis $(\boldsymbol{w}^i)_i$ and $(\phi^j)_j$ of the spaces $\boldsymbol{V}$ and $H^2_0(\Omega)$, respectively, let $\boldsymbol{V}^m$ and W^m be the finite-dimensional subspaces spanned by

$$\{\boldsymbol{w}^1, \dots, \boldsymbol{w}^m\} \quad \text{and} \quad \{\phi^1, \dots, \phi^m\}$$

respectively. Given $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\varphi_0 \in H^2$, for each $m \ge 1$, we seek an approximate solution $(\boldsymbol{u}_m, \varphi_m)$, such that $\boldsymbol{u}_m : [0,T] \mapsto \boldsymbol{V}^m$ and $\varphi_m = \widetilde{\varphi} + \widehat{\varphi}_m$, where $\widetilde{\varphi}$ is an adequate lifting function of the boundary data φ_1, φ_2 and $\widehat{\varphi}_m : [0,T] \mapsto W^m$, which satisfies the following variational formulation a.e. $t \in (0,T)$:

$$\begin{cases} (\partial_t \boldsymbol{u}_m(t), \boldsymbol{v}_m) + ((\boldsymbol{u}_m(t) \cdot \nabla)\boldsymbol{u}_m(t), \boldsymbol{v}_m) + \nu(\nabla \boldsymbol{u}_m(t), \nabla \boldsymbol{v}_m) \\ \qquad -(w_m(t)\nabla\varphi_m(t), \boldsymbol{v}_m) = 0 \quad \forall\, \boldsymbol{v}_m \in \boldsymbol{V}^m, \\ (\partial_t \varphi_m(t), e_m) + (\boldsymbol{u}_m(t) \cdot \nabla \varphi_m(t), e_m) + (w_m(t), e_m) \\ \qquad = (\partial_t \varphi_m(t), e_m), \quad \forall\, e_m \in W^m, \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} = P_m(\boldsymbol{u}_0), \quad \varphi_m(0) = \varphi_{0m} = Q_m(\varphi_0) \quad \text{in } \Omega. \end{cases} \tag{38}$$

Here, $P_m : \boldsymbol{H} \mapsto \boldsymbol{V}^m$ denotes the projection from $\boldsymbol{H}$ onto $\boldsymbol{V}^m$; $Q_m : L^2 \mapsto W^m$ the projection from L^2 onto W^m; and the Euler-Lagrange equation $\Delta^2\varphi_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m)$ has been projected into W^m by taking

$$w_m := Q_m(\Delta^2\varphi_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m)).$$

In particular, $\boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ in $\boldsymbol{L}^2$ and $\varphi_{0m} \to \varphi_0$ in H^2 (as $m \to 0$). If we write

$$\boldsymbol{u}_m(t) = \sum_{i=1}^{m} \xi_{i,m}(t)\boldsymbol{w}^i \quad \text{and} \quad \varphi_m(t) = \sum_{j=1}^{m} \zeta_{j,m}(t)\phi^j,$$

then (38) can be rewritten as a first-order ordinary differential system (in normal form), associated to the unknowns $(\xi_{i,m}(t), \zeta_{j,m}(t))$. By proceeding in an analogous way to [10] and [3] (local existence, a priori estimates, and tending towards the limit where the nonlinear terms are controlled by compactness), the existence of weak solutions $(\boldsymbol{u},\varphi)$ of (1)-(7) in $(0,+\infty)$ can be proved, which are also strong solutions (and unique) in $(T^*_{reg},+\infty)$ for a sufficiently long-time $T^*_{reg} \geq 0$. Observe that T^*_{reg} can be obtained by applying Theorem 2 to $\Phi^m(t) = \|\boldsymbol{u}^m\|_1^2 + |\partial_t\varphi^m|_2^2$, and by taking into account that T^* given in Theorem 2 is independent of m.

