On the Recognition of Tori Embedded in $\mathbb{R}^3$.

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Abstract Many known algorithms allow us to topologically recognize surfaces in 3D images. However, none of them permits us to distinguish different types of embeddings of surfaces. In this paper, we restrict our study to the case of embedded tori, and focus on their recognition up to isotopy. We recall the mathematical definition of isotopy, then we define the two key elements which will enable us to classify an embedded torus in $\mathbb{R}^3$ up to isotopy : its state and the knot associated with it. At the end of the paper, we bring up two algorithms which aim at finding the isotopy type of an embedded torus, by determining its state and computing its associated knot.
Keywords torus, embedding, isotopy, solid torus, knots, 3D images.

1 Introduction

In medical imaging for instance, it would be helpful to have algorithms able to fully recognize surfaces in 3D images. Numerous programs computing topological properties such as the Euler characteristic, the genus, and the homology of a surface already exist (see for instance [Mun84], [DG98] and [DE95]). They enable computers to recognize surfaces in images up to homeomorphism. It means that these programs can tell us if a given surface in the image is a sphere, a torus, a projective plane... However, a single surface can be embedded in very different ways in an image. For instance, both surfaces of Figure 1 are orientable surfaces of genus 2. Topologically, they are the same surface, and yet by virtue of their position in $\mathbb{R}^3$ they are different, the second one is not just a deformation of the first one.

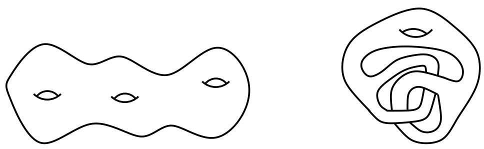


Figure 1: The same surface embedded in two different ways in $\mathbb{R}^3$.

That is why algorithms allowing us to recognize embedded surfaces "up to deformation" would be of interest. We will restrict our study to the case of embedded tori. Our purpose is thus to recognize tori in 3D images "up to deformation".

The paper is organized as follows. In section 2 basic notions necessary to the understanding of the rest of the article are recalled. Section 3 deals with recognition of embedded tori: it contains a definition and a description of the different states of tori embedded in $\mathbb{R}^3$, and it introduces the notion of knot associated with a torus. Finally, section 4 mentions briefly two algorithms that have been implemented in order to recognize embedded tori.

2 Preliminaries

2.1 Conventions and notations

In this paper, maps and spaces will be assumed to be piecewise-linear (denoted by P.L.). It means that each space is a simplicial complex, and that each map, after some subdivision of its domain and range, sends simplexes linearly onto simplexes (see [Hud69], and [RS72]). This assumption is natural, since we deal with images, and necessary to use several theorems as Dehn's lemma, the loop theorem, and hence the solid torus theorem (see [Rol76]).

If Σ is a topological manifold with boundary, we will denote by $\mathrm{Bd}(\Sigma)$ its boundary, and by $\mathrm{Int}(\Sigma)$ its interior.

Given $n \in \mathbb{N}$ (the set of non-negative integers), we will use the classical notation of the n-dimensional unit sphere : $S^n = \{x \in \mathbb{R}^{n+1}, \|x\| = 1\}$. Moreover, the n-dimensional unit disc will be denoted by $D^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$.

Let X a be topological space, $E \subset X$ and $n \in \mathbb{N}$. Then we say that E is a n-sphere if E is homeomorphic to S^n, that E is a n-disc if E is homeomorphic to D^n, and that E is a disc if E is a 2-disc.

2.2 Alexandroff one-point compactification

This compactification will be useful to apply results valid for S^3 to $\mathbb{R}^3$.

Alexandroff compactification extends a noncompact topological space to a compact one, by adding a single point. More precisely, let X be a locally compact topological space. The Alexandroff extension of X is the space $\tilde{X} = X \cup \{x\}$ where $x \notin X$, endowed with the topology whose open sets are the open sets of X together with the sets of the form $\{x\} \cup (X \setminus K)$ where K is a compact subset of X. Then $\tilde{X}$ is compact (see [Kel75]). The point x is usually called *point at infinity* and is denoted by ∞.

