

# Counting excellent discrete Morse functions on compact orientable surfaces

R. Ayala, D. Fernández-Ternero and J.A. Vilches

Dpto. de Geometría y Topología, Universidad de Sevilla, 41080, Sevilla, SPAIN  
rdayala@us.es,desamfer@us.es,vilches@us.es

**Abstract** We obtain the number of non-homologically equivalent excellent discrete Morse functions defined on compact orientable surfaces. This work is a continuation of the study which has been done in [2, 4] for graphs.

**Keywords** compact orientable triangulated surface; critical simplex; discrete gradient vector field.

## 1 Introduction

At the end of the last century, R. Forman [6] developed a discrete version of Morse theory that turned out to be a fruitful and efficient method for the study of the topology of discrete objects, which play a central role in many different fields of pure and applied mathematics. Essentially, a discrete Morse function on a simplicial complex is a way to assign a real number to each simplex, without any continuity hypothesis, in such a way that the natural order given by the dimension of the simplices is respected, except at most in one (co)face of the given simplex. As in the smooth setting, discrete Morse theory allows to establish a link between the topology of the level sets of the function and its critical simplices. This provides a useful tool to deal with shape recognition problems by methods that are grounded in Morse theory: Reeb graphs, persistent homology or Morse Shape descriptors in the sense of [1].

The critical simplices of any discrete Morse function defined on a triangulation  $K$  of a surface are of two kinds: the essential ones which are linked to the Betti numbers of  $|K|$  and the superfluous ones which represent the appearance of a connected component or a 1-cycle which will disappear later. Although the induced gradient vector field by a discrete Morse function codifies the qualitative properties of this function, it does not give us information about the essential (or superfluous) nature of critical simplices. Thus, the gradient field does not allow us to perceive the noise level of the function. While every discrete Morse function defined on any compact connected and orientable surface has a unique critical vertex and a unique critical 2-simplex, which are those simplices where the function reach its global minimum and maximum respectively, we distinguish between essential and superfluous edges by means of the notion of persistence pairs. With this purpose, we associate with each pair  $(K, f)$  three finite sequences of positive integers, called the homological sequences of the function. Their values correspond to the Betti numbers of the natural filtration induced by the critical values of  $f$ . Thus, the homology sequences capture the process of birth and death of homology classes through the filtration. This allows us to compare the evolutions of the level subsets of two discrete Morse functions on a closed orientable surface: they are called homologically equivalent if their homological sequences coincide.

Inspired by the results of Nicolaescu [11] on the number of smooth functions on the 2-sphere, we study the size of the set of discrete Morse functions defined on a compact orientable surface with a given number of critical simplices. The size of this set is equal to the number of different

possibilities of placing the births and deaths for a given number of persistence pairs. We establish links between the homological sequences of such functions and certain reticular walks in  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , whose number is computed. It is worthwhile to mention that the proof is constructive, in the sense that it indicates precisely the procedure to obtain an excellent discrete Morse function from sequences satisfying certain conditions.

## 2 Preliminaries

In this paper we are going to deal with discrete Morse functions defined on compact triangulated surfaces. We introduce here the basic notions of Discrete Morse theory, (see [7] for more details). A **discrete Morse function** on a finite simplicial complex  $K$  is a function  $f : K \rightarrow \mathbb{R}$  such that, for any  $p$ -simplex  $\sigma \in K$ :

$$(M1) \text{ card}\{\tau^{(p+1)} > \sigma/f(\tau) \leq f(\sigma)\} \leq 1.$$

$$(M2) \text{ card}\{v^{(p-1)} < \sigma/f(v) \geq f(\sigma)\} \leq 1.$$

A  $p$ -simplex  $\sigma \in K$  is said to be a **critical simplex** with respect to  $f$  if:

$$(C1) \text{ card}\{\tau^{(p+1)} > \sigma/f(\tau) \leq f(\sigma)\} = 0.$$

$$(C2) \text{ card}\{v^{(p-1)} < \sigma/f(v) \geq f(\sigma)\} = 0.$$

A value of a discrete Morse function on a critical simplex is called **critical value**.