Remark 9 *The differential inequality (36)has been obtained with Φ depending on $\boldsymbol{u}$ and $\partial_t\varphi$. Another possibility could be to deduce a similar differential inequality for a Φ depending on $\boldsymbol{u}$ and w (instead of for $\partial_t\varphi$). To this end, the computations could be: take $\partial_t w$ as a test function in the w-equation* (3)*, derive the φ-equation (4) with respect to t and take $\partial_t\varphi$ as a test function. Adding both equalities to (32) the term $(\partial_t\varphi, \partial_t w)$ is cancelled, thereby arriving at the following inequality instead of (33):*

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + |\partial_t\Delta\varphi|_2^2 = -(\boldsymbol{u}\cdot\nabla\varphi, \partial_t w) + (\partial_t\boldsymbol{f}_\varepsilon(\nabla\varphi), \partial_t\nabla\varphi). \tag{39}$$

Nevertheless, we do not know how to estimate the convective term $(\boldsymbol{u}\cdot\nabla\varphi, \partial_t w)$ in order to deduce a differential inequality such as in (36).

3.4 Convergence at infinite time

We recall the definition of the elastic energy:

$$E_e(\varphi(t)) = \int_\Omega \left(\frac{1}{2}|\Delta\varphi(t)|^2 + F_\varepsilon(\nabla\varphi(t))\right)$$

and the kinetic and total energy is also defined as:

$$E_k(\boldsymbol{u}(t)) = \frac{1}{2}\int_\Omega |\boldsymbol{u}(t)|^2, \qquad E(\boldsymbol{u}(t),\varphi(t)) = E_k(\boldsymbol{u}(t)) + E_e(\varphi(t)).$$

Theorem 10 *Assume that* $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{H} \times H^2$. *Let* $(\boldsymbol{u}(t), \varphi(t), w(t))$ *be a weak solution of (1)-(7) in* $(0, +\infty)$ *which is a strong solution in* $(T^*_{reg}, +\infty)$ *for some* $T^*_{reg} > 0$, *then there exists a number* $E_\infty \geq 0$ *such that the total energy satisfies*

$$E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{ as } t \uparrow +\infty. \tag{40}$$

Moreover, the following convergences hold:

$$\boldsymbol{u}(t) \to 0 \text{ in } \boldsymbol{H}^1_0 \quad \text{and} \quad w(t) \to 0 \text{ in } L^2 \quad \text{ as } t \uparrow +\infty. \tag{41}$$

Proof. The (decreasing) convergence of the energy given in (40) is easy to deduce from energy equality (25) (observe (12)). By applying Lemma 1 for $\Phi(t) := \|\boldsymbol{u}\|_1^2 + |\partial_t \varphi|_2^2$, we obtain $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\partial_t \varphi(t) \to 0$ in L^2. Finally; from (37), $w(t) \to 0$ in L^2 holds. ∎

Let S be the set of equilibrium points of (1)-(4):

$$S = \{(0, \overline{\varphi}) : \overline{\varphi} \in H^4(\Omega), \ \Delta^2 \overline{\varphi} - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \overline{\varphi}) = 0, \ \overline{\varphi}|_{\partial\Omega} = \varphi_1, \ \partial_{\mathrm{n}} \overline{\varphi}|_{\partial\Omega} = \varphi_2\}.$$

On the other hand, the ω-limit set of a global weak solution, $(\boldsymbol{u}, \varphi)$, associated to the initial data, $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{H} \times H^2$, is defined as follows:

$$\omega(\boldsymbol{u}_0, \varphi_0) = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{V} \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}.$$

Theorem 11 *Under the assumptions of Theorem 10,* $\omega(\boldsymbol{u}_0, \varphi_0)$ *is non-empty and* $\omega(\boldsymbol{u}_0, \varphi_0) \subset S$. *Moreover, for any* $(0, \overline{\varphi}) \in S$ *such that* $(0, \overline{\varphi}) \in \omega(\boldsymbol{u}_0, \varphi_0)$, *then* $E_e(\overline{\varphi}) = E_\infty$ *holds.*

Proof. The proof is divided into two steps.