Its is well known that for $n \in \mathbb{N}^*$, the Alexandroff extension of $\mathbb{R}^n$ is an n-sphere. For instance, if $n = 2$, S^2 can be seen as $\mathbb{R}^2$ glued with a point at infinity (see Figure 2). The case which will be useful in this paper is $n = 3$.

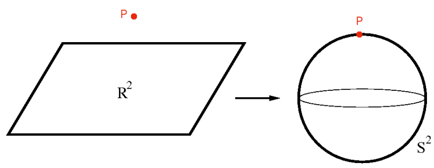


Figure 2: The Alexandroff extension of $\mathbb{R}^2$.

2.3 Isotopy

Our goal is to recognize embedded tori "up to deformation". Thus we need to define mathematically what we mean by deformation. This deformation is conveyed by the notion of *isotopy* (see [Cro04]).

We firstly define isotopy of maps :

Definition 1. *Let X, Y be topological spaces. Two* homeomorphisms $\varphi_0, \varphi_1 : X \to Y$ *are said to be* isotopic *if there exists a continuous map $\Phi : X \times [0,1] \to Y$ such that :*

1. $\Phi(x, 0) = \varphi_0(x)\ \forall x \in X$
2. $\Phi(x, 1) = \varphi_1(x)\ \forall x \in X$
3. *For all $t \in [0,1]$, the partial map $\varphi_t : X \to Y$ defined by $\varphi_t(x) = \Phi(x,t)$ is a homeomorphism.*

Thus the notion of isotopy is similar to homotopy, but stronger since isotopy requires the partial functions to be homeomorphisms.

Example 1. *Let $X = Y = [-1, 1]$ and let f be given by* $f(x) = \begin{cases} 3x + 2 & \text{if } x \in \left[-1, -\frac{1}{2}\right] \\ \frac{1}{2}x + \frac{2}{3} & \text{if } x \in \left[-\frac{1}{2}, 1\right]. \end{cases}$

Then f is isotopic to identity ($id_{[-1,1]}$) through the map (see Figure 3) :

$$\begin{array}{rcl} \Phi : \quad [-1,1] \times [0,1] & \to & [-1,1] \\ (x,t) & \mapsto & \begin{cases} \left(\frac{t+2}{2-t}\right) x + \left(\frac{2t}{2-t}\right) & \text{if } x \in \left[-1, -\frac{1}{2}t\right] \\ \left(\frac{2-t}{t+2}\right) x + \left(\frac{2t}{t+2}\right) & \text{if } x \in \left[-\frac{1}{2}t, 1\right]. \end{cases} \end{array}$$

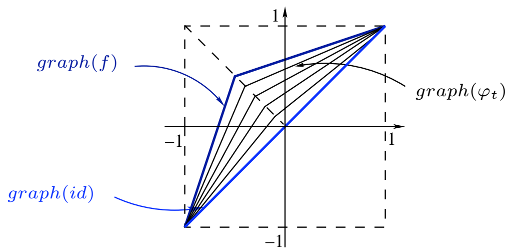


Figure 3: f is isotopic to $id_{[-1,1]}$.

We can now state a geometrical definition of isotopy (often called *ambient isotopy*):

Definition 2. *Let X be a topological space. Two subsets $X_1, X_2 \subset X$ are* (ambient) isotopic *in X if there exists a homeomorphism $h : X \to X$ isotopic to the identity map id_X of X, and such that $h(X_1) = X_2$.*

Geometrically, this means that X_1 and X_2 are isotopic if and only if X_1 can be continuously deformed inside X to obtain X_2, without cutting.

For instance, in Figure 4, surfaces (a) and (b) are isotopic in $\mathbb{R}^3$ (the second one can be continuously deformed to obtain the first one), while (a) and (c) are not isotopic in $\mathbb{R}^3$ (to obtain (a) from (c), we need to cut it).

