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REGULARITY OF PULLBACK ATTRACTORS AND ATTRACTION IN H^1 IN ARBITRARILY LARGE FINITE INTERVALS FOR 2D NAVIER-STOKES EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper we strengthen some results on the existence and properties of pullback attractors for a non-autonomous 2D Navier-Stokes model with infinite delay. Actually we prove that under suitable assumptions, and thanks to regularity results, the attraction also happens in the H^1 norm for arbitrarily large finite intervals of time. Indeed, from comparison results of attractors we establish that all these families of attractors are in fact the same object. The tempered character of these families in H^1 is also analyzed.

1. **Introduction and statement of the problem.** The appearance of delay effects in partial differential equations that model fluid flows has been intensively treated during the last few decades. For instance, this type of effects are considered in the constitutive equations of the "finite-linear" theory of viscoelasticity when the movement is close to steady states, in models of simple materials with a perturbation of the Newtonian part with a viscoelastic part given by a functional of the history of the displacement gradient, applied to the study of polymeric liquids, K-BKZ theory in analogy to hyperelasticity, Curtiss-Bird fluids, Jeffreys flows, etcetera (e.g. cf. [25, 26, 13, 20, 21, 22, 23, 10, 9, 19] and the references therein).

Therefore, the long-time behaviour of these problems is a meaningful task: stability of equilibria, bifurcations, and attractors among many other questions.

Besides the above, in many physical experiments, the inclusion of measurement devices may incorporate additional external forces to the model including also delay effects (see e.g. [15] for a wind tunnel experiment).

In this context, we should mention a sequence of papers introduced by Caraballo and Real (cf. [2, 3, 4]) where Navier-Stokes models including external force terms with finite delay were treated. Namely, under suitable assumptions they obtained existence and uniqueness of solutions, global exponential decay to the stationary solution, and finally existence of attractors.

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Joseé Real died on January 27th, 2012. J.G.-L. and P.M.-R. would like to dedicate this paper to his memory.

We are interested in a non-autonomous Navier-Stokes model which was introduced in [18] and that includes force terms that incorporate infinite-delay effects.

Our aim is to strengthen the results of that paper, studying, among other questions, the asymptotic behaviour of solutions (namely, the existence of pullback attractors) and their regularity properties.

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set with smooth enough boundary $\partial\Omega$, and consider the following functional Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f(t) + g(t, u_t) \quad \text{in } \ \Omega \times (\tau, \infty), \\ \text{div } u \ = 0 \quad \text{in } \ \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \ \partial\Omega \times (\tau, \infty), \\ u(x, \tau + s) = \phi(x, s), \quad x \in \Omega, \ s \in (-\infty, 0], \end{cases} \tag{1}$$

where we assume that $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, $\tau \in \mathbb{R}$ is a given initial time, f is a non-delayed external force field, g is another external force containing some hereditary characteristics, $\phi(x, s - \tau)$ is the initial datum in the interval of time $(-\infty, \tau]$, and for each $t \geq \tau$, we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$.

The structure of the paper is as follows. In the rest of this section, we establish some functional spaces to state the problem in an abstract form, basic properties and estimates of the involved operators, and the notions of weak and strong solutions. In Section 2 we present some existence and uniqueness results, which improve some of the obtained previously in [18], some additional estimates on these solutions, and continuity properties. Section 3 is devoted to recalling briefly some abstract results on non-autonomous dynamical systems and the existence of minimal pullback attractors for a given universe (a class of families of time-depending sets with certain tempered conditions), and relations between several families of these objects. Finally, in Section 4 we establish our main results, which, roughly speaking, show attraction in a higher norm and prove the relationship among all these attractors.

To start with, we consider the following usual function spaces. Let

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \text{div } u \ = 0 \right\},$$

and let H be the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot, \cdot)$, where for u, $v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) \, dx.$$

Also, V will be the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot, \cdot))$, where for u, $v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality between V' and V. We consider every element $h \in H$ as an element of V', given by the equality

$\langle h, v\rangle = (h, v)$ for all $v \in V$. Then, it follows that $V \subset H \subset V'$, where the injections are dense and continuous, and, in fact, compact.

Define the operator $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$ for all $u,\ v \in V$. Let us denote $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent, and $D(A)$ is compactly and densely injected in V.

Let us define

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for all functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b makes sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$.

Some useful properties concerning b that we will use throughout the paper are the following (see [24] or [28]): there exists a constant $C_1 > 0$, depending only on Ω, such that (recall that we are in dimension two)

$$|b(u, v, w)| \le C_1 |u|^{1/2} |Au|^{1/2} \|v\| |w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H,$$

and

$$|b(u, v, w)| \le C_1 |Au| \|v\| |w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H.$$

There are several phase spaces which allow us to deal with infinite delays (cf. [11, 12]). For instance, for a given $\gamma > 0$, we may consider the space

$$C_\gamma(H) = \left\{\varphi \in C((-\infty, 0]; H) \ : \ \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in H\right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\gamma = \sup_{s\in(-\infty,0]} e^{\gamma s} |\varphi(s)|.$$

We will use the above space, and for the term g, in which the delay is present, we assume that $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^2$ satisfies

($g1$) For any $\xi \in C_\gamma(H)$, the mapping $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is measurable.

($g2$) $g(\cdot, 0) = 0$.

($g3$) There exists a constant $L_g > 0$ such that for any $t \in \mathbb{R}$ and all $\xi,\ \eta \in C_\gamma(H)$,

$$|g(t, \xi) - g(t, \eta)| \le L_g \|\xi - \eta\|_\gamma.$$

An example of an operator satisfying assumption ($g3$) was given in [18].

We assume that $f \in L^2_{loc}(\mathbb{R}; V')$ and $\phi \in C_\gamma(H)$ with $\gamma > 0$, and we define what we understand by a weak solution to (1).

Definition 1.1. A weak solution to (1) is a function $u \in C((-\infty, T]; H) \cap L^2(\tau, T; V)$ for all $T > \tau$, with $u_\tau = \phi$, and such that for all $v \in V$,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = \langle f(t), v\rangle + (g(t, u_t), v), \qquad (2)$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Remark 1. If u is a weak solution to (1), then u satisfies the energy equality

$$|u(t)|^2+2\nu\int_s^t \|u(r)\|^2dr = |u(s)|^2+2\int_s^t \big[\langle f(r),u(r)\rangle+(g(r,u_r),u(r))\big]dr\ \forall\,\tau\le s\le t.$$

A notion of more regular solution is also suitable for problem (1).

Definition 1.2. A strong solution to (1) is a weak solution u to (1) such that $u\in L^2(\tau,T;D(A))\cap L^\infty(\tau,T;V)$ for all $T>\tau$.

Remark 2. If $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, and u is a strong solution to (1), then $u'\in L^2(\tau,T;H)$ for all $T>\tau$, and so $u\in C([\tau,\infty);V)$. In this case the following second energy equality holds:

$$\begin{aligned}&\|u(t)\|^2+2\nu\int_s^t|Au(r)|^2\,dr+2\int_s^t b(u(r),u(r),Au(r))\,dr\\ =\ &\|u(s)\|^2+2\int_s^t(f(r)+g(r,u_r),Au(r))\,dr\quad\forall\,\tau\le s\le t.\end{aligned}\tag{3}$$

2. **Existence of solutions and related properties.** In this section we generalize some results from [18] (see also [16]). Namely, we establish existence of weak and strong solutions for problem (1) and some related properties when $u_\tau\in C_\gamma(H)$ and additional assumptions are satisfied.

Let us denote by $\lambda_1=\min_{v\in V\setminus\{0\}}\|v\|^2/|v|^2>0$ the first eigenvalue of the Stokes operator A.