Given  $c \in \mathbb{R}$  the **level subcomplex**  $K(c)$  is the subcomplex of  $K$  consisting of all simplices  $\tau$  with  $f(\tau) \leq c$ , as well as all of their faces, that is,

$$K(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma \leq \tau} \sigma$$

**Theorem 2.1.** [7] *Let  $f$  be a discrete Morse function defined on a finite simplicial 2-complex  $K$  and let  $b_p$  be the  $p$ -th Betti number of  $K$  with  $p = 0, 1, 2$ . Then:*

$$(I1) m_p(f) - m_{p-1}(f) + \dots \pm m_0 \geq b_p - b_{p-1} + \dots \pm b_0,$$

$$(I2) m_p(f) \geq b_p,$$

$$(I3) m_0(f) - m_1(f) + m_2(f) = b_0 - b_1 + b_2 = \chi(K),$$

where  $m_p(f)$  denotes the number of critical  $p$ -simplices of  $f$  on  $K$ .

Given a discrete Morse function defined on a finite simplicial complex  $K$ , we say that a pair of simplices  $(\alpha^{(p)} < \beta^{(p+1)})$  is in the **gradient vector field** induced by  $f$  if and only if  $f(\alpha^{(p)}) \leq f(\beta^{(p+1)})$ .

Given a gradient vector field  $V$  on  $K$ , a  **$V$ -path** is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \dots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}, \dots,$$

such that, for each  $i \geq 0$ , the pair  $(\alpha_i^{(p)} < \beta_i^{(p+1)}) \in V$  and  $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$ .

**Theorem 2.2.** [7] *A discrete vector field  $V$  is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed  $V$ -paths.*

Notice that the numbers of critical  $p$ -simplices of a discrete Morse function  $f$  are determined by its induced gradient vector field  $V_f$ . However, the changes on the topology of their level subcomplexes are not detected by  $V_f$ .

Two discrete Morse functions  $f$  and  $g$  defined on a simplicial complex  $K$  are **equivalent** if every pair of simplices  $\alpha^{(p)}$  and  $\beta^{(p+1)}$  in  $K$  such that  $\alpha^{(p)} < \beta^{(p+1)}$  verify that

$$f(\alpha) < f(\beta) \quad \text{if and only if} \quad g(\alpha) < g(\beta)$$

The next result states that any two equivalent discrete Morse functions have the same gradient vector field and conversely.

**Theorem 2.3.** [3] *Two discrete Morse functions  $f$  and  $g$  defined on a simplicial complex  $K$  are equivalent if and only if  $f$  and  $g$  induce the same gradient vector field.*

### 3 Excellent discrete Morse functions on a compact orientable surface

As we have just seen in Section 2, the changes on the topology of the level subcomplexes of a simplicial complex are lost if we deal with gradient vector fields. Thus, it does not seem reasonable for our purposes to consider two discrete Morse functions defined on a simplicial complex as indistinguishable if they are equivalent, that is, if they induce the same gradient vector field. For this reason, it is convenient to deal with functions whose critical values are different and so, we assume that two such functions are “equal” if their level subcomplexes have the same homology.

**Definition 3.1.** A discrete Morse function is called **excellent** if all its critical values are different.

**Definition 3.2.** Two excellent discrete Morse functions  $f$  and  $g$  defined on a finite simplicial complex  $K$  with critical values  $a_0 < a_1 < \dots < a_{m-1}$  and  $c_0 < c_1 < \dots < c_{m-1}$  respectively will be called **homologically equivalent** if for all  $i = 0, \dots, m-1$  the level subcomplexes  $K(a_i)$  and  $K(c_i)$  have the same Betti numbers.