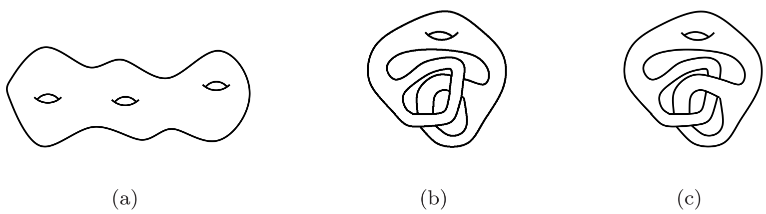


Figure 4: Surfaces (a) and (b) are isotopic in $\mathbb{R}^3$, whereas (a) and (c) are not.

2.4 Embedded tori

Since this article deals with embedded tori, we now quickly recall what is an embedded torus. By *torus*, noted T^2, we mean any space homeomorphic with the product $S^1 \times S^1$. The picture to keep in mind is the surface of a doughnut.

Let Σ be a topological manifold (with or without boundary). An *embedding* of Σ in a topological space E is a map $p : \Sigma \to E$ which is a homeomorphism onto its image. Thus, an *embedded torus* is a topological space which can be written as $p(S^1 \times S^1)$ for some embedding p. Consequently two embedded tori in $\mathbb{R}^3$ are always homeomorphic, but not necessarily isotopic in $\mathbb{R}^3$. For instance, the two embedded tori of Figure 5 are not isotopic. The difference between them comes from their embedding in $\mathbb{R}^3$, hence from the ambiant space.

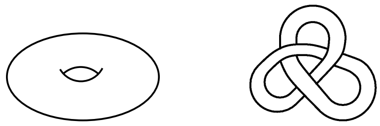


Figure 5: Two non isotopic embedded tori in $\mathbb{R}^3$.

Therefore, to distinguish two embedded tori, we study their complement in $\mathbb{R}^3$.

3 Recognition of tori embedded in $\mathbb{R}^3$

3.1 Embedded tori in $\mathbb{R}^3$ and solid tori

By *solid torus*, denoted by T_p^2, we mean any space homeomorphic with the product $S^1 \times D^2$. This time, the picture to keep in mind is the whole doughnut.

If T_p is a solid torus embedded in $\mathbb{R}^3$, then its boundary is an embedded torus in $\mathbb{R}^3$. Then, the question that comes to mind is whether each torus embedded in $\mathbb{R}^3$ is the boundary of an embedded solid torus in $\mathbb{R}^3$. The following fundamental results will help us to answer this question. These results are stated for tori embedded in S^3, and we will apply them to $\mathbb{R}^3$ through Alexandroff compactification.

Given a torus T embedded in S^3, the Jordan-Brouwer theorem ensures that $S^3 \setminus T$ consists of two open connected components, where the boundary of each component is T. The following result, adapted from [Rol76], provides a sufficient condition for the closure of one of these two components to be a solid torus.

Proposition 1. *Let T be a torus embedded in S^3, C one of the two connected components of $S^3 \setminus T$, and $A = C \cup T = \overline{C}$. Assume that there exists a disc $D \subset A$ such that*

- *$Int(D) \subset C$,*
- *$Bd(D) \subset T$,*
- *$Bd(D)$ is non-trivial in $\pi_1(T)$ (the fundamental group of T).*

Then A is a solid torus.

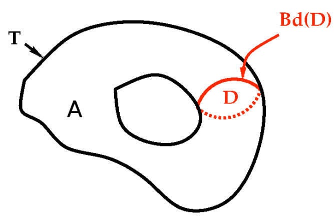


Figure 6: The existence of the disc D ensures that A is a solid torus

Figure 6 provides an illustration of this result.

One of the consequences of this proposition is the following fundamental result. Reader shall refer to [Rol76] for a proof.

Theorem 1. **(Solid torus theorem)** *Let T be a torus embedded in S^3 and C_1, C_2 the two connected components of $S^3 \setminus T$. Then $\overline{C_1}$ or $\overline{C_2}$ is a solid torus.*

In other words, any torus embedded in S^3 bounds a solid torus on at least one side.

