REGULAR TIME-REPRODUCTIVE SOLUTIONS FOR GENERALIZED BOUSSINESQ MODEL WITH NEUMANN BOUNDARY CONDITIONS FOR TEMPERATURE

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Abstract. The aim of this work is to prove existence of regular time reproductive solutions for a generalized Boussinesq model (with nonlinear diffusion for velocity and temperature). The main idea is to obtain higher regularity (of H^3 type) for temperature than for velocity (of H^2 type), using specifically the Neumann boundary condition for temperature.

1. Introduction

Assume a bounded, regular open set Ω in $\mathbb{R}^N$ ($N = 2$ or 3). This paper is concerned with some equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are explicit functions depending on temperature. The involved equations are

$$(1) \qquad \begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \alpha \boldsymbol{g}\theta + \nabla p = \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \theta - \nabla \cdot (k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0, \end{cases}$$

in $\Omega \times [0,\infty)$, where

- $\boldsymbol{u}(x,t) \in \mathbb{R}^N$ denotes the velocity of the fluid at point $x \in \Omega$ and time $t \in [0,+\infty)$.
- $p(x,t) \in \mathbb{R}$ is the (hydrostatic) pressure.
- $\theta(x,t) \in \mathbb{R}$ is the temperature.
- $\boldsymbol{g}(x,t) \in \mathbb{R}^N$ denotes the gravitational field and $\alpha > 0$ is a constant associated to the coefficient of volume expansion.
- $\boldsymbol{f}(x,t) \in \mathbb{R}^N$ denotes the resulting of external forces.
- $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is the kinematic viscosity.
- $k(\cdot) : \mathbb{R} \to \mathbb{R}$ is the thermal conductivity.

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We will search for a triplet $\{\boldsymbol{u}, p, \theta\}$ regular reproductive solution of (1) in $\Omega\times[0,\infty)$, together the follows Dirichlet-Neumann boundary conditions:

$$\boldsymbol{u}=0, \qquad \partial_n\theta = 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{2}$$

and the time reproductive condition:

$$\boldsymbol{u}(0)=\boldsymbol{u}(T), \qquad \theta(0)=\theta(T) \qquad \text{in } \Omega. \tag{3}$$

Existence and uniqueness of the initial value problem related to (1) and with Dirichlet's boundary conditions for velocity and temperature, was proved by Lorca and Boldrini in [5]. The stationary problem is studied in [6] for bounded domains and in [10] for exterior domains. On the other hand, A.C. Moretti, M.A. Rojas-Medar and M.D. Rojas-Medar [8] proved existence of reproductive weak solutions in exterior domains. The classical Boussinesq model, where ν and k are positives constants, has been analyzed in great extent, see for instance, Morimoto [9], Óeda [11].

The arguments used in [5] in order to obtain regular solution (and uniqueness) are not valid to find reproductivity since the initial conditions play a fundamental role. Our contribution in this paper is to obtain higher order estimates for the temperature than in [5]; namely in [5] $H^2(\Omega)$ regularity is obtained for velocity and temperature, but now we will arrive at $H^3(\Omega)$ regularity for the temperature. Consequently, a reproductive condition for time derivative of temperature also holds, i.e. $\partial_t\theta(0)=\partial_t\theta(T)$. In addition, the arguments used in this paper are remarkably simpler than the used ones in [5]. By the contrary, now the regularity obtained for the solution is not sufficient to prove uniqueness.

Notation.

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p\geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(X)$ the Banach space $L^p(0,T;X)$. Also, boldface letters will be used for vectorial spaces, for instance $\boldsymbol{L}^2 = L^2(\Omega)^N$.
- The L^p norm is denoted by $|\cdot|_p$, $1\leq p\leq\infty$. The H^m norm is denoted by $\|\cdot\|_m$
- We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{v}\in C_0^\infty(\Omega)^N$ satisfying $\nabla\cdot\boldsymbol{v}=0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,
$$\begin{aligned} \boldsymbol{H} &= \{\boldsymbol{u}\in\boldsymbol{L}^2;\ \nabla\cdot\boldsymbol{u}=0,\ \boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\}, \\ \boldsymbol{V} &= \{\boldsymbol{u}\in\boldsymbol{H}^1;\ \nabla\cdot\boldsymbol{u}=0,\ \boldsymbol{u}=0 \text{ on } \partial\Omega\} \end{aligned}$$
- It is easy deduce that $\dfrac{d}{dt}\displaystyle\int_\Omega \theta(x,t)=0$ from the convection-diffusion equation for θ. Then, we can fix $\displaystyle\int_\Omega\theta=0$. Therefore, let us consider the following spaces
$$H_N^k = \left\{\theta\in H^k;\ \frac{\partial\theta}{\partial n}=0 \text{ on } \partial\Omega\,,\ \int_\Omega\theta = 0\right\}$$
where $k=2,3$. Hence, H_N^k is a closed subspace of H^k. Consequently $|\Delta\theta|_2$ is equivalent to $\|\theta\|_2$ in H_N^2 and $|\nabla\Delta\theta|_2$ is equivalent to $\|\theta\|_3$ in H_N^3 ([12]).

