

Pullback attractors for a $2D$ –Navier-Stokes model in an infinite delay case

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Abstract

We prove the existence of solutions for a Navier-Stokes model in dimension two with an external force containing infinite delay effects in the weighted space $C_\gamma(H)$. After that, under additional suitable assumptions we prove the existence and uniqueness of a stationary solution and the exponential decay of the solution of the evolutionary problem to this stationary solution. Finally, we study the existence of pullback attractors for the dynamical system associated to the problem under more general assumptions.

Keywords: Navier-Stokes equations; infinite delays; pullback attractors.

Mathematics Subject Classifications (2000): 35K55, 35Q30, 34D45, 34K20.

1 Introduction and statement of the problem

The Navier-Stokes equations govern the motion of usual fluids like water, air, oil, etc. These equations have been the object of numerous works (e.g. see [17, 23] and references cited therein), since the first paper was published by Leray [16].

On other hand, delay terms appear naturally for instance as effects in wind tunnels experiments (cf. [18]). Very recently, Caraballo & Real [4, 5, 6] developed a full theory of existence, stability of solutions and global attractors for Navier-Stokes models including some hereditary characteristics in several ways (fixed, variable and distributed delays). This study has been continued by some other authors, e.g. [22, 9, 19, 21].

While in other fields, such as reaction-diffusion equations (cf. [24]), infinite-delay effects have been considered, to our knowledge it has not been studied so thoroughly for the Navier-Stokes equations yet.

Our purpose is to study the following problem. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set with boundary Γ (not necessarily smooth) and consider (arbitrary) values $\tau < T$ in \mathbb{R} and the following functional Navier-Stokes problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i} = f(t) - \nabla p + g(t, u_t) & \text{in } (\tau, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (\tau, T) \times \Omega, \\ u = 0 & \text{on } (\tau, T) \times \Gamma, \\ u(\tau + r, x) = \phi(r, x), & r \in (-\infty, 0], x \in \Omega. \end{cases} \quad (1)$$

where we assume that $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, f a non-delayed external force field, g another external force containing some hereditary characteristic and $\phi(s - \tau)$ the initial datum in the interval of time $(-\infty, \tau]$.

The structure of the paper is the following: in this section we introduce some functional spaces useful for the establishment of the abstract variational formulation of the problem, and some assumptions on the delay operator (and an example of the existence of such kind of operators, as well). In Section 2 we prove the existence and uniqueness of solution for (1) by an energy method among other compactness arguments; we also analyze continuity properties of the solutions with respect to initial data. In Section 3 a simpler model –the stationary problem– is studied, proving the existence and uniqueness of a stationary solution and the exponential decay of the solutions of the evolutionary problem toward the stationary solution under suitable assumptions as well. Finally, Section 4 is devoted to generalize the above results on asymptotic behaviour, proving under more general assumptions the existence of pullback attractors in two different frameworks, those of fixed bounded sets and in a universe of time-dependent families defined by a tempered condition.

To start with, we consider some usual abstract spaces. Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\},$$

and let H be the closure of \mathcal{V} in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product (\cdot, \cdot) , where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^2 \int_{\Omega} u_j(x) v_j(x) dx.$$

Also, V will be the closure of \mathcal{V} in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot, \cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous, and, in fact, compact. We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality between

V' and V . Define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall u, v, w \in V.$$

Let X be a Banach space. The notation $B_X(a, r)$ will be used to denote the open ball of center a and radius r in X . Given a function $u : (-\infty, T] \rightarrow X$, for each $t \leq T$ we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t+s)$, $s \in (-\infty, 0]$.

There are several phase spaces which allow us to deal with infinite delays (cf. [11]). One of them is to consider, for any $\gamma > 0$, the space

$$C_{\gamma}(H) = \left\{ \varphi \in C((-\infty, 0]; H) : \exists \lim_{s \rightarrow -\infty} e^{\gamma s} \varphi(s) \in H \right\},$$

which is a Banach space with the norm

$$\|\varphi\|_{\gamma} := \sup_{s \in (-\infty, 0]} e^{\gamma s} |\varphi(s)|.$$

Hereafter we will use this phase space (with a suitable $\gamma > 0$) for our problem.

In order to state the problem in the correct framework, let us first establish some initial assumptions on some terms in the equation.

We will assume that $f \in L^2(\tau, T; V')$.

For the term g , in which the delay is present, we assume that $g : [\tau, T] \times C_{\gamma}(H) \rightarrow (L^2(\Omega))^2$ satisfies

(g1) For any $\xi \in C_{\gamma}(H)$ the mapping $[\tau, T] \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is measurable.

(g2) $g(\cdot, 0) = 0$.

(g3) There exists a constant $L_g > 0$ such that for any $t \in [\tau, T]$ and all $\xi, \eta \in C_{\gamma}(H)$,

$$|g(t, \xi) - g(t, \eta)| \leq L_g \|\xi - \eta\|_{\gamma}.$$

Remark 1. (i) Condition (g2) is not really a restriction, since otherwise, if $|g(\cdot, 0)| \in L^2(\tau, T)$, we could redefine $\hat{f}(t) = f(t) + g(t, 0)$ and $\hat{g}(t, \cdot) = g(t, \cdot) - g(t, 0)$. In this way the problem is exactly the same and \hat{f} and \hat{g} satisfy the required assumptions.

(ii) Conditions (g2) and (g3) imply that

$$|g(t, \xi)| \leq L_g \|\xi\|_{\gamma},$$

so that $|g(\cdot, \xi)| \in L^{\infty}(\tau, T)$.

An example of an operator satisfying assumption (g3) is given here.

Example 2. We will consider on the Banach space $C_\gamma(H)$ the operator $g : [\tau, T] \times C_\gamma(H) \rightarrow (L^2(\Omega))^2$ defined as follows: for all $t \in [\tau, T]$, $\xi \in C_\gamma(H)$ we take $g(t, \xi)$ as the element of $(L^2(\Omega))^2$ given by

$$g(t, \xi)(x) := \int_{-\infty}^0 G(t, s, \xi(s)(x)) ds, \text{ for a.e. } x \in \Omega,$$

where the function $G : [\tau, T] \times \mathbb{R}_- \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies the following assumptions:

(a) $G(t, s, 0) = 0 \quad \forall (t, s) \in [\tau, T] \times \mathbb{R}_-$.

(b) There exists a function $\kappa : \mathbb{R}_- \rightarrow \mathbb{R}_+$ such that

$$\|G(t, s, u) - G(t, s, v)\|_{\mathbb{R}^2} \leq \kappa(s) \|u - v\|_{\mathbb{R}^2} \quad \forall u, v \in \mathbb{R}^2, \forall (t, s) \in [\tau, T] \times \mathbb{R}_-.$$

(c) The function κ satisfies that $\kappa(\cdot)e^{-(\gamma+\vartheta)\cdot} \in L^2(\mathbb{R}_-)$ for certain $\vartheta > 0$.

We check now that g satisfies the assumption given in (g3), and also, using (a) above, we obtain that it is well defined as a map with values in $(L^2(\Omega))^2$:

$$\begin{aligned} & |g(t, \xi) - g(t, \eta)|^2 \\ & \leq \int_{\Omega} \left(\int_{-\infty}^0 \kappa(s) \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2} ds \right)^2 dx \\ & = \int_{\Omega} \left(\int_{-\infty}^0 \kappa(s) e^{-(\gamma+\vartheta)s} e^{(\gamma+\vartheta)s} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2} ds \right)^2 dx \\ & \leq \int_{\Omega} \left(\int_{-\infty}^0 \kappa^2(s) e^{-2(\gamma+\vartheta)s} ds \right) \left(\int_{-\infty}^0 e^{2(\gamma+\vartheta)s} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 ds \right) dx \\ & = \left(\int_{-\infty}^0 \kappa^2(s) e^{-2(\gamma+\vartheta)s} ds \right) \int_{\Omega} \left(\int_{-\infty}^0 e^{2(\gamma+\vartheta)s} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 ds \right) dx \\ & = \|\kappa(\cdot)e^{-(\gamma+\vartheta)\cdot}\|_{L^2(\mathbb{R}_-)}^2 \int_{-\infty}^0 \int_{\Omega} e^{2(\gamma+\vartheta)s} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 dx ds \\ & \leq \|\kappa(\cdot)e^{-(\gamma+\vartheta)\cdot}\|_{L^2(\mathbb{R}_-)}^2 \left[\sup_{s \in (-\infty, 0]} e^{2\gamma s} \int_{\Omega} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 dx \right] \int_{-\infty}^0 e^{2\vartheta s} ds \\ & = \|\kappa(\cdot)e^{-(\gamma+\vartheta)\cdot}\|_{L^2(\mathbb{R}_-)}^2 \|\xi - \eta\|_{\gamma}^2 \frac{1}{2\vartheta} \\ & = L_g^2 \|\xi - \eta\|_{\gamma}^2. \end{aligned}$$

Now we will give the definition of weak solutions for problem (1).

Definition 3. A weak solution of (1) in the interval $(-\infty, T]$, with initial datum $\phi \in C_\gamma(H)$, is a function $u \in C((-\infty, T]; H) \cap L^2(\tau, T; V)$ with $u_\tau = \phi$ such that for all $v \in V$, and $t \in (\tau, T)$,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = \langle f(t), v \rangle + (g(t, u_t), v), \quad (2)$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, T)$.

Remark 4. If u is a weak solution of (1) in the sense given above, then u satisfies an energy equality. Namely:

$$\begin{aligned} & |u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr \\ = & |u(s)|^2 + 2 \int_s^t [\langle f(r), u(r) \rangle + (g(r, u_r), u(r))] dr \quad \forall s, t \in [\tau, T]. \end{aligned}$$

2 Existence of solutions

In this section we establish the existence of weak solutions for (1) by a compactness method using a Faedo-Galerkin scheme. Let us denote

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2} > 0.$$

Our main result is the following

Theorem 5. Take γ such that $\nu\lambda_1 < 2\gamma$. If $f \in L^2(\tau, T; V')$, $g : [\tau, T] \times C_\gamma(H) \rightarrow (L^2(\Omega))^2$ satisfying the assumptions (g1)–(g3), and $\phi \in C_\gamma(H)$ are given, then there exists a unique weak solution of (1).

