Manuscript submitted to
AIMS' Journals
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org
pp. **000–000**

PULLBACK ATTRACTORS FOR 2D-NAVIER-STOKES EQUATIONS WITH DELAYS IN CONTINUOUS AND SUB-LINEAR OPERATORS

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(Communicated by Aim Sciences)

Abstract. We obtain a result of existence of solutions to the 2D-Navier-Stokes model with delays, when the forcing term containing the delay is sub-linear and only continuous. As a consequence of the continuity assumption the uniqueness of solutions does not hold in general. We use then the theory of multi-valued dynamical system to establish the existence of attractors for our problem in several senses and establish relations among them.

1. **Introduction and statement of the problem.** The Navier-Stokes equations govern the motion of usual fluids like water, air, oil, etc. These equations have been the object of numerous works [14, 22] and references cited therein.

On other hand, delay terms appear naturally for instance as effects in wind tunnels experiments (cf. [15]). Very recently, Caraballo & Real [8, 9, 10] developed a full theory of existence, stability of solutions and global attractors for Navier-Stokes models including some hereditary characteristics in several ways (fixed, variable and distributed delays) for bounded domains and with uniqueness of solutions. This study has continued by some other authors, e.g. [21] for the study of exponential decay of solutions, or [11] for the existence of solutions and [16] for the existence of attractors for some delayed version on unbounded domains.

Nevertheless, in all the above cited cases, uniqueness of solutions allow to apply the classical results of Dynamical Systems, while the case of non-uniqueness or unknown, as the celebrated problem for dimension three, requires a different treatment. In this sense, we may cite the results by Ball [2] for the 3D deterministic case, or [18] for the 3D stochastic case.

However, even without going to such complicated situation of dimension three, the case of a 2D-model with force term that is only continuous (and therefore with the same problem of uniqueness for the solutions) does not seem to be treated.

2000 *Mathematics Subject Classification.* Primary: 35B40; 35Q30; 37L30.

Key words and phrases. Navier-Stokes equations, delay terms, pullback attractors, tempered attractors.

Partially supported by Ministerio de Ciencia e Innovación (Spain), project MTM2008-00088 and Junta de Andalucía, grant P07-FQM-02468.

More exactly, we aim to consider in an arbitrary interval $[\tau,+\infty)\subset\mathbb{R}$ the following functional Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f(t,u(t-\rho(t))) \text{ in } (\tau,+\infty)\times\Omega,\\ \operatorname{div} u=0 \text{ in } (\tau,+\infty)\times\Omega,\\ u=0 \text{ on } (\tau,+\infty)\times\partial\Omega,\\ u(\tau,x)=u^0(x),\ x\in\Omega,\\ u(\tau+t,x)=\phi(t,x),\ t\in(-h,0),\ x\in\Omega, \end{cases} \tag{1}$$

where the set $\Omega\subset\mathbb{R}^2$ is open, bounded and connected, $\nu>0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, u^0 is the initial velocity field, f is the external force term and contains some memory effects during a fixed interval of time of length $h>0$, being ρ an adequate given delay function, and ϕ the initial datum on the interval $(-h,0)$.

The goal of this paper is to study existence of solutions and of attractors for such model. The structure of the paper is the following. In this section we introduce some abstract functional spaces, necessary for the variational statement of the problem. In Section 2 we prove our first main result on existence of at least one solution for (1) using a compactness method. In Section 3 we recall some recent abstract results on existence of pullback attractors, which will be applied to our case in Section 4 combining again the compactness method with suitable decaying energy functionals. These results will provide attractors in several senses, roughly speaking in phase-spaces of continuous and of some $L^{\tilde p}$ functions taking values in a Hilbert space, and for both the universe of fixed bounded sets (of the respective cited phase-spaces) and in a universe defined by a tempered growth condition. The relation among all of them is also established.

To start, we consider the following usual abstract spaces:

$$\mathcal{V}=\left\{u\in\left(C_0^\infty(\Omega)\right)^2:\operatorname{div} u\ =0\right\},$$

$H=$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$ where for $u,v\in(L^2(\Omega))^2$,

$$(u,v)=\sum_{j=1}^{2}\int_\Omega u_j(x)v_j(x)\mathrm{d}x,$$

$V=$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for $u,v\in(H_0^1(\Omega))^2$,

$$((u,v))=\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\mathrm{d}x.$$

It follows that $V\subset H\equiv H'\subset V'$, where the injections are dense and compact. We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality $\langle V',V\rangle$.

Define the operator $A:V\to V'$ as

$$\langle Au,v\rangle:=((u,v))\quad\forall u,v\in V.$$

Define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x \quad \forall\, u,v,w \in V,$$

and the operator $B : V \times V \to V'$ as

$$\langle B(u,v), w\rangle = b(u,v,w) \quad \forall u,v,w \in V.$$

Some useful properties concerning the form b and the operator B, that we will use in the next sections, are the following,

$$\begin{aligned} b(u,v,w) &= -b(u,w,v) \quad \forall u,v,w \in V, \\ b(u,v,v) &= 0 \quad \forall u,v \in V. \end{aligned} \tag{2}$$

On the other hand, using that $\|u\|_{(L^4(\Omega))^2} \le C|u|^{1/2}\|u\|^{1/2}$, we also have the estimate

$$\begin{aligned} |b(u,v,w)| &= |-b(u,w,v)| \\ &\le \|u\|_{(L^4(\Omega))^2}\|w\|\|v\|_{(L^4(\Omega))^2} \\ &\le C\|u\|^{1/2}|u|^{1/2}\|w\|\|v\|^{1/2}|v|^{1/2} \quad \forall u,v,w \in V, \end{aligned}$$

so it yields

$$\|B(u,v)\|_* \le C\|u\|^{1/2}|u|^{1/2}\|v\|^{1/2}|v|^{1/2} \quad \forall u,v \in V. \tag{3}$$

Now, let us establish some assumptions for (1).

We assume that the given delay function satisfies $\rho \in C^1([0,+\infty);[0,h])$, and there exists a constant ρ_* satisfying

$$\rho'(t) \le \rho_* < 1 \quad \forall\, t \ge 0. \tag{4}$$

Moreover, we assume that $f : [\tau,+\infty)\times H \to H$ satisfies the following assumptions:

(c1): $f(\cdot,v) : [\tau,+\infty) \to H$ is measurable for all $v \in H$,

(c2): $f(t,\cdot) : H \to H$ is continuous for all $t \ge \tau$,

(c3): There exist functions α, $\beta : [\tau,+\infty) \to [0,+\infty)$, with $\alpha \in L^p(\tau,T)$ and $\beta \in L^1(\tau,T)$ for all $T > \tau$, for some $1 \le p \le +\infty$, such that for any $v \in H$,

$$|f(t,v)|^2 \le \alpha(t)|v|^2 + \beta(t),\ \forall t \ge \tau.$$

Let us also assume that the initial data satisfy the following condition

(c4): $\phi \in L^{2p'}(-h,0;H)$, $u^0 \in H$.

