Some Considerations about Geometric Algebras in relation with Visibility in Computer Graphics

Ángel F. Tenorio[1] and Lilian Aveneau[2]

[1] Applied Mathematics Division. Pablo de Olavide University, Spain.
aftenorio@upo.es
[2] XLIM/SIC, CNRS. Université de Poitiers, France
lilian.aveneau@univ-poitiers.fr.

Abstract. We give some considerations about the use of geometric algebras in the context of visibility, showing some advantages and disadvantages for their use as the underlying framework. We emphasize the use of conformal geometric algebra since, among other reasons, it allows us to study easily the visibility for flat varieties and, due to the same algebraic expression of hyper-spheres and linear varieties, the results might be generalized to non-flat objects.

1 Clifford or geometric algebras

Clifford algebras or, as they are usually known, geometric algebras (GA from here on) appear with the introduction of a new geometric product between vectors generalizing and combining both the scalar product and the wedge product.

The first GA dates from the middle of 19th century, when Grassmann [6] defined the exterior algebra via the wedge product and Hamilton [7] introduced the quaternion algebra as a generalization of his algebraic notation of complex numbers. Later, Clifford [4] gave the definition of GA, generalizing them by modification of the wedge product into the geometric product. From a naive approach, the idea behind GA is to generalize the algebraic notation of quaternions over $\mathbb{R}$ by defining abstract vector roots for -1 and $+1$, satisfying anti-commutativity for the geometric product of two roots.

However, GA was not used until the 1960s when Hestenes [8] attempted to unify the mathematical language in Physics and Geometry, taking advantage of geometric product. Some years later, Hestenes [9] introduced the conformal geometric algebra (CGA from here on) for the n-dimensional Euclidean space. This algebraic framework has been usefully applied to several scientific fields, as robotics and computer vision.

GA can be axiomatically defined in many ways, but we use here the following naive formulation: a *geometric algebra* consists of endowing a vector space $\mathbb{R}^m$ with an associative inner law being distributive over vector addition, as well as obtaining a real number as result of the square for vectors (i.e. $a^2 \in \mathbb{R}$, $\forall a \in \mathbb{R}^n$). This inner product, named *geometric* or *Clifford product*, corresponds to a bilinear form over $\mathbb{R}^m$.

The square of a vector via the geometric product is not necessarily positive: this product admits vectors with negative square, contrary to what happens with the usual scalar product (only with positive squares). Thus, GA can be of mixed signature, including Grassmann algebra and quaternion algebra.

From geometric product, both scalar ($\cdot$) product and exterior ($\wedge$) product can be retrieved as $\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba})$ and $\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba})$, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$. Consequently, geometric product can be expressed as $\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$.

Wedging k vectors together, a *k-blade* is obtained. Hence, 0-blades are scalars; 1-blades are vectors; 2-blades are wedges of two vectors; and so on. The number of vectors in the blade is called *grade*, being less or equal to the dimension of the vector space $\mathbb{R}^m$ in virtue of the skew-symmetry of the wedge.

Linear combinations of k-blades are named *k-vectors*. The set $\bigwedge^k(\mathbb{R}^m)$ of k-vectors is a $\mathbb{R}$-vector space with basis $\left\{\wedge_{h=1}^k e_{i_h}\right\}_{1\leq i_1<\cdots<i_k\leq m}$, where $\{e_i\}_{i=1}^m$ is a basis of $\mathbb{R}^m$. Analogously, linear combinations of blades (not necessarily of the same grade) are called *multivectors*. These latter form the graded $\mathbb{R}$-vector space $\bigwedge(\mathbb{R}^m)$, named Grassmann algebra (since it is endowed with the wedge product) and spanned by blades of grade $0\leq k\leq m$ as $\bigwedge(\mathbb{R}^m)=\bigoplus_{k=0}^m\bigwedge^k(\mathbb{R}^m)$. Naturally, the geometric product can be extended from vectors to multivectors, providing generalizations for inner and wedge products of multivectors: if $M,N\in\bigwedge(\mathbb{R}^m)$, the geometric product MN is another multivector, with its blades split according to the grade. The inner product $M\cdot N$ corresponds to minimal grade, and the wedge $M\wedge N$ to the maximal one.