This gives rise to our definition of the three states of embedded tori in $\mathbb{R}^3$.

3.2 The three states of tori embedded in $\mathbb{R}^3$

Keeping in mind that our main concern is to recognize tori in 3-dimensional images, we now focus on the case of $\mathbb{R}^3$. Contrary to knot theory, which is unchanged in S^3 and in $\mathbb{R}^3$, the theory of torus embeddings is more complicated in $\mathbb{R}^3$. As S^3 is the Alexandroff extension of $\mathbb{R}^3$, we can use what has already been done in this case.

Definition 3. *Let T be a torus embedded in $\mathbb{R}^3$. Let us consider the Alexandroff extension $S^3 = \mathbb{R}^3 \cup \{p_\infty\}$ of $\mathbb{R}^3$. Let C_1, C_2 the connected components of $S^3 \setminus T$. Assume that $p_\infty \in C_1$. Three cases may happen :*

1. *both $\overline{C_1}$ and $\overline{C_2}$ are solid tori, T is then said* canonical*, or* standard*,*
2. *only $\overline{C_2}$ is a solid torus. : we say that T is a* knotted torus*,*
3. *only $\overline{C_1}$ is a solid torus : T is then called a* knotted anti-torus.

Figure 7 shows these three states of embedded tori. In each case, the component $\overline{C_1}$ is white, while $\overline{C_2}$ is shaded.

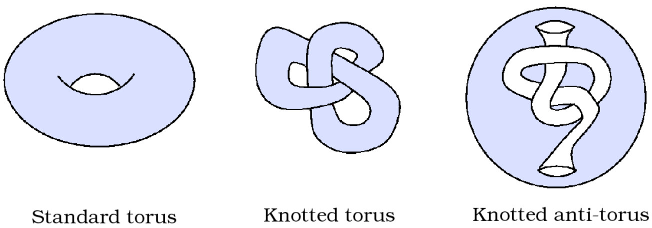


Figure 7: The three states of tori embedded in $\mathbb{R}^3$

We could state a similar definition of the different states of tori embedded in S^3, but there would be only two possibilities, due to the fact that if T is a torus embedded in S^3 then both connected components of $S^3 \setminus T$ are bounded. Let C_1, C_2 be the connected components of $S^3 \setminus T$. Then two cases may occur : either both $\overline{C_1}$ and $\overline{C_2}$ are solid tori (we say that T is *canonical* in S^3), or only one of them is a solid torus (T is then said *standard* in S^3).

Since we want to recognize tori up to isotopy, this definition of the three states of embedded tori in $\mathbb{R}^3$ needs to fit with isotopy, and fortunately it is the case :

Proposition 2. *Two isotopic embedded tori in $\mathbb{R}^3$ have the same state (both are canonical, both are knotted, or both are knotted anti-tori).*

Proof : Let T_1 and T_2 be two isotopic embedded tori in $\mathbb{R}^3$, C_1, C_2 the two connected components of $\mathbb{R}^3 \setminus T_1$ and C_1', C_2' those of $\mathbb{R}^3 \setminus T_2$. Assume that C_1 and C_1' are the bounded ones. Since T_1 and T_2 are isotopic, there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ isotopic to identity and such that $h(T_1) = T_2$.

As C_1 is connected and h is a homeomorphism, $h(C_1)$ is connected. Moreover, $h(C_1) \subset (\mathbb{R}^3 \setminus T_2)$, since h is a bijection such that $h(T_1) = T_2$. Therefore, $h(C_1)$ is included in one of the connected components of $\mathbb{R}^3 \setminus T_2$: $h(C_1) \subset C_1'$ or $h(C_1) \subset C_2'$.