Proof. The uniqueness of solution can be obtained in the following way: consider two solutions, u and v of (1) with the same initial data, and denote $w = u - v$. We note that

$$\begin{aligned} & |b(u(t), u(t), u(t) - v(t)) - b(v(t), v(t), u(t) - v(t))| \\ = & |b(u(t), u(t), u(t) - v(t)) \mp b(v(t), u(t), u(t) - v(t)) - b(v(t), v(t), u(t) - v(t))| \\ = & |b(w(t), u(t), w(t))|, \end{aligned}$$

where we have used the standard property $b(v, w, w) = 0$. We shall take into account the estimate

$$\|u\|_{(L^4(\Omega))^2}^2 \leq 2^{-1/2} |u| \|u\|, \quad (3)$$

used already in [9], which is a slight improvement of Lemma 3.3 in [23, p.291] (the proof follows the same lines of that lemma using a remark given in [15]). Then we have

$$\begin{aligned} |b(w, u, w)| & \leq \|w\|_{(L^4(\Omega))^2}^2 \|u\| \\ & \leq 2^{-1/2} |w| \|w\| \|u\| \quad \forall u, w \in V. \end{aligned}$$

Then, from the equation satisfied by w and the integration by parts formula, we obtain for all $t \in [\tau, T]$ that

$$\begin{aligned} & |w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds \\ = & -2 \int_\tau^t b(w(s), u(s), w(s)) ds + 2 \int_\tau^t (g(s, u_s) - g(s, v_s), w(s)) ds \\ \leq & 2^{1/2} \int_\tau^t |w(s)| \|w(s)\| \|u(s)\| ds + 2L_g \int_\tau^t \|w_s\|_\gamma |w(s)| ds. \end{aligned}$$

Observe that $w(\theta) = 0$ if $\theta \leq \tau$. Therefore,

$$\begin{aligned}\|w_s\|_\gamma &= \sup_{\theta \leq 0} e^{\gamma\theta} |w(s + \theta)| \\ &\leq \sup_{\theta \in [\tau-s, 0]} |w(s + \theta)|, \quad \text{for } \tau \leq s \leq T.\end{aligned}$$

So, it yields that

$$\begin{aligned}& |w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds \\ &\leq 2^{1/2} \int_\tau^t |w(s)| \|w(s)\| \|u(s)\| ds + 2L_g \int_\tau^t \sup_{r \in [\tau, s]} |w(r)| |w(s)| ds \\ &\leq \nu \int_\tau^t \|w(s)\|^2 ds + \frac{1}{2\nu} \int_\tau^t \|u(s)\|^2 |w(s)|^2 ds + 2L_g \int_\tau^t \sup_{r \in [\tau, s]} |w(r)|^2 ds,\end{aligned}$$

where we have used the Young inequality. Now we deduce that

$$\sup_{r \in [\tau, t]} |w(r)|^2 \leq \left(\frac{1}{2\nu} + 2L_g \right) \int_\tau^t (1 + \|u(s)\|^2) \sup_{r \in [\tau, s]} |w(r)|^2 ds,$$

whence the Gronwall lemma finishes the proof of uniqueness.

For the existence, we split the proof in several steps.

Step 1: A Galerkin scheme. Let us consider $\{v_j\} \subset V$, the orthonormal basis of H of all the eigenfunctions of the Stokes problem in Ω with homogeneous Dirichlet boundary conditions. Denote $V_m = \text{span}[v_1, \dots, v_m]$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j) v_j$.

Define also

$$u^m(t) = \sum_{j=1}^m \alpha_{m,j}(t) v_j,$$

where the upper script m will be used instead of (m) for short since no confusion is possible with powers of u , and where the coefficients $\alpha_{m,j}$ are required to satisfy the following system:

$$\begin{aligned}& \frac{d}{dt} (u^m(t), v_j) + \nu ((u^m(t), v_j)) + b(u^m(t), u^m(t), v_j) \\ &= \langle f(t), v_j \rangle + (g(t, u_t^m), v_j), \quad 1 \leq j \leq m,\end{aligned}\tag{4}$$

and where the equations are understood in the sense of $\mathcal{D}'(\tau, T)$, and the initial conditions are $u^m(\tau + s) = P_m \phi(s)$ for $s \in (-\infty, 0]$.

The above system of ordinary functional differential equations with infinite delay fulfills the conditions for existence and uniqueness of local solution of [12, Th.1.1, p.36].

Next, we will deduce a priori estimates that assure that the solutions do exist for all time $[\tau, T]$.

Step 2: A priori estimates.

Multiplying (4) by $\alpha_{m,j}$, summing up and using Poincaré's inequality, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |u^m(t)|^2 + \frac{\nu\lambda_1}{2} |u^m(t)|^2 + \frac{\nu}{2} \|u^m(t)\|^2 \\
& \leq \langle f(t), u^m(t) \rangle + (g(t, u_t^m), u^m(t)) \\
& \leq \|f(t)\|_* \|u^m(t)\| + L_g \|u_t^m\|_\gamma |u^m(t)| \\
& \leq \frac{\nu}{4} \|u^m(t)\|^2 + \frac{\|f(t)\|_*^2}{\nu} + L_g \|u_t^m\|_\gamma^2.
\end{aligned}$$

Hence

$$\begin{aligned}
& |u^m(t)|^2 + \int_\tau^t e^{-\nu\lambda_1(t-s)} \frac{\nu}{2} \|u^m(s)\|^2 ds \\
& \leq e^{-\nu\lambda_1(t-\tau)} |u(\tau)|^2 + \int_\tau^t 2e^{-\nu\lambda_1(t-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds. \tag{5}
\end{aligned}$$

Further

$$\begin{aligned}
\|u_t^m\|_\gamma^2 & \leq \max \left\{ \sup_{\theta \in (-\infty, \tau-t]} e^{2\gamma\theta} |\phi(\theta + t - \tau)|^2, \right. \\
& \quad \sup_{\theta \in [\tau-t, 0]} \left[e^{2\gamma\theta - \nu\lambda_1(t-\tau+\theta)} |u(\tau)|^2 \right. \\
& \quad \left. \left. + e^{2\gamma\theta} \int_\tau^{t+\theta} 2e^{-\nu\lambda_1(t+\theta-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds \right] \right\}.
\end{aligned}$$

On the one hand

$$\begin{aligned}
\sup_{\theta \in (-\infty, \tau-t]} e^{\gamma\theta} |\phi(\theta + t - \tau)| & = \sup_{\theta \leq 0} e^{\gamma(\theta - (t-\tau))} |\phi(\theta)| \\
& = e^{-\gamma(t-\tau)} \|\phi\|_\gamma.
\end{aligned}$$

On the other, as we are assuming that $\nu\lambda_1 < 2\gamma$,

$$\sup_{\theta \in [\tau-t, 0]} e^{2\gamma\theta - \nu\lambda_1(t-\tau+\theta)} |u(\tau)|^2 \leq e^{-\nu\lambda_1(t-\tau)} |u(\tau)|^2$$

and

$$\begin{aligned}
& \sup_{\theta \in [\tau-t, 0]} e^{2\gamma\theta} \int_\tau^{t+\theta} 2e^{-\nu\lambda_1(t+\theta-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds \\
& \leq \int_\tau^t 2e^{-\nu\lambda_1(t-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds.
\end{aligned}$$

Collecting these inequalities we deduce

$$\|u_t^m\|_\gamma^2 \leq e^{-\nu\lambda_1(t-\tau)} \|\phi\|_\gamma^2 + \int_\tau^t 2e^{-\nu\lambda_1(t-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds.$$

By the Gronwall lemma we have

$$\|u_t^m\|_\gamma^2 \leq e^{-(\nu\lambda_1 - 2L_g)(t-\tau)} \|\phi\|_\gamma^2 + 2\nu^{-1} \int_\tau^t e^{-(\lambda_1\nu - 2L_g)(t-s)} \|f(s)\|_*^2 ds.$$

Then we obtain the following estimates: there exists a constant C , depending on some constants of the problem (namely, λ_1 , ν , L_g and f), and on τ , T and $R > 0$, such that

$$\|u_t^m\|_\gamma^2 \leq C(\tau, T, R) \quad \forall t \in [\tau, T], \|\phi\|_\gamma \leq R, \quad \forall m \geq 1. \quad (6)$$

In particular, this implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau, T; H). \quad (7)$$

Now, it follows from (5) and (6) that

$$\begin{aligned} & e^{-\nu\lambda_1(T-\tau)} \frac{\nu}{2} \int_\tau^T \|u^m(s)\|^2 ds \\ & \leq \int_\tau^T e^{-\nu\lambda_1(T-s)} \frac{\nu}{2} \|u^m(s)\|^2 ds \\ & \leq |u(\tau)|^2 + \int_\tau^T 2e^{-\nu\lambda_1(T-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s^m\|_\gamma^2) ds \\ & \leq R^2 + \int_\tau^T 2e^{-\nu\lambda_1(T-s)} (\|f(s)\|_*^2/\nu + L_g C(\tau, T, R)) ds, \end{aligned} \quad (8)$$

so that we conclude the existence of another constant (relabelled the same) $C(\tau, T, R)$ such that

$$\|u^m\|_{L^2(\tau, T; V)}^2 \leq C(\tau, T, R) \quad \forall m. \quad (9)$$

Then, as (3) implies

$$\begin{aligned} |b(u, u, v)| &= |-b(u, v, u)| \\ &\leq \|u\|_{(L^4(\Omega))^2} \|v\| \\ &\leq 2^{-1/2} \|u\| \|u\| \|v\|, \quad \forall u, v \in V, \end{aligned}$$

from (4) we obtain

$$\|(u^m)'\|_* \leq \nu \|u^m\| + 2^{-1/2} \|u^m\| \|u^m\| + \|f\|_* + \lambda_1^{-1/2} |g(t, u_t^m)|,$$

which combined with Remark 1, (6), (7) and (9) implies that

$$\{(u^m)'\} \text{ is bounded in } L^2(\tau, T; V'). \quad (10)$$

Step 3: Approximation in $C_\gamma(H)$ of the initial datum. For the initial datum $\phi \in C_\gamma(H)$, we have used the projections in the Galerkin scheme in Step 1. Let us check that

$$P_m \phi \rightarrow \phi \quad \text{in } C_\gamma(H). \quad (11)$$

Indeed, if not, there would exist $\varepsilon > 0$ and a subsequence, that we relabel the same, such that

$$e^{\gamma\theta_m} |P_m\phi(\theta_m) - \phi(\theta_m)| > \varepsilon. \quad (12)$$

One can assume that $\theta_m \rightarrow -\infty$. Otherwise, if $\theta_m \rightarrow \theta$, then $P_m\phi(\theta_m) \rightarrow \phi(\theta)$, since $|P_m\phi(\theta_m) - \phi(\theta)| \leq |P_m\phi(\theta_m) - P_m\phi(\theta)| + |P_m\phi(\theta) - \phi(\theta)| \rightarrow 0$ as $m \rightarrow +\infty$. But with $\theta_m \rightarrow -\infty$ as $m \rightarrow +\infty$, if we denote $x = \lim_{\theta \rightarrow -\infty} e^{\gamma\theta}\phi(\theta)$, we obtain that

$$\begin{aligned} & e^{\gamma\theta_m} |P_m\phi(\theta_m) - \phi(\theta_m)| \\ &= |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - e^{\gamma\theta_m}\phi(\theta_m)| \\ &\leq |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - P_mx| + |P_mx - x| + |x - e^{\gamma\theta_m}\phi(\theta_m)| \rightarrow 0. \end{aligned}$$

This is a contradiction with (12), so (11) holds.