Some interpolation inequalities. We will use the following classical interpolation and Sobolev inequalities (for $3D$ domains):

$$|v|_6 \le C\|v\|_1, \quad |v|_3 \le |v|_2^{1/2}\|v\|_1^{1/2}$$

and

$$|v|_\infty \le C\|v\|_1^{1/2}\|v\|_2^{1/2}$$

In this work, it will be useful to use the following result (see [5]):

Lemma 1. *Let $\boldsymbol{u} \in \boldsymbol{V} \cap \boldsymbol{H}^2$ and consider the Helmholtz decomposition of $-\Delta \boldsymbol{u}$, i.e. $-\Delta \boldsymbol{u} = A\boldsymbol{u} + \nabla q$, where $q \in H^1$ is taken such that $\int_\Omega q\, dx = 0$ and A is the Stokes operator. Then,*

$$\|q\|_1 \le C|A\boldsymbol{u}|_2.$$

Moreover, for every $\delta > 0$ there exists a positive constant C_δ (independent of $\boldsymbol{u}$) such that

$$|q|_2 \le C_\delta|\nabla \boldsymbol{u}|_2 + \delta|A\boldsymbol{u}|_2.$$

2. The Main Result

Definition 1. *It will be said that $(\boldsymbol{u}, p, \theta)$ is a regular solution of (1)–(3) in $(0,T)$, if*

$$\boldsymbol{u} \in L^2(\boldsymbol{H}^2) \cap L^\infty(\boldsymbol{H}^1) \qquad \text{and} \qquad \partial_t \boldsymbol{u} \in L^2(\boldsymbol{L}^2),$$
$$p \in L^2(H^1),$$
$$\theta \in L^2(H^3_N) \cap L^\infty(H^2_N) \qquad \text{and} \qquad \partial_t \theta \in L^2(H^1_N),$$

satisfying (1) a.e. in $(0,T)\times\Omega$, boundary conditions (2) and time reproductivity conditions (3) in the sense of spaces $\boldsymbol{V}$ and H^2_N respectively.

Notice that we have imposed higher regularity for θ than for $\boldsymbol{u}$.

Theorem 1. *Let $T > 0$ and Ω a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with a boundary of class $C^{2,1}$. Let the functions $\nu \in C^1(\mathbb{R})$ and $k \in C^2(\mathbb{R})$ such that*

$$0 < \nu_{min} \le \nu(s) \le \nu_{max} \quad 0 < k_{min} \le k(s) \le k_{max} \quad \text{in } \mathbb{R},$$

and ν', k', k'' are bounded in $\mathbb{R}$ (i.e. $|\nu'(s)| \le \nu'_{max}$, $|k'(s)| \le k'_{max}$, $|k''(s)| \le k''_{max}$). Assume that $\boldsymbol{f} \in L^2(\boldsymbol{L}^2)$ and $\boldsymbol{g} \in L^\infty(\boldsymbol{L}^2)$ and

$$\|\boldsymbol{f}\|_{L^2(L^2)} \le \delta$$

for δ small enough, then there exists a regular (and small) reproductive solution of (1)–(3) in $(0,T)$. Moreover, this solution also verifies $\partial_t\theta(0) = \partial_t\theta(T)$.

Remark: *The uniqueness of solutions furnished by previous Theorem remains open, because higher regularity for the velocity is necessary. To obtain H^3 regularity for the velocity seem complicated because the argument made in the proof of Lemma 3 in order to get H^3 regularity is based in the Neumann condition, but we have Dirichlet condition for $\boldsymbol{u}$.*

The proof of this Theorem will be given in Section 5. The method is based on the Galerkin approximation with spectral basis (defined in Section 3) and some differential inequalities in regular norms given in Section 4.

3. The Galerkin Initial-Boundary Problem

Let $\{\phi_i\}_{i\geq 1}$ and $\{\varphi_i\}_{i\geq 1}$ "special" basis of $\boldsymbol{V}$ and $\boldsymbol{H}^1_0(\Omega)$, respectively, formed by eigenfunctions of the Stokes and the Poisson problems following:

$$
(4) \qquad \left\{ \begin{array}{rcll} -\Delta\phi_i &=& \lambda_i\phi_i & \text{in } \Omega \\ \phi_i &=& 0 & \text{on } \partial\Omega \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\Delta\varphi_i &=& \mu_i\varphi_i & \text{in } \Omega \\ \partial_n\varphi_i &=& 0 & \text{on } \partial\Omega, \end{array} \right.
$$

with $\|\phi_i\|_1 = 1$, $\|\varphi_i\|_1 = 1$ for all i and $\int_\Omega \varphi_i = 0$. Let $\boldsymbol{V}^m$ and W^m be the finite-dimensional subspaces spanned by $\{\phi_1, \phi_2, \dots, \phi_m\}$ and $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ respectively.