**Definition 3.3.** Let  $f$  be an excellent discrete Morse function defined on a connected compact triangulated surface  $K$  with  $m$  critical simplices and critical values  $a_0, a_1, \dots, a_{m-1}$ . We denote the level subcomplexes  $K(a_i)$  by  $K_i$  for all  $i = 0, 1, \dots, m-1$ . The **homological sequences** of  $f$  are the three sequences  $B_0, B_1, B_2 : \{0, 1, \dots, m-1\} \rightarrow \mathbb{N}$  containing the homological information of the level subcomplexes  $K_0, K_1, \dots, K_{m-1}$ , that is,  $B_p(i) = b_p(K_i) = \dim(H_p(K_i))$  for each  $i = 0, 1, \dots, m-1$  and  $p = 0, 1, 2$ .

*Remark 3.4.* The homological sequences of  $f$  satisfy:

1.  $|B_p(i+1) - B_p(i)| = 0$  or  $1$ , for each  $i = 0, 1, \dots, m-2$  and  $p = 0, 1, 2$ ;
2.  $B_0(0) = B_0(m-2) = B_0(m-1) = b_0 = 1$ ,  $B_0(i) > 0$ ;
3.  $B_1(0) = 0$ ,  $B_1(m-2) = B_1(m-1) = b_1$ ;
4.  $B_2(i) = 0$ , for each  $i = 0, 1, \dots, m-2$ ,  $B_2(m-1) = 1$ .

**Lemma 3.5.** *For each  $i = 0, 1, \dots, m-2$  it holds one and only one of the following identities:*

$$(H0^+) \quad B_0(i+1) = B_0(i) + 1.$$

$$(H0^-) \quad B_0(i+1) = B_0(i) - 1.$$

$$(H1^+) \quad B_1(i+1) = B_1(i) + 1.$$

$$(H1^-) \quad B_1(i+1) = B_1(i) - 1.$$

It is interesting to point out that identities  $(H0^+)$  and  $(H0^-)$  of the above Lemma reveal the appearance of a new connected component and the join of two connected components, respectively, in the process of obtention of  $K$  by level subcomplexes. Analogously, identities  $(H1^+)$  and  $(H1^-)$  reveal the creation of a new 1-cycle of  $K$  and the destruction of a 1-cycle by gluing a triangle in this process, respectively.

Notice that two excellent discrete Morse functions are homologically equivalent if and only if their homological sequences are the same.

*Remark 3.6.* We can regard the sequence of the points  $(B_0(i) - 1, B_1(i))$ ,  $i = 0, 1, \dots, m - 2$  in the plane as a walk on  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  of  $m - 2$  steps, each in a direction North, South, East or West, from  $(0, 0)$  to  $(0, b_1)$  (see Figure 1). For example, the walk in Figure 1 corresponds to a excellent discrete Morse function defined on the torus with the following homological sequences:

$$\begin{aligned} B_0 &: 1, 2, 2, 3, 4, 4, 4, 5, 6, 6, 5, 4, 3, 2, 2, 1, 1, 1 \\ B_1 &: 0, 0, 1, 1, 1, 2, 3, 3, 3, 2, 2, 2, 2, 2, 3, 3, 2, 2 \\ B_2 &: 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \end{aligned}$$

The number  $\omega'_{m-2}(0, b_1)$  of such different walks was obtained in [8].

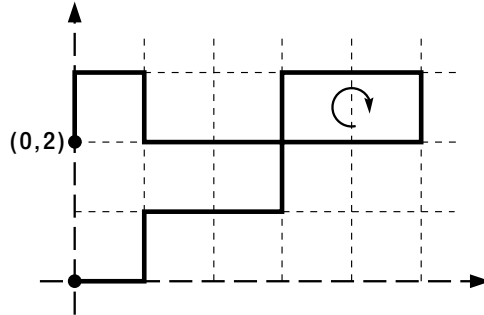


Figure 1: A walk of 16 steps from  $(0, 0)$  to  $(0, 2)$ .

## 4 Persistent Morse Homology and the essential and superfluous critical simplices

Now we shall consider two different kinds of critical simplices: the essential critical simplices are those critical simplices which arise forced by the topology of the considered surface and the non-essential critical simplices are the remaining critical simplices, that is, those critical simplices which produce temporary changes in the topology of the level subcomplexes during the process of building the surface given by the excellent discrete Morse function defined on it.