Now we consider the functional formulation of problem (1), namely

$$\begin{aligned} &u(t)+\int_\tau^t (\nu Au(s)+B(u(s),u(s)))ds = u^0+\int_\tau^t f(s,u(s-\rho(s)))ds,\ \forall\, t \ge \tau, \\ &u(\tau+t) = \phi(t),\ a.e.\, t \in (-h,0). \end{aligned} \tag{5}$$

2. **Existence of solution.**

Definition 1. A solution of (1) is a function

$$u \in L^{2p'}(\tau-h,T;H) \cap L^2(\tau,T;V) \cap L^\infty(\tau,T;H) \quad \text{for all } T > \tau,$$

such that $u(\tau+t)$ coincides with $\phi(t)$ in $(-h,0)$ and satisfies the equation from (5) in V', for all $t \ge \tau$.

Remark 1. Observe that if u is a solution of (5) then in particular $Au+B(u,u)\in L^2(\tau,T;V')$ for all $T>\tau$. Moreover, if we define $\tilde{\alpha}(t)=g\circ\theta^{-1}(t)$, where $\theta:[\tau,+\infty)\to[-\rho(\tau),+\infty)$ is the differentiable and nonnegative strictly increasing function given by $\theta(s)=s-\rho(s)$, we obtain

$$\begin{aligned}
&\int_\tau^T |f(t,u(t-\rho(t)))|^2dt\\
\leq\;& \int_\tau^T \alpha(t)|u(t-\rho(t))|^2dt+\int_\tau^T\beta(t)dt\\
\leq\;& \frac{1}{1-\rho_*}\int_{\tau-\rho(\tau)}^{T-\rho(T)}\tilde{\alpha}(t)|u(t)|^2dt+\int_\tau^T\beta(t)dt\\
\leq\;& \frac{1}{1-\rho_*}\left(\int_{-\rho(\tau)}^0\tilde{\alpha}(t+\tau)|\phi(t)|^2dt+\int_\tau^T\tilde{\alpha}(t)|u(t)|^2dt\right)+\int_\tau^T\beta(t)dt,
\end{aligned}$$

and therefore, taking into account that $\tilde{\alpha}\in L^p(-\rho(\tau),T)$ for all $T>\tau$, we have that $f(t,u(t-\rho(t)))$ belongs to $L^2(\tau,T;H)$ for all $T>\tau$.

Therefore, from (5) we deduce that the derivative u' belongs to $L^2(\tau,T;V')$ for all $T>\tau$, and this fact and $u\in L^2(\tau,T;V)$ for all $T>\tau$ imply that

$$u\in C([\tau,+\infty);H),$$

and satisfies the energy equality

$$\frac{d}{dt}|u(t)|^2+2\nu\langle Au(t),u(t)\rangle=2(f(t,u(t-\rho(t))),u(t)),$$

in the distributions sense on $(\tau,+\infty)$.

Theorem 1. *Under the assumptions above, there exists a solution u to problem (5).*

Proof. We prove our result by a Faedo-Galerkin scheme and compactness method. Without loss of generality in the sequel we assume that $\tau=0$. For a different value τ we only have to proceed by translation.

Step 1. The approximating sequence. Consider the Hilbert basis of H formed by the eigenfunctions $\{v_k\}_{k\geq1}$ of A, i.e. $Av_k=\lambda_kv_k$ (with $\{\lambda_k\}_{k\geq1}\subset(0,+\infty)$). Indeed, these elements allow to define the operator $P_mv=\sum_{j=1}^m(v_k,v)v_k$, which is the orthogonal projection of H and of V in V_m :=span$[v_1,\ldots,v_m]$ with their respective norms.

Denote $u^m(t)=\sum_{k=1}^m\gamma_{mk}(t)v_k$, where $\gamma_{mk}(t)=(u^m(t),v_k)$, $k=1,2,...,m$, are unknown real functions satisfying the finite-dimensional problem

$$\begin{cases}
(u^m(t),v_k)+\nu\displaystyle\int_0^t\langle Au^m(s),v_k\rangle ds+\int_0^t\langle B(u^m(s),u^m(s)),v_k\rangle ds\\
=(u^0,v_k)+\displaystyle\int_0^t(f(s,u^m(s-\rho(s))),v_k)ds,\ t\geq0,\quad\forall\,1\leq k\leq m,\\
u^m(t)=\phi^m(t),\quad a.e.\,t\in(-h,0),
\end{cases}\tag{6}$$

with $\phi^m(t)=P_m\phi(t)$.

For the (local) well-posedness of this finite-dimensional delay problem see [12, Sec.2.6, p.58]. Next step will provide estimates which imply that the solutions are well-defined in the whole $[0, +\infty)$.

Step 2. Estimates for the approximating sequence. By (6), we have

$$\begin{aligned} & \frac{d}{dt}(u^m(t), v_k) + \nu\langle Au^m(t), v_k\rangle + \langle B(u^m(t), u^m(t)), v_k\rangle \\ = \; & (f(t, u^m(t-\rho(t))), v_k) \quad \text{a.e. } t > 0, \text{ for all } 1 \le k \le m. \end{aligned} \tag{7}$$

Multiplying (7) by $(u^m(t), v_k)$, summing from $k = 1$ to $k = m$, and using (2), we easily obtain

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 = 2(f(t, u^m(t-\rho(t))), u^m(t)) \quad a.e.\ t > 0. \tag{8}$$

Now observe that by (c3) and the Young inequality,

$$\begin{aligned} & 2(f(t, u^m(t-\rho(t))), u^m(t)) \\ \le \; & 2|f(t, u^m(t-\rho(t)))||u^m(t)| \\ \le \; & 2(\alpha^{1/2}(t)|u^m(t-\rho(t))| + \beta^{1/2}(t))|u^m(t)| \\ \le \; & \nu(1-\rho_*)|u^m(t-\rho(t))|^2 + \left(1 + \frac{\alpha(t)}{\nu(1-\rho_*)}\right)|u^m(t)|^2 + \beta(t). \end{aligned}$$

Using this inequality in (8), and observing that

$$\begin{aligned} \int_0^t |u^m(s-\rho(s))|^2 ds \; & \le \; \frac{1}{1-\rho_*}\int_{-\rho(0)}^{t-\rho(t)} |u^m(s)|^2 ds \\ & \le \; \frac{1}{1-\rho_*}\left(\int_{-h}^0 |\phi(s)|^2 ds + \int_0^t |u^m(s)|^2 ds\right), \end{aligned}$$

we obtain

$$\begin{aligned} & |u^m(t)|^2 + 2\nu\int_0^t \|u^m(s)\|^2 ds \\ \le \; & |u^0|^2 + \int_0^T \beta(s)ds + \nu\int_{-h}^0 |\phi(s)|^2 ds + \int_0^t \left(1 + \nu + \frac{\alpha(s)}{\nu(1-\rho_*)}\right)|u^m(s)|^2 ds, \end{aligned}$$

for all $0 \le t \le T$.