For each $m \geq 1$, given $\boldsymbol{u}_{0m} \in \boldsymbol{V}^m$ and $\theta_{0m} \in W^m$, we seek an approximate solution $(\boldsymbol{u}_m, \theta_m)$, with $\boldsymbol{u}_m : [0,T] \mapsto \boldsymbol{V}^m$ and $\theta_m : [0,T] \mapsto W^m$, verifying the following variational formulation a.e. in $t \in (0,T)$:

$$
(5) \qquad \left\{ \begin{array}{l} (\partial_t \boldsymbol{u}_m(t), \mathbf{v}_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{u}_m(t), \mathbf{v}_m) + (\nu(\theta_m(t))\nabla \boldsymbol{u}_m(t), \nabla \mathbf{v}_m) \\ \qquad -(\alpha\theta_m(t)\boldsymbol{g}, \mathbf{v}_m) - (\boldsymbol{f}, \mathbf{v}_m) = 0 \qquad \forall\, \mathbf{v}_m \in \boldsymbol{V}^m \\ (\partial_t\theta_m(t), e_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\theta_m(t), e_m) \\ \qquad +(k(\theta_m(t))\nabla\theta_m(t), \nabla e_m) = 0 \qquad \forall\, e_m \in W^m \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \quad \theta_m(0) = \theta_{0m}, \end{array} \right.
$$

If we put

$$
\boldsymbol{u}_m(t) = \sum_{j=1}^m \xi_{i,m}(t)\phi_i \quad \text{and} \quad \theta_m(t) = \sum_{j=1}^m \zeta_{i,m}(t)\varphi_i,
$$

then (5) can be rewritten as a first order ordinary differential system (in normal form) associated to the unknowns $(\xi_{i,m}(t), \zeta_{i,m}(t))$. Then, one has existence of a maximal solution (defined in some interval $[0,\tau_m) \subset [0,T]$) of the related Cauchy problem. Moreover, from a priori estimates (independent on m) which will be obtained below, in particular one has that $\tau_m = T$. Finally, using regularity of the chosen spectral basis, uniqueness of approximate solution holds ([2]).

4. Differential inequalities in regular norms

In the sequel, δ and ε will denote some constants sufficiently small. By C we will denote different constants, independent on data and δ and ε.

Lemma 2. *For each $\delta, \varepsilon > 0$ sufficiently small, there exists a constant $K = K(\delta,\varepsilon) > 0$ such that*

$$
\frac{d}{dt}\int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \nu_{min}\|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 \leq \delta\|\partial_t\theta_m\|_1^2
$$
$$
+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 + K(\|\boldsymbol{u}_m\|_1^6 + \|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + \|\boldsymbol{g}\|^2_{L^\infty(L^2)}\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2)
$$

Proof.

First, taking $\boldsymbol{v} = A\boldsymbol{u}_m$ as test function in the $\boldsymbol{u}$-system of (5) (A is the Helmholtz operator mentioned in Lemma 1) one has

$$
(6) \qquad \begin{array}{c} (\partial_t \boldsymbol{u}_m, A\boldsymbol{u}_m) - (\nabla\cdot(\nu(\theta_m)\nabla\boldsymbol{u}_m), A\boldsymbol{u}_m) + ((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) \\ -\alpha(\boldsymbol{g}\theta_m, A\boldsymbol{u}_m) = (\boldsymbol{f}, A\boldsymbol{u}_m) \end{array}
$$

We can write the first term as

$$
(\partial_t \boldsymbol{u}_m, A\boldsymbol{u}_m) = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_1^2
$$

The second term of (6) is split as follows (using the Helmholtz decomposition $\Delta \boldsymbol{u} = -A\boldsymbol{u} + \nabla q$),

$$\begin{aligned} -(\nabla \cdot (\nu(\theta_m)\nabla \boldsymbol{u}_m), A\boldsymbol{u}_m) &= (\nu(\theta_m)A\boldsymbol{u}_m, A\boldsymbol{u}_m) \\ &+ (\nu(\theta_m)\nabla q, A\boldsymbol{u}_m) - (\nu'(\theta_m)\nabla\theta_m \nabla \boldsymbol{u}_m, A\boldsymbol{u}_m). \end{aligned}$$

Taking into account that

$$\begin{aligned} (\nu(\theta_m)\nabla q, A\boldsymbol{u}_m) &= -(q, \nabla \cdot (\nu(\theta_m)A\boldsymbol{u}_m)) \\ &= -(q, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m) - (q, \nu(\theta_m)\nabla \cdot A\boldsymbol{u}_m) \\ &= -(q, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m) \end{aligned}$$

since $\nabla \cdot A\boldsymbol{u}_m = 0$, hence the second term of (6) remains

$$\begin{aligned} -(\nabla \cdot (\nu(\theta_m)\nabla \boldsymbol{u}_m), A\boldsymbol{u}_m) &= (\nu(\theta_m)A\boldsymbol{u}_m, A\boldsymbol{u}_m) \\ &- (q, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m) - (\nu'(\theta_m)\nabla\theta_m \nabla \boldsymbol{u}_m, A\boldsymbol{u}_m) \end{aligned}$$

Then, (6) can also be written as follows (using $\nu(\theta_m) \geq \nu_{min} > 0$)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_1^2 + \nu_{min}\|\boldsymbol{u}_m\|_2^2 \leq -((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) - \alpha(\boldsymbol{g}\theta_m, A\boldsymbol{u}_m) \\ &+(q, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m) + (\nu'(\theta_m)\nabla\theta_m\nabla\boldsymbol{u}_m, A\boldsymbol{u}_m) + (\boldsymbol{f}, A\boldsymbol{u}_m) \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{7}$$