Step 4: Energy method and compactness results. Now we combine some well-known compactness results with an energy method to pass to the limit in a subsequence of $\{u^m\}$ to obtain a solution of (1).

From the assumptions on the operator g and Step 2 we deduce using the Compactness Theorem (cf. [17]) that there exist a subsequence (which we relabel the same) $\{u^m\}$, an element $u \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V)$ with $u' \in L^2(\tau, T; V')$, and $\xi \in L^2(\tau, T; (L^2(\Omega))^2)$ such that

$$\begin{cases} u^m \overset{*}{\rightharpoonup} u & \text{weakly star in } L^\infty(\tau, T; H), \\ u^m \rightharpoonup u & \text{weakly in } L^2(\tau, T; V), \\ (u^m)' \rightharpoonup u' & \text{weakly in } L^2(\tau, T; V'), \\ u^m \rightarrow u & \text{strongly in } L^2(\tau, T; H) \\ g(\cdot, u^m) \rightharpoonup \xi & \text{weakly in } L^2(\tau, T; (L^2(\Omega))^2). \end{cases} \quad (13)$$

Observe that $u \in C([\tau, T]; H)$ (cf. [23, p.261]).

Using (13) we also deduce that we can assume that

$$u^m(t) \rightarrow u(t) \quad \text{in } H \text{ a.e. } t \in (\tau, T), \quad (14)$$

which is not enough for our purposes.

However, we can obtain convergence for all $t \in [\tau, T]$ with a little more effort and in a more general sense. Observe that

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r) dr \quad \text{in } V', \quad \forall s, t \in [\tau, T],$$

and by (10) we have that $\{u^m\}$ is equi-continuous on $[\tau, T]$ with values in V' .

Since the injection of V into H is compact, by [1, Th.VI.4, p.90] the injection of H into V' is compact too. So, from (6) and the equi-continuity in V' , by the Ascoli-Arzelà theorem we have that

$$u^m \rightarrow u \quad \text{in } C([\tau, T]; V'). \quad (15)$$

This, jointly with (6), allows us to claim that for any sequence $\{t_m\} \subset [\tau, T]$, with $t_m \rightarrow t$, one has

$$u^m(t_m) \rightharpoonup u(t) \quad \text{weakly in } H, \quad (16)$$

where we have used (15) in order to identify which is the weak limit.

Hereafter we will denote again as u the function defined by ϕ in $(-\infty, \tau]$ pasted with the above limit in $[\tau, T]$.

Our goal now is to prove that in fact

$$u^m \rightarrow u \quad \text{in } C([\tau, T]; H). \quad (17)$$

If it were not so, then, taking into account that $u \in C([\tau, T]; H)$, there would exist $\varepsilon > 0$, a value $t_0 \in [\tau, T]$ and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau, T]$ with $\lim_{m \rightarrow +\infty} t_m = t_0$ such that

$$|u^m(t_m) - u(t_0)| \geq \varepsilon \quad \forall m.$$

To prove that this is absurd, we will use an energy method.

Observe that the following energy inequality holds for all u^m :

$$\begin{aligned} & \frac{1}{2}|u^m(t)|^2 + \frac{\nu}{2} \int_s^t \|u^m(r)\|^2 dr \\ & \leq \int_s^t \langle f(r), u^m(r) \rangle dr + \frac{1}{2}|u^m(s)|^2 + C(t-s), \quad \forall s, t \in [\tau, T], \end{aligned} \quad (18)$$

where $C = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |g(r, u_r^m)|^2 dr \leq D(t-s), \quad \text{for all } \tau \leq s \leq t \leq T, \quad (19)$$

by (g2), (g3) and (6). On the other hand, observe that by (13), passing to the limit in (4), we have that $u \in C([\tau, T]; H)$ is a solution of a similar problem to (1), namely,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = \langle f(t), v \rangle + (\xi(t), v), \quad \forall v \in V,$$

fulfilled with the initial data $u(\tau) = \phi(0)$. Therefore, it satisfies the energy equality

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr = |u(s)|^2 + 2 \int_s^t \langle f(r), u(r) \rangle + (\xi, u(r)) dr, \quad \forall s, t \in [\tau, T].$$

On other hand, from last convergence in (13) we deduce that

$$\begin{aligned} \int_s^t |\xi(r)|^2 dr & \leq \liminf_{m \rightarrow +\infty} \int_s^t |g(r, u_r^m)|^2 dr \\ & \leq D(t-s), \quad \forall \tau \leq s \leq t \leq T. \end{aligned}$$

So, u also satisfies inequality (18) with the same constant C .

Now, consider the functions $J_m, J : [\tau, T] \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J_m(t) & = \frac{1}{2}|u^m(t)|^2 - \int_\tau^t \langle f, u^m(r) \rangle dr - Ct, \\ J(t) & = \frac{1}{2}|u(t)|^2 - \int_\tau^t \langle f, u(r) \rangle dr - Ct, \end{aligned}$$

with C the constant given in (18). From (18) and the analogous inequality for u , it is clear that J_m and J are non-increasing (and continuous) functions. Moreover, by (14),

$$J_m(t) \rightarrow J(t) \text{ a.e. } t \in [\tau, T]. \quad (20)$$

Now we are ready to prove that

$$u^m(t_m) \rightarrow u(t_0) \text{ in } H. \quad (21)$$

Firstly, recall from (16) that

$$u^m(t_m) \rightharpoonup u(t_0) \text{ weakly in } H. \quad (22)$$

So, we have that

$$|u(t_0)| \leq \liminf_{m \rightarrow +\infty} |u^m(t_m)|.$$

Therefore, if we show that

$$\limsup_{m \rightarrow +\infty} |u^m(t_m)| \leq |u(t_0)|, \quad (23)$$

we obtain that $\lim_{m \rightarrow +\infty} |u^m(t_m)| = |u(t_0)|$, which jointly with (22) implies (21).

Now, observe that the case $t_0 = \tau$ follows directly from Step 3 and (18) with $s = \tau$. So, we may assume that $t_0 > \tau$. This is important, since we will approach this value t_0 from the left by a sequence $\{\tilde{t}_k\}$, i.e. $\lim_{k \rightarrow +\infty} \tilde{t}_k \nearrow t_0$, being $\{\tilde{t}_k\}$ values where (20) holds. Since $u(\cdot)$ is continuous at t_0 , for any $\varepsilon > 0$ there is k_ε such that

$$|J(\tilde{t}_k) - J(t_0)| < \varepsilon/2, \quad \forall k \geq k_\varepsilon.$$

On other hand, taking $m \geq m(k_\varepsilon)$ such that $t_m > \tilde{t}_{k_\varepsilon}$, as J_m is non-increasing and for all \tilde{t}_k the convergence (20) holds, one has that

$$J_m(t_m) - J(t_0) \leq |J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| + |J(\tilde{t}_{k_\varepsilon}) - J(t_0)|,$$

and obviously, taking $m \geq m'(k_\varepsilon)$, it is possible to obtain $|J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| < \varepsilon/2$. It can also be deduced from (13) that

$$\int_\tau^{t_m} \langle f, u^m(r) \rangle dr \rightarrow \int_\tau^{t_0} \langle f, u(r) \rangle dr,$$

so we conclude that (23) holds. Thus, (21) and finally (17) are also true, as we wanted to check.

This also implies, thanks to Step 3, that

$$u_t^m \rightarrow u_t \text{ in } C_\gamma(H) \quad \forall t \leq T.$$

Indeed,

$$\begin{aligned} & \sup_{\theta \leq 0} e^{\gamma\theta} |u^m(t + \theta) - u(t + \theta)| \\ &= \max \left\{ \sup_{\theta \in (-\infty, \tau-t]} e^{\gamma\theta} |P_m \phi(\theta + t - \tau) - \phi(\theta + t - \tau)|, \right. \\ & \quad \left. \sup_{\theta \in [\tau-t, 0]} e^{\gamma\theta} |u^m(t + \theta) - u(t + \theta)| \right\} \\ &\leq \max \left\{ e^{\gamma(\tau-t)} \|P_m \phi - \phi\|_\gamma, \max_{\theta \in [\tau, t]} |u^m(\theta) - u(\theta)| \right\} \rightarrow 0. \end{aligned}$$

Therefore, we identify the weak limit ξ from (13), and indeed, from the above convergence and since g satisfies (g3), we have that

$$g(\cdot, u^m) \rightarrow g(\cdot, u.) \quad \text{in } L^2(\tau, T; (L^2(\Omega))^2).$$

Thus, we can pass to the limit finally in (4) concluding that u solves (1). \square

Proposition 6 (Continuity of solutions w.r.t. initial data). *Under the assumptions of Theorem 5, the solutions obtained for (1) are continuous with respect to the initial condition.*

Namely, denoting u^i , for $i = 1, 2$, the corresponding solution to initial data $\phi^i \in C_\gamma(H)$, the following estimates hold

$$\begin{aligned} & \max_{r \in [\tau, t]} |u^1(r) - u^2(r)|^2 \\ & \leq \left(|\phi^1(0) - \phi^2(0)|^2 + \frac{Lg}{2\gamma} \|\phi^1 - \phi^2\|_\gamma^2 \right) e^{\int_\tau^t (3Lg + \frac{1}{4\nu} \|u^1(s)\|^2) ds}, \end{aligned} \quad (24)$$

$$\|u_t^1 - u_t^2\|_\gamma^2 \leq \left(1 + \frac{Lg}{2\gamma} \right) \|\phi^1 - \phi^2\|_\gamma^2 e^{\int_\tau^t (3Lg + \frac{1}{4\nu} \|u^1(s)\|^2) ds}. \quad (25)$$

Proof. Consider the equations satisfied by u^i for $i = 1$ and 2 , acting on the element $u^1 - u^2$, and take the difference. This gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u^1(t) - u^2(t)|^2 + \nu \|u^1(t) - u^2(t)\|^2 \\ & + b(u^1(t), u^1(t), u^1(t) - u^2(t)) - b(u^2(t), u^2(t), u^1(t) - u^2(t)) \\ & = (g(t, u_t^1) - g(t, u_t^2), u^1 - u^2). \end{aligned}$$

Arguing as in the proof of Theorem 5 and using (3) we have

$$\begin{aligned} & |b(u^1(t), u^1(t), u^1(t) - u^2(t)) - b(u^2(t), u^2(t), u^1(t) - u^2(t))| \\ & = |b(u^1(t) - u^2(t), u^1(t), u^1(t) - u^2(t))| \\ & \leq 2^{-1/2} |u^1(t) - u^2(t)| \|u^1(t) - u^2(t)\| \|u^1(t)\|. \end{aligned}$$