- Assume that $h(C_1) \subset C'_1$. By the same argument, we have $h^{-1}(C'_1) \subset C_1$ or $h^{-1}(C'_1) \subset C_2$. If we had $h^{-1}(C'_1) \subset C_2$, as $h(C_1) \subset C'_1$, it would imply that $C_1 = h^{-1}(h(C_1)) \subset C_2$, which is absurd. Consequently $h^{-1}(C'_1) \subset C_1$, and so $h(C_1) = C'_1$.
- If $h(C_1) \subset C'_2$, the same proof shows that $h(C_1) = C'_2$.

Thus we have $h(C_1) = C'_1$ or $h(C_1) = C'_2$. And yet $\overline{C_1}$ is bounded, hence compact. Thus $h(\overline{C_1})$ is compact too, therefore $h(C_1) \subset h(\overline{C_1})$ is bounded. Consequently $h(C_1) = C'_1$, and in the same way $h(C_2) = C'_2$. Since h is a homeomorphism, this proves that for $i \in \{1, 2\}$, C_i is a solid torus if and only if C'_i is, hence the announced result. □

However, it is obvious that two tori of the same state are not always isotopic. Figure 8 gives an example of two knotted tori that are not isotopic. Therefore knowing the state (canonical, knotted torus or knotted anti-torus) is not sufficient to tell two embedded tori apart, and we will have to add extra information.

Figure 8: Two knotted tori that are not isotopic.

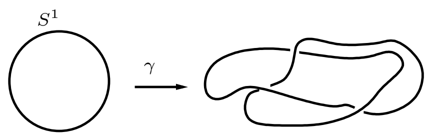


Figure 9: A knot is an embedding of the unit sphere S^1.

3.3 Knot associated with an embedded torus in $\mathbb{R}^3$

In order to distinguish two embedded tori of same state, we are going to associate a knot with each embedded torus. First of all, let us recall what is a knot:

Definition 4. *A* knot *in $\mathbb{R}^3$ (respectively S^3) is a map $\gamma : S^1 \to \mathbb{R}^3$ (respectively $\gamma : S^1 \to S^3$) which is a homeomorphism onto its image.*

In other words, a *knot* is an embedding of the unit sphere S^1 (see Figure 9). We will sometimes refer the image of the map γ as a *knot*.

A knot can be naturally associated with a solid torus, thanks to the following result :

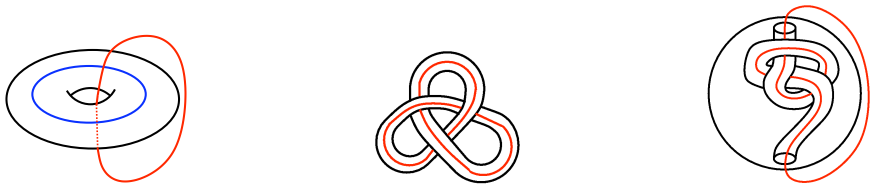
Proposition 3. *Let T_p be a solid torus embedded in S^3. Then T_p admits a deformation retraction to a knot in S^3 (unique up to isotopy).*

This means that a solid torus in S^3 shrinks continuously into a knot. This result enables us to define the knot associated with a torus embedded in $\mathbb{R}^3$.

Definition 5. *Let T be a torus embedded in $\mathbb{R}^3$, $S^3 = \mathbb{R}^3 \cup \{p_\infty\}$ the Alexandroff extension of $\mathbb{R}^3$, and C_1, C_2 the connected components of $S^3 \setminus T$. Assume that $p_\infty \in C_1$.*

1. *If $\overline{C_1}$ and $\overline{C_2}$ are solid tori (i.e. T is standard), then both admit a deformation retraction to the trivial knot, and the* knot associated with *the embedded torus T is the trivial knot (see Figure 10).*
2. *Otherwise, only one of the $\overline{C_i}$ is a solid torus, and the knot to which $\overline{C_i}$ retracts is called* knot associated with *T (see Figure 10).*

Thus, the recognition of a torus T embedded in $\mathbb{R}^3$ consists of two steps : firstly we have to determine its state (standard, knotted torus or knotted anti-torus), and then we need to find and recognize its associated knot. This last step involves knot theory, several algorithms are available (for example, the Jones polynomial is easy to compute, see [Kau91]). Even if they unfortunately can't distinguish all knots, since they are based on knots invariants, they are efficient with common knots. Our main concern is thus the first step and the very begining of the second step of knot recognition : finding the state of an embedded torus and computing its associated knot.