The first two terms and the last term on the right hand side of (15) are bounded respectively by

$$I_1 \leq \delta\|\boldsymbol{u}_m\|_2^2 + C_\delta\|\boldsymbol{u}_m\|_1^6, \qquad I_2 \leq \delta\|\boldsymbol{u}_m\|_2^2 + C_\delta|\boldsymbol{g}|_2^2\|\theta_m\|_2^2$$

and

$$I_5 \leq \delta\|\boldsymbol{u}_m\|_2^2 + C_\delta|\boldsymbol{f}|_2^2.$$

In order to estimate the third term we use the Lemma 1 (and $|\nu'(\theta_m)| \leq \nu'_{max}$)

$$\begin{aligned} |I_3| &= |(q, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m)| \leq \nu'_{max}|q|_3|\nabla\theta_m|_6|A\boldsymbol{u}_m\|_2 \\ &\leq C|q|_2^{1/2}\|q\|_1^{1/2}\|\theta_m\|_2\|\boldsymbol{u}_m\|_2 \leq C(C_\varepsilon\|\boldsymbol{u}_m\|_1^{1/2} + \varepsilon\|\boldsymbol{u}_m\|_2^{1/2})\|\boldsymbol{u}_m\|_2^{3/2}\|\theta_m\|_2 \\ &\leq C_\varepsilon\|\boldsymbol{u}_m\|_1^{1/2}\|\boldsymbol{u}_m\|_2^{3/2}\|\theta_m\|_2 + \varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 \\ &\leq \delta\|\boldsymbol{u}_m\|_2^2 + C_{\varepsilon,\delta}\|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + \varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2. \end{aligned}$$

In what concerns to the fourth term,

$$\begin{aligned} |I_4| &= |(\nu'(\theta_m)\nabla\theta_m\nabla\boldsymbol{u}_m, A\boldsymbol{u}_m)| \leq \nu'_{max}|\nabla\theta_m|_6|\nabla\boldsymbol{u}_m|_3|A\boldsymbol{u}_m|_2 \\ &\leq C\|\theta_m\|_2\|\boldsymbol{u}_m\|_1^{1/2}\|\boldsymbol{u}_m\|_2^{3/2} \leq \delta\|\boldsymbol{u}_m\|_2^2 + C_\delta\|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4. \end{aligned}$$

Consequently, choosing δ small enough we obtain that

$$\begin{aligned} \frac{d}{dt}\|\boldsymbol{u}_m\|_1^2 + \nu_{min}\|\boldsymbol{u}_m\|_2^2 &\leq C_\varepsilon(\|\boldsymbol{u}_m\|_1^6 + \|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + |\boldsymbol{g}|_2^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2) \\ &+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 \end{aligned} \tag{8}$$

On the other hand, using $\partial_t\boldsymbol{u}_m$ as a test function in the $\boldsymbol{u}$-system of (5), one obtains

$$\begin{aligned} &(\partial_t\boldsymbol{u}_m, \partial_t\boldsymbol{u}_m) + (\nu(\theta_m)\nabla\boldsymbol{u}_m, \partial_t\nabla\boldsymbol{u}_m) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \partial_t\boldsymbol{u}_m) \\ &-\alpha(\boldsymbol{g}\theta_m, \partial_t\boldsymbol{u}_m) = (\boldsymbol{f}, \partial_t\boldsymbol{u}_m). \end{aligned} \tag{9}$$

By taking into account that the second term in (9) can be written as

$$(\nu(\theta_m)\nabla \boldsymbol{u}_m, \partial_t \nabla \boldsymbol{u}_m) = \frac{1}{2}\frac{d}{dt}(\nu(\theta_m)\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m) - \frac{1}{2}(\partial_t(\nu(\theta_m))\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m),$$

we deduce from (9) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \nu(\theta_m)|\nabla \boldsymbol{u}_m|^2 + |\partial_t \boldsymbol{u}_m|_2^2 &\le -((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m, \partial_t \boldsymbol{u}_m)\\ &+\alpha(\boldsymbol{g}\theta_m, \partial_t \boldsymbol{u}_m) + \frac{1}{2}(\partial_t(\nu(\theta_m))\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m) + (\boldsymbol{f}, \partial_t \boldsymbol{u}_m)\\ &:= J_1 + J_2 + J_3 + J_4\end{aligned} \tag{10}$$

The first two terms and the last term on the right side of (10) are bounded respectively, by

$$J_1 \le \delta(|\partial_t \boldsymbol{u}_m|_2^2 + \|\boldsymbol{u}_m\|_2^2) + C_\delta \|\boldsymbol{u}_m\|_1^6, \qquad J_2 \le \delta|\partial_t \boldsymbol{u}_m|_2^2 + C_\delta |\boldsymbol{g}|_2^2 \|\theta_m\|_2^2,$$

and

$$J_4 \le \delta|\partial_t \boldsymbol{u}_m|_2^2 + C_\delta|\boldsymbol{f}|_2^2.$$

Lastly, we go into detail the third term:

$$\begin{aligned}|J_3| &= |(\nu'(\theta_m)\partial_t(\theta_m)\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m)| \le \nu'_{max}|\partial_t\theta_m|_6 |\nabla \boldsymbol{u}_m\|_3 |\nabla \boldsymbol{u}_m\|_2\\ &\le C\|\partial_t\theta_m\|_1 \|\boldsymbol{u}_m\|_1^{3/2}\|\boldsymbol{u}_m\|_2^{1/2} \le \delta(\|\partial_t\theta_m\|_1^2 + \|\boldsymbol{u}_m\|_2^2) + C_\delta\|\boldsymbol{u}_m\|_1^6\end{aligned}$$

Consequently, choosing δ small enough,

$$\begin{aligned}\frac{d}{dt}\int_\Omega \nu(\theta_m)|\nabla \boldsymbol{u}_m|^2 + |\partial_t \boldsymbol{u}_m|_2^2 &\le \delta(\|\partial_t\theta_m\|_1^2 + \|\boldsymbol{u}_m\|_2^2)\\ &+C_\delta(\|\boldsymbol{u}_m\|_1^6 + |\boldsymbol{g}|_2^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2)\end{aligned} \tag{11}$$

Finally, (8) and (11) prove the Lemma.