Thus, by the Lipschitz assumption on g , and the fact that for $s \in [\tau, t]$ one has

$$\begin{aligned} \|u_s^1 - u_s^2\|_\gamma & = \sup_{\theta \leq 0} e^{\gamma\theta} |u^1(s + \theta) - u^2(s + \theta)| \\ & = \max \left\{ \sup_{\theta \in (-\infty, \tau - s]} e^{\gamma\theta} |\phi^1(s - \tau + \theta) - \phi^2(s - \tau + \theta)|, \right. \\ & \quad \left. \sup_{\theta \in [\tau - s, 0]} e^{\gamma\theta} |u^1(s + \theta) - u^2(s + \theta)| \right\} \\ & \leq \max \left\{ e^{\gamma(\tau - s)} \|\phi^1 - \phi^2\|_\gamma, \max_{\theta \in [\tau, s]} |u^1(\theta) - u^2(\theta)| \right\}, \end{aligned} \quad (26)$$

we conclude that for all $t \in [\tau, T]$

$$\begin{aligned} \frac{1}{2}|u^1(t) - u^2(t)|^2 &\leq \frac{1}{2}|\phi^1(0) - \phi^2(0)|^2 + L_g \|\phi^1 - \phi^2\|_\gamma \int_\tau^t e^{\gamma(\tau-s)} |u^1(s) - u^2(s)| ds \\ &\quad + L_g \int_\tau^t |u^1(s) - u^2(s)| \max_{\theta \in [\tau, s]} |u^1(\theta) - u^2(\theta)| ds \\ &\quad + \frac{1}{8\nu} \int_\tau^t \|u^1(s)\|^2 |u^1(s) - u^2(s)|^2 ds. \end{aligned}$$

If we now substitute t by $r \in [\tau, t]$ and consider the maximum when varying this r , from the above we can conclude that

$$\begin{aligned} \max_{r \in [\tau, t]} |u^1(r) - u^2(r)|^2 &\leq |\phi^1(0) - \phi^2(0)|^2 + \frac{L_g}{2\gamma} \|\phi^1 - \phi^2\|_\gamma^2 \\ &\quad + \int_\tau^t \left(3L_g + \frac{1}{4\nu} \|u^1(s)\|^2 \right) \max_{r \in [\tau, s]} |u^1(r) - u^2(r)|^2 ds. \end{aligned}$$

Hence, by the Gronwall lemma we obtain (24).

Finally, (25) follows from (24) and (26). \square

Proposition 7 (Continuity of solutions w.r.t. initial time). *Under the assumptions of Theorem 5, the solutions obtained for (1) are continuous with respect to the initial time, i.e. let us denote $u(\cdot; s, \phi)$ the solution of (1) with initial time s . Then, for each $t \in [\tau, T]$ and $\phi \in C_\gamma(H)$ fixed, the mapping $[\tau, t] \ni s \mapsto u_t(\cdot; s, \phi) \in C_\gamma(H)$ is continuous.*

Proof. Fix one value $s_0 \in (\tau, T)$. We will prove the continuity of the above application in two steps, from the left and from the right (the extremal cases s_0 equal to τ or T are analogous).

Step 1: Let us start proving the continuity of the map from the right. Assume that $s_0 < s$ and let $t \geq s$ and $\phi \in C_\gamma(H)$ be fixed. Then by (25) we get

$$\|u_t(\cdot; s_0, \phi) - u_t(\cdot; s, \phi)\|_\gamma^2 \leq \left(1 + \frac{L_g}{2\gamma}\right) \|u_s(\cdot; s_0, \phi) - \phi\|_\gamma^2 e^{\int_s^t (3L_g + \frac{1}{4\nu} \|u(r; s_0, \phi)\|^2) dr}. \quad (27)$$

We note that

$$\|u_s(\cdot; s_0, \phi) - \phi\|_\gamma \rightarrow 0 \quad \text{as } s \searrow s_0. \quad (28)$$

Indeed,

$$\begin{aligned} \|u_s(\cdot; s_0, \phi) - \phi\|_\gamma &= \sup_{\theta \leq 0} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)| \\ &= \max \left\{ \sup_{\theta \in (-\infty, -\tilde{T}]} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)|, \right. \\ &\quad \left. \sup_{\theta \in [-\tilde{T}, 0]} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)| \right\}. \end{aligned}$$

Now, in view of the definition of the space $C_\gamma(H)$, and particularly by the fact that there exists $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta)$ in H , for any $\varepsilon > 0$ one can find $\bar{T} > 0$ and $0 < \bar{s}$, both depending on ε , such that

$$\begin{aligned} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)| &= e^{\gamma\theta} |\phi(s - s_0 + \theta) - \phi(\theta)| \\ &\leq \varepsilon \quad \text{if } s - s_0 \leq \bar{s}, \theta \leq -\bar{T} - \bar{s} = -\tilde{T}. \end{aligned}$$

Thus, for any $s \leq \bar{s} + s_0$,

$$\sup_{\theta \in (-\infty, -\tilde{T}]} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)| \leq \varepsilon.$$

On the other hand, $u(\cdot; s_0, \phi) \in C([-T, t], H)$ and $u(s_0 + \theta) = \phi(\theta)$, so that by uniform continuity there exists $\eta \leq \bar{s}$, depending on ε , such that

$$\sup_{\theta \in [-\tilde{T}, 0]} e^{\gamma\theta} |u(s + \theta; s_0, \phi) - \phi(\theta)| \leq \varepsilon \quad \text{if } |s - s_0| < \eta.$$

Hence, using (28) in (27), we deduce that $\|u_t(\cdot; s_0, \phi) - u_t(\cdot; s, \phi)\|_\gamma \rightarrow 0$ as $s \searrow s_0$.

Step 2: Let us now prove the continuity of the map from the left. Assume that $s < s_0$ and let $\phi \in C_\gamma(H)$, $t \geq s_0$ be fixed. Again by (25) we get

$$\|u_t(\cdot; s_0, \phi) - u_t(\cdot; s, \phi)\|_\gamma^2 \leq \left(1 + \frac{Lg}{2\gamma}\right) \|\phi - u_{s_0}(\cdot; s, \phi)\|_\gamma^2 e^{\int_{s_0}^t (3Lg + \frac{1}{4\nu} \|u(r; s_0, \phi)\|^2) dr}. \quad (29)$$

We will prove that

$$\|\phi - u_{s_0}(\cdot; s, \phi)\|_\gamma \rightarrow 0. \quad (30)$$

If not, we can obtain $\varepsilon > 0$ and a subsequence u^{s_n} , $s_n \nearrow s_0$, such that

$$\|u_{s_0}(\cdot; s_n, \phi) - \phi\|_\gamma \geq \varepsilon \text{ for any } n. \quad (31)$$

Let us denote $v^s(\cdot) = u(\cdot + s - s_0)$. Then v^s is a weak solution of (1) with initial data $v_{s_0}^s = \phi$ and with $f(r)$ and $g(r, \cdot)$ replaced by $f^s(r) = f(r + s - s_0)$ and $g^s(r, \cdot) = g(r + s - s_0, \cdot)$, respectively. It is well known that $f^s \rightarrow f$ in $L^2(s_0, T; V')$ as $s \rightarrow s_0$ for any $T > s_0$ (cf. [8, Ch.IV]). Arguing as in the proof of Theorem 5 we obtain the existence of a subsequence of $\{v^{s_n}\}$ (relabelled the same) and a function $v(\cdot)$ such that $v^{s_n} \rightarrow v$ in $C([s_0, T], H)$. Since $v(s_0) = \phi(0)$, we will see that this implies that

$$\sup_{\theta \in [-\tilde{T}, 0]} |u(s_0 + \theta; s_n, \phi) - \phi(\theta)| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (32)$$

for all $\tilde{T} > 0$. Indeed, if not, then there would exist $\varepsilon > 0$ and a sequence $\theta_n \rightarrow \theta_0$ such that

$$|u(s_0 + \theta_n; s_n, \phi) - \phi(\theta_0)| \geq \varepsilon \text{ for all } n. \quad (33)$$

We can split the sequence $\{\theta_n\}_n$ in two subsequences and assume that either $\theta_n \geq s_n - s_0$ or $\theta_n < s_n - s_0$ for all n . If $\theta_n \geq s_n - s_0$, then $\theta_0 = 0$ and

$$\|u(s_0 + \theta_n; s_n, \phi) - \phi(0)\| = \|v^{s_n}(2s_0 - s_n + \theta_n) - \phi(0)\| \rightarrow 0,$$

as $n \rightarrow \infty$. If $\theta_n < s_n - s_0$, then

$$\|u(s_0 + \theta_n; s_n, \phi) - \phi(\theta_0)\| = \|\phi(s_0 - s_n + \theta_n) - \phi(\theta_0)\| \rightarrow 0.$$

Hence, we obtain a contradiction with (33).

Now, we note that for any $\tilde{T} > 0$, we may split

$$\|u_{s_0}(\cdot; s_n, \phi) - \phi\|_\gamma = \max \left\{ \sup_{\theta \in (-\infty, -\tilde{T}]} e^{\gamma\theta} |\phi(s_0 - s_n + \theta) - \phi(\theta)|, \sup_{\theta \in [-\tilde{T}, 0]} e^{\gamma\theta} |u(s_0 + \theta; s_n, \phi) - \phi(\theta)| \right\}.$$

Fix some $\bar{s} \geq s_0 - s_n$. Arguing as before for any $\varepsilon > 0$ one can find \bar{T} , depending on ε , such that

$$e^{\gamma\theta} |\phi(s_0 - s_n + \theta) - \phi(\theta)| \leq \varepsilon \quad \text{if } \theta \leq -\bar{T} - \bar{s} = -\tilde{T}.$$

Thus using (32) we deduce that $\|u_{s_0}(\cdot; s_n, \phi) - \phi\|_\gamma \rightarrow 0$, which contradicts (31). Therefore, we obtain (30), which jointly with (29), implies that $\|u_t(\cdot; s_0, \phi) - u_t(\cdot; s, \phi)\|_\gamma \rightarrow 0$ as $s \nearrow s_0$. \square

3 Stationary solutions and their stability

In this section we are interested in proving that problem (1), with some obvious restrictions, admits stationary solutions, and under additional assumptions, that in fact the stationary solution is unique and is globally asymptotically exponentially stable.