(a) The knot associated with a standard torus in $\mathbb{R}^3$ is the trivial knot.

(b) Knot associated with a knotted torus.

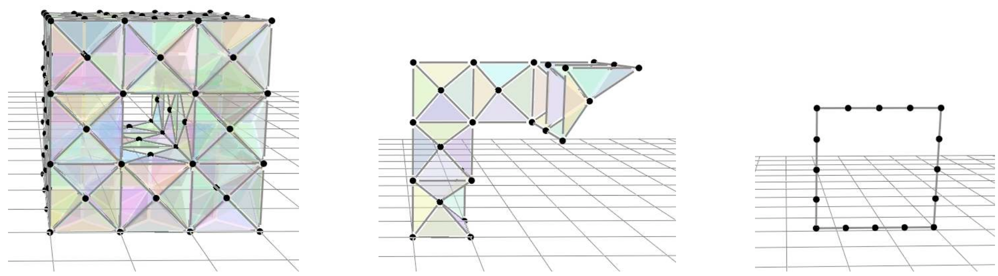
(c) Knot associated with a knotted anti-torus

Figure 10: Knots associated with tori embedded in $\mathbb{R}^3$.

4 Implementation

In this article, the choice was made to focus on theory. The two algorithms that we already have implemented are prototypes intended to show the effectiveness of the theoretical study. Therefore, they are briefly described, without getting down to the specifics (formal description, complexity, memory occupation ...).

The first one aims at finding the state of a torus T embedded in $\mathbb{R}^3$. It is based on an algorithm from [DPF08] which computes the three dimensional minimal generalized map homologous to a 3D object in order to find its homology group generators. We isolated a part of this algorithm and adapted it to simplicial complexes. Given our definition of the state of an embedded torus, determining it amounts to telling if the connected components of $\mathbb{R}^3 \setminus T$ are, or not, solid tori. This algorithm starts with a simplicial complex K whose boundary is T, and may ensure that this complex is a solid torus by finding a disc which satisfies the conditions of Proposition 1. For instance, applied to the simplicial complex of Figure 11 (a), it gives the disc shown in (b). This proves that this simplicial complex is a solid torus. However, even if K is a solid torus, the algorithm may not be able to find such a disc. Thus this algorithm enables us to find the state of some embedded tori, but not of all of them.



(a) A simplicial complex which is a solid torus

(b) The first algorithm applied to the solid torus of (a) gives a disc

(c) The second algorithm applied to the solid torus of (a) returns a knot

Figure 11: The two algorithms applied to an example of solid torus.

The point of the second algorithm is twofold : to determine the state of an embedded torus T and, at the same time, to compute its associated knot. It starts with a simplicial complex K whose boundary is T, and rectracts, shrinks it as much as possible. We have proved that if the result of the algorithm is a knot, then K is a solid torus, and this knot is the one associated with

T. For example, when we apply it to the simplicial complex of Figure 11 (a), we obtain the knot (c). This shows, once again, that the starting simplicial complex is a solid torus. We are now trying to prove that if K is a solid torus, then the algorithm necessarily gives a knot. If this were true, the algorithm would be able to recognize solid tori, thus to determine the state of any embedded torus. In this case, it would allow us to determine both the state of embedded tori and their associated knots.

5 Conclusion and outlook

Our algorithms can determine the state and the knot associated with *some* embedded tori. If we manage to prove that applied to any solid torus, the second one always returns a knot, then we will be able to find the state and the knot associated with *each* embedded torus. Using algorithms from knot theory, we may then recognize this associated knot, and thus obtain a complete description of the embedded torus. There currently does not exist algorithm able to determine all knots, however existing ones are very efficient with common knots.

A future (difficult) task will be to generalize this study to embedded surfaces of genus greater than one.

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