Theorem 2.1. *Assume that $f\in L^2_{loc}(\mathbb{R};V')$, $\gamma>0$, and $g:\mathbb{R}\times C_\gamma(H)\to(L^2(\Omega))^2$ satisfying the assumptions (g1)–(g3), are given. Then, for any $\tau\in\mathbb{R}$ and $\phi\in C_\gamma(H)$, there exists a unique weak solution $u=u(\cdot;\tau,\phi)$ to (1), and the following estimates hold for all $t\ge\tau$, and any $\mu\in(0,\nu)$ such that $(\nu-\mu)\lambda_1\le\gamma$:*

$$\|u_t\|_\gamma^2\le e^{-2((\nu-\mu)\lambda_1-L_g)(t-\tau)}\|\phi\|_\gamma^2+\mu^{-1}\int_\tau^t e^{-2((\nu-\mu)\lambda_1-L_g)(t-s)}\|f(s)\|_*^2\,ds,\tag{4}$$

$$\mu\int_\tau^t\|u(s)\|^2\,ds\le e^{2L_g(t-\tau)}\|\phi\|_\gamma^2+\mu^{-1}e^{2L_gt-2(\nu-\mu)\lambda_1\tau}\int_\tau^t e^{2((\nu-\mu)\lambda_1-L_g)s}\|f(s)\|_*^2\,ds.\tag{5}$$

Moreover, if $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, it holds that u is a strong solution in the sense that $u\in C([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;D(A))$ for all $\varepsilon>0$ and any $T>\tau+\varepsilon$. If besides $u(\tau)\in V$, then u is properly a strong solution, i.e. $u\in C([\tau,T];V)\cap L^2(\tau,T;D(A))$ for all $T>\tau$.

Proof. The existence and uniqueness of weak solution was stated in [18, Theorem 5]. There, for the existence of a solution, the additional assumption $2\gamma>\nu\lambda_1$ was made. The fact that this assumption is unnecessary can be seen as follows.

Denote by $\{v_j\}\subset V$ the Hilbert basis of H of all the normalized eigenfunctions of the Stokes operator A.

Consider the Galerkin approximations $u^m(t)=\sum_{j=1}^m\alpha_{m,j}(t)v_j$, which are the solutions of the system

$$\left\{\begin{array}{l}\dfrac{d}{dt}(u^m(t),v_j)+\nu((u^m(t),v_j))+b(u^m(t),u^m(t),v_j)\\ =\langle f(t),v_j\rangle+(g(t,u^m_t),v_j),\quad\text{in }\mathcal{D}'(\tau,\infty),\quad 1\le j\le m,\\ u^m(\tau+s)=\sum_{j=1}^m(\phi(s),v_j)v_j\quad\text{for }s\in(-\infty,0].\end{array}\right.\tag{6}$$

Multiplying the $j-$equation in (6) by $\alpha_{m,j}(t)$, and summing from $j=1$ to $j=m$, one has

$$\begin{aligned}\frac{d}{dt}|u^m(t)|^2+2\nu\|u^m(t)\|^2 &= 2\langle f(t),u^m(t)\rangle+2(g(t,u^m_t),u^m(t))\\ &\leq \nu\|u^m(t)\|^2+\nu^{-1}\|f(t)\|_*^2+2L_g\|u^m_t\|_\gamma^2, \quad \text{a.e. } t>\tau,\end{aligned}$$

and therefore,

$$|u^m(t)|^2+\nu\int_\tau^t\|u^m(s)\|^2\,ds\leq|u(\tau)|^2+\int_\tau^t\left(\|f(s)\|_*^2/\nu+2L_g\|u^m_s\|_\gamma^2\right)ds \quad \forall\, t\geq\tau. \tag{7}$$

Thus,

$$\begin{aligned}\|u^m_t\|_\gamma^2 &\leq \max\Bigg\{\sup_{\theta\in(-\infty,\tau-t]}e^{2\gamma\theta}|\phi(t+\theta-\tau)|^2,\\ &\qquad \sup_{\theta\in[\tau-t,0]}\left(e^{2\gamma\theta}|u(\tau)|^2+e^{2\gamma\theta}\int_\tau^{t+\theta}\left(\|f(s)\|_*^2/\nu+2L_g\|u^m_s\|_\gamma^2\right)ds\right)\Bigg\}\\ &\leq \max\Bigg\{\sup_{\theta\in(-\infty,\tau-t]}e^{2\gamma\theta}|\phi(t+\theta-\tau)|^2,\\ &\qquad |u(\tau)|^2+\int_\tau^t\left(\|f(s)\|_*^2/\nu+2L_g\|u^m_s\|_\gamma^2\right)ds\Bigg\} \quad \forall\, t\geq\tau,\end{aligned} \tag{8}$$

and therefore, observing that

$$\begin{aligned}\sup_{\theta\in(-\infty,\tau-t]}e^{\gamma\theta}|\phi(t+\theta-\tau)| &= \sup_{\theta\leq 0}e^{\gamma(\theta-(t-\tau))}|\phi(\theta)|\\ &= e^{-\gamma(t-\tau)}\|\phi\|_\gamma\\ &\leq \|\phi\|_\gamma,\end{aligned}$$

and $|u(\tau)|=|\phi(0)|\leq\|\phi\|_\gamma$, we deduce from (8) that

$$\|u^m_t\|_\gamma^2\leq\|\phi\|_\gamma^2+\int_\tau^t\left(\|f(s)\|_*^2/\nu+2L_g\|u^m_s\|_\gamma^2\right)ds \quad \forall\, t\geq\tau.$$

Thus, by Gronwall's lemma, we have

$$\|u^m_t\|_\gamma^2\leq e^{2L_g(t-\tau)}\left(\|\phi\|_\gamma^2+\nu^{-1}\int_\tau^t\|f(s)\|_*^2\,ds\right) \quad \forall\, t\geq\tau.$$

Using this inequality and (7), one also obtains that there exists a constant C, depending on some constants of the problem (namely, ν, L_g and f), and on τ, T and $R>0$, such that

$$\|u^m_t\|_\gamma^2\leq C(\tau,T,R) \quad \forall t\in[\tau,T],\ \|\phi\|_\gamma\leq R,\ \forall m\geq 1,$$
$$\|u^m\|_{L^2(\tau,T;V)}^2\leq C(\tau,T,R) \quad \forall m.$$

Now, the proof of the existence of weak solution follows as in [18].

Estimates (4) and (5) were proved in [18, Lemma 17] for the particular case $\mu=\nu/2$. For the general case, the proof is as follows.

Take μ such that $0<\mu<\nu$. By the energy equality, one has

$$\begin{aligned}\frac{d}{dt}|u(t)|^2+2\nu\|u(t)\|^2 &= 2\langle f(t),u(t)\rangle+2(g(t,u_t),u(t))\\ &\leq 2\|f(t)\|_*\|u(t)\|+2L_g\|u_t\|_\gamma|u(t)|\\ &\leq \mu\|u(t)\|^2+\mu^{-1}\|f(t)\|_*^2+2L_g\|u_t\|_\gamma^2, \quad \text{a.e. } t>\tau.\end{aligned}$$

Thus,

$$\frac{d}{dt}|u(t)|^2 + 2(\nu-\mu)\lambda_1|u(t)|^2 + \mu\|u(t)\|^2 \le \mu^{-1}\|f(t)\|_*^2 + 2L_g\|u_t\|_\gamma^2, \quad \text{a.e. } t > \tau,$$

and therefore,

$$\begin{aligned} &|u(t)|^2 + \mu\int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}\|u(s)\|^2\,ds \\ &\le e^{-2(\nu-\mu)\lambda_1(t-\tau)}|u(\tau)|^2 + \int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}\big(\|f(s)\|_*^2/\mu + 2L_g\|u_s\|_\gamma^2\big)ds \ \forall\, t \ge \tau. \end{aligned} \tag{9}$$

Consequently,

$$\begin{aligned} \|u_t\|_\gamma^2 \le \max\Bigg\{ &\sup_{\theta\in(-\infty,\tau-t]} e^{2\gamma\theta}|\phi(t+\theta-\tau)|^2, \\ &\sup_{\theta\in[\tau-t,0]}\bigg(e^{2\gamma\theta-2(\nu-\mu)\lambda_1(t+\theta-\tau)}|u(\tau)|^2 \\ &+e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-2(\nu-\mu)\lambda_1(t+\theta-s)}\big(\|f(s)\|_*^2/\mu + 2L_g\|u_s\|_\gamma^2\big)\,ds\bigg)\Bigg\} \quad \forall\, t \ge \tau. \end{aligned}$$

Let us assume that moreover μ satisfies $(\nu-\mu)\lambda_1 \le \gamma$.