From this inequality and Gronwall lemma one has that

$$\{u^m\} \text{ is bounded in } \; L^2(0,T;V) \cap L^\infty(0,T;H) \quad \text{for any } T > 0. \tag{9}$$

Finally, taking into account (9), from Remark 1 and the fact that, by the choice of the basis, $\|P_m\|_{\mathcal{L}(V)} \le 1$ for all $m \ge 1$, we deduce that $\{(u^m)'\}$ is bounded in $L^2(0,T;V')$.

Step 3. Passing to the limit.

From the above estimates, the compactness of the injection of V in H, and the Aubin theorem (see [14] or [20] or [1]) we obtain that there exist a subsequence of u^m (that we relabel the same) and a function $u \in L^{2p'}(-h,T;H) \cap L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, with $u = \phi$ in $(-h, 0)$, and the derivative $u' \in L^2(0,T;V')$ for all $T > 0$, such that, among other things,

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;V), \text{ for all } T > 0, \tag{10}$$

$$u^m \to u \quad \text{in } L^{2p'}(-h,0;H) \text{ and in } L^2(0,T;H), \text{ for all } T > 0,$$

and

$$u^m(t) \to u(t), \quad \text{in } H \text{ a.e. } t > -h. \tag{11}$$

By (10), it is evident that $Au^m \rightharpoonup Au$ weakly in $L^2(0,T;V')$.

On the other hand, reasoning as in [14, p.76], we have that

$$\langle B(u^m(t),u^m(t)),v_k\rangle \rightharpoonup \langle B(u(t),u(t)),v_k\rangle \ \text{ weakly in } L^2(0,T), \quad \forall\, k \geq 1.$$

Finally, observe that by (11), (c2) and (4), for any $T > 0$

$$f(t,u^m(t-\rho(t))) \to f(t,u(t-\rho(t))) \text{ in } H \quad \text{a.e. in } (0,T), \tag{12}$$

and as $\{u^m\}$ is bounded in $L^\infty(0,T;H)$, by (c3) we have

$$|f(t,u^m(t-\rho(t)))|^2 \leq \begin{cases} \alpha(t)C^2 + \beta(t) & \text{if } t-\rho(t) > 0, \\ \alpha(t)|\phi(t-\rho(t))|^2 + \beta(t) & \text{if } t-\rho(t) < 0, \end{cases} \tag{13}$$

with $C = \sup\limits_{m\geq 1} \|u^m\|_{L^\infty(0,T;H)}$.

From (12) and (13) we obtain that

$$f(t,u^m(t-\rho(t))) \to f(t,u(t-\rho(t))) \ \text{ in } L^2(0,T;H).$$

It is now easy to pass to the limit and to prove that u is a solution of (5). □

Remark 2. The uniqueness of u is not guaranteed unless we assume additional assumptions on f. For example, in Theorem 1, if we assume that f satisfies (c1), (c3), (c4), and it exists $\gamma : [0,+\infty) \to [0,+\infty)$ such that $\gamma \in L^1(0,T)$ for all $T > 0$ and

(c5) for all $R > 0$ exists $L(R) > 0$ such that if $|u| \leq R, |v| \leq R$, then

$$|f(t,u) - f(t,v)| \leq L(R)\gamma^{1/2}(t)|u-v| \quad \forall\, t \geq 0,$$

then we can assure the uniqueness of solution to problem (5).

3. **Recent abstract results on pullback attractors.** In order to analyze the existence of pullback attractors for our model, we need to recall briefly some results on the abstract theory.

The results in this section are a combination of two difficulties (involved in our model). On the one hand, we aim to study non-autonomous dynamical systems as they appear in [5, 17], but in these references the framework is single-valued (uniqueness of solution holds). On the other hand, some classical multi-valued results on dynamical systems (see e.g. [19]) are stated in an autonomous framework. We recall that some sort of results for the combination of these ingredients appear in [6, 7] but not completely adapted to the sharp conditions involving a family depending on time. Our results here are related to those in [3], although the presentation there is random instead of deterministic (which is our case); some of them, but for a universe without relating it with that of fixed bounded sets, are also exposed in [4]. Since proofs are very similar to those in [17], for the sake of brevity we omit them.

Consider a metric space (X,d) and denote by $\mathcal{P}(X)$ the class of nonempty subsets of X. As usual, let us denote by $\operatorname{dist}(C_1,C_2)$ the Hausdorff semidistance between C_1 and C_2, i.e.

$$\operatorname{dist}(C_1,C_2) = \sup_{x\in C_1} \inf_{y\in C_2} d(x,y) \quad \text{for } C_1,\, C_2 \subset X.$$

Definition 2. A multi-valued process U is a family of mappings $U(t,\tau) : X \to \mathcal{P}(X)$ for any pair $\tau \leq t$ of real numbers, such that it satisfies

$$U(t,\tau)x \subset U(t,r)U(r,\tau)x, \quad \forall x \in X, \forall \tau \leq r \leq t.$$

If the above relation is not only an inclusion but an equality, we say that the multi-valued process is strict.

Remark 3. In our model, the multi-valued process will be strict. The case of only an inclusion in Definition 2 is useful, for instance, when dealing with 3D-Navier-Stokes equations, e.g. cf. [13].