Step 1: *It can been seen that* $\omega(\boldsymbol{u}_0, \varphi_0) \neq \emptyset$ *and* $\omega(\boldsymbol{u}_0, \varphi_0) \subset S$.

From weak estimates, $(\boldsymbol{u}, \varphi) \in L^\infty(0, +\infty; \boldsymbol{H} \times H^2)$, hence there exists $\{t_n\} \uparrow +\infty$ and $(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H} \times H^2$ such that $(\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty)$ weakly in $\boldsymbol{H} \times H^2$. From (41), $\boldsymbol{u}_\infty = 0$ and $\boldsymbol{u}(t_n) \to 0$ in $\boldsymbol{H}^1_0$. On the other hand, φ_∞ will be a weak solution of the equilibrium equation $\Delta^2 \varphi_\infty - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi_\infty) = 0$. Indeed, since $\nabla \varphi(t_n) \to \nabla \varphi_\infty$ a.e. in Ω, then

$$\boldsymbol{f}_\varepsilon(\nabla \varphi(t_n)) \to \boldsymbol{f}_\varepsilon(\nabla \varphi_\infty) \text{ a.e. in } \Omega$$

and, by using the weak estimate $\|\varphi(t_n)\|_2 \leq C$, then

$$|\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n))|_{6/5} \leq C(|\nabla \varphi(t_n)|_6^2 + 1)|D^2 \varphi(t_n)|_2 \leq C(\|\varphi(t_n)\|_2^2 + 1)\|\varphi(t_n)\|_2 \leq C,$$

hence

$$\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n)) \to \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi_\infty) \text{ weakly in } L^{6/5}(\Omega).$$

By taking into account that $\varphi(t_n) \to \varphi_\infty$ weakly in H^2 and $w(t) \to 0$ (strongly) in L^2 as $t \to +\infty$, it suffices to take limits in (23) as $\{t_n\} \uparrow +\infty$ to illustrate that φ_∞ is a weak solution of the equilibrium equation

$$\Delta^2 \varphi_\infty - \nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi_\infty) = 0. \tag{42}$$

This step finishes by proving the convergence $\varphi(t_n) \to \varphi_\infty$ in H^4. Indeed, from (4), (10) and (23), it is now that

$$\|\varphi(t_n)\|_4 \le C(|\Delta^2 \varphi(t_n)|_2 + 1) \le C(|\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n))|_2 + |w(t_n)|_2 + 1). \tag{43}$$

On the other hand, by using the interpolation inequalities $|\nabla \varphi|_\infty \le \|\varphi\|_2^{1/2} \|\varphi\|_3^{1/2}$ and $\|\varphi\|_3 \le \|\varphi\|_2^{1/2} \|\varphi\|_4^{1/2}$, and the weak estimate $\|\varphi(t_n)\|_2 \le C$, we obtain

$$|\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n))|_2 \le C(\|\varphi(t_n)\|_2 \|\varphi(t_n)\|_3 + 1)\|\varphi(t_n)\|_2 \le C(\|\varphi(t_n)\|_4^{1/2} + 1) \le \delta \|\varphi(t_n)\|_4 + C/\delta.$$

The application of the latter inequality for a sufficiently small $\delta > 0$ in (43) yields

$$\|\varphi(t_n)\|_4 \le C. \tag{44}$$

Moreover, from the weak estimates and (44), it is easy to attain the bound

$$\|\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n))\|_1 \le C.$$

By compactness, $\nabla \cdot \boldsymbol{f}_\varepsilon(\nabla \varphi(t_n))$ converges strongly in $L^2(\Omega)$, for at least an equally labelled subsequence. Therefore, by again using (23), $\Delta^2 \varphi(t_n) \to \Delta^2 \varphi(t_n)$ converges strongly in $L^2(\Omega)$, and hence $\varphi(t_n) \to \varphi_\infty$ converges strongly in $H^4(\Omega)$.