Lemma 3. *For each $\delta > 0$ small enough, there exists $C_\delta > 0$ such that*

$$\begin{aligned}&\frac{d}{dt}(\|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2) + k_{min}(\|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2)\\ &\le \delta|\partial_t \boldsymbol{u}_m\|_2^2 + C_\delta(\|\theta_m\|_2^6 + \|\theta_m\|_2^4|\partial_t\theta_m|_2^2 + \|\theta_m\|_2^2\|\boldsymbol{u}_m\|_1^4)\end{aligned}$$

□

Proof.

Differenting respect to the time the θ_m-equation of (5) and using $\partial_t\theta_m$ as test function, one obtains

$$\frac{1}{2}\frac{d}{dt}|\partial_t\theta_m|_2^2 + (\partial_t(k(\theta_m)\nabla\theta_m), \partial_t\nabla\theta_m) + (\partial_t \boldsymbol{u}_m\cdot\nabla\theta_m, \partial_t\theta_m) = 0 \tag{12}$$

since $(\boldsymbol{u}_m\cdot\nabla\partial_t\theta_m, \partial_t\theta_m) = 0$.

By taking into account that the second term in (12) can be split as

$$(\partial_t(k(\theta_m)\nabla\theta_m), \partial_t\nabla\theta_m) = (k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m) + (k(\theta_m)\partial_t\nabla\theta_m, \partial_t\nabla\theta_m),$$

we deduce from (12) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_t\theta_m|_2^2 + k_{min}|\partial_t\nabla\theta_m|_2^2 \quad \le \quad & -(k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m)\\ & -(\partial_t \boldsymbol{u}_m\cdot\nabla\theta_m, \partial_t\theta_m).\end{aligned} \tag{13}$$

Bounding both terms on the right hand side of (13) ($k'_{max} = \max|k'|$):

$$|(k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m)| \le k'_{max}|\nabla\theta_m|_6|\partial_t\theta_m|_3|\partial_t\nabla\theta_m|_2$$
$$\le C\|\theta_m\|_2|\partial_t\theta_m|_2^{1/2}\|\partial_t\theta_m\|_1^{3/2} \le \delta\|\partial_t\theta_m\|_1^2 + C_\delta\|\theta_m\|_2^4|\partial_t\theta_m|_2^2$$

and

$$|(\partial_t\boldsymbol{u}_m\cdot\nabla\theta_m, \partial_t\theta_m)| \le |\partial_t\boldsymbol{u}_m|_2|\nabla\theta_m|_6|\partial_t\theta_m|_3$$
$$\le C|\partial_t\boldsymbol{u}_m|_2\|\theta_m\|_2|\partial_t\theta_m|_2^{1/2}\|\partial_t\theta_m\|_1^{1/2}$$
$$\le \delta(\|\partial_t\theta_m\|_1^2 + |\partial_t\boldsymbol{u}_m|_2^2) + C_\delta\|\theta_m\|_2^4|\partial_t\theta_m|_2^2$$

we obtain, for δ small enough,

$$\frac{d}{dt}|\partial_t\theta_m|_2^2 + k_{min}\|\partial_t\theta_m\|_1^2 \le \delta|\partial_t\boldsymbol{u}_m|_2^2 + C_\delta\|\theta_m\|_2^4|\partial_t\theta_m|_2^2 \tag{14}$$

Now, using $\Delta^2\theta_m$ as test function ($\Delta^2\theta_m \in W^m$ thanks to the election of spectral basis) and integrating by parts in all terms (boundary terms vanish since $(\nabla\Delta\theta_m\cdot\boldsymbol{n})|_{\partial\Omega} = 0$), one obtains:

$$-(\partial_t\nabla\theta_m, \nabla\Delta\theta_m) + (\nabla[\nabla\cdot(k(\theta_m)\nabla\theta_m)], \nabla\Delta\theta_m) - (\nabla(\boldsymbol{u}\cdot\nabla\theta_m), \nabla\Delta\theta_m) = 0. \tag{15}$$

Notice that if Dirichlet boundary condition is imposed for the temperature θ, the boundary terms do not vanish in the integration by parts and we can not obtain the previous inequalities.