**Definition 4.1.** Let  $f$  be an excellent discrete Morse function defined on a connected compact triangulated surface  $K$  with critical values  $a_0 < \dots < a_{m-1}$ . We say that a critical vertex  $v$  is an **essential vertex** if  $f(v)$  is the global minimum of  $f$  on  $K$ , that is,  $f(v) = a_0$ . We say that a critical triangle  $T$  is an **essential triangle** if  $f(T)$  is the global maximum of  $f$  on  $K$ , that is,  $f(T) = a_{m-1}$ . Otherwise, if a critical vertex or triangle is not an essential one, we say that it is a **superfluous or cancellable** vertex or triangle.

Notice that it is straightforward to prove that a critical vertex or triangle is essential if and only if it represents the basic element of  $H_0(K)$  or  $H_2(K)$ , respectively. But it is not so simple introduce the concepts of cancellable and essential critical edges. If we only take into account

the critical values, it is not possible to distinguish those critical edges which complete a 1-cycle representing a basic element of  $H_1(K)$  not considered until this point. We will use the notion of homological persistence to detect the changes of the homology groups in the sequence of the level subcomplexes and so, we will be able to determine if a critical edge is essential or not.

Let  $f$  be an excellent discrete Morse function defined on a connected compact triangulated surface  $K$  with  $m$  critical simplices and critical values  $a_0, a_1, \dots, a_{m-1}$ . Let  $K_i, i = 0, 1, \dots, m-1$  be the level subcomplexes of  $K$  by  $f$ . For  $0 \leq p < q \leq m-1$ , let  $i_*^{p,q} : H_*(K_p) \rightarrow H_*(K_q)$  denote the homomorphism induced by inclusion  $K_p \subset K_q$ .

**Definition 4.2.** Let  $\sigma$  be a critical simplex such that  $f(\sigma) = a_i$ . The **predecessor** of  $\sigma$  is the critical simplex  $\sigma_-$  such that  $f(\sigma_-) = a_{i-1}$ .

Let  $\sigma$  and  $\tau$  be critical simplices of dimension  $d$  and  $d+1$ , respectively, such that  $f(\sigma) = a_p < f(\tau) = a_q$  (since  $K$  is the dimension 2, it is only possible that  $d = 0$  or 1). If we consider the predecessors of  $\sigma$  and  $\tau$  we obtain the sequence

$$H_d(K_{p-1}) \xrightarrow{i_d^{p-1,p}} H_d(K_p) \xrightarrow{i_d^{p,q-1}} H_d(K_{q-1}) \xrightarrow{i_d^{q-1,q}} H_d(K_q)$$

induced by the inclusions  $K_{p-1} \subset K_p \subseteq K_{q-1} \subset K_q$ .

**Definition 4.3.** We say that a equivalence class  $h \in H_d(K_p)$  is **born at**  $\sigma$  if  $h \notin \text{Im} \left( i_d^{p-1,p} \right)$ . Moreover, we say that a equivalence class  $h \in H_d(K_p)$  that is born at  $\sigma$  **dies entering**  $\tau$  if  $i_d^{p,q}(h) \in \text{Im} \left( i_d^{p-1,q} \right)$  and  $i_d^{p,q-1}(h) \notin \text{Im} \left( i_d^{p-1,q-1} \right)$ .

**Definition 4.4.** If there exists a equivalence class  $h$  that is born at  $\sigma$  and dies entering  $\tau$ , then  $(\sigma, \tau)$  is called a **persistence pair**.

Notice that the essential vertex  $v$  is not in any persistence pair. It can be justified since the equivalence class that is born in  $v$  never dies. Analogously we can prove that an essential critical triangle is not in any persistence pair. We shall use this idea in the next definition.

**Definition 4.5.** Let  $f$  be an excellent discrete Morse function defined on a connected compact triangulated surface  $K$ . We say that a critical edge  $e$  is an **superfluous or cancellable edge** if there exists a cancellable vertex or a cancellable triangle such that, together with  $e$ , give rise to a persistence pair. Otherwise, if a critical edge is not a cancellable one, we say that it is an **essential edge**.