Consider given a family $\widehat{D}_0 = \{D_0(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 3. A multi-valued process U is $\widehat{D}_0-$asymptotically compact if for any $t \in \mathbb{R}$, any sequences $\{\tau_n\}$ with $\tau_n \to -\infty$, $\{x_n\}$ with $x_n \in D_0(\tau_n)$, and $\{\xi_n\}$ with $\xi_n \in U(t,\tau_n)x_n$, the sequence $\{\xi_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s\leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau)D_0(\tau)}, \quad \forall t \in \mathbb{R}. \tag{14}$$

Proposition 1. *If U is a multi-valued process $\widehat{D}_0-$asymptotically compact, then the sets $\Lambda(\widehat{D}_0, t)$ defined by (14) are nonempty compact subsets of X.*

Moreover, $\Lambda(\widehat{D}_0, t)$ attracts in a pullback sense to $\widehat{D}_0$ at time t, i.e.

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0.$$

Indeed, it is the minimal closed set with this property.

Definition 4. The family $\widehat{D}_0 = \{D_0(t) \ : \ t \in \mathbb{R}\}$ is said pullback-absorbing for a multi-valued process U if for every $t \in \mathbb{R}$ and every bounded subset B of X, there exists $\tau(t,B) \leq t$ such that

$$U(t,\tau)B \subset D_0(t), \ \forall \tau \leq \tau(t,B).$$

Proposition 2. *Consider a multi-valued process U and a family $\widehat{D}_0 = \{D_0(t) \ : \ t \in \mathbb{R}\}$ which is pullback-absorbing for U. Assume also that U is $\widehat{D}_0-$asymptotically compact. Then for any bounded set B of X it holds that*

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B, \Lambda(\widehat{D}_0, t)) = 0.$$

Definition 5. Let be given a multi-valued process U. A family $\mathcal{A} = \{\mathcal{A}(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback attractor for U if $t \in \mathbb{R}$ the set $\mathcal{A}(t)$ is compact and attracts at time t to every bounded set B of X in pullback sense, i.e.

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B, \mathcal{A}(t)) = 0.$$

It is clear that with the above weak definition of a pullback attractor, it has not to be unique. However, it can be considered unique in the sense of minimal, i.e. the minimal closed family with such property. In this way, we have the following result.

Theorem 2. *Consider a multi-valued process U and a family $\widehat{D}_0$ which is pullback absorbing for U, and assume that U is $\widehat{D}_0-$asymptotically compact. Then, for any bounded subset B of X and any $t \in \mathbb{R}$, the set*

$$\Lambda(B,t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)B}$$

is a nonempty compact subset contained in $\Lambda(\widehat{D}_0,t)$, which attracts to B in a pullback sense. Indeed, it is the minimal closed set with this property.

Moreover,

$$\mathcal{A}(t) = \overline{\bigcup_{B \ \text{bounded}} \Lambda(B,t)}$$

is a pullback attractor (contained in $\Lambda(\widehat{D}_0,t)$).

Corollary 1. *Under the assumptions of Theorem 2, if there exists a time $T \in \mathbb{R}$ such that $\cup_{t\le T} D_0(t)$ is bounded, then*

$$\mathcal{A}(t) = \overline{\bigcup_{B \ \text{bounded}} \Lambda(B,t)} = \Lambda(\widehat{D}_0,t) \ \forall t \le T.$$

If we introduce a continuity assumption, we are able to precise an additional property. Namely, while in the single-valued case the continuity of the flow provides (non-autonomous) invariance, in the multi-valued case, the most natural notion of continuity, upper semi continuity, provides negatively invariance of the omega limit sets and the attractor. Concretely, we have the following

Definition 6. A multi-valued process U on X is said upper semi continuous (u.s.c. for short) if for all $t \ge \tau$, the mapping $U(t,\tau)$ is u.s.c. from X into $\mathcal{P}(X)$, i.e., given a converging sequence $x_n \to x$ in X, for any sequence $\{y_n\}$ such that $y_n \in U(t,\tau)x_n$ for all n, there exists a subsequence of $\{y_n\}$ converging in X to an element of $U(t,\tau)x$.

Proposition 3. *Consider a family $\widehat{D}_0$ and a multi-valued process U which is $\widehat{D}_0-$asymptotically compact and u.s.c. Then, the family $\{\Lambda(\widehat{D}_0,t) \ : \ t \in \mathbb{R}\}$ is negatively invariant, i.e.*

$$\Lambda(\widehat{D}_0,t) \subset U(t,\tau)\Lambda(\widehat{D}_0,\tau) \ \forall t \ge \tau.$$

For any bounded set B of X, the family $\{\Lambda(B,t) \ : \ t \in \mathbb{R}\}$ is also negatively invariant. Finally, the family

$$\mathcal{A}(t) = \overline{\bigcup_{B \text{bounded}} \Lambda(B,t)}$$

is also negatively invariant.

Now we briefly recall the analogous concept to assure the existence of attractor in a given universe. Let us consider $\mathcal{D}$ a class of sets parameterized in time, $\widehat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 7. A multi-valued process U is pullback $\mathcal{D}-$asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequences $\{\tau_n\}$ with $\tau_n \le t$ and $\tau_n \to -\infty$, $\{x_n\}$ with $x_n \in D(\tau_n)$, and $\{\xi_n\}$ with $\xi_n \in U(t,\tau_n)x_n$, this last sequence $\{\xi_n\}$ is relatively compact.

Proposition 4. *Let be given a universe $\mathcal{D}$. If U is a multi-valued process $\mathcal{D}-$asymptotically compact, then for any $\widehat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}, t)$ is a nonempty compact set of X that attracts to $\widehat{D}$ at time t in a pullback sense. Indeed it is the minimal closed set with such property. If U is also strong-weak u.s.c., then $\{\Lambda(\widehat{D}, t) \ : \ t \in \mathbb{R}\}$ is negatively invariant.*

Definition 8. A family $\widehat{D}_0 = \{D_0(t) \ : \ t \in \mathbb{R}\}$ is said pullback $\mathcal{D}-$absorbing for U if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists $\tau(\widehat{D}, t) \leq t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t) \ \forall \tau \leq \tau(\widehat{D}, t).$$

This definition may help to weaken the assumptions of the above proposition as we show now (although in applications there is usually no difference on obtaining $\widehat{D}_0-$asymptotic compactness and $\mathcal{D}-$asymptotic compactness when $\widehat{D}_0 \in \mathcal{D}$).