Integrating by parts the first term of (15) (again the boundary term vanishes since $(\partial_t\nabla\theta_m\cdot\mathbf{n})|_{\partial\Omega} = 0$), the term remains $\dfrac{1}{2}\dfrac{d}{dt}|\Delta\theta_m|_2^2$. The second term is

$$(\nabla[\nabla\cdot(k(\theta_m)\nabla\theta_m)], \nabla\Delta\theta_m) = (k''(\theta_m)(\nabla\theta_m)^3, \nabla\Delta\theta_m)$$
$$+2(k'(\theta_m)\nabla^2\theta_m\nabla\theta_m, \nabla\Delta\theta_m)$$
$$+(k'(\theta_m)\nabla\theta_m\Delta\theta_m, \nabla\Delta\theta_m) + (k(\theta_m)\nabla\Delta\theta_m, \nabla\Delta\theta_m).$$

Hence, we deduce of (15) that ($|k''(\theta_m)| \le k''_{max} = \max|k''|$)

$$\frac{1}{2}\frac{d}{dt}|\Delta\theta_m|_2^2 + k_0|\nabla\Delta\theta_m|_2^2 \le k''_{max}|((\nabla\theta_m)^3, \nabla\Delta\theta_m)|$$
$$+2k'_{max}|(\nabla^2\theta_m\nabla\theta_m, \nabla\Delta\theta_m)| + k'_{max}|(\nabla\theta_m\Delta\theta_m, \nabla\Delta\theta_m)|$$
$$+|(\nabla\boldsymbol{u}_m\nabla\theta_m, \nabla\Delta\theta_m)| + |(\boldsymbol{u}_m\nabla^2\theta_m, \nabla\Delta\theta_m)|$$
$$:= L_1 + L_2 + L_3 + L_4 + L_5.$$

Replacing in the above inequality the following estimations

$$L_1 \le C|\nabla\theta_m|_6^3|\nabla\Delta\theta_m|_2 \le \delta\|\theta_m\|_3^2 + C_\delta\|\theta_m\|_2^6$$

$$L_2 \le C|\nabla^2\theta_m|_3|\nabla\theta_m|_6|\nabla\Delta\theta_m|_2 \le C\|\theta_m\|_2^{3/2}\|\theta_m\|_3^{3/2} \le \delta\|\theta_m\|_3^2 + C_\delta\|\theta_m\|_2^6$$

$$L_3 \le C|\nabla\theta_m|_6|\Delta\theta_m|_3|\nabla\Delta\theta_m|_2 \le C\|\theta_m\|_2^{3/2}\|\theta_m\|_3^{3/2} \le \delta\|\theta_m\|_3^2 + C_\delta\|\theta_m\|_2^6$$

$$L_4 \le C|\nabla\boldsymbol{u}_m|_2|\nabla\theta_m|_\infty|\nabla\Delta\theta_m|_2 \le C\|\boldsymbol{u}_m\|_1\|\theta_m\|_2^{1/2}\|\theta_m\|_3^{3/2}$$
$$\le \delta\|\theta_m\|_3^2 + C_\delta\|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2$$

$$L_5 \le C|\boldsymbol{u}_m|_6|\nabla^2\theta_m|_3|\nabla\Delta\theta_m|_2 \le C\|\boldsymbol{u}_m\|_1\|\theta_m\|_2^{1/2}\|\theta_m\|_3^{3/2}$$
$$\le \delta\|\theta_m\|_3^2 + C_\delta\|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2$$

we get, taking δ small enough,

$$\frac{d}{dt}\|\theta_m\|_2^2 + k_{min}\|\theta_m\|_3^2 \le C_\delta(\|\theta_m\|_2^6 + \|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2) \tag{16}$$

Finally, (14) added to (16) proves the Lemma.

5. Proof of Theorem 2.2

If we denote

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2$$

$$\Psi_m(t) = \|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 + \|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2$$

taking an adequate balance between inequalities from Lemmas 2 and 3 in order to vanish the term $||\boldsymbol{g}||^2_{L^\infty(L^2)}||\theta_m||_2^2$ at the right hand-side, one has

$$\begin{cases} \Phi_m' + C\Psi_m \le \varepsilon\Psi_m\Phi_m^{1/2} + C_0(t) + D\Phi_m^3 \\ \Phi_m(0) = \Phi_{m0} \end{cases} \tag{17}$$

where $C, D > 0$ are constant and $C_0(t)$ is a positive function depending on data $\boldsymbol{f}$. Concretely, $C_0(t) = C_0|\boldsymbol{f}|_2^2$.

Let $\Phi_m(0) \le \delta$ for δ a small enough constant (that we will specify latter).

First step: If $\Phi_m(0) \le \delta$ and $\|\boldsymbol{f}\|_{L^2(L^2)} \le \delta$, then $\Phi_m(t) < 2\delta$ $\forall t \in [0,T]$.