Step 2: *If $(0, \overline{\varphi}) \in \omega(\boldsymbol{u}_0, \varphi_0)$ then $E(0, \overline{\varphi}) = E_e(\overline{\varphi}) = E_\infty$ (E_∞ given in Theorem 10).*

From the definition of $\omega(\boldsymbol{u}_0, \varphi_0)$, there exists $\{t_n\} \uparrow +\infty$ such that $(\boldsymbol{u}(t_n), \varphi(t_n)) \to (0, \overline{\varphi})$ in $\boldsymbol{H}^1 \times H^4$ as $n \uparrow +\infty$. In particular,

$$\lim_{n \to +\infty} E(\boldsymbol{u}(t_n), \varphi(t_n)) = E_e(\overline{\varphi}).$$

Finally, from (40) and the uniqueness of the limit, one has $E_e(\overline{\varphi}) = E_\infty$. ■

Although the set of critical points $\overline{\varphi}$ (with the same elastic energy) might even be a continuum of functions, the uniqueness of limit of the whole trajectory of $\varphi(t)$ can be deduced.

Theorem 12 *Under the hypotheses of Theorem 11, there exists $\overline{\varphi} \in H^4$ such that $\varphi(t) \to \overline{\varphi}$ in H^4 as $t \uparrow +\infty$, i.e. $\omega(\boldsymbol{u}_0, \varphi_0) = \{(0, \overline{\varphi})\}$.*

Proof. Let $(0, \overline{\varphi}) \in \omega(\boldsymbol{u}_0, \varphi_0) \subset S$, i.e, there exists $t_n \uparrow +\infty$ such that $\boldsymbol{u}(t_n) \to 0$ in $\boldsymbol{H}^1$ and $\varphi(t_n) \to \overline{\varphi}$ in H^4.

Without any loss of generality, it can be assumed that $E(\boldsymbol{u}(t),\varphi(t)) > E(0,\overline{\varphi})(= E_\infty)$ for all t, because otherwise, if there some $\widetilde{t} > 0$ exists such that $E(\boldsymbol{u}(\widetilde{t}),\varphi(\widetilde{t})) = E(0,\overline{\varphi})$, then, from the energy equality (25) for each $t \geq \widetilde{t}$,

$$E(\boldsymbol{u}(t),\varphi(t)) = E(0,\overline{\varphi}), \quad |\nabla \boldsymbol{u}(t)|_2^2 = 0 \quad \text{and} \quad |w(t)|_2^2 = 0.$$

Therefore, $\boldsymbol{u}(t) = 0$ and $w(t) = 0$. In particular, by using the w-equation, then $\partial_t\varphi(t) = 0$, and hence $\varphi(t) = \overline{\varphi}$ for each $t \geq \widetilde{t}$. In this situation the convergence of the φ-trajectory is trivial.

The proof is now divided into three steps.

Step 1: *Assuming there exists $t_\star > T^*_{reg}$ such that*

$$\|\varphi(t) - \overline{\varphi}\|_3 \leq \beta \quad \text{and} \quad |\boldsymbol{u}(t)|_2 \leq 1 \quad \forall\, t \geq t_\star$$

where the solution is strong in $(T^*_{reg}, +\infty)$ and $\beta > 0$ is the constant appearing in Lemma 5 (of Lojasiewicz-Simon's type), then the following inequalities hold:

$$\frac{d}{dt}\Big((E(\boldsymbol{u}(t),\varphi(t)) - E(0,\overline{\varphi}))^\theta\Big) + C\,\theta\,(|\nabla \boldsymbol{u}(t)|_2 + |w(t)|_2) \leq 0, \quad \forall\, t \geq t_\star \tag{45}$$

$$\int_{t_0}^{t_1} |\partial_t\varphi|_2 \leq \frac{C}{\theta}(E(\boldsymbol{u}(t_0),\varphi(t_0)) - E(0,\overline{\varphi})))^\theta, \qquad \forall\, t_1 > t_0 \geq t_\star, \tag{46}$$

where $\theta \in (0,1/2]$ is the constant appearing in Lemma 5.