The restrictions we must impose to give sense to an stationary solution are that $f \in V'$ and g are now autonomous, i.e. without dependence on time. When necessary, we will identify an element $w \in H$ as the constant function of time with value w , which is an element of $C_\gamma(H)$. So, consider the following equation,

$$\frac{du}{dt} + \nu Au + B(u) = f + g(u_t) \quad \forall t \geq 0, \quad (34)$$

where A is the Stokes operator. By a stationary solution to (34) we mean an element $u^* \in V$ such that

$$\nu((u^*, v)) + b(u^*, u^*, v) = \langle f, v \rangle + (g(u^*), v) \quad \forall v \in V. \quad (35)$$

Theorem 8. *Under the above assumptions and notation, if $\lambda_1^{-1}L_g < \nu$, then:*

- (a) *The problem (34) admits at least one stationary solution, which indeed belongs to $D(A)$ if $f \in (L^2(\Omega))^2$.*

Moreover, any such stationary solution (say u^) satisfies the estimate*

$$(\nu - \lambda_1^{-1}L_g)\|u^*\| \leq \|f\|_*. \quad (36)$$

- (b) *If besides*

$$(2\lambda_1)^{-1/2}\|f\|_* < (\nu - \lambda_1^{-1}L_g)^2, \quad (37)$$

then the stationary solution of (34) is unique.

Proof. It is analogous to that in [5, Th.3.1]. To be precise, in that reference an abstract constant C_Ω corresponding to the continuous injection of $H_0^1(\Omega)$ into $L^4(\Omega)$ appears. This constant, which can be obtained from (3), is made explicit here, namely $C_\Omega = (2\lambda_1)^{-1/4}$ (see also [23, 15, 9]). \square

Theorem 9. *Assume the assumptions in Theorem 5 with g and f independent of time, and suppose moreover that (37) holds. Then there exists a value $0 < \lambda < 2\gamma$ such that, denoting for brevity $u(t) = u(t; 0, \phi)$ the solution of (1) with $\tau = 0$ and $\phi \in C_\gamma(H)$, and $w(t) = u(t) - u^*$, with u^* the unique stationary solution given by Theorem 8, the following estimates hold for all $t \geq 0$:*

$$|w(t)|^2 \leq e^{-\lambda t} \left(|w(0)|^2 + \frac{L_g}{2\gamma - \lambda} \|\phi - u^*\|_\gamma^2 \right), \quad (38)$$

$$\|w_t\|_\gamma^2 \leq \max \left\{ e^{-2\gamma t} \|\phi - u^*\|_\gamma^2, e^{-\lambda t} \left(|w(0)|^2 + \frac{L_g}{2\gamma - \lambda} \|\phi - u^*\|_\gamma^2 \right) \right\}. \quad (39)$$

Proof. Considering equations (34) for $u(t)$ and (35) for u^* , one has

$$\frac{d}{dt}(w(t), v) + \nu((w(t), v)) + b(u(t), u(t), v) - b(u^*, u^*, v) = (g(u_t) - g(u^*), v).$$

From the energy equality and the fact that

$$b(u(t), u(t), w) - b(u^*, u^*, w) = b(w(t), u^*, w(t)),$$

combined with the Lipschitz condition on g , and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, we obtain

$$\frac{d}{dt}(e^{\lambda t} |w(t)|^2) \leq e^{\lambda t} (\lambda |w(t)|^2 - 2\nu \|w(t)\|^2 + 2|b(w(t), w(t), u^*)| + 2L_g \|w_t\|_\gamma |w(t)|).$$

From (36) and estimating b in the same way as in Proposition 6 we obtain

$$|b(w(t), u^*, w(t))| \leq \frac{(2\lambda_1)^{-1/2}}{\nu - \lambda_1^{-1} L_g} \|f\|_* \|w(t)\|^2.$$

Hence, using a Young inequality with $\delta > 0$ to be fixed later on, we conclude that

$$\begin{aligned} & \frac{d}{dt}(e^{\lambda t} |w(t)|^2) \\ & \leq e^{\lambda t} (-2\nu + \lambda \lambda_1^{-1} + \frac{2(2\lambda_1)^{-1/2}}{\nu - \lambda_1^{-1} L_g} \|f\|_* + \delta \lambda_1^{-1} L_g) \|w(t)\|^2 + \frac{L_g}{\delta} e^{\lambda t} \|w_t\|_\gamma^2. \end{aligned}$$

Therefore, integrating from 0 to t , we have

$$\begin{aligned} e^{\lambda t} |w(t)|^2 & \leq |w(0)|^2 + \frac{L_g}{\delta} \int_0^t e^{\lambda s} \|w_s\|_\gamma^2 ds \\ & \quad + (-2\nu + \lambda \lambda_1^{-1} + \frac{2(2\lambda_1)^{-1/2}}{\nu - \lambda_1^{-1} L_g} \|f\|_* + \delta \lambda_1^{-1} L_g) \int_0^t e^{\lambda s} \|w(s)\|^2 ds. \quad (40) \end{aligned}$$

In order to control the term $\int_0^t e^{\lambda s} \|w_s\|_\gamma^2 ds$, we proceed as follows.

$$\begin{aligned}
& \int_0^t e^{\lambda s} \sup_{\theta \leq 0} e^{2\gamma\theta} |w(s+\theta)|^2 ds \\
&= \int_0^t e^{\lambda s} \max\left\{ \sup_{\theta \leq -s} e^{2\gamma\theta} |w(s+\theta)|^2, \sup_{\theta \in [-s, 0]} e^{2\gamma\theta} |w(s+\theta)|^2 \right\} ds \\
&= \int_0^t \max\left\{ e^{-(2\gamma-\lambda)s} \|\phi - u^*\|_\gamma^2, \sup_{\theta \in [-s, 0]} e^{(2\gamma-\lambda)\theta} e^{\lambda(s+\theta)} |w(s+\theta)|^2 \right\} ds.
\end{aligned}$$

So, if $\lambda \leq 2\gamma$, using the above equality in (40), we obtain

$$\begin{aligned}
e^{\lambda t} |w(t)|^2 &\leq |w(0)|^2 + \frac{L_g}{\delta} \|\phi - u^*\|_\gamma^2 \int_0^t e^{(\lambda-2\gamma)s} ds \\
&\quad + \left(-2\nu + \lambda\lambda_1^{-1} + \frac{2(2\lambda_1)^{-1/2} \|f\|_*}{\nu - \lambda_1^{-1} L_g} + \delta\lambda_1^{-1} L_g + L_g(\lambda_1\delta)^{-1} \right) \\
&\quad \times \int_0^t \max_{r \in [0, s]} (e^{\lambda r} \|w(r)\|^2) ds.
\end{aligned}$$

Observe that the (optimal) choice of $\delta = 1$ makes $\delta + \delta^{-1}$ minimum and therefore the coefficient of the last integral is negative with a suitable choice of $\lambda \in (0, 2\gamma)$ by (37). So, we can omit this term and deduce that

$$e^{\lambda t} |w(t)|^2 \leq |w(0)|^2 + \frac{L_g}{2\gamma - \lambda} (1 - e^{(\lambda-2\gamma)t}) \|\phi - u^*\|_\gamma^2,$$

whence (38) follows.

Finally, (39) can be deduced in the following way:

$$\begin{aligned}
\|w_t\|_\gamma^2 &= \sup_{\theta \leq 0} e^{2\gamma\theta} |w(t+\theta)|^2 \\
&= \max\left\{ \sup_{\theta \in (-\infty, -t]} e^{2\gamma\theta} |\phi(t+\theta) - u^*|^2, \max_{\theta \in [-t, 0]} e^{2\gamma\theta} |w(t+\theta)|^2 \right\} \\
&= \max\left\{ e^{-2\gamma t} \|\phi - u^*\|_\gamma^2, \max_{\theta \in [-t, 0]} e^{2\gamma\theta} |w(t+\theta)|^2 \right\}.
\end{aligned}$$

The second term can be estimated using (38) and that $e^{(2\gamma-\lambda)\theta} \leq 1$ when $\theta \leq 0$. \square

4 Existence of pullback attractors

Our goal in this section is to prove more general results on the asymptotic behaviour of problem (1) than those shown in the above section.

Namely, we will establish for a suitable semi process related to problem (1) that we can assure the existence of a pullback attractor under less restrictive assumptions than those in Theorem 8 and Theorem 9. Observe that if the terms f and g in (1) do not depend on time, the problem becomes autonomous and the notions of pullback attractor and global attractor lead in fact to the same object.

In fact, we split the analysis in two subsections. The first one is devoted to pullback attractors of fixed bounded sets, which is the most usual framework. In a second subsection, we extend the previous result to the more recent framework of pullback attractors in a universe of families of time dependent sets with a tempered growth condition, following the ideas of [2, 20].

4.1 Pullback attractor for fixed bounded sets

In order to proceed, we start with some standard notions related to Dynamical Systems.

Definition 10. *Given a metric space (X, d) , a semi process U on X is a bi-parametric family of mappings $U(t, \tau) : X \rightarrow X$ for $-\infty < \tau \leq t < +\infty$, with the following properties:*

- (i) $U(t, \tau) \in C(X; X)$ for all $t \geq \tau$.
- (ii) $U(\tau, \tau) = Id$ (the identity map) for all $\tau \in \mathbb{R}$.
- (iii) $U(t, \tau) = U(t, r)U(r, \tau)$ for all $-\infty < \tau \leq r \leq t < +\infty$.

Next we recall some useful concepts in order to study the asymptotic behaviour of a semi process. (Although slightly different from the classical ones, this allows us to simplify significantly some requirements in order to ensure the existence of attractors, as can be seen e.g. in [3, Th.17].)

Definition 11. *For a semi process U defined on a metric space (X, d) , a family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ of subsets of X is said to be **pullback absorbing for bounded sets** if for any bounded set B of X , and any t there exists a time $\tau(B, t)$ such that*

$$U(t, \tau)B \subset B_0(t) \quad \forall \tau \leq \tau(B, t).$$

The semi process U is said to be \widehat{B}_0 -**pullback asymptotically compact** if for any t , and any sequences $\{\tau_n\}, \{x_n\} \subset X$ with $\tau_n \leq t$, $\lim_{n \rightarrow +\infty} \tau_n = -\infty$, and $x_n \in B_0(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .

Definition 12. *Finally, a family $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is said to be a pullback attractor for the semi process U if*

- (i) Each $\mathcal{A}(t)$ is a compact subset of X for all $t \in \mathbb{R}$.
- (ii) It is invariant, i.e.

$$U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t) \quad \forall \tau \leq t,$$

- (iii) It attracts bounded sets in a pullback sense, i.e. given a bounded B of X ,

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)B, \mathcal{A}(t)) = 0, \quad \forall t \in \mathbb{R},$$

where $\text{dist}(C_1, C_2)$ denotes the Hausdorff semi-distance between two sets C_1 and C_2 , that is,

$$\text{dist}(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} d(x, y).$$

The proof of the following result, which is a slight variant of Theorem 1.1 in [7], can be found in [20].

Theorem 13. Consider a family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ of nonempty subsets of X and a process U on X that is \widehat{B}_0 -asymptotically compact, and assume also that \widehat{B}_0 is pullback-absorbing for U . Then, the family of sets $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ given by

$$\mathcal{A}(t) = \overline{\bigcup_{B \text{ bounded}} \Lambda(B, t)}^X, \quad (41)$$

where

$$\Lambda(B, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B}^X \quad \forall t \in \mathbb{R},$$

is a pullback attractor for U .