Indeed, by an absurd argument, let T^* be the first value in $[0,T]$ such that $\Phi_m(T^*) = 2\delta$, hence

$$\Phi_m(T^*) = 2\delta \qquad \text{and} \qquad \Phi_m(s) < 2\delta \quad \forall s \in [0,T^*).$$

Moreover, there exists a Poincaré constant $C_p > 0$ such that $\Phi_m(t) \le C_p\Psi_m(t)$. Then for ε small enough we have

$$C\Psi_m - \varepsilon\Psi_m\Phi_m^{1/2} \ge C\Psi_m - \varepsilon\Psi_m(2\delta)^{1/2} \ge \bar{C}\Psi_m \ge \frac{\bar{C}}{C_p}\Phi_m \equiv \tilde{C}\Phi_m.$$

The above inequality together (17) lead:

$$\begin{cases} \Phi_m' + \tilde{C}\Phi_m \le C_0(t) + D\Phi_m^3 \\ \Phi_m(0) = \Phi_{m0}. \end{cases} \tag{18}$$

in $[0,T^*]$. Then, $\Phi_m' + \tilde{C}\Phi_m \le C_0(t) + 4\delta^2 D\Phi_m$ in $[0,T^*]$. We can find δ such that $\tilde{C} - 4\delta^2 D \ge \bar{C}$ being $\bar{C}$ a positive constant. Therefore,

$$\Phi_m' + \bar{C}\Phi_m \le C_0(t) \quad \text{in } [0,T^*]$$

hence

$$(e^{\bar{C}t}\Phi_m)' \le e^{\bar{C}t}C_0(t) \quad \text{in } [0,T^*]. \tag{19}$$

Integrating in $[0,T^*]$ one finds:

$$e^{\bar{C}T^*}\Phi_m(T^*) \le \Phi_m(0) + \int_0^{T^*} e^{\bar{C}t}C_0(t),$$

hence

$$\Phi_m(T^*) \le \delta e^{-\bar{C}T^*} + \int_0^{T^*} C_0(t).$$

We can choose $\displaystyle\int_0^{T^*} C_0(t)$ small enough such that the right side is smaller that 2δ (for example $\displaystyle\int_0^{T^*} C_0(t) \le \delta$). Thus, we arrive at a contradiction.

Second step: If $\Phi_m(0)$ and $\|\boldsymbol{f}\|_{L^2(L^2)}$ are small enough, then $\Phi_m(T) \le \Phi_m(0)$

Now, as $\Phi_m(t) < 2\delta \; \forall t \in [0,T]$, we can repeat the above argument and to obtain (18) in $[0,T]$. Therefore, integrating (19) in $[0,T]$ we arrive at

$$\Phi_m(T) \le \Phi_m(0) e^{-\bar{C}T} + \int_0^T C_0(s).$$

Again, for $\displaystyle\int_0^t C_0(t)$ small enough (for example $\displaystyle\int_0^{T^*} C_0(t) \le \delta(1-e^{-\bar{C}T})$) one obtains that $\Phi_m(T) \le \Phi_m(0)$.

Third step: Existence of approximate reproductive solution

Given $(\boldsymbol{u}_{m0}, \theta_{m0}) \in V^m \times W^m$, we define the map

$$\begin{array}{rcl} L^m : [0,T] & \mapsto & \mathbb{R}^m \times \mathbb{R}^m \\ t & \mapsto & (\xi_{1m}(t), ..., \xi_{mm}(t), \zeta_{1m}(t), ..., \zeta_{mm}(t)) \end{array}$$

where $(\xi_{1m}(t), ..., \xi_{mm}(t))$ and $(\zeta_{1m}(t), ..., \zeta_{mm}(t))$ are coefficients of $\boldsymbol{u}_m(t)$ and $\theta_m(t)$ respect to V^m and W^m respectively, being $(\boldsymbol{u}_m(t), \theta_m(t))$ the (unique) approximate solution of (5) corresponding to the initial data $(\boldsymbol{u}_{m0}, \theta_{m0})$.

Now, varying the initial data $(\boldsymbol{u}_{m0}, \theta_{m0})$, we are going to define a new map

$$\mathcal{R}^m : \bar{B} \subset \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^m \times \mathbb{R}^m$$

as follows: given $L_0^m \in \mathbb{R}^m \times \mathbb{R}^m$, we define $\mathcal{R}^m(L_0^m) = L^m(T)$, where $L^m(t)$ is related to the solution of problem (5) with initial data $L_0^m (= L^m(0))$ and

$$\bar{B} = \{(\xi_{1m}, ..., \xi_{mm}, \zeta_{1m}, ..., \zeta_{mm}) := L_0^m : \Phi_m(0) \le \delta\}.$$

By uniqueness of approximate solution of problem (5), this map is well-defined. Moreover, using regularity of the corresponding ordinary differential system (equivalent to (5)), this map is continuous. By the second step, $\mathcal{R}^m$ apply $\bar{B}$ into $\bar{B}$ and $\bar{B}$ is a closed, convex and compact set. Consequently, Brouwer Theorem implies the existence of fixed point of $\mathcal{R}^m$, which give us existence of reproductive Galerkin solution.

Four step: Pass to the limit in reproductive approximate solutions

If the data of the problem are small, thanks to the first step we have

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t \theta_m|_2^2 \le 2\delta$$

and (19). Therefore, the following uniformly bounds hold:

$$(\boldsymbol{u}_m) \text{ in } L^\infty(H^1) \cap L^2(H^2),$$

$$(\theta_m) \text{ in } L^\infty(H_N^2) \cap L^2(H_N^3),$$

$$(\partial_t \boldsymbol{u}_m) \text{ in } L^2(L^2),$$

$$(\partial_t \theta_m) \text{ in } L^\infty(L^2) \cap L^2(H^1).$$

Using compactness results for time spaces with values in Banach spaces with the triplet $H^2 \hookrightarrow H^1 \hookrightarrow L^2$ and $H^3 \hookrightarrow H^2 \hookrightarrow H^1$, one has

$$(\boldsymbol{u}_m) \text{ is relatively compact in } L^2(H^1)$$

and

$$(\theta_m) \text{ is relatively compact in } L^2(H^2).$$

In fact, this compactness is sufficient in the pass to the limit in (5) in order to control the nonlinear terms.