On the one hand,

$$\begin{aligned} \sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}|\phi(t+\theta-\tau)| &= \sup_{\theta\le 0} e^{\gamma(\theta-(t-\tau))}|\phi(\theta)| \\ &= e^{-\gamma(t-\tau)}\|\phi\|_\gamma \\ &\le e^{-(\nu-\mu)\lambda_1(t-\tau)}\|\phi\|_\gamma. \end{aligned}$$

On the other hand,

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta-2(\nu-\mu)\lambda_1(t+\theta-\tau)}|u(\tau)|^2 \le e^{-2(\nu-\mu)\lambda_1(t-\tau)}|u(\tau)|^2$$

and

$$\begin{aligned} &\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-2(\nu-\mu)\lambda_1(t+\theta-s)}\big(\|f(s)\|_*^2/\mu + 2L_g\|u_s\|_\gamma^2\big)\,ds \\ \le\ &\int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}\big(\|f(s)\|_*^2/\mu + 2L_g\|u_s\|_\gamma^2\big)\,ds. \end{aligned}$$

Collecting these inequalities we deduce

$$\|u_t\|_\gamma^2 \le e^{-2(\nu-\mu)\lambda_1(t-\tau)}\|\phi\|_\gamma^2 + \int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}\big(\|f(s)\|_*^2/\mu + 2L_g\|u_s\|_\gamma^2\big)\,ds \quad \forall\, t \ge \tau.$$

Then, by Gronwall's lemma we conclude that (4) holds.

Now, from (9), (4), and Fubini's theorem, we conclude (5).

The final part of the theorem is a consequence of well-known regularity results, taking into account the fact that if $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, then the function $\hat f$ defined by $\hat f(t) = f(t) + g(t,u_t)$, $t > \tau$, belongs to $L^2(\tau,T;(L^2(\Omega))^2)$ for all $T > \tau$. □

Remark 3. It must be observed that estimate (4) also holds for the Galerkin approximations u^m, and that, among others (see [18, Theorem 5]), the following convergences hold for any $T > \tau$:

$$\begin{cases} u^m \to u & \text{strongly in } C([\tau,T];H), \\ u^m \rightharpoonup u & \text{weakly in } L^2(\tau,T;V), \\ (u^m)' \rightharpoonup u' & \text{weakly in } L^2(\tau,T;V'), \\ u_t^m \to u_t & \text{strongly in } C_\gamma(H)\ \forall\, t \in [\tau,T]. \end{cases}$$

Let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \le t\}$.

Definition 2.2. A process U on a metric space (X, d_X) is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau) = \mathrm{Id}_X$ for all $\tau \in \mathbb{R}$, and the following concatenation property holds: $U(t,r)U(r,\tau) = U(t,\tau)$ for any $\tau \le r \le t$.

From Theorem 2.1 we deduce that we may define a family of processes or dynamical systems associated to problem (1) (one of them was already introduced in [18, Proposition 16]).

For any $h \ge 0$, let us denote by

$$C_\gamma^{h,V}(H) = \{\varphi \in C_\gamma(H) : \varphi|_{[-h,0]} \in B([-h,0];V)\}, \tag{10}$$

where $B([-h,0];V)$ is the space of bounded functions from $[-h,0]$ into V. The space $C_\gamma^{h,V}(H)$ is a Banach space with the norm

$$\|\varphi\|_{\gamma,h,V} = \|\varphi\|_\gamma + \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|.$$

Corollary 1. *Assume that $f \in L^2_{loc}(\mathbb{R};V')$, $\gamma > 0$, and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^2$ satisfying assumptions $(g1)$–$(g3)$, are given. Then, the bi-parametric family of mappings $U(t,\tau) : C_\gamma(H) \to C_\gamma(H)$, with $t \ge \tau$, defined by*

$$U(t,\tau)\phi = u_t, \tag{11}$$

where $u(\cdot;\tau,\phi)$ is the unique weak solution to (1), is a process on $C_\gamma(H)$. Moreover, $U(t,\tau)$ maps bounded sets of $C_\gamma(H)$ into bounded sets of $C_\gamma(H)$.

If in addition $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, then for any $h \ge 0$, the family of mappings $U(t,\tau)|_{C_\gamma^{h,V}(H)}$, with $t \ge \tau$, is also a well defined process on $C_\gamma^{h,V}(H)$.

The following result can be obtained analogously to [8, Proposition 5.1] (see also [6]), with the natural changes in the delay norms, but the proof is included here just for the sake of completeness.

Proposition 1. *Assume that $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, $\gamma > 0$, and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^2$ satisfying the assumptions $(g1)$–$(g3)$, are given. Then, for any bounded set $B \subset C_\gamma(H)$, one has:*

(i) The set of weak solutions $\{u(\cdot;\tau,\phi) : \phi \in B\}$ is bounded in $L^\infty(\tau+\varepsilon,T;V)$ for any $\varepsilon > 0$ and any $T > \tau+\varepsilon$.

(ii) Moreover, if $\{\phi(0) : \phi \in B\}$ is bounded in V, then $\{u(\cdot;\tau,\phi) : \phi \in B\}$ is bounded in $L^\infty(\tau,T;V)$ for all $T > \tau$.

Proof. By the second energy equality, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 + b(u(\theta),u(\theta),Au(\theta)) \\ = \ & (f(\theta) + g(\theta,u_\theta), Au(\theta)) \\ \le \ & \frac{2}{\nu}(|f(\theta)|^2 + |g(\theta,u_\theta)|^2) + \frac{\nu}{4}|Au(\theta)|^2, \quad \text{a.e. } \theta > \tau, \end{aligned}$$

where we have used Young's inequality.

The trilinear term b can be estimated as

$$\begin{aligned}|b(u(\theta),u(\theta),Au(\theta))| &\leq C_1|u(\theta)|^{1/2}\|u(\theta)\||Au(\theta)|^{3/2}\\ &\leq \frac{\nu}{4}|Au(\theta)|^2+C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4,\end{aligned}$$

where

$$C^{(\nu)}=27C_1^4(4\nu^3)^{-1}. \tag{12}$$

This, combined with the above and the properties of g, gives

$$\frac{d}{d\theta}\|u(\theta)\|^2+\nu|Au(\theta)|^2\leq\frac{4}{\nu}|f(\theta)|^2+2C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4+\frac{4L_g^2}{\nu}\|u_\theta\|_\gamma^2,\quad \text{a.e. } \theta>\tau. \tag{13}$$

Integrating, in particular we deduce that for all $\tau<s\leq r$

$$\|u(r)\|^2\leq\|u(s)\|^2+\frac{4}{\nu}\int_s^r|f(\theta)|^2\,d\theta+2C^{(\nu)}\int_s^r|u(\theta)|^2\|u(\theta)\|^4\,d\theta+\frac{4L_g^2}{\nu}\int_s^r\|u_\theta\|_\gamma^2\,d\theta.$$