Theorem 3. *Assume that $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for a multi-valued process U, which is also $\widehat{D}_0-$asymptotically compact. Then all thesis in Proposition 4 hold.*

Moreover, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) \ : \ t \in \mathbb{R}\}$ given by $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t)$ satisfies the following properties:

1. *For each $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is compact.*
2. *$\mathcal{A}_{\mathcal{D}}$ attracts pullback to any $\widehat{D} \in \mathcal{D}$.*
3. *If U is u.s.c., $\mathcal{A}_{\mathcal{D}}$ is negatively invariant.*
4. *$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}$.*
5. *If $\widehat{D}_0 \in \mathcal{D}$, $\mathcal{A}_{\mathcal{D}}$ is the minimal family of closed sets that attracts pullback to elements of $\mathcal{D}$.*
6. *If $\widehat{D}_0 \in \mathcal{D}$, each $D_0(t)$ is closed and the universe $\mathcal{D}$ is inclusion-closed, then $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and it is the only family of $\mathcal{D}$ satisfying properties 1, 2 and 3 above.*

Corollary 2. *Assume that $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for a multi-valued process U, which is also $\widehat{D}_0-$asymptotically compact. If $\mathcal{D}$ contains the families of fixed bounded sets, then $\mathcal{A} = \{\mathcal{A}(t) \ : \ t \in \mathbb{R}\}$ with*

$$\mathcal{A}(t) = \overline{\bigcup_{B \ \text{bounded}} \Lambda(B, t)}$$

is well-defined, it is the minimal pullback attractor of bounded sets, and $\mathcal{A}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.

Corollary 3. *Under the assumptions of Corollary 2, if there exists $T \in \mathbb{R}$ such that $\cup_{t \leq T} D_0(t)$ is bounded, then $\mathcal{A}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.*

4. Existence of attractors.

Our goal in this section is to analyze the asymptotic behaviour of the problem (1). As long as the delay operator may contain explicit terms depending on time, we will seek for conditions to assure the existence of pullback attractors.

We carry at the same time the analysis in two senses, according to the abstract theory introduced in Section 3. The first one is devoted to pullback attractors of fixed bounded sets, which is the most usual framework, but with the peculiarity that uniqueness is unknown and so the approach is multi-valued. Second aspect is concerned with how to extend the previous results to the more recent framework of pullback attractors in a universe of families of time dependent sets with a tempered growth condition.

First at all, we have to extend the assumptions given in Section 1, which roughly speaking are now assumed to be satisfied in any interval of the form $[\tau, +\infty)$ (since we need now to be able to take limits when $\tau \to -\infty$).

Hereafter we assume the following assumptions for f :

(c1'): $f(\cdot, v) : \mathbb{R} \to H$ is measurable for all $v \in H$,
(c2'): $f(t, \cdot) : H \to H$ is continuous for all $t \in \mathbb{R}$,
(c3'): There exist nonnegative functions α, $\beta : \mathbb{R} \to [0, +\infty)$, with $\alpha \in L^p_{\mathrm{loc}}(\mathbb{R})$ for some $1 \leq p \leq +\infty$, and $\beta \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that for any $v \in H$,

$$|f(t, v)|^2 \leq \alpha(t)|v|^2 + \beta(t), \ \forall t \geq \tau.$$

We have different options for constructing a multi-valued semiflow, for instance $M_H^{2p'} = H \times L^{2p'}(-h, 0; H)$ and $C_H = C([-h, 0]; H)$ are valid choices as phase-spaces.

Denote $D(\tau, u^0, \phi)$ the set of global solutions of (1) in $[\tau, +\infty)$ with initial datum $(u^0, \phi) \in M_H^{2p'}$.

Then, after Theorem 1 we may define two processes, $(M_H^{2p'}, \{U(\cdot, \cdot)\})$ as

$$U(t, \tau)(u^0, \phi) = \{u_t \ : \ u \in D(\tau, u^0, \phi)\}, \quad \forall\, (u^0, \phi) \in M_H^{2p'},$$

and $(C_H, \{U(\cdot, \cdot)\})$ as

$$U(t, \tau)\phi = \{u_t \ : \ u \in D(\tau, \phi(0), \phi)\} \quad \forall\, \phi \in C_H.$$

Indeed, thanks to the regularity of the problem, the asymptotic behaviour of both processes will be *the same*, as we will see below.

From now on, for any $\delta > 0$, we denote

$$\kappa_\delta(t, s) = (\nu\lambda_1 - \delta)(t - s) - \frac{e^{\nu\lambda_1 h}}{\nu\lambda_1(1 - \rho_*)} \int_s^t \alpha(\sigma) d\sigma \quad \forall\, t, s \in \mathbb{R}. \tag{15}$$

It is easy to see that

$$-\kappa_\delta(t, s) = \kappa_\delta(0, t) - \kappa_\delta(0, s) \quad \forall\, t, s \in \mathbb{R}, \tag{16}$$

and if $0 < \delta < \nu\lambda_1$, then

$$\kappa_\delta(0, r) \leq \kappa_\delta(0, t) + (\nu\lambda_1 - \delta)h \quad \forall r \in [t - h, t]. \tag{17}$$

Lemma 1. *Under the assumptions (c1')-(c3'), for any $\delta > 0$, any $(u^0, \phi) \in M_H^{2p'}$ and any $u \in D(\tau, u^0, \phi)$, it hold*

$$|u(t)|^2 \leq \left(|u^0|^2 + \nu\lambda_1 \int_{-h}^0 e^{\nu\lambda_1 s}|\phi(s)|^2 ds\right) e^{-\kappa_\delta(t,\tau)} + \delta^{-1} \int_\tau^t e^{-\kappa_\delta(t,s)}\beta(s) ds. \tag{18}$$

and

$$\begin{aligned} &\nu \int_\tau^t e^{\nu\lambda_1 s}\|u(s)\|^2 ds \\ \leq \ &\left\{\left(|u^0|^2 + \nu\lambda_1 \int_{-h}^0 e^{\nu\lambda_1 s}\|\phi(s)\|^2 ds\right) e^{-\kappa_\delta(0,\tau)} + \delta^{-1} \int_\tau^t e^{-\kappa_\delta(0,s)}\beta(s) ds\right\} \\ &\times \left[e^{\nu\lambda_1 t + \kappa_\delta(0,t)} + \nu\lambda_1\delta^{-1} + \nu\lambda_1 \int_0^t e^{\nu\lambda_1 s + \kappa_\delta(0,s)}\, ds\right], \end{aligned} \tag{19}$$

for all $t \geq \tau$.