Notice that it is easy to prove that a critical edge  $e$  is essential if and only if  $e$  is completing a 1-cycle which represents a basic element of  $H_1(K)$  not considered until this point.

## 5 Counting excellent discrete Morse functions on a compact orientable surface

We present one result which computes the number of all non-homologically equivalent excellent discrete Morse functions with a given number of critical elements defined on a connected compact orientable triangulated surface.

**Theorem 5.1.** *The number of homology equivalence classes of excellent discrete Morse functions with  $m = b_0 + b_1 + b_2 + 2k$  critical elements on a connected compact orientable surface  $M$  is:*

$$\omega'_{m-2}(0, b_1) = \binom{m-2}{k} \binom{m}{k} - \binom{m}{k+1} \binom{m-2}{k-1}.$$

*Sketch of proof.* By Remark 3.6, the number of homology equivalence classes of excellent discrete Morse functions with  $m$  critical simplices is less than or equal to  $\omega'_{m-2}(0, b_1)$ .

Now, we are going to prove the equality. Let  $M$  be a connected compact orientable surface and let  $B_0, B_1$  and  $B_2$  be sequences satisfying Remark 3.4 and Lemma 3.5. We obtain an excellent discrete Morse function  $f$  on  $M$  with these homological sequences as follows: First, we consider a particular triangulation of  $M$  and we choose the  $m$  simplices which will be the critical elements of the Morse function. Next, we construct a discrete vector field  $V$  with no non-trivial closed  $V$ -paths and not containing the simplices chosen. Finally we obtain an excellent discrete Morse function  $f$  on  $M$  whose homological sequences are the given ones and such that the gradient vector field induced by  $f$  is  $V$ .  $\square$

*Remark 5.2.* For  $S^2$  ( $b_1 = 0$ ) this calculation was done in the differentiable setting in [11] obtaining the same number of these equivalence classes:

$$\omega'_{m-2}(0, 0) = C_m C_{m+1}$$

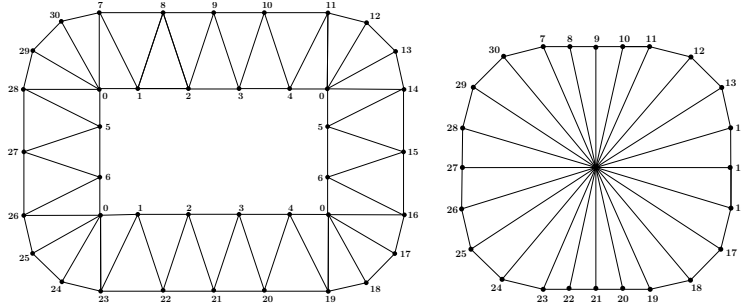
where  $C_k = \frac{1}{k+1} \binom{2k}{k}$  denotes the  $k$ -th Catalan number.

In the following example we show some of the constructions carried out in the the proof of the above theorem:

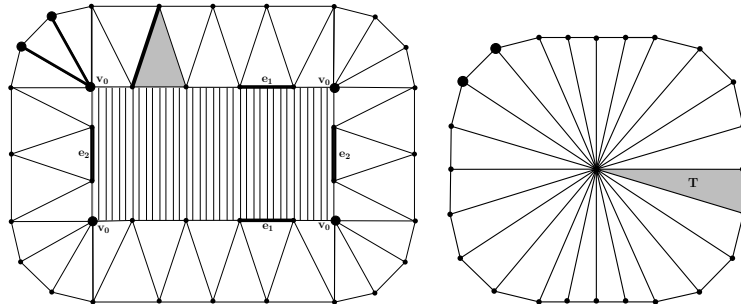
**Example 5.3.** Let us define an excellent discrete Morse function on the torus with  $m = 10$  critical elements and whose homological sequences are:

$$\begin{aligned} B_0 &: 1, 2, 2, 3, 3, 3, 2, 2, 1, 1 \\ B_1 &: 0, 0, 1, 1, 2, 3, 3, 2, 2, 2 \\ B_2 &: 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \end{aligned}$$

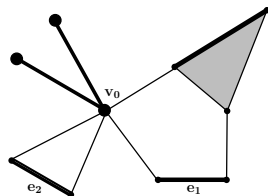
First, we consider a particular triangulation of the torus: we take two triangulated discs, one of these with a hole where we realize the usual identifications, and we glue both discs by their boundaries.