Remark 14. Observe that the pullback attractor defined by (41) is the minimal family of closed sets that attracts all bounded sets, i.e. if $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t)\}$ also attracts bounded sets in a pullback sense and $\tilde{\mathcal{A}}(t)$ is closed for all t , then $\mathcal{A}(t) \subset \tilde{\mathcal{A}}(t)$.

We add here a result concerning the connectedness of the global attractor.

Proposition 15. Let X be a connected metric space. Assume that the semi process U satisfies additionally that for every t and $x \in X$ the map $(-\infty, t] \ni \tau \rightarrow U(t, \tau)x$ is continuous. If U possesses a pullback attractor \mathcal{A} , then $\mathcal{A}(t)$ is connected for every $t \in \mathbb{R}$.

Proof. Assume that for some t the set $\mathcal{A}(t)$ is not connected. Then $\mathcal{A}(t) = A_1 \cup A_2$, where A_i are compact non-empty disjoint sets, and there exist two open disjoint sets U_i for which $A_i \subset U_i$.

We define the following disjoint sets:

$$X_i = \{x \in X_i : U(t, \tau)x \subset U_i, \forall \tau \leq T(x)\}.$$

We shall obtain a contradiction if we prove that X_i are non-empty open sets and $X_1 \cup X_2 = X$, as X is connected.

First we state that $X_1 \cup X_2 = X$. For an arbitrary $x \in X$ there exists $T = T(x)$ such that $B := \bigcup_{\tau \leq T} U(t, \tau)x \subset U_1 \cup U_2$. The continuity of $(-\infty, t] \ni \tau \rightarrow U(t, \tau)x$ implies that B is connected. Hence, either $B \subset U_1$ or $B \subset U_2$. Thus, x belongs to one of the sets X_i .

On the other hand, each X_i is non-empty. Indeed, as A_i is non-empty, there exists at least one bounded set B such that $U_i \cap \Lambda(B, t) \neq \emptyset$. There exists $T(B)$ for which $U(t, \tau)B \subset U_1 \cup U_2$ for all $\tau \leq T$. On the other hand, $U_i \cap \Lambda(B, t) \neq \emptyset$ implies the existence of a sequence $y_n = U(t, \tau_n)x_n \in U(t, \tau_n)B$, $\tau_n \rightarrow -\infty$, such that $y_n \in U_i$. If we take n for which $\tau_n \leq T$, then the continuity of $(-\infty, t] \ni \tau \rightarrow U(t, \tau)x$ implies that $x_n \in X_i$. Thus, X_i is non-empty.

Finally, we check that X_i are open sets. Let $x \in X_i$ and B be a bounded neighborhood of x . There exists $T \geq 0$ such that $U(t, \tau)B \subset U_1 \cup U_2$, and also $U(t, \tau)x \in U_1$, $\forall \tau \leq T$. Since the map $x \rightarrow U(t, T)x$ is continuous, there exists a neighborhood $O \subset B$ of x such that $U(t, T)O$ belongs to U_1 . Arguing as before for each $y \in O$, we get $U(t, \tau)y \in U_1$, for all $\tau \leq T$. Hence $O \subset X_1$. Since x is arbitrary, X_i is open. \square

Now, by the results of Section 2 we are able to define correctly a semi process U and obtain some properties according to the above results.

Proposition 16. Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and that $g : \mathbb{R} \times C_\gamma(H) \rightarrow (L^2(\Omega))^2$ satisfies the assumptions (g1)–(g3) for all $\tau < T$. Suppose also that $\nu\lambda_1 < 2\gamma$. Then, the bi-parametric family of maps $U(t, \tau) : C_\gamma(H) \rightarrow C_\gamma(H)$, with $\tau \leq t$, given by

$$U(t, \tau)\phi = u_t$$

where u is the unique solution of (1), defines a semi process on $C_\gamma(H)$.

Proof. It is a consequence of Theorem 5 and Proposition 6. \square

Lemma 17. Under the assumptions of Proposition 16, the following estimates hold for a solution to (1) for all $t \geq \tau$:

$$\|u_t\|_\gamma^2 \leq e^{-(\nu\lambda_1 - 2L_g)(t-\tau)} \|\phi\|_\gamma^2 + \frac{2}{\nu} \int_\tau^t e^{-(\nu\lambda_1 - 2L_g)(t-s)} \|f(s)\|_*^2 ds, \quad (42)$$

$$\begin{aligned} \frac{\nu}{2} \int_\tau^t \|u(s)\|^2 ds &\leq e^{\nu\lambda_1(t-\tau)} |u(\tau)|^2 + e^{2L_g(t-\tau)} \|\phi\|_\gamma^2 + 2\nu^{-1} e^{-\nu\lambda_1\tau} \int_\tau^t e^{\nu\lambda_1 s} \|f(s)\|_*^2 ds \\ &\quad + 2\nu^{-1} e^{2L_g t - \nu\lambda_1\tau} \int_\tau^t e^{(\nu\lambda_1 - 2L_g)s} \|f(s)\|_*^2 ds. \end{aligned} \quad (43)$$

Proof. The uniform estimates that we require for the solutions which define the process U are analogous to those provided in the proof of Theorem 5, but there with the Galerkin approximations.

For the sake of brevity, we only sketch the main ideas:

Multiplying (2) by u , by the Young inequality, splitting the term related to the delay in the initial datum and the evolution of the solution for $t \geq \tau$, and using that $\nu\lambda_1 < 2\gamma$, we arrive to

$$\|u_t\|_\gamma^2 \leq e^{-\nu\lambda_1(t-\tau)} \|\phi\|_\gamma^2 + \int_\tau^t 2e^{-\nu\lambda_1(t-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s\|_\gamma^2) ds.$$

By the Gronwall lemma we conclude that (42) holds.

In the middle of the above manipulations we have omitted a positive term in the left hand side, namely $\|u\|_{L^2(\tau, T; V)}^2$. Indeed, analogously as we did in (8), we have

$$e^{-\nu\lambda_1(t-\tau)} \frac{\nu}{2} \int_\tau^t \|u(s)\|^2 ds \leq |u(\tau)|^2 + \int_\tau^t 2e^{-\nu\lambda_1(t-s)} (\|f(s)\|_*^2/\nu + L_g \|u_s\|_\gamma^2) ds.$$

Combining this with estimate (42), applying the Fubini theorem and rearranging coefficients, we conclude (43). \square

From now on we will assume that

$$2L_g < \nu\lambda_1, \quad (44)$$

and

$$\int_{-\infty}^0 e^{(\nu\lambda_1 - 2L_g)s} \|f(s)\|_*^2 ds < +\infty. \quad (45)$$

Remark 18. If we assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$, assumption (45) is equivalent to

$$\int_{-\infty}^t e^{-(\nu\lambda_1 - 2L_g)(t-s)} \|f(s)\|_*^2 ds < +\infty, \quad \forall t \in \mathbb{R}.$$

Corollary 19. Under the assumptions of Proposition 16, if moreover conditions (44) and (45) are satisfied, then the family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$, with $B_0(t) = B_{C_\gamma(H)}(0, \rho(t))$, where

$$\rho^2(t) = 1 + 2\nu^{-1} \int_{-\infty}^t e^{-(\nu\lambda_1 - 2L_g)(t-s)} \|f(s)\|_*^2 ds, \quad (46)$$

is pullback absorbing for bounded sets for the semi process U .

Proof. It follows immediately from Lemma 17. \square

Proposition 20. Under the assumptions of Corollary 19, the semi process U is \widehat{B}_0 -asymptotically compact.

Proof. Our proof relies on an energy method, analogous to that employed in Step 4 in the proof of Theorem 5. Again, to avoid unnecessary repetitions, we only sketch the main ideas.

Let $t_0 \in \mathbb{R}$, $u^n(\cdot)$ be a sequence of solutions in their respective intervals $[\tau_n, t_0]$, with initial data $\phi^n \in B_0(\tau_n) = B_{C_\gamma(H)}(0, \rho(\tau_n))$, where $\tau_n \rightarrow -\infty$ as $n \rightarrow +\infty$. Consider the sequence $\xi^n = u^n_{t_0}$. Then we will prove that this sequence is relatively compact in $C_\gamma(H)$.

Step 1: For brevity let us denote

$$\sigma = \nu\lambda_1 - 2L_g.$$

Consider two arbitrary values $0 < \bar{T} < T$. We will prove that $\xi^n|_{[-\bar{T}, 0]}$ is relatively compact in $C([-\bar{T}, 0]; H)$. It follows from (42) and (45) that there exists $n_0(t_0, T)$ such that $\tau_n \leq t_0 - T$ for $n \geq n_0(t_0, T)$, and

$$\|u_t^n\|_\gamma^2 \leq R(t_0, T) \quad \forall t \in [t_0 - T, t_0], \quad \forall n \geq n_0(t_0, T), \quad (47)$$

where

$$R(t_0, T) = 1 + \frac{2}{\nu} e^{-\sigma(t_0 - T)} \int_{-\infty}^{t_0} e^{\sigma s} \|f(s)\|_*^2 ds,$$

so that

$$\begin{aligned} |u^n(t)|^2 &\leq R(t_0, T), \quad \forall t \in [t_0 - T, t_0], \quad \forall n \geq n_0(t_0, T), \\ \|u^n_{t_0 - T}\|_\gamma^2 &\leq R(t_0, T), \quad \forall n \geq n_0(t_0, T). \end{aligned} \quad (48)$$

Let $y^n(\cdot) = u^n(\cdot + t_0 - T)$. Then, for each $n \geq 1$ such that $\tau_n < t_0 - T$, the function $y^n(\cdot)$ is a solution on $[0, T]$ of a similar problem to (1), namely with f replaced by $\tilde{f}(s) = f(t_0 - T + s)$ and g replaced by $\tilde{g}(s, \cdot) = g(t_0 - T + s, \cdot)$, and with $y_0^n = u^n_{t_0 - T}$, $y_T^n = u^n_{t_0} = \xi^n$. Then $\|y_0^n\|_\gamma$ satisfies the estimate in (48), for all $n \geq n_0(t_0, T)$. From (43) we have

$$\|y^n\|_{L^2(0, T; V)}^2 \leq K(t_0, T).$$

Hence, $\{y^n\}_n$ is bounded in $L^\infty(0, T; H)$ and $L^2(0, T; V)$, and $\{(y^n)'\}_n$ is bounded in $L^2(0, T; V')$. Thus, up to a subsequence (relabelled the same), for some function $y(\cdot)$ we have that

$$\begin{aligned} y^n &\overset{*}{\rightharpoonup} y && \text{weakly star in } L^\infty(0, T; H), \\ y^n &\rightharpoonup y && \text{weakly in } L^2(0, T; V), \\ (y^n)' &\rightharpoonup y' && \text{weakly in } L^2(0, T; V'), \\ y^n &\rightarrow y && \text{strongly in } L^2(0, T; H), \\ y^n(t) &\rightarrow y(t) && \text{a.e. } t \in (0, T). \end{aligned}$$

Moreover, reasoning as in the proof of Theorem 5 we obtain

$$y^n(t_n) \rightharpoonup y(t_0) \text{ weakly in } H,$$

if $t_n \rightarrow t_0 \in [0, T]$.