Proof. Consider $u \in D(\tau, u^0, \phi)$. By the energy equality and the Poincaré inequality, we deduce that

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 + \nu\|u(t)\|^2 \leq 2(f(t, u(t-\rho(t))), u(t)).$$

So, using the Young inequality we arrive to

$$\begin{aligned} & \frac{d}{dt}(e^{\nu\lambda_1 t}|u(t)|^2) + \nu e^{\nu\lambda_1 t}\|u(t)\|^2 \\ \leq\ & 2e^{\nu\lambda_1 t}|f(t, u(t-\rho(t)))||u(t)| \\ \leq\ & 2e^{\nu\lambda_1 t}(\alpha^{1/2}(t)|u(t-\rho(t))| + \beta^{1/2}(t))|u(t)| \\ \leq\ & C_*^{-1}e^{\nu\lambda_1 t}|u(t-\rho(t))|^2 + (C_*\alpha(t) + \delta)\, e^{\nu\lambda_1 t}|u(t)|^2 \\ & +\delta^{-1}e^{\nu\lambda_1 t}\beta(t), \end{aligned} \tag{20}$$

where for short we have denoted

$$C_* = \frac{e^{\nu\lambda_1 h}}{\nu\lambda_1(1-\rho_*)}.$$

Taking into account that

$$\begin{aligned} & \int_\tau^t e^{\nu\lambda_1 s}|u(s-\rho(s))|^2 ds \\ \leq\ & \frac{e^{\nu\lambda_1 h}}{1-\rho_*}\int_{\tau-h}^t e^{\nu\lambda_1 r}|u(r)|^2 dr \\ =\ & \frac{e^{\nu\lambda_1 h}}{1-\rho_*}\left(e^{\nu\lambda_1\tau}\int_{-h}^0 e^{\nu\lambda_1 r}|\phi(r)|^2 dr + \int_\tau^t e^{\nu\lambda_1 r}|u(r)|^2 dr\right), \end{aligned}$$

integrating (20) in $[\tau, t]$, we obtain

$$\begin{aligned} & e^{\nu\lambda_1 t}|u(t)|^2 + \nu\int_\tau^t e^{\nu\lambda_1 s}\|u(s)\|^2 ds \\ \leq\ & e^{\nu\lambda_1\tau}C_\tau + \int_\tau^t (C_*\alpha(r) + \delta + \nu\lambda_1)e^{\nu\lambda_1 r}|u(r)|^2 dr \\ & +\delta^{-1}\int_\tau^t e^{\nu\lambda_1 r}\beta(r)dr, \end{aligned} \tag{21}$$

where again for short we have denoted

$$C_\tau = |u^0|^2 + \nu\lambda_1\int_{-h}^0 e^{\nu\lambda_1 r}|\phi(r)|^2 dr.$$

Applying the Poincaré inequality and the Gronwall lemma, we may conclude that (18) holds.

For the proof of (19), observe that using (18) in (21), we also deduce that

$$
\begin{aligned}
&\nu \int_\tau^t e^{\nu\lambda_1 s}\|u(s)\|^2 ds \\
\leq\ & e^{\nu\lambda_1\tau} C_\tau + \delta^{-1}\int_\tau^t e^{\nu\lambda_1 s}\beta(s)ds \\
&+\int_\tau^t (C_*\alpha(s)+\delta+\nu\lambda_1) \\
&\quad\times\left[C_\tau e^{\nu\lambda_1\tau} e^{\int_\tau^s (C_*\alpha(r)+\delta)dr} + \delta^{-1}\int_\tau^s e^{\nu\lambda_1 r+\int_r^s (C_*\alpha(\sigma)+\delta)d\sigma}\beta(r)dr\right] ds.
\end{aligned}
\tag{22}
$$

Now, observe that

$$
\int_\tau^t (C_*\alpha(s)+\delta)e^{\int_\tau^s (C_*\alpha(r)+\delta)dr}\, ds = e^{\int_\tau^t (C_*\alpha(r)+\delta)dr} - 1, \tag{23}
$$

and

$$
\begin{aligned}
& e^{\nu\lambda_1\tau}\int_\tau^t e^{\int_\tau^s (C_*\alpha(r)+\delta)dr}\, ds \\
=\ & e^{-\kappa_\delta(0,\tau)}\int_\tau^t e^{\delta s + C_*\int_0^s\alpha(r)dr}\, ds \\
\leq\ & e^{-\kappa_\delta(0,\tau)}\left[\int_{-\infty}^0 e^{\delta s + C_*\int_0^s\alpha(r)dr}\, ds + \int_0^t e^{\delta s + C_*\int_0^s\alpha(r)dr}\, ds\right] \\
\leq\ & e^{-\kappa_\delta(0,\tau)}\left[\int_{-\infty}^0 e^{\delta s}\, ds + \int_0^t e^{\delta s + C_*\int_0^s\alpha(r)dr}\, ds\right] \\
=\ & e^{-\kappa_\delta(0,\tau)}\left[\delta^{-1} + \int_0^t e^{\nu\lambda_1 s+\kappa_\delta(0,s)}\, ds\right].
\end{aligned}
\tag{24}
$$

On the other hand, applying the Fubini theorem and integrating, we have

$$
\begin{aligned}
& \int_\tau^t (C_*\alpha(s)+\delta+\nu\lambda_1)\left[\int_\tau^s e^{\nu\lambda_1 r+\int_r^s (C_*\alpha(\sigma)+\delta)d\sigma}\beta(r)dr\right] ds \\
=\ & \int_\tau^t (C_*\alpha(s)+\delta+\nu\lambda_1)e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}\left[\int_\tau^s e^{-\kappa_\delta(0,r)}\beta(r)dr\right] ds \\
=\ & \int_\tau^t e^{-\kappa_\delta(0,r)}\beta(r)\left[\int_r^t (C_*\alpha(s)+\delta+\nu\lambda_1)e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}ds\right] dr \\
=\ & \int_\tau^t e^{-\kappa_\delta(0,r)}\beta(r)\Big[e^{\int_0^t (C_*\alpha(\sigma)+\delta)d\sigma} - e^{\int_0^r (C_*\alpha(\sigma)+\delta)d\sigma} \\
&\qquad\qquad + \nu\lambda_1\int_r^t e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}ds\Big] dr.
\end{aligned}
\tag{25}
$$

Finally, observe that

$$
\begin{aligned}
\int_r^t e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}ds &\leq \int_{-\infty}^t e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}ds \\
&\leq \delta^{-1} + \int_0^t e^{\int_0^s (C_*\alpha(\sigma)+\delta)d\sigma}ds.
\end{aligned}
\tag{26}
$$

From (22)-(26) we deduce (19). □

Hereafter we assume that

$$\limsup_{t\to-\infty} \frac{1}{t}\int_0^t \alpha(r)dr = \bar{\alpha} \in [0,+\infty), \tag{27}$$

and there exists $\delta > 0$ such that

$$\frac{\bar{\alpha}e^{\nu\lambda_1 h}}{\nu\lambda_1(1-\rho_*)} + \delta < \nu\lambda_1, \tag{28}$$

and β satisfies

$$\int_{-\infty}^{0} e^{-\kappa_\delta(0,s)}\beta(s)ds < +\infty, \tag{29}$$

where $\kappa_\delta(t,s)$ is the function given by (15).