By Gronwall's lemma we obtain that for all $\tau<s\leq r$

$$\begin{aligned}\|u(r)\|^2 &\leq \left(\|u(s)\|^2+\frac{4}{\nu}\int_s^r|f(\theta)|^2\,d\theta+\frac{4L_g^2}{\nu}\int_s^r\|u_\theta\|_\gamma^2\,d\theta\right)\\ &\qquad\times\exp\left(2C^{(\nu)}\int_s^r|u(\theta)|^2\|u(\theta)\|^2\,d\theta\right).\end{aligned} \tag{14}$$

Integrating once more with respect to $s\in(\tau,r)$ yields

$$\begin{aligned}&(r-\tau)\|u(r)\|^2\\ \leq\ &\left(\int_\tau^T\|u(s)\|^2\,ds+\frac{4(T-\tau)}{\nu}\int_\tau^T|f(\theta)|^2\,d\theta+\frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T\|u_\theta\|_\gamma^2\,d\theta\right)\\ &\times\exp\left(2C^{(\nu)}\int_\tau^T|u(\theta)|^2\|u(\theta)\|^2\,d\theta\right)\quad\forall\,\tau<r\leq T.\end{aligned}$$

In particular, for $\tau+\varepsilon\leq r\leq T$, it holds

$$\begin{aligned}\|u(r)\|^2\leq\ &\frac{1}{\varepsilon}\left(\int_\tau^T\|u(s)\|^2\,ds+\frac{4(T-\tau)}{\nu}\int_\tau^T|f(\theta)|^2\,d\theta+\frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T\|u_\theta\|_\gamma^2\,d\theta\right)\\ &\times\exp\left(2C^{(\nu)}\int_\tau^T|u(\theta)|^2\|u(\theta)\|^2\,d\theta\right).\end{aligned}$$

Taking into account (4) and (5), the claim (i) is proved.

The proof of claim (ii) is simpler. If $\phi(0)$ belongs to V, then from (13) one deduces that for all $\tau\leq r\leq T$,

$$\|u(r)\|^2\leq\|u(\tau)\|^2+\frac{4}{\nu}\int_\tau^r|f(\theta)|^2\,d\theta+2C^{(\nu)}\int_\tau^r|u(\theta)|^2\|u(\theta)\|^4\,d\theta+\frac{4L_g^2}{\nu}\int_\tau^r\|u_\theta\|_\gamma^2\,d\theta.$$

Therefore, one may apply directly Gronwall's lemma and proceed analogously as before to conclude (ii). □

One ingredient in order to obtain pullback attractors below is that the dynamical system be closed (cf. [7]). We obtain a stronger property here: the process U is continuous in the several phase spaces that we defined above.

Proposition 2. *Assume that* $f \in L^2_{loc}(\mathbb{R}; V')$, $\gamma > 0$, *and* $g : \mathbb{R}\times C_\gamma(H) \to (L^2(\Omega))^2$ *satisfying the assumptions* $(g1)$*–*$(g3)$*, are given. Let us denote* $u = u(\cdot;\tau,\phi)$ *and* $v = v(\cdot;\tau,\psi)$ *the weak solutions for (1) corresponding to initial data* ϕ *and* $\psi \in C_\gamma(H)$*. Then, the following continuity properties hold:*

(i) For any $\tau \le t$,

$$\|u_t - v_t\|^2_\gamma \le \Big(1 + \frac{L_g}{2\gamma}\Big)\|\phi-\psi\|^2_\gamma \, exp\Big(\int_\tau^t \big(3L_g + \frac{1}{4\nu}\|u(s)\|^2\big)\, ds\Big),$$

and in particular the mapping $U(t,\tau) : C_\gamma(H) \to C_\gamma(H)$*, defined by (11), is continuous.*

(ii) If $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, $\phi(0) \in V$ *and* $\psi(0) \in V$*, then*

$$\begin{aligned}\|u(s) - v(s)\|^2 \;\le\; & \Big(\|\phi(0) - \psi(0)\|^2 + \frac{L_g^2}{\nu}\int_\tau^t \|u_\theta - v_\theta\|^2_\gamma \, d\theta\Big) \\ & \times exp\Big[\int_\tau^t \Big(2C^{(\nu)}\lambda_1^{-1}\|u(\theta)\|^4 + \frac{2C_1^2}{\nu}|v(\theta)||Av(\theta)|\Big) d\theta\Big] \;\; \forall \tau \le s \le t,\end{aligned}$$

where $C^{(\nu)}$ *is given in (12).*

In particular, for all $h \ge 0$ *and any* $\tau \le t$*, the mapping* $U(t,\tau) : C^{h,V}_\gamma(H) \to C^{h,V}_\gamma(H)$ *defined by (11), is continuous.*

Proof. Claim (i) was proved in [18, Proposition 6]. Observe that the assumption $2\gamma > \nu\lambda_1$ appearing in [18] was not really used.

Claim (ii) follows analogously as in [8, Proposition 5.2] with the natural changes in the delay norms. □

3. **Abstract results on minimal pullback attractors.**

In this section we recall some basic definitions and main results that we will use later about properties required of a process for a non-autonomous dynamical system in order to have a (minimal) pullback attractor.

These results can be found in [7] and [17] (see also [1]), so here we only reproduce the statements for the sake of completeness.

In this section, we consider fixed a metric space (X, d_X).

From Proposition 2 we know that the processes for our problem are continuous (in the sense that for any pair $\tau \le t$, $U(t,\tau) : X \to X$ is continuous). However, it is worth pointing out that the theory of attractors for dynamical systems can be developed with more relaxed assumptions. Namely, the following definition is weaker than asking for the process to be strong-weak (also known as norm-to weak) continuous, and of course weaker than asking U to be continuous.

Definition 3.1. A process U on X is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 3.2. We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0,t) = \bigcap_{s\le t}\overline{\bigcup_{\tau\le s} U(t,\tau)D_0(\tau)}^X \quad \forall\, t\in\mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\operatorname{dist}_X(\mathcal{O}_1,\mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\operatorname{dist}_X(\mathcal{O}_1,\mathcal{O}_2) = \sup_{x\in\mathcal{O}_1}\inf_{y\in\mathcal{O}_2} d_X(x,y).$$

Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t\in\mathbb{R}\}\subset\mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.3. A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D}\in\mathcal{D}$.

It is said that $\widehat{D}_0 = \{D_0(t) : t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t\in\mathbb{R}$ and any $\widehat{D}\in\mathcal{D}$, there exists a $\tau_0(\widehat{D},t)\le t$ such that

$$U(t,\tau)D(\tau)\subset D_0(t) \quad \forall\,\tau\le\tau_0(\widehat{D},t).$$

With the above definitions, we may establish the main result of this section (cf. [7, Theorem 3.11]).

Theorem 3.4. *Consider a closed process $U:\mathbb{R}^2_d\times X\to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t\in\mathbb{R}\}\subset\mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t\in\mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},t)}^X$, has the following properties:

- *(a) for any $t\in\mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t)\subset\Lambda(\widehat{D}_0,t)$,*
- *(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e. $\lim_{\tau\to-\infty}\operatorname{dist}_X(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D}\in\mathcal{D}$, and any $t\in\mathbb{R}$,*
- *(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $(t,\tau)\in\mathbb{R}^2_d$,*
- *(d) if $\widehat{D}_0\in\mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0,t)\subset\overline{D_0(t)}^X$ for all $t\in\mathbb{R}$.*

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t\in\mathbb{R}\}\in\mathcal{D}$, $\lim\limits_{\tau\to-\infty}\operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_{\mathcal{D}}(t)\subset C(t)$.

Remark 4. Under the assumptions of Theorem 3.4, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}$-attractor for the process U.

If $\mathcal{A}_{\mathcal{D}}\in\mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}}\in\mathcal{D}$ is to have that $\widehat{D}_0\in\mathcal{D}$, the set $D_0(t)$ is closed for all $t\in\mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e. if $\widehat{D}\in\mathcal{D}$, and $\widehat{D}' = \{D'(t) : t\in\mathbb{R}\}\subset\mathcal{P}(X)$ with $D'(t)\subset D(t)$ for all t, then $\widehat{D}'\in\mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result (where $\mathcal{A}_{\mathcal{D}_F(X)}$ is the original pullback attractor of [5]).