Also, by (g3) and (47) we obtain

$$\int_0^t |\tilde{g}(s, y_s^n)|^2 ds \leq Ct,$$

where $C > 0$ does not depend either on n or t , and also

$$\begin{aligned} \tilde{g}(\cdot, y^n) &\rightharpoonup \xi \text{ weakly in } L^2(0, T; (L^2(\Omega))^2), \\ \int_s^t |\tilde{g}(r, y_r^n)|^2 dr &\leq C(t-s), \\ \int_s^t |\xi(s)|^2 ds &\leq \liminf_{n \rightarrow +\infty} \int_s^t |\tilde{g}(r, y_r^n)|^2 dr \leq C(t-s), \forall 0 \leq s \leq t \leq T. \end{aligned}$$

Then, in a standard way, one can prove that $y(\cdot)$ is the unique weak solution to the problem

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla)u = -\nabla p + \tilde{f}(t) + \xi(t), \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0, x) = y(0, x), x \in \Omega. \end{cases} \quad (49)$$

Multiplying the equation in (1) by u^n , the equation in (49) by y , and integrating we obtain the energy inequality

$$\frac{1}{2}|z(t)|^2 + \nu \int_s^t \|z(r)\|^2 dr \leq \frac{1}{2}|z(s)|^2 + \int_s^t \langle \tilde{f}(r), z(r) \rangle dr + C(t-s), \quad 0 \leq s \leq t \leq T,$$

where $z = y^n$ or $z = y$. Then the maps $J_n, J : [0, T] \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(t) &= \frac{1}{2}|y(t)|^2 - \int_0^t \langle \tilde{f}(r), y(r) \rangle dr - Ct, \\ J_n(t) &= \frac{1}{2}|y^n(t)|^2 - \int_0^t \langle \tilde{f}(r), y^n(r) \rangle dr - Ct, \end{aligned}$$

are non-increasing and continuous.

Analogously as we did in Step 4 of the proof in Theorem 5, for a fixed $t_0 > 0$, using a sequence $\{\tilde{t}_k\}$ with $\tilde{t}_k \nearrow t_0$, we are able to establish the convergence of the norms,

and therefore, jointly with the weak convergence already proved, deduce that $y^n \rightarrow y$ in $C([\delta, T]; H)$, for any $\delta > 0$.

Now, as we had $T > \bar{T}$, we obtain that $\xi^n \rightarrow \psi$ in $C([-T, 0]; H)$, where $\psi(s) = y(s+T)$, for $s \in [-T, 0]$. Repeating the same procedure for $2\bar{T}$, $3\bar{T}$, etc. for a diagonal subsequence (relabelled the same) we can obtain a continuous function $\psi : (-\infty, 0] \rightarrow H$ and a subsequence such that $\xi^n \rightarrow \psi$ in $C([-T, 0]; H)$ on every interval $[-T, 0]$.

Moreover, since for a fixed $T > 0$, $u^n(s+t_0)$, with $s \in [-T, 0]$, satisfies the estimate (48) for any $n \geq n_0(t_0, T)$, it is clear that we also have

$$|\psi(s)|^2 \leq 1 + Me^{\sigma T}, \quad \forall s \in [-T, 0], \quad \forall T > 0, \quad (50)$$

where

$$M = \frac{2}{\nu} e^{-\sigma t_0} \int_{-\infty}^{t_0} e^{\sigma s} \|f(s)\|_*^2 ds.$$

Step 2: We claim that ξ^n converges to ψ in $C_\gamma(H)$. Indeed, we have to see that for every $\epsilon > 0$ there exists n_ϵ such that

$$\sup_{s \in (-\infty, 0]} |\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \quad \forall n \geq n_\epsilon. \quad (51)$$

Fix $T_\epsilon > 0$ such that $\max\{e^{-2\gamma T_\epsilon}, Me^{\sigma(\sigma-2\gamma)T_\epsilon}\} \leq \epsilon/8$, and take $n_\epsilon \geq n_0(t_0, T_\epsilon)$ such that

$$|\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \quad \forall s \in [-T_\epsilon, 0], \quad \text{and} \quad \tau_n \leq t_0 - T_\epsilon, \quad \forall n \geq n_\epsilon.$$

(This is possible since the convergence of ξ^n to ψ holds in compact intervals of time.) So, in order to prove (51) we only have to check that

$$\sup_{s \in (-\infty, -T_\epsilon]} |\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \quad \forall n \geq n_\epsilon.$$

By (50) and the choice of T_ϵ , it is not difficult to check that for all $k \in \mathbb{N} \cup \{0\}$ that for all $s \in [-(T_\epsilon + k + 1), -(T_\epsilon + k)]$ it holds that

$$\begin{aligned} e^{2\gamma s} |\psi(s)|^2 &\leq e^{-2\gamma(T_\epsilon+k)} (1 + Me^{\sigma(T_\epsilon+k+1)}) \\ &= e^{-2\gamma T_\epsilon} e^{-2\gamma k} + Me^{\sigma(\sigma-2\gamma)T_\epsilon} e^{k(\sigma-2\gamma)} \\ &\leq \epsilon/8 + \epsilon/8 \\ &= \epsilon/4. \end{aligned}$$

So, it suffices to prove the following:

$$\sup_{s \in (-\infty, -T_\epsilon]} e^{2\gamma s} |\xi^n(s)|^2 \leq \epsilon/4 \quad \forall n \geq n_\epsilon.$$

We remember that ξ^n has two parts:

$$\xi^n(s) = \begin{cases} \phi^n(s+t_0-\tau_n), & \text{if } s \in (-\infty, \tau_n-t_0), \\ u^n(s+t_0), & \text{if } s \in [\tau_n-t_0, 0]. \end{cases}$$

Thus, the proof is finished if we prove that

$$\max \left\{ \sup_{s \in (-\infty, \tau_n-t_0)} e^{2\gamma s} |\phi^n(s+t_0-\tau_n)|^2, \sup_{s \in [\tau_n-t_0, -T_\epsilon]} e^{2\gamma s} |u^n(s+t_0)|^2 \right\} \leq \epsilon/4.$$

The first term above can be estimated as follows:

$$\begin{aligned}
\sup_{s \leq \tau_n - t_0} e^{2\gamma s} |\phi^n(s + t_0 - \tau_n)|^2 &= \sup_{s \leq \tau_n - t_0} e^{2\gamma(s+t_0-\tau_n)} e^{2\gamma(\tau_n-t_0)} |\phi^n(s + t_0 - \tau_n)|^2 \\
&= e^{2\gamma(\tau_n-t_0)} \|\phi^n\|_\gamma^2 \\
&\leq e^{2\gamma(\tau_n-t_0)} \rho^2(\tau_n) \\
&\leq e^{2\gamma(\tau_n-t_0)} + M e^{(2\gamma-\sigma)(\tau_n-t_0)} \leq \epsilon/4,
\end{aligned}$$

thanks to the choice of n_ϵ . And finally, for the second term, we have

$$\begin{aligned}
\sup_{s \in [\tau_n - t_0, -T_\epsilon]} e^{2\gamma s} |u^n(s + t_0)|^2 &= \sup_{\theta \in [\tau_n - t_0 + T_\epsilon, 0]} e^{2\gamma(\theta - T_\epsilon)} |u^n(t_0 - T_\epsilon + \theta)|^2 \\
&\leq e^{-2\gamma T_\epsilon} \|u_{t_0 - T_\epsilon}^n\|_\gamma^2 \\
&\leq e^{-2\gamma T_\epsilon} R(t_0, T_\epsilon) \\
&= e^{-2\gamma T_\epsilon} + M e^{(\sigma - 2\gamma)T_\epsilon} \\
&\leq \epsilon/4,
\end{aligned}$$

where we have used (48) with $T = T_\epsilon$. \square

Joining all the above statements we may conclude our main result of this section.

Theorem 21. *Assume that $f \in L_{\text{loc}}^2(\mathbb{R}; V')$ satisfying (45) and $g : \mathbb{R} \times C_\gamma(H) \rightarrow (L^2(\Omega))^2$ satisfying the assumptions (g1)–(g3) are given. Also, suppose that $2L_g < \nu\lambda_1 < 2\gamma$. Then, the semi process U defined in $C_\gamma(H)$ associated to (1) has a pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}$. Moreover, every $\mathcal{A}(t)$ is connected in $C_\gamma(H)$.*

Proof. The existence of the pullback attractor is a direct consequence of Theorem 13, Proposition 16, Corollary 19, and Proposition 20. The connectedness follows from Propositions 15 and 7 and the fact that the space $C_\gamma(H)$ is connected. \square

4.2 Pullback \mathcal{D} –attractor

In this section we briefly recall some concepts of Dynamical Systems when the universe of objects that can be attracted is not fixed but composed of time-dependent families (e.g. cf. [2, 20]).

This approach is useful in some situations when the proof of existence of the pullback attractor in the sense of Definition 11 is unclear. Although we have obtained successfully Theorem 21, we will check that with the same effort we may obtain also a pullback attractor in this new sense.

Let be given \mathcal{D} , a nonempty class of sets parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\}$, with $D(t) \subset X$ for all $t \in \mathbb{R}$. The main concepts, adapted to those in Definition 11, are the following.

Definition 22. *The semi process U on X is said to be pullback \mathcal{D} –asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, any sequence $\tau_n \rightarrow -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .*

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is pullback \mathcal{D} –absorbing for the process U if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Remark 23. *If there exists a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ such that the semi process U on X is pullback \widehat{D}_0 -asymptotically compact (in the sense given in Definition 11), and besides \widehat{D}_0 is pullback \mathcal{D} -absorbing for the process U , then U is pullback \mathcal{D} -asymptotically compact. Indeed, given any $\widehat{D} \in \mathcal{D}$, one only has to introduce a sequence $\tilde{\tau}_m \rightarrow -\infty$ and choose suitably a diagonal sequence of times $\{\tau_{n(m)}\}_m$ related to \widehat{D} to ensure that the process maps initial data of some $D(\tau_{n(m)})$ in the appropriate time sections $D_0(\tilde{\tau}_m)$.*

Now, we have the following result (cf. [20, Th.18] and also [2, Th.7]):

Theorem 24. *Assume that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is pullback \mathcal{D} -absorbing for U and also that U is pullback \widehat{D}_0 -asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau) D_0(\tau)}^X, \quad t \in \mathbb{R},$$

has the following properties:

- (a) the set $\mathcal{A}_{\mathcal{D}}(t)$ is compact for any $t \in \mathbb{R}$;
- (b) $\mathcal{A}_{\mathcal{D}}$ is pullback \mathcal{D} -attracting, i.e.