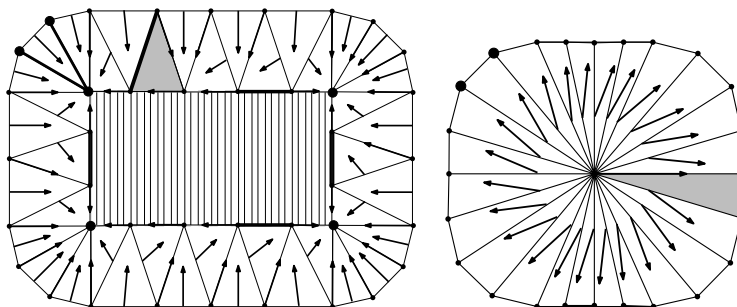
Notice that we triangulate the torus to have enough simplices to choose in a convenient way the  $m$  simplices which will be the critical elements of the Morse function. We select a triangle  $T$  in the right disc and one vertex  $v_0$  and two edges  $e_1$  and  $e_2$  in the internal boundary of the left disc to will be the essential critical elements. We take the remaining selected simplices in the left disc as it can be seen in the pictures below (the triangles in gray and the vertices and edges in dark):



Next, we take into account the collapses  $K - \{T\} \searrow L$  and  $L \searrow L - S$  to construct a discrete vector field  $V$  with no non-trivial closed  $V$ -paths and not containing the simplices chosen, where  $S$  is the set of the superfluous simplices and  $L$  is the subcomplex in the following picture:

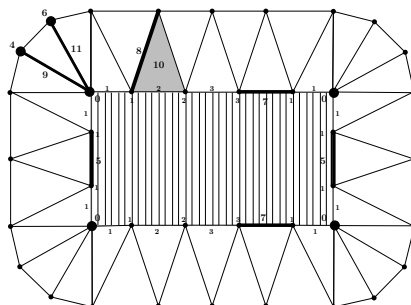


The discrete vector field  $V$  is denoted by arrows in the following pictures:

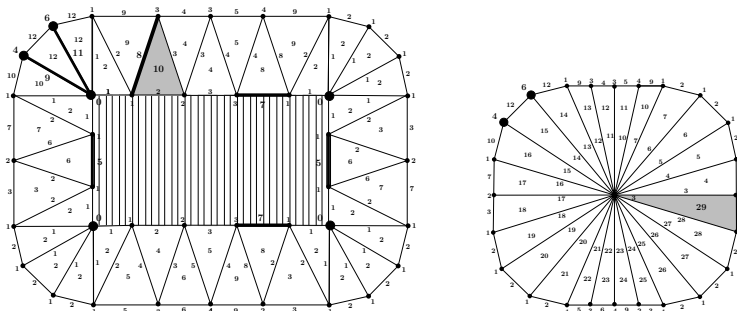


In several steps we define an excellent discrete Morse function on the subcomplex  $L$  with  $m - 1 = 9$  critical elements and whose homological sequences are:

$$\begin{aligned} B_0 &: 1, 2, 2, 3, 3, 3, 2, 2, 1 \\ B_1 &: 0, 0, 1, 1, 2, 3, 3, 2, 2 \\ B_2 &: 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{aligned}$$



Next, we can extend this function to  $K - \{T\}$  by the discrete vector field  $V$ . Finally, we define  $f$  on the triangle  $T$  as the greatest critical value of  $f$ .



As we can see in the last picture, the excellent Morse function  $f$  has the given homological sequences.

*Remark 5.4.* The procedure to obtain an excellent discrete Morse function from sequences that will be their homological sequences is linear. We consider a triangulation of the surface whose number of simplices is a linear function of  $m$ . From such triangulation we consider particular collapses and by using them, we get the discrete vector field  $V$  in linear time. Finally, each simplex is visited once when we define the discrete Morse function.

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