Corollary 2. *Under the assumptions of Theorem 3.4, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.*

Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \le T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \le T$.

The following result allows us to compare two attractors for a same process in different phase spaces under appropriate assumptions.

Theorem 3.5. *Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be two metric spaces such that $X_1 \subset X_2$ with continuous injection, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that we have a map U that acts as a process in both cases, i.e., $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$ is a process.*

For each $t \in \mathbb{R}$, let us denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \qquad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence of the respective topology.

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Suppose moreover that the two following conditions are satisfied:

(i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$,

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t^*_{\widehat{D}_1} \le t$ (both possibly depending on t and $\widehat{D}_2$), such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \le t^*_{\widehat{D}_1}$ there exists a $\tau_s \le s$ such that $U(s, \tau) D_2(\tau) \subset D_1(s)$ for all $\tau \le \tau_s$.

Then, under all the conditions above, $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Remark 5. In the preceding theorem, if instead of assumption (ii) we consider the following condition:

(ii') for any $\widehat{D}_2 \in \mathcal{D}_2$ and any sequence $\tau_n \to -\infty$, there exist another family $\widehat{D}_1 \in \mathcal{D}_1$ and another sequence $\tau'_n \to -\infty$ with $\tau'_n \ge \tau_n$ for all n, such that U is pullback $\widehat{D}_1$-asymptotically compact, and

$$U(\tau'_n, \tau_n) D_2(\tau_n) \subset D_1(\tau'_n) \quad \forall\, n,$$

then, with a similar proof, one can obtain that the equality $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ also holds for all $t \in \mathbb{R}$.

Observe that a sufficient condition for (ii') is that for each $t \in \mathbb{R}$, there exists $T = T(t) > 0$ such that for any $\widehat{D}_2 \in \mathcal{D}_2$, there exists a $\widehat{D}_1 \in \mathcal{D}_1$ satisfying that U is pullback $\widehat{D}_1$-asymptotically compact, and $U(\tau + T, \tau) D_2(\tau) \subset D_1(\tau + T)$ for all $\tau < t - T$.

4. Pullback attractors for 2D Navier-Stokes equations with infinite delay and their relation.

In the context of pullback $\mathcal{D}$-attractors, applications usually involve a concrete universe. Namely, and having in mind (4), the two first of the following families were already used as universes in [18] (the first one for $\mu = \nu/2$). The rest of the families are related to our goal of improving the regularity of the attractor, and combine the Banach space $C_\gamma(H)$ with the space $C_\gamma^{h,V}(H)$ given in (10).

Definition 4.1. For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(C_\gamma(H))$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_\gamma(H))$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{\varphi\in D(\tau)}\|\varphi\|_\gamma^2\right) = 0.$$

Accordingly to the notation introduced in the previous section, $\mathcal{D}_F(C_\gamma(H))$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_\gamma(H)$.

For any $\sigma > 0$ and $h \geq 0$, we will also denote by $\mathcal{D}_\sigma^{h,V}(C_\gamma(H))$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma(C_\gamma(H))$ such that for any $t \in \mathbb{R}$ and for any $\varphi \in D(t)$, it holds that $\varphi|_{[-h,0]} \in B([-h,0];V)$.

Analogously, we will denote by $\mathcal{D}_F^{h,V}(C_\gamma(H))$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_\gamma(H)$ such that for any $\varphi \in D$, it holds that $\varphi|_{[-h,0]} \in B([-h,0];V)$.

Finally, we will denote by $\mathcal{D}_F(C_\gamma^{h,V}(H))$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_\gamma^{h,V}(H)$.

Remark 6. The chain of inclusions for the universes in the above definition is the following:

$$\mathcal{D}_F(C_\gamma^{h,V}(H)) \subset \mathcal{D}_F^{h,V}(C_\gamma(H)) \subset \mathcal{D}_\sigma^{h,V}(C_\gamma(H)) \subset \mathcal{D}_\sigma(C_\gamma(H)),$$

and

$$\mathcal{D}_F(C_\gamma^{h,V}(H)) \subset \mathcal{D}_F^{h,V}(C_\gamma(H)) \subset \mathcal{D}_F(C_\gamma(H)) \subset \mathcal{D}_\sigma(C_\gamma(H)),$$

for all $\sigma > 0$ and any $h \geq 0$.

It must also be pointed out that $\mathcal{D}_\sigma(C_\gamma(H))$ and $\mathcal{D}_\sigma^{h,V}(C_\gamma(H))$ are inclusion-closed, which will be important (cf. Remark 4).

Hereon, let us assume that

$$\text{there exists } 0 < \mu < \nu \text{ such that } L_g < (\nu-\mu)\lambda_1 \leq \gamma \tag{15}$$

and

$$\int_{-\infty}^0 e^{\sigma_\mu s}\|f(s)\|_*^2\,ds < \infty, \tag{16}$$

where

$$\sigma_\mu = 2((\nu-\mu)\lambda_1 - L_g). \tag{17}$$

As an immediate consequence of (4) we have the following

Proposition 3. *Let $\gamma > 0$, g satisfying assumptions $(g1)$–$(g3)$, and $f \in L^2_{loc}(\mathbb{R};V')$ be given. Assume that (15) and (16) hold. Then, the family $\widehat{D}_{0,\mu} = \{D_{0,\mu}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_\gamma(H))$, with $D_{0,\mu}(t) = \overline{B}_{C_\gamma(H)}(0,\rho_\mu(t))$, the closed ball in $C_\gamma(H)$ of center zero and radius $\rho_\mu(t)$, where*

$$\rho_\mu^2(t) = 1 + \mu^{-1}\int_{-\infty}^t e^{-\sigma_\mu(t-s)}\|f(s)\|_*^2\,ds,$$

is pullback $\mathcal{D}_{\sigma_\mu}(C_\gamma(H))$-absorbing for the process $U : \mathbb{R}_d^2 \times C_\gamma(H) \to C_\gamma(H)$ defined by (11). Moreover, $\widehat{D}_{0,\mu} \in \mathcal{D}_{\sigma_\mu}(C_\gamma(H))$.

From above, we have the following slight improvement of [18, Theorem 28].

Theorem 4.2. *Under the assumptions of Proposition 3, there exist the minimal pullback $\mathcal{D}_F(C_\gamma(H))$-attractor $\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}$ and the minimal pullback $\mathcal{D}_{\sigma_\mu}(C_\gamma(H))$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}$ for the process U associated to (1), and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t) \subset \overline{B}_{C_\gamma(H)}(0, \rho_\mu(t)) \quad \forall\, t \in \mathbb{R}$$

and

$$\lim_{t\to-\infty} \left(e^{\sigma_\mu t} \sup_{v \in \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t)} \|v\|_\gamma^2 \right) = 0.$$

Remark 7. If we also assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, from the invariance of both pullback attractors, and the regularity property stated in Theorem 2.1, it turns out that

$$\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t) \subset C((-\infty, 0]; V) \quad \forall\, t \in \mathbb{R}.$$

We establish now some results on absorbing properties of $U : \mathbb{R}^2_d \times C^{h,V}_\gamma(H) \to C^{h,V}_\gamma(H)$. The first one is a consequence of Proposition 3.

Proposition 4. *Let $\gamma > 0$ and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^2$ satisfying assumptions $(g1)$–$(g3)$ be given. Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ and that there exists $0 < \mu < \nu$ such that $L_g < (\nu - \mu)\lambda_1 \le \gamma$, and*

$$\int_{-\infty}^0 e^{\sigma_\mu s} |f(s)|^2\, ds < \infty, \tag{18}$$

where σ_μ is given by (17).