Now, we go to pass to the limit in reproductive conditions. From estimations of θ_m in $L^\infty(H^2)$ and $(\partial_t\theta_m)$ in $L^2(H^1)$ and using the triplet of spaces $H^2 \hookrightarrow H^1 \hookrightarrow H^1$, one has that θ_m is relatively compact in $C([0,T];H^1)$, hence $\theta_m(T) \to \theta(T)$ and $\theta_m(0) \to \theta(0)$ strongly in $H^1(\Omega)$. Since $\theta_m(T) = \theta_m(0)$, then $\theta(T) = \theta(0)$ in $H^1(\Omega)$. Finally, since $\theta_m(T)$ and $\theta_m(0)$ are bounded in $H^2(\Omega)$, we have that $\theta(T) = \theta(0)$ in $H^2(\Omega)$.

The argument for $\boldsymbol{u}$ is similar, hence one deduces $\boldsymbol{u}(T) = \boldsymbol{u}(0)$ in $H^1(\Omega)$.

To prove $\partial_t\theta(0) = \partial_t\theta(T)$ we go to consider the orthogonal projector $P_m : H^1 \to W^m$

$$P_m(g) = \sum_{k=1}^{m} (\nabla g, \nabla\varphi_i)\varphi_i, \qquad \forall g \in H^1$$

then $\|P_m\|_{\mathcal{L}(H^1,H^1)} \leq 1$ and $\|P_m\|_{\mathcal{L}((H^1)',(H^1)')} \leq 1$ (see [4]). Since

$$P_m(g) = \sum_{k=1}^{m} \mu_i(g,\varphi_i)\varphi_i, \qquad \forall g \in H^1$$

then P_m is also the orthogonal projector from L^2 to W^m respect to the L^2 scalar product. Therefore

$$(P_m(g),\varphi_i) = (g,\varphi_i) \quad \forall\, i, \quad \forall\, g \in H^1(\Omega),$$

hence the Galerkin equation for θ_m can be written as

$$\partial_t\theta_m = P_m\left(-\boldsymbol{u}_m \cdot \nabla\theta_m + \nabla\cdot(k(\theta_m)\nabla\theta_m)\right)$$

Differenting respect to the time the θ_m-equation in (5),

$$\partial_{tt}\theta_m = P_m\left(-\partial_t\boldsymbol{u}_m \cdot \nabla\theta_m - \boldsymbol{u}_m \cdot \nabla\partial_t\theta_m + \nabla\cdot(k'(\theta_m)\partial_t\theta_m\nabla\theta_m + k(\theta_m)\partial_t\nabla\theta_m)\right)$$

In particular

$$\begin{aligned} ||\partial_{tt}\theta_m||_{(H^1)'} &\leq ||-\partial_t\boldsymbol{u}_m \cdot \nabla\theta_m - \boldsymbol{u}_m \cdot \nabla\partial_t\theta_m||_{(L^6)'} \\ &\quad + ||\nabla\cdot(k'(\theta_m)\partial_t\theta_m\nabla\theta_m + k(\theta_m)\partial_t\nabla\theta_m)||_{(H^1)'} \\ &\leq |\partial_t\boldsymbol{u}_m|_2|\nabla\theta_m|_3 + |\boldsymbol{u}_m|_3|\nabla\partial_t\theta_m|_2 + |k'(\theta_m)\partial_t\theta_m\nabla\theta_m + k(\theta_m)\partial_t\nabla\theta_m|_2 \end{aligned}$$

The terms on the right hand-side of previous inequality will be bounded in $L^2(0,T)$. Indeed, $\partial_t\boldsymbol{u}_m$ is bounded in $L^2(L^2)$ and $\nabla\theta_m$ in $L^\infty(H^1)$, hence

$$|\partial_t\boldsymbol{u}_m|_2|\nabla\theta_m|_3 \quad \text{is bounded in} \quad L^2(0,T)$$

Using that $\boldsymbol{u}_m$ is bounded in $L^\infty(H^1)$ and $\partial_t\nabla\theta_m$ in $L^2(L^2)$, one has

$$|\boldsymbol{u}_m|_3|\nabla\partial_t\theta_m|_2 \quad \text{is bounded in} \quad L^2(0,T)$$

Using that $\partial_t\theta_m$ and $\nabla\theta_m$ are bounded in $L^4(L^3)$ and $L^\infty(H^1)$ respectively, one has that $k'(\theta_m)\partial_t\theta_m\nabla\theta_m$ is bounded in $L^2(L^2)$. Finally, $k(\theta_m)\partial_t\nabla\theta_m$ is bounded in $L^2(L^2)$.

Consequently, $\|\partial_{tt}\theta_m\|_{(H^1)'}$ is uniformly bounded in $L^2(0,T)$. This along with $\partial_t\theta_m$ is uniformly bounded in $L^\infty(L^2)$ gives that $\partial_t\theta_m$ is relatively compact in $C([0,T];(H^1)')$, which is suffices to prove $\partial_t\theta(0) = \partial_t\theta(T)$.

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