Indeed, the energy equality (25) can be written as

$$\frac{d}{dt}(E(\boldsymbol{u}(t),\varphi(t)) - E_\infty) + C\left(|\nabla \boldsymbol{u}(t)|_2^2 + |w(t)|_2^2\right) = 0.$$

Therefore, by taking the time derivative of the (strictly positive) function

$$H(t) := (E(\boldsymbol{u}(t),\varphi(t)) - E_\infty)^\theta > 0,$$

we obtain

$$\frac{dH(t)}{dt} + \theta(E(\boldsymbol{u}(t),\varphi(t)) - E_\infty)^{\theta-1}C(\big|\nabla \boldsymbol{u}(t)|_2^2 + |w(t)|_2^2\big) = 0. \tag{47}$$

On the other hand, by recalling that the unique critical point of the kinetic energy is $\boldsymbol{u} = 0$, and by taking into account that $|E_k(\boldsymbol{u}) - E_k(0)| = \frac{1}{2}|\boldsymbol{u}|_2^2$ and since $2(1-\theta) > 1$ and $|\boldsymbol{u}(t)|_2 \leq 1$, then

$$|E_k(\boldsymbol{u}(t)) - E_k(0)|^{1-\theta} = \frac{1}{2^{1-\theta}}|\boldsymbol{u}(t)|_2^{2(1-\theta)} \leq C|\boldsymbol{u}(t)|_2 \quad \forall\, t \geq t_\star.$$

Therefore, by using the Lojasiewicz-Simon inequality (given in Lemma 5):

$$(E(\boldsymbol{u}(t),\varphi(t)) - E_\infty)^{1-\theta} \leq |E_k(\boldsymbol{u}(t)) - E_k(0)|^{1-\theta} + |E_e(\varphi(t)) - E_e(\overline{\varphi})|^{1-\theta} \leq C(|\boldsymbol{u}(t)|_2 + |w(t)|_2),$$

and hence, by using the Poincare inequality:

$$(E(\boldsymbol{u}(t),\varphi(t))-E_\infty)^{\theta-1} \geq C(|\nabla \boldsymbol{u}(t)|_2+|w(t)|_2)^{-1} \quad \forall\, t\geq t_\star \tag{48}$$

From (47) and (48), we obtain

$$\frac{dH(t)}{dt}+\theta\, C(|\nabla \boldsymbol{u}(t)|_2+|w(t)|_2)\leq 0, \quad \forall\, t\geq t_\star$$

and (45) is proved. Integrating (45) into $[t_0,t_1]$ (for any $t_1>t_0\geq t_\star$) yields

$$(E(\boldsymbol{u}(t_1),\varphi(t_1))-E_\infty)^{\theta}+\theta\, C\int_{t_0}^{t_1}(|\nabla \boldsymbol{u}(t)|_2+|w(t)|_2)dt\leq (E(\boldsymbol{u}(t_0),\varphi(t_0))-E_\infty)^{\theta}. \tag{49}$$

On the other hand, since $\partial_t\varphi+\nabla\cdot(\boldsymbol{u}\otimes\varphi)-w=0$, then, by using the weak estimate $\|\varphi(t)\|_2\leq C$, it can be deduced that

$$|\partial_t\varphi|_2\leq C(\|\boldsymbol{u}\otimes\varphi\|_1+|w|_2)\leq C(|\nabla\boldsymbol{u}|_2+|w|_2)$$

By applying this inequality in (49), we obtain (46).