Then, for any $h \ge 0$, the family $\widehat{D}_{0,\mu,h} = \{D_{0,\mu,h}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C^{h,V}_\gamma(H))$, with

$$D_{0,\mu,h}(t) = D_{0,\mu}(t) \cap C^{h,V}_\gamma(H),$$

is a family of closed sets of $C^{h,V}_\gamma(H)$ and is pullback $\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$-absorbing for the process $U : \mathbb{R}^2_d \times C^{h,V}_\gamma(H) \to C^{h,V}_\gamma(H)$. Moreover, $\widehat{D}_{0,\mu,h} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$.

Lemma 4.3. *Under the assumptions of Proposition 4, for any $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_\gamma(H))$ and any $r > h$, the family $\widehat{D}^{(r)} = \{D^{(r)}(\tau) \ : \ \tau \in \mathbb{R}\}$, where $D^{(r)} = U(\tau+r, \tau)D(\tau)$, for any $\tau \in \mathbb{R}$, belongs to $\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$.*

Proof. From (4), we deduce

$$\sup_{\psi \in D^{(r)}(\tau)} (e^{\sigma_\mu \tau} \|\psi\|_\gamma^2) \le e^{-\sigma_\mu r} \sup_{\phi \in D(\tau)} (e^{\sigma_\mu \tau} \|\phi\|_\gamma^2) + (\mu\lambda_1)^{-1} \int_\tau^{\tau+r} e^{\sigma_\mu s} |f(s)|^2 ds,$$

which jointly with the regularity property in Theorem 2.1 and (18), conclude the proof. □

Now, we establish several estimates in finite intervals of time when the initial time is sufficiently shifted in a pullback sense.

Lemma 4.4. *Under the assumptions of Proposition 4, for any $t \in \mathbb{R}$, $h \ge 0$ and $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_\gamma(H))$, there exists $\tau_1(\widehat{D}, t, h) < t-h-2$ and functions $\{\rho_i\}_{i=1}^4$ depending* *on t and h, such that for any $\tau \le \tau_1(\widehat{D}, t, h)$ and any $\phi^\tau \in D(\tau)$, it holds*

$$\begin{cases} |u(r;\tau,\phi^\tau)|^2 \le \rho_1(t) & \forall r \in [t-h-2, t], \\ \|u(r;\tau,\phi^\tau)\|^2 \le \rho_2(t) & \forall r \in [t-h-1, t], \\ \nu \displaystyle\int_{r-1}^{r} |Au(\theta;\tau,\phi^\tau)|^2 \, d\theta \le \rho_3(t) & \forall r \in [t-h, t], \\ \displaystyle\int_{r-1}^{r} |u'(\theta;\tau,\phi^\tau)|^2 \, d\theta \le \rho_4(t) & \forall r \in [t-h, t], \end{cases} \tag{19}$$

where

$$\rho_1(t) = 1 + (\mu\lambda_1)^{-1} e^{-\sigma_\mu(t-h-2)} \int_{-\infty}^{t} e^{\sigma_\mu s} |f(s)|^2 \, ds,$$

$$\begin{aligned} \rho_2(t) &= \left\{ \nu^{-1}\rho_1(t)(1 + L_g^2(4 + 2\nu^{-1}\lambda_1^{-1})) + \nu^{-1}(4 + 2\nu^{-1}\lambda_1^{-1}) \int_{t-h-2}^{t} |f(\theta)|^2 \, d\theta \right\} \\ &\quad \times \exp\left\{ 2\nu^{-1} C^{(\nu)} \rho_1(t) \left[\rho_1(t)(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2) + 2\nu^{-1}\lambda_1^{-1} \int_{t-h-2}^{t} |f(\theta)|^2 d\theta \right] \right\}, \\ \rho_3(t) &= \rho_2(t) + 4\nu^{-1} \int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta + 2C^{(\nu)} \rho_1(t)\rho_2^2(t) + 4L_g^2 \nu^{-1} \rho_1(t), \\ \rho_4(t) &= \nu\rho_2(t) + 4 \int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta + 2C_1^2 \nu^{-1} \rho_2(t)\rho_3(t) + 4L_g^2 \rho_1(t), \end{aligned}$$

with σ_μ given by (17), and $C^{(\nu)}$ defined in (12).

Proof. Let $\tau_1(\widehat{D}, t, h) < t - h - 2$ be such that

$$e^{-\sigma_\mu(t-h-2)} e^{\sigma_\mu \tau} \|\phi^\tau\|_\gamma^2 \le 1 \quad \forall \tau \le \tau_1(\widehat{D}, t, h), \ \phi^\tau \in D(\tau).$$

Consider fixed $\tau \le \tau_1(\widehat{D}, t, h)$ and $\phi^\tau \in D(\tau)$.

The first estimate in (19) follows directly from (4), using the definition of the norm $\|\cdot\|_\gamma$ and the increasing character of the exponential.

Now, for the rest of the estimates, let us consider again the Galerkin approximations already used in Theorem 2.1, and denote for short $u^m(r) = u^m(r;\tau,\phi^\tau)$.

Multiplying each equation of (6) by $\alpha_{m,j}(t)$ and summing from $j = 1$ to m, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 &= (f(t) + g(t, u_t^m), u^m(t)) \\ &\le \frac{1}{\nu\lambda_1}(|f(t)|^2 + |g(t, u_t^m)|^2) + \frac{\nu}{2}\lambda_1 |u^m(t)|^2, \text{ a.e. } t > \tau, \end{aligned}$$

where we have used Young inequality. Now, from the properties of g, we obtain

$$\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 \le \frac{2}{\nu\lambda_1}(|f(t)|^2 + L_g^2 \|u_t^m\|_\gamma^2), \quad \text{a.e. } t > \tau.$$

Integrating, in particular we deduce that

$$\nu \int_{r-1}^{r} \|u^m(\theta)\|^2 \, d\theta \le |u^m(r-1)|^2 + \frac{2}{\nu\lambda_1} \int_{r-1}^{r} (|f(\theta)|^2 + L_g^2 \|u_\theta^m\|_\gamma^2) \, d\theta \quad \forall \tau \le r - 1. \tag{20}$$

Now, observe that (4) and the estimates obtained in the proof of Proposition 1 also hold for u^m.

From (14), integrating with respect to $s \in (r-1, r)$, and using (4), we obtain

$$\|u^m(r)\|^2 \leq \Big[\int_{r-1}^{r} \|u^m(s)\|^2 \, ds + 4\nu^{-1} \int_{r-1}^{r} |f(\theta)|^2 \, d\theta$$

$$+4L_g^2 \nu^{-1} \int_{r-1}^{r} \Big(e^{-\sigma_\mu(\theta-\tau)} \|\phi^\tau\|_\gamma^2 + (\mu\lambda_1)^{-1} \int_{\tau}^{\theta} e^{-\sigma_\mu(\theta-\eta)} |f(\eta)|^2 d\eta \Big) d\theta \Big]$$

$$\times \exp\Big(2C^{(\nu)} \int_{r-1}^{r} |u^m(\theta)|^2 \|u^m(\theta)\|^2 \, d\theta \Big) \quad \forall \tau \leq r-1.$$

From this, jointly with (20) and the first estimate in (19), which holds exactly the same for the approximations u^m, one deduces

$$\|u^m(r;\tau,\phi^\tau)\|^2 \leq \rho_2(t) \quad \forall r \in [t-h-1, t]. \tag{21}$$

From this inequality and Remark 3, we deduce that

$$u^m \overset{*}{\rightharpoonup} u(\cdot;\tau,\phi^\tau) \quad \text{weakly-star in } L^\infty(t-h-1,t;V).$$

So, taking inferior limit when m goes to infinity in (21), and using the fact that $u(\cdot;\tau,\phi^\tau) \in C([t-h-1,t];V)$, we obtain the second estimate in (19).