Remark 4. (i) Sufficient conditions to assure (27) and (28) for some $\delta > 0$, are $\alpha \in L^\infty(-\infty,0)$ and

$$-\nu\lambda_1 + \frac{e^{\nu\lambda_1 h}}{\nu\lambda_1(1-\rho_*)}\|\alpha\|_{L^\infty(-\infty,0)} < 0.$$

(ii) Another different sufficient condition, just using Young inequality, to assure (27) and (28) for some $\delta > 0$, is that $\alpha \in L^q(-\infty,0)$ for some $q \in [1,+\infty)$. In this case, $\bar{\alpha} = 0$.

After the above assumptions, we introduce the definition of the two natural tempered universes that will play an essential role in the following.

Definition 9 (Tempered universes)**.** Let $\mathcal{R}_\delta$ be the set of all functions $r : \mathbb{R} \to [0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{-\kappa_\delta(0,t)}r^2(t) = 0.$$

We will denote by $\mathcal{D}^\delta_{M_H^{2p'}}$ the class of all families $\widehat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(M_H^{2p'})$ such that $D(t) \subset \overline{B}_{M_H^{2p'}}(0, r_{\widehat{D}}(t))$, for some $r_{\widehat{D}} \in \mathcal{R}_\delta$.

Analogously, we will denote by $\mathcal{D}^\delta_{C_H}$ the class of all families $\widehat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that $D(t) \subset \overline{B}_{M_H^{2p'}}(0, r_{\widehat{D}}(t))$, for some $r_{\widehat{D}} \in \mathcal{R}_\delta$.

Remark 5. Notice that $\mathcal{D}^\delta_{C_H} \subset \mathcal{D}^\delta_{M_H^{2p'}}$ and that both are inclusion-closed. Moreover, thanks to (27) and (28), for any fixed bounded set $B_0 \subset M_H^{2p'}$, the family $\widehat{B} = \{B(t) \equiv B_0 \ : \ t \in \mathbb{R}\}$ is contained in $\mathcal{D}^\delta_{M_H^{2p'}}$ and analogous observation w.r.t. $\mathcal{D}^\delta_{C_H}$ if $B_0 \subset C_H$. In other words, the universe of fixed bounded sets is contained in the universes $\mathcal{D}^\delta_{C_H}$ and $\mathcal{D}^\delta_{M_H^{2p'}}$, and so the results that hold for these two tempered universes also hold for the universe of fixed bounded sets.

Proposition 5. *Under the assumptions (c1')-(c3'), assume that also* (27)-(29) *hold. Then, for any family* $\widehat{B} = \{B(t) \ : \ t \in \mathbb{R}\} \in \mathcal{D}^\delta_{M_H^{2p'}}$ *and any* $t \in \mathbb{R}$*, there exists* $\tau(\widehat{B},t) \le t$ *such that any function* $u \in D(\tau,u^0,\phi)$*, with* $\tau \le \tau(\widehat{B},t)$ *and*

$(u^0,\phi)\in B(\tau)$, satisfies that $|u(t)|\le R_H(t)$, where $R_H(\cdot)$ is the positive continuous function given by

$$R_H^2(t)=1+\delta^{-1}\int_{-\infty}^t e^{-\kappa_\delta(t,s)}\beta(s)ds\quad\forall\, t\in\mathbb{R}.$$

Proof. The result is a consequence of the definition of $\mathcal{D}^\delta_{M_H^{2p'}}$, Lemma 1 and (16). □

Corollary 4. *Under the assumptions of Proposition 5, consider the family $\widehat{D}_0=\{D_0(t)\ :\ t\in\mathbb{R}\}\subset\mathcal{P}(C_H)$ given by*

$$D_0(t)=\overline{B}_{C_H}(0,\widetilde{R}_H(t))\qquad\text{and}\qquad \widetilde{R}_H(t)=\max_{r\in[t-h,t]}R_H(r)\quad\forall\, t\in\mathbb{R}.$$

Then, $\widehat{D}_0\in\mathcal{D}^\delta_{C_H}$ and it is $\mathcal{D}^\delta_{M_H^{2p'}}-$pullback absorbing for the process $(M_H^{2p'},U)$, and therefore $\mathcal{D}^\delta_{C_H}-$pullback absorbing for (C_H,U), and also when considering the universe of fixed bounded sets.

Proof. It is an immediate consequence of Proposition 5 and (17). □

Proposition 6. *Under the assumptions of Proposition 5, the processes $(M_H^{2p'},U)$ and (C_H,U) are $\widehat{D}_0-$asymptotically compact.*

Proof. Fix a value $t_0\in\mathbb{R}$ and consider a sequence $\{\tau_n\}\subset(-\infty,t_0-2h]$ with $\tau_n\to-\infty$, and a sequence $\{u^n\}$ with $u^n\in D(\tau_n,\phi_n(0),\phi_n)$ and $\phi_n\in D_0(\tau_n)$. We will see that $\{u^n_{t_0}\}$ is relatively compact in C_H, so the result will be proved.

By Lemma 1, and since by (3) one has that

$$\|(u^n)'\|_*\le\nu\|u^n\|+\|b(u^n,u^n,\cdot)\|_*+\lambda_1^{-1/2}|f(t,u^n(t-\rho(t)))|,$$

we obtain uniform estimates, independently of n, for $\{u^n\}$ and $\{(u^n)'\}$ in suitable spaces such that there exists $u\in L^\infty(t_0-2h,t_0;H)\cap L^2(t_0-2h,t_0;V)$ with $u'\in L^2(t_0-h,t_0;V')$, and a subsequence, relabelled the same, such that the following convergences hold:

$$\begin{cases} u^n\overset{*}{\rightharpoonup}u & \text{weakly-star in } L^\infty(t_0-2h,t_0;H),\\ u^n\rightharpoonup u & \text{weakly in } L^2(t_0-2h,t_0;V),\\ (u^n)'\rightharpoonup u & \text{weakly in } L^2(t_0-h,t_0;V'),\\ u^n\to u & \text{strongly in } L^2(t_0-2h,t_0;H),\\ u^n(s)\to u(s) & \text{strongly in } H\quad a.e.\ s\in(t_0-2h,t_0). \end{cases}\tag{30}$$