Step 2: *There exists a sufficiently large n_0 such that $t_{n_0}\geq T^*_{reg}$ and $\|\varphi(t)-\overline{\varphi}\|_3\leq\beta$ and $|\boldsymbol{u}(t)|_2\leq 1$ for all $t\geq t_{n_0}$.*

The bound $|\boldsymbol{u}(t)|_2\leq 1$ is based on $\boldsymbol{u}(t)\to 0$ in $\boldsymbol{H}^1_0$ given in (41). We now focus on the bound for $\|\varphi(t)-\overline{\varphi}\|_3$. Since $\varphi(t_n)\to\overline{\varphi}$ in H^4 and $E(\boldsymbol{u}(t_n),\varphi(t_n))\to E_\infty=E_e(\overline{\varphi})$, then for any $\varepsilon\in(0,\beta)$, there exists an integer $N(\varepsilon)$ such that, for all $n\geq N(\varepsilon)$,

$$\|\varphi(t_n)-\overline{\varphi}\|_3\leq\varepsilon \quad \text{ and } \quad \frac{1}{\theta}(E_e(\boldsymbol{u}(t_n),\varphi(t_n))-E_\infty)^{\theta}\leq\varepsilon \tag{50}$$

For each $n\geq N(\varepsilon)$, we define

$$\bar{t}_n:=\sup\{t: t>t_n,\, \|\varphi(s)-\overline{\varphi}\|_3<\beta \quad \forall s\in[t_n,t)\}.$$

It suffices to prove that $\bar{t}_{n_0}=+\infty$ for some n_0. Assume by contradiction that $t_n<\bar{t}_n<+\infty$ for all n. Observe that $\|\varphi(\bar{t}_n)-\overline{\varphi}\|_3=\beta$ and $\|\varphi(t)-\overline{\varphi}\|_3<\beta$ for all $t\in[t_n,\bar{t}_n)$. From Step 1, for all $t\in[t_n,\bar{t}_n]$, from (46) and (50) we obtain

$$\int_{t_n}^{\bar{t}_n}|\partial_t\varphi|_2\leq C\varepsilon, \quad \forall\, n\geq N(\varepsilon).$$

Therefore,

$$|\varphi(\bar{t}_n)-\overline{\varphi}|_2\leq|\varphi(t_n)-\overline{\varphi}|_2+\int_{t_n}^{\bar{t}_n}|\partial_t\varphi|_2\leq(1+C)\varepsilon,$$

which implies that $\lim_{n\to+\infty}|\varphi(\bar{t}_n)-\overline{\varphi}|_2=0$. Since φ is bounded in $L^\infty(t^*,+\infty;H^4)$, $(\varphi(t))_{t\geq t^*}$ is relatively compact in H^3. Therefore, there exists a subsequence of $\varphi(\bar{t}_n)$, which is still denoted as $\varphi(\bar{t}_n)$, that converges to $\overline{\varphi}$ in H^3. Hence, for a sufficiently large n, $\|\varphi(\bar{t}_n) - \overline{\varphi}\|_3 < \beta$, which contradicts the definition of $\bar{t}_n$.

Step 3: *There exists a unique $\overline{\varphi}$ such that $\varphi(t) \to \overline{\varphi}$ in H^4 as $t \uparrow +\infty$.*

By using Steps 1 and 2, from (46) it is deduced that, for all $t_1 > t_0 \geq t_{n_0}$,

$$|\varphi(t_1) - \varphi(t_0)|_2 \leq \int_{t_0}^{t_1} |\partial_t \varphi|_2 \to 0, \quad \text{as } t_0, t_1 \to +\infty.$$

Therefore, $(\varphi(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in L^2 as $t \uparrow +\infty$, and hence the L^2-convergence of the whole trajectory is deduced, i.e. there exists a unique $\overline{\varphi} \in L^2$ such that $\varphi(t) \to \overline{\varphi}$ in L^2 as $t \uparrow +\infty$. Finally, the strong H^4-convergence by sequences of $\varphi(t)$ proved in Step 1 of Theorem 11, yields $\varphi(t) \to \overline{\varphi}$ in H^4. ■

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