On other hand, from (13), and using again (4), we also obtain

$$\nu \int_{r-1}^{r} |Au^m(\theta)|^2 \, d\theta$$

$$\leq \|u^m(r-1)\|^2 + 4\nu^{-1} \int_{r-1}^{r} |f(\theta)|^2 \, d\theta + 2C^{(\nu)} \int_{r-1}^{r} |u^m(\theta)|^2 \|u^m(\theta)\|^4 \, d\theta$$

$$+4L_g^2 \nu^{-1} \int_{r-1}^{r} \Big(e^{-\sigma_\mu(\theta-\tau)} \|\phi^\tau\|_\gamma^2 + (\mu\lambda_1)^{-1} \int_{\tau}^{\theta} e^{-\sigma_\mu(\theta-s)} |f(s)|^2 \, ds \Big) d\theta$$

for all $\tau \leq r-1$. Therefore,

$$\nu \int_{r-1}^{r} |Au^m(\theta;\tau,\phi^\tau)|^2 \, d\theta \leq \rho_3(t) \quad \forall r \in [t-h, t]. \tag{22}$$

From Remark 3, (22), and the uniqueness of solutions, we deduce that

$$u^m \rightharpoonup u(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1,r;D(A)) \ \ \forall r \in [t-h,t].$$

Thus, taking inferior limit when m goes to infinity in (22), we obtain the third inequality in (19).

Finally, multiplying each equation in (6) by $\alpha'_{m,j}(t)$ and summing from $j=1$ to m, we obtain

$$|(u^m)'(\theta)|^2 + \frac{\nu}{2}\frac{d}{d\theta}\|u^m(\theta)\|^2 + b(u^m(\theta), u^m(\theta), (u^m)'(\theta))$$

$$= (f(\theta), (u^m)'(\theta)) + (g(\theta, u^m_\theta), (u^m)'(\theta)), \quad \text{a.e. } \theta > \tau.$$

Since

$$\begin{aligned}
|(f(\theta), (u^m)'(\theta))| &\leq \frac{1}{8}|(u^m)'(\theta)|^2 + 2|f(\theta)|^2, \\
|(g(\theta, u^m_\theta), (u^m)'(\theta))| &\leq \frac{1}{8}|(u^m)'(\theta)|^2 + 2|g(\theta, u^m_\theta)|^2, \\
|b(u^m(\theta), u^m(\theta), (u^m)'(\theta))| &\leq C_1 |Au^m(\theta)| \|u^m(\theta)\| |(u^m)'(\theta)| \\
&\leq \frac{1}{4}|(u^m)'(\theta)|^2 + C_1^2 |Au^m(\theta)|^2 \|u^m(\theta)\|^2,
\end{aligned}$$

we obtain that

$$|(u^m)'(\theta)|^2 + \nu \frac{d}{d\theta}\|u^m(\theta)\|^2 \le 4|f(\theta)|^2 + 4|g(\theta, u^m_\theta)|^2 + 2C_1^2|Au^m(\theta)|^2\|u^m(\theta)\|^2$$

a.e. $\theta > \tau$. From the properties of g, (4), and integrating above, we conclude

$$\begin{aligned}
&\int_{r-1}^{r} |(u^m)'(\theta)|^2\, d\theta \\
&\le \nu\|u^m(r-1)\|^2 + 4\int_{r-1}^{r} |f(\theta)|^2\, d\theta + 2C_1^2 \int_{r-1}^{r} |Au^m(\theta)|^2\|u^m(\theta)\|^2\, d\theta \\
&\quad +4L_g^2 \int_{r-1}^{r} \left(e^{-\sigma_\mu(\theta-\tau)}\|\phi^\tau\|_\gamma^2 + (\mu\lambda_1)^{-1}\int_\tau^\theta e^{-\sigma_\mu(\theta-s)}|f(s)|^2\, ds \right) d\theta \ \forall \tau \le r-1.
\end{aligned}$$

From (21) and (22) we deduce that

$$\int_{r-1}^{r} |(u^m)'(\theta;\tau,\phi^\tau)|^2\, d\theta \le \rho_4(t) \quad \forall\, r \in [t-h, t]. \tag{23}$$

From Remark 3, (23), and the uniqueness of solutions, we deduce that

$$(u^m)' \rightharpoonup u'(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1, r; H) \ \ \forall\, r \in [t-h, t].$$

Thus, taking inferior limit when m goes to infinity in (23), we obtain the fourth inequality in (19). □

Now, we can prove the asymptotic compactness of the process U restricted to the space $C_\gamma^{h,V}(H)$. The proof relies on an energy method with continuous functions, and is similar to that in [18] but using the energy equality (3) (see also [7, Lemma 4.13]); we reproduce it here just for the sake of completeness.

Lemma 4.5. *Under the assumptions of Proposition 4, and for any $h \ge 0$, the process $U : \mathbb{R}_d^2 \times C_\gamma^{h,V}(H) \to C_\gamma^{h,V}(H)$ is pullback $\mathcal{D}_{\sigma_\mu}^{h,V}(C_\gamma(H))$-asymptotically compact.*

Proof. Since the asymptotic compactness in the norm of $C_\gamma(H)$ was already established in Theorem 4.2, we only must care about the sup norm in $B([-h,0];V)$. So, let us fix $t \in \mathbb{R}$, a family $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_{\sigma_\mu}^{h,V}(C_\gamma(H))$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{\phi^{\tau_n}\} \subset C_\gamma^{h,V}(H)$, with $\phi^{\tau_n} \in D(\tau_n)$ for all n.

For short, let us denote $u^n(\cdot) = u(\cdot;\tau_n,\phi^{\tau_n})$. It is enough to prove that the sequence $\{u^n(t+\cdot)\}$ is relatively compact in $C([-h,0];V)$.

By the asymptotic compactness in the norm of $C_\gamma(H)$, we may assume without loss of generality that there exists $\xi \in C_\gamma(H)$ such that

$$u_t^n \to \xi \quad \text{strongly in } C_\gamma(H). \tag{24}$$

Denote $u(t+r) = \xi(r)$ for all $r \in (-\infty, 0]$.

From Lemma 4.4 we know that there exists a value $\tau_1(\widehat{D}, t, h) < t-h-2$ such that the subsequence $\{u^n : \tau_n \le \tau_1(\widehat{D}, t, h)\}$ is bounded in $L^\infty(t-h-1, t; V) \cap L^2(t-h-1, t; D(A))$ with $\{(u^n)'\}$ bounded in $L^2(t-h-1, t; H)$.

Using the Aubin-Lions compactness lemma (e.g. cf. [14]), and taking into account (24), we may ensure that $u \in L^\infty(t-h-1, t; V) \cap L^2(t-h-1, t; D(A))$ with

$u' \in L^2(t-h-1,t;H)$, and for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-h-1,t;V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-h-1,t;D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-h-1,t;H), \\ u^n \to u & \text{strongly in } L^2(t-h-1,t;V), \\ u^n(s) \to u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-h-1,t). \end{cases} \tag{25}$$

Indeed, $u \in C([t-h-1,t];V)$ satisfies, thanks to (24) and (25), the equation (2) in $(t-h-1,t)$.

From the boundedness of $\{u^n\}$ in $C([t-h-1,t];V)$, we have that for any sequence $\{s_n\} \subset [t-h-1,t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{26}$$

where we have used (24) to identify the weak limit. We will prove that

$$u^n \to u \quad \text{strongly in } C([t-h,t];V), \tag{27}$$

using an energy method for continuous functions analogous to that employed, for instance, in [18, 7].