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau) D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D};$$

- (c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t, \tau) \mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$;
- (d) the following equality holds:

$$\mathcal{A}_{\mathcal{D}}(t) = \bigcup_{\widehat{D} \in \mathcal{D}} \overline{\Lambda(\widehat{D}, t)}^X \quad \text{for } t \in \mathbb{R}.$$

The family $\mathcal{A}_{\mathcal{D}}$, called the global pullback \mathcal{D} -attractor for the semi process U , is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau) B(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 25. (i) *If we assume that $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family \mathcal{D} is inclusion-closed (i.e. if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t , then $\widehat{D}' \in \mathcal{D}$), then the pullback \mathcal{D} -attractor $\mathcal{A}_{\mathcal{D}}$ belongs to \mathcal{D} , and is the only family in \mathcal{D} satisfying properties (a)–(c) above.*

(ii) *When Theorem 24 can be applied, and the universe \mathcal{D} contains all bounded subsets of X , then the family \mathcal{A} given by (41) is a pullback attractor for the semi process U in the sense of Definition 11, and the following relation exists between \mathcal{A} and the attractor $\mathcal{A}_{\mathcal{D}}$ given in Theorem 24:*

$$\mathcal{A}(t) \subset \mathcal{A}_{\mathcal{D}}(t), \quad \forall t \in \mathbb{R}.$$

If, moreover, for some $T \in \mathbb{R}$ the set $\cup_{t \leq T} D_0(t)$ is bounded, then

$$\mathcal{A}(t) = \mathcal{A}_{\mathcal{D}}(t), \quad \forall t \leq T.$$

Proposition 26. *Assume that the conditions of Theorem 24 hold and also that $\mathcal{A}_{\mathcal{D}}(t) \subset B_0(t)$, $\forall t \in \mathbb{R}$, where $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and $B_0(t)$ is connected for every t . Then $\mathcal{A}_{\mathcal{D}}(t)$ is connected for every $t \in \mathbb{R}$.*

Proof. Assume that for some t the set $\mathcal{A}_{\mathcal{D}}(t)$ is not connected. Then $\mathcal{A}_{\mathcal{D}}(t) = A_1 \cup A_2$, where A_i are compact non-empty disjoint sets, and there exist two open disjoint sets U_i for which $A_i \subset U_i$.

We have that $\mathcal{A}(t) = U(t, \tau)\mathcal{A}(\tau) \subset U(t, \tau)B_0(\tau)$, for all $\tau \leq t$. Since $B_0(\tau)$ is connected and $x \rightarrow U(t, \tau)x$ is continuous, $U(t, \tau)B_0(\tau)$ is connected as well. Thus, $U(t, \tau)B_0(\tau) \cap U_i \neq \emptyset$, for $i = 1, 2$, implies that $U_1 \cup U_2$ cannot contain $U(t, \tau)B_0(\tau)$ for any $\tau \leq t$. Then there exist sequences $\{\xi_n\}_n$, $\{x_n\}_n$, and $\{\tau_n\}_n$ such that $\xi_n \in U(t, \tau_n)x_n$, $\tau_n \rightarrow -\infty$, $x_n \in B_0(\tau_n)$, and with $\xi_n \notin U_1 \cup U_2$. As U is pullback \mathcal{D} -asymptotically compact, we obtain passing to a subsequence (relabelled the same) that $\xi_n \rightarrow \xi \notin U_1 \cup U_2$. But this is a contradiction, as $\xi \in \Lambda(t, \widehat{B}_0) \subset \mathcal{A}_{\mathcal{D}}(t)$. \square

We specify now for our problem (1) what is our choice of universe. Having in mind the estimates obtained in the last section, we consider the (tempered) universe \mathcal{D} of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_\gamma(H))$ such that $D(t) \subset B_{C_\gamma(H)}(0, r_{\widehat{D}}(t))$ for some function $r_{\widehat{D}} : \mathbb{R} \rightarrow (0, +\infty)$ satisfying the tempered condition

$$\lim_{t \rightarrow -\infty} e^{(\nu\lambda_1 - 2L_g)t} r_{\widehat{D}}^2(t) = 0.$$

The results that we obtain in this new framework are the following:

Proposition 27. *Under the assumptions of Corollary 19, the family \widehat{B}_0 defined by (46) belongs to \mathcal{D} and is pullback \mathcal{D} -absorbing for the semi process U .*

In particular, the semi process U is pullback \mathcal{D} -asymptotically compact.

Proof. Again the first part of the result follows from Lemma 17, more precisely from (42). The second part is a consequence of the above, Proposition 20 and Remark 23. \square

As a consequence of the above result and Remark 25, we conclude our second main result of this section.

Theorem 28. *Under the assumptions of Theorem 21, there exists a pullback \mathcal{D} -attractor $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t)\}$, which belongs to \mathcal{D} , and is in fact the unique global pullback \mathcal{D} -attractor for the semi process U on $C_\gamma(H)$ associated to (1). Moreover, $\mathcal{A}_{\mathcal{D}}(t)$ is connected and the following relation holds:*

$$\mathcal{A}(t) \subset \mathcal{A}_{\mathcal{D}}(t), \quad \forall t \in \mathbb{R}.$$

Proof. It is a consequence of Theorem 24, Remark 25, Proposition 26 and Proposition 27. We note that in order to apply Proposition 26 we can take the pullback \mathcal{D} -absorbing family \widehat{D}_0 as \widehat{B}_0 , as $\mathcal{A}_{\mathcal{D}}(t) \subset D_0(t)$ is a consequence of $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and the invariance property of $\mathcal{A}_{\mathcal{D}}$. \square

Remark 29. *If additionally, we assume that*

$$\sup_{r \leq 0} \int_{-\infty}^r e^{-(\nu\lambda_1 - 2L_g)(r-s)} \|f(s)\|_*^2 ds < +\infty,$$

then $\mathcal{A}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$, just taking into account Remark 25 (ii).

4.3 The autonomous case

Let us consider briefly the matter of the existence of the global attractor when functions f and g do not depend on the time variable t , i.e. in the autonomous case.

In this case we define a continuous (with respect to the initial datum ϕ) semigroup of operators $S : \mathbb{R}^+ \times C_\gamma(H) \rightarrow C_\gamma(H)$ by denoting

$$S(t)\phi = u_t,$$

u being the unique solution to (1) corresponding to the initial datum $\phi \in C_\gamma(H)$. It is easy to see that $S(t)\phi = U(t, 0)\phi = U(t + \tau, \tau)\phi$, for any $\tau \in \mathbb{R}$.

We recall that the compact set \mathcal{A} is said to be a global attractor for \mathcal{A} if it is invariant (i.e. $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$) and attracts every bounded set B of $C_\gamma(H)$, that is,

$$\text{dist}(S(t)B, \mathcal{A}) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Assume that the conditions of Corollary 19 are fulfilled (but in this autonomous case (45) is trivially satisfied).

From estimate (42) we obtain that the ball

$$B_0 = \{u \in C_\gamma(H) : \|u\|_\gamma \leq \rho\},$$

with $\rho^2 = 1 + \frac{2\|f\|_*^2}{\nu(\lambda_1\nu - 2L_g)}$, is absorbing, that is, for any bounded set B there is $T(B)$ such that $S(t)B \subset B_0$ as soon as $t \geq T$. Also, it follows that the set $\cup_{t \geq 0} S(t)B$ is bounded for any bounded B .

On the other hand, for any $t_n \rightarrow +\infty$ and $\phi^n \in B$, a bounded set of $C_\gamma(H)$, the sequence

$$S(t_n)\phi^n = U(t_n, 0)\phi^n = U(0, -t_n)\phi^n$$

is relatively compact by Proposition 27. Hence, S is asymptotically compact.

Then, it follows from standard theorems that the semigroup S possesses a global connected attractor \mathcal{A} (see e.g. [14] for the existence of the attractor and [10] for the connectedness).

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References

- [1] H. Brézis, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, 1983.
- [2] T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.* **64** (2006), 484–498.

- [3] T. Caraballo, P. Marín-Rubio, and J. Valero, Attractors for differential equations with unbounded delays, *J. Differential Equations* **239** (2007), 311–342.
- [4] T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), 2441–2453.
- [5] T. Caraballo and J. Real, Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), 3181–3194.
- [6] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations* **205** (2004), 271–297.
- [7] H. Crauel, A. Debussche, and F. Flandoli, Random attractors, *J. Dynam. Differential Equations* **9** (1997), no. 2, 307–341.
- [8] H. Gajewsky, K. Gröger, and K. Zacharias, *Nichtlineare operatorgleichungen und operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.
- [9] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.* **64** (2006), 1100–1118.
- [10] M. Gobbino and M. Sardella, On the connectedness of attractors for dynamical systems, *J. Differential Equations* **133** (1997), 1–14.
- [11] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11–41.
- [12] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.
- [13] A. V. Kapustyan and J. Valero, Weak and strong attractors for the 3D Navier-Stokes system, *J. Differential Equations* **240** (2007), 249–278.
- [14] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.
- [15] O. A. Ladyzhenskaya, First boundary value problem for the Navier-Stokes equations in domains with nonsmooth boundaries, *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992), 253–258.
- [16] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82.
- [17] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier- Villars, Paris, 1969.
- [18] A. Z. Manitius, Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulation, *IEEE Trans. Automat. Control* **29** (1984), 1058–1068.
- [19] P. Marín-Rubio and J. Real, Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains, *Nonlinear Anal.* **67** (2007), 2784–2799.

- [20] P. Marín-Rubio and J. Real, On the relation between two different concepts of pull-back attractors for non-autonomous dynamical systems, *Nonlinear Anal.* **71** (2009), 3956–3963.
- [21] G. Planas and E. Hernández, Asymptotic behaviour of two-dimensional time-delayed Navier-Stokes equations, *Discrete Contin. Dyn. Syst.* **21** (2008), 1248–1288.
- [22] T. Taniguchi, The exponential behavior of Navier-Stokes equations with time delay external force, *Discrete Contin. Dyn. Syst.* **12** (2005), 997–1018.
- [23] R. Temam, *Navier-Stokes equations, Theory and Numerical Analysis*, 2nd. ed., North Holland, Amsterdam, 1979.
- [24] Z. J. Zhao, Q. K. Song, and J. Y. Zhang, Exponential periodicity and stability of neural networks with reaction-diffusion terms and both variable and unbounded delays, *Comput. Math. Appl.* **51** (2006), 475–486.