GA is used as framework of projective geometry in robotics and computer vision because linear dependency can be expressed using the wedge operator, without using coordinates. For a general review on GA, the reader can consult [5, Chapters 2 & 4].

2 Conformal geometric algebra

Next, we must recall that, using Grassman algebra as framework, the distance cannot be appropriately retrieved from vectors involved without using coordinates. On the contrary, CGA allows to retrieve the distance between two geometric points as the scalar product of their vector representations. Hence, CGA is nowadays used in mechanics and computer vision. Chapter 10 in [5] gives a general overview on CGA.

Let $\mathfrak{G}_m=\mathbb{R}^m$ denote the m-dimensional geometrical space, being identified with its underlying vector space. Any point $p\in\mathfrak{G}_m$ is faithfully represented as a unitary vector in the unit hyper-sphere $S^m\subset\mathbb{R}^{m+1}$ via stereographic projection from south pole s and with $\mathfrak{G}_m$ as the equatorial hyperplane:

$$S:\mathfrak{G}_m\to S^m: x\to S(x)=\frac{2x}{1+x^2}-\frac{1-x^2}{1+x^2}e_0,$$

where e_0 is the unitary vector from origin $o\in\mathfrak{G}_m$ to pole s, and orthogonal to $\mathbb{R}^m$. Thus, s represents the point at infinity of $\mathbb{R}^m$, whereas the north pole represents origin o. Since $S(x)$ is not homogeneous, wedge product cannot express linear dependence contrary to Grassmann algebra in projective framework.

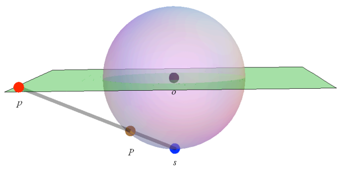

Fig. 1. Example of stereographic projection for $\mathbb{R}^2$.

Homogeneity is obtained by introducing an additional vector $e_{m+1}\notin\mathbb{R}^{m+1}$, orthogonal to $\mathfrak{G}_m\oplus e_0$ and with negative signature (i.e. $e_{m+1}^2=-1$). Hence, these representations belong to the vector space $\mathbb{R}^{m+1,1}$ of dimension $m+2$, with $m+1$ basis vectors of positive signature and one of negative signature.

This faithful representation $\Phi:\mathfrak{G}_m\to\mathbb{R}^{m+1,1}$ is given by $\Phi(x)=X=S(x)+e_{m+1}$, where $X\in\mathbb{R}^{m+1,1}$ is a non-zero null vector (i.e. $X\neq 0$ and $X^2=0$). Hence, Φ is an homogeneous representation not depending on scalar multiplication. This allows to re-scale the vector as $X = 2x - (1-x^2)e_0 + (1+x^2)e_{m+1}$. However, neither e_0 nor e_{m+1} have geometrical meaning and, hence, null vectors $n = e_0 + e_{m+1}$ and $\bar{n} = e_0 - e_{m+1}$ are defined because they respectively represent the point at infinity and the origin o of $\mathfrak{G}_m$. Using them, the *standard conformal representation* of the geometrical point $x \in \mathfrak{G}_m$ is obtained as $X = F(x) = \frac{1}{2}(2x + x^2 n - \bar{n})$, providing the condition $X \cdot n = -1$.

3 Visibility using (conformal) geometric algebras

A very common problem in robotics and computer graphics consists in characterizing (and computing) the visibility between geometrical objects. This problem can be reduced to study if two given points are visible from each other or not; which can be geometrically translated into knowing if the line segment relying them stabs other geometrical objects or not. Most of papers deal with this problem only in particular cases, but very few consider an arbitrary dimension and continuous groups of points.

There are several tools for the theoretical study of visibility. The visibility complex is one of the most usual constructs; though it is only defined for dimension 2 [11] and 3 [2]. Despite its theoretical applicability, many problems appear to implement the 3-dimensional visibility complex, becoming impracticable even for simple geometries.

Visibility in dimension 3 can be computed using line spaces as in [1, 10], via working with Plücker coordinates, which involves to consider lines as points in a 5-dimensional projective space. These coordinates are quite difficult to use in proofs. Anyway, visibility calculus using line spaces provides efficient methods but with two serious drawbacks: a) only low-complexity geometrical models can be used due to memory usage and b) heuristics must be applied in practice due to theoretical background lacks.