Indeed, if (27) is false, there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-h,t]$, without loss of generality converging to some t_*, and such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall\, n \geq 1. \tag{28}$$

Recall that by (26) we have

$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{29}$$

On the other hand, using the energy equality (3) for u and all u^n, and reasoning as for the obtention of (13), we have that for all $t-h-1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} &\|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2\,dr \\ &\leq \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4\,dr + \frac{4}{\nu}\int_{s_1}^{s_2} |f(r)|^2\,dr + \frac{4L_g^2}{\nu}\int_{s_1}^{s_2} \|u^n_r\|_\gamma^2\,dr, \end{aligned}$$

and

$$\begin{aligned} &\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2\,dr \\ &\leq \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4\,dr + \frac{4}{\nu}\int_{s_1}^{s_2} |f(r)|^2\,dr + \frac{4L_g^2}{\nu}\int_{s_1}^{s_2} \|u_r\|_\gamma^2\,dr. \end{aligned}$$

In particular, we can define the functions

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u^n(r)|^2 \|u^n(r)\|^4\,dr - \frac{4}{\nu}\int_{t-h-1}^{s} |f(r)|^2\,dr \\ &\quad - \frac{4L_g^2}{\nu}\int_{t-h-1}^{s} \|u^n_r\|_\gamma^2\,dr, \\ J(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u(r)|^2 \|u(r)\|^4\,dr - \frac{4}{\nu}\int_{t-h-1}^{s} |f(r)|^2\,dr \\ &\quad - \frac{4L_g^2}{\nu}\int_{t-h-1}^{s} \|u_r\|_\gamma^2\,dr. \end{aligned}$$

These are continuous functions on $[t-h-1,t]$, and from the above inequalities, both J_n and J are non-increasing. Moreover, by (24) and (25), we have

$$J_n(s) \to J(s) \quad \text{a.e. } s \in (t-h-1,t).$$

Thus, there exists a sequence $\{\tilde{t}_k\} \subset (t-h-1,t_*)$ such that $\tilde{t}_k \to t_*$, when $k \to \infty$, and

$$\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \forall\, k.$$

Fix an arbitrary value $\delta > 0$. From the continuity of J, there exists k_δ such that

$$|J(\tilde{t}_k) - J(t_*)| < \delta/2 \quad \forall\, k \geq k_\delta.$$

Now consider $n(k_\delta)$ such that for all $n \geq n(k_\delta)$ it holds

$$t_n \geq \tilde{t}_{k_\delta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| < \delta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n \geq n(k_\delta)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\delta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta. \end{aligned}$$

This yields that

$$\limsup_{n\to\infty} J_n(t_n) \leq J(t_*),$$

and therefore, by (24) and (25),

$$\limsup_{n\to\infty} \|u^n(t_n)\| \leq \|u(t_*)\|,$$

which joined to (29) and (26) implies that $u^n(t_n) \to u(t_*)$ strongly in V, in contradiction with (28). Thus, (27) is proved as desired. □

Now, we can establish our main result.

Theorem 4.6. *Let $\gamma > 0$ and g satisfying assumptions $(g1)$–$(g3)$ be given. Assume that there exists $0 < \mu < \nu$ such that $L_g < (\nu-\mu)\lambda_1 \leq \gamma$, and $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$ satisfies (18). Then, for any $h \geq 0$, the process U on $C_\gamma^{h,V}(H)$ defined by (11) possesses a minimal pullback $\mathcal{D}_{\sigma_\mu}^{h,V}(C_\gamma(H))$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{h,V}(C_\gamma(H))}$, a minimal pullback $\mathcal{D}_F^{h,V}(C_\gamma(H))$-attractor $\mathcal{A}_{\mathcal{D}_F^{h,V}(C_\gamma(H))}$, and a minimal pullback $\mathcal{D}_F(C_\gamma^{h,V}(H))$-attractor $\mathcal{A}_{\mathcal{D}_F(C_\gamma^{h,V}(H))}$. Moreover, the following relations hold:*

$$\begin{aligned} \mathcal{A}_{\mathcal{D}_F(C_\gamma^{h,V}(H))}(t) &\subset \mathcal{A}_{\mathcal{D}_F^{h,V}(C_\gamma(H))}(t) \\ &\subset \mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) \\ &\subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{h,V}(C_\gamma(H))}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t) \\ &\subset C((-\infty,0];V) \quad \forall\, t \in \mathbb{R}, \end{aligned} \tag{30}$$

and for any family $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_\gamma(H))$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_{C_\gamma^{h,V}(H)}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t)) = 0 \quad \forall\, t \in \mathbb{R}. \tag{31}$$

Finally, if in addition f satisfies

$$\sup_{s\leq 0}\left(e^{-\sigma_\mu s}\int_{-\infty}^{s} e^{\sigma_\mu \theta}|f(\theta)|^2\, d\theta\right) < \infty, \tag{32}$$

then all attractors in (30) coincide, $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$ *and it is tempered in* $C^{h,V}_\gamma(H)$*, in the sense that*

$$\lim_{t\to-\infty}\left(e^{\sigma_\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t)}\|v\|^2_{\gamma,h,V}\right)=0. \tag{33}$$

Proof. Let us fix $h \ge 0$. The existence of $\mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))}$ is a consequence of Theorem 3.4, since the process U on $C^{h,V}_\gamma(H)$ is continuous (cf. Proposition 2 (ii)) and therefore closed, the existence of a pullback absorbing family was given by Proposition 4, and in Lemma 4.5 we have proved the pullback $\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$-asymptotic compactness.

The existence of the pullback attractors $\mathcal{A}_{\mathcal{D}^{h,V}_F(C_\gamma(H))}$ and $\mathcal{A}_{\mathcal{D}_F(C^{h,V}_\gamma(H))}$ follows from the above facts, and the inclusions $\mathcal{D}_F(C^{h,V}_\gamma(H)) \subset \mathcal{D}^{h,V}_F(C_\gamma(H)) \subset \mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$.

In (30), the chain of inclusions follows from Corollary 2, Theorem 3.5, and Remark 6. The equality is a consequence of Theorem 3.5 and Remark 5, by using Lemma 4.3 with $T = r = h + 1$. The last inclusion was observed in Remark 7.

The property (31) is a consequence of Lemma 4.3, since for any $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_\gamma(H))$ and any $\tau < t - h - 1$,

$$\begin{aligned} & \operatorname{dist}_{C^{h,V}_\gamma(H)}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t)) \\ = \; & \operatorname{dist}_{C^{h,V}_\gamma(H)}(U(t,\tau+h+1)(U(\tau+h+1,\tau)D(\tau)), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))}(t)) \\ = \; & \operatorname{dist}_{C^{h,V}_\gamma(H)}(U(t,\tau+h+1)D^{(h+1)}(\tau), \mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))}(t)). \end{aligned}$$

The coincidence of all attractors in (30) under the additional assumption (32) holds by applying once more Theorem 3.5, Proposition 3, and the second estimate in (19), since (32) is equivalent to

$$\sup_{s\le 0}\int_{s-1}^{s}|f(\theta)|^2\,d\theta < \infty. \tag{34}$$

The fact that $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_\gamma(H))} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))$ is a consequence of Theorem 3.4 and remarks 4 and 6.

The tempered condition (33) of the attractor comes from (32) (and therefore (34)) and the expressions of $\rho_\mu(t)$ and $\rho_2(t)$. □

Remark 8. Observe that, under the assumptions of Theorem 4.6, one has that $\mathcal{A}_{\mathcal{D}^{h_1,V}_{\sigma_\mu}(C_\gamma(H))} \equiv \mathcal{A}_{\mathcal{D}^{h_2,V}_{\sigma_\mu}(C_\gamma(H))}$ for any $h_1, h_2 \ge 0$, i.e., the pullback attractor $\mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_\gamma(H))}$ is independent of $h \ge 0$.

Actually, if f also satisfies (32), then $\mathcal{A}_{\mathcal{D}_F(C^{h,V}_\gamma(H))} \equiv \mathcal{A}_{\mathcal{D}^{h,V}_F(C_\gamma(H))}$ is independent of h.

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