These problems can be avoided with an appropriate definition of line space. Thus, Charneau et al. [3] gave such a definition via GA, getting essential improvements in visibility calculation and formalizing previous heuristics as theorems in [1]. Hence, visibility for flats can be calculated in the geometrical space $\mathfrak{G}_m$ with $m \geq 2$.

Using GA in this context is due to its availability to handle geometrical objects: any linear (affine) variety in $\mathfrak{G}_m$ can be identified with a wedge of vectors in $\mathfrak{G}_m$. Moreover, passing from a dimension to other is quite easy via the inner and outer products, the first to loose information and the second to gain. Indeed, the inner product checks orthogonality and the outer one encodes linear dependency, which makes possible to span a well-defined vector space with all projective lines. However, to study Euclidean geometry, CGA is more appropriate, because previous properties are preserved after adaptation; the inner product retrieves the distance between points; the wedge product represents both linear (affine) varieties and hyper-spheres (including circles); and Euclidean transformations (rotations, translations, reflections, homotheties or inversions) are expressed using rotors (i.e. conformal rotations).

To study visibility using CGA as framework, Tenorio et al. [12] have extended the notion of line space to CGA, which should make possible to determine the visibility between flat and non-flat geometrical objects. Using the richness of conformal rotations, visibility might be computed for objects, which are rotated, translated or even re-scaled. Besides of the line space in CGA, the authors looked for how to generalize previous works on Grassmann Algebra. Thus, they showed that the Grassmannian (i.e. the smallest linear variety containing all Euclidean lines) in CGA was a vector subspace of $\bigwedge^3 \left(\mathbb{R}^{m+1,1}\right)$, but containing also circles since they are expressed as 3-vectors of three points in CGA and lines are degenerate circles in with one of the points being the point n at infinity. Hence, CGA might be used to calculate visibility between lines and spherical objects. Finally, they also characterized relative position of lines and affine linear varieties using two GA operations (wedge and left-contraction) as well as characterizing lines stabbing convex faces. Next, we reproduce the main results:

Theorem 1 (Relative Position [12]). Let L and H be respectively an Euclidean line and an affine linear variety of $\mathfrak{G}_m$, both embedded in $\mathbb{R}^{m+1,1}$, with $m \geq 2$, and let $\bar{L}$ and $\bar{H}$ be such that $\bar{L} = e_0 \rfloor L$ and $\bar{H} = e_0 \rfloor H$.

1. $((e_0 \wedge e_{m+1}) \rfloor L) \wedge H = 0 \Longrightarrow L$ and H are parallel.
2. $((e_0 \wedge e_{m+1}) \rfloor L) \wedge H \neq 0$:
   (a) $\bar{L} \wedge \bar{H} \wedge n = 0 \Longrightarrow L$ and H intersect in a singleton.
   (b) $\bar{L} \wedge \bar{H} \wedge n \neq 0 \Longrightarrow L$ and H are skew varieties.

Proposition 1 (Lines stabbing convex faces [12]). Let F be a convex $(m-1)$-face of $\mathfrak{G}_m$, supported by the hyperplane $\mathcal{H}_F$ of $\mathfrak{G}_m$ and bounded by the $(m-2)$-dimensional affine linear varieties $\{f_i\}_{i=1}^{r}$. The varieties $\{f_i\}_{i=1}^{r}$ only have two orientations such that, for any Euclidean line L, $L \cap F$ is a singleton (i.e. L stabs F) if and only if the following three conditions hold:

1. $\bar{L} \wedge \overline{\mathcal{H}_F} \wedge n = 0$, where $\bar{L} = e_0 \rfloor L$ and $\overline{\mathcal{H}_F} = e_0 \rfloor \mathcal{H}_F$.
2. $((e_0 \wedge e_{m+1}) \rfloor L) \wedge \mathcal{H}_F \neq 0$.
3. $(\bar{L} \wedge f_i \geq 0, \forall i = 1, \ldots, r)$ or $(\bar{L} \wedge f_i \leq 0, \forall i = 1, \ldots, r)$.

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