By the assumptions on f, analogously as in Theorem 1, we also have that

$$f(s,u^n(s-\rho(s))\to f(s,u(s-\rho(s))\quad\text{strongly in } H\ a.e.\ s\in(t_0-h,t_0),$$

and by the uniform estimate in $L^\infty(t_0-2h,t_0;H)$, and the Lebesgue theorem,

$$f(s,u^n(s-\rho(s)))\to f(s,u(s-\rho(s)))\quad\text{strongly in } L^2(t_0-h,t_0;H).$$

Then we can deduce that $u\in C([t_0-h,t_0];H)$ and

$$\begin{aligned} & u(t)+\int_{t_0-h}^t(\nu Au(s)+B(u(s),u(s)))\,ds\\ =\ & u(t_0-h)+\int_{t_0-h}^t f(s,u(s-\rho(s)))\,ds,\quad\forall\, t\in[t_0-h,t_0]. \end{aligned}$$

On other hand, the uniform estimate of $\{(u^n)'\}$ in $L^2(t_0-h,t_0;V')$ means that $\{u^n\}$ is equi-continuous in V' in the interval $[t_0-h,t_0]$, and since $\{u^n\}$ is uniformly bounded in $L^\infty(t_0-h,t_0;H)$ (indeed in $C([t_0-h,t_0];H)$), by the Ascoli-Arzelà theorem, in particular, we conclude that

$$u^n \to u \quad \text{strongly in } C([t_0-h,t_0];V'). \tag{31}$$

Again from the uniform bound for $\{u^n\}$ in $C([t_0-h,t_0];H)$, we know that

$$u^n(s) \rightharpoonup u(s) \quad \text{weakly in } H \quad \forall s\in[t_0-h,t_0],$$

where we have used (31) to identify the weak limit. Indeed, using the same argument we have a stronger property:

$$u^n(s_n) \rightharpoonup u(s) \quad \text{weakly in } H,\ \forall\{s_n\}\subset[t_0-h,t_0]\ :\ s_n\to s\in[t_0-h,t_0]. \tag{32}$$

Our goal is to show that

$$u^n \to u \quad \text{strongly in } C([t_0-h,t_0];H).$$

Using that $u\in C([t_0-h,t_0];H)$, if the above convergence does not hold, then there would exist a value $\varepsilon>0$, a sequence (relabelled the same) $\{t_n\}\subset[t_0-h,t_0]$, and $t_*\in[t_0-h,t_0]$ with $t_n\to t_*$ such that $\|u^n(t_n)-u(t_*)\|\geq\varepsilon$ for all $n\geq1$. However, we will see that $u^n(t_n)\to u(t_*)$ in H. To prove this last claim, since we have (32), we only need the convergence of the norms above, i.e. $|u^n(t_n)|\to|u(t_*)|$ as $n\to+\infty$.

Observe that by (32) we know that

$$|u(t_*)|\leq\liminf_{n\to+\infty}|u^n(t_n)|.$$

So, we have to prove that

$$\limsup_{n\to+\infty}|u^n(t_n)|\leq|u(t_*)|. \tag{33}$$

To check this, we use an energy method (cf. e.g. [13]).

From the energy equality, we have that

$$\begin{aligned}&\frac{1}{2}|z(t)|^2+\nu\int_s^t\|z(r)\|^2dr \qquad (34)\\ =\;&\frac{1}{2}|z(s)|^2+\int_s^t(f(r,z(r-\rho(r))),z(r))dr,\quad\forall\, t_0-h\leq s\leq t\leq t_0,\end{aligned}$$

where z can be u or any u^n.

Consider now the continuous functions defined for $t\in[t_0-h,t_0]$:

$$\begin{aligned}J(t)&=\frac{1}{2}|u(t)|^2-\int_{t_0-h}^t(f(r,u(r-\rho(r))),u(r))dr,\\ J_n(t)&=\frac{1}{2}|u^n(t)|^2-\int_{t_0-h}^t(f(r,u^n(r-\rho(r))),u^n(r))dr.\end{aligned}$$

From the equality (34), it is clear that J and J_n are non-increasing functions. Moreover, by the convergences (30) above, we have that

$$J_n(t)\to J(t)\quad a.e.\ t\in(t_0-h,t_0). \tag{35}$$

We are now ready to prove that (33) holds.

Assume that $t_*>t_0-h$. This is not a restriction, because if necessary we can modify all the argument to an interval $[t_0-h-1,t_0]$. Now, consider a sequence

$\{\tilde{t}_k\} \subset (t_0 - h, t_*)$, with $\tilde{t}_k \to t_*$, such that (35) holds for $t = \tilde{t}_k$. Fix an arbitrarily small value $\epsilon > 0$. By continuity of J,

$$\exists k_\epsilon \ : \ |J(\tilde{t}_k) - J(t_*)| < \epsilon/2, \ \forall k \geq k_\epsilon.$$

Take now $n(k_\epsilon)$ such that

$$t_n \geq \tilde{t}_{k_\epsilon} \quad \text{and} \quad |J_n(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| < \epsilon/2 \quad \forall n \geq n(k_\epsilon).$$

Then, we conclude that for $n \geq n(k_\epsilon)$

$$J_n(t_n) - J(t_*) \leq |J_n(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| + |J(\tilde{t}_{k_\epsilon}) - J(t_*)| < \epsilon.$$

This gives (33) as desired. □

Proposition 7. *Under the assumptions of Proposition 5, the processes* $(M_H^{2p'}, U)$ *and* (C_H, U) *are* $\mathcal{D}^\delta_{M_H^{2p'}}$*−asymptotically compact and* $\mathcal{D}^\delta_{C_H}$*−asymptotically compact respectively.*

Proof. The ideas used in the proof of Proposition 6 are valid for any family in $\mathcal{D}^\delta_{M_H^{2p'}}$, so the result follows. □

Proposition 8. *Under the assumptions of Proposition 5, the processes* $(M_H^{2p'}, U)$ *and* (C_H, U) *are upper semi continuous and* $U(t,\tau) : M_H^{2p'} \to \mathcal{P}(M_H^{2p'})$ *and* $U(t,\tau) : C_H \to \mathcal{P}(C_H)$ *have compact values in their respective topologies.*

Proof. Observe that in order to apply the theoretical results of Section 3 we only need u.s.c. of the process U.

Indeed, the u.s.c. of $(M_H^{2p'}, U)$ follows from similar arguments to those used for the Galerkin sequence in the proof of Theorem 1.

However, using the same energy-procedure as in Lemma 6, we are able to prove that in $[\tau, t]$ any set of solutions (with a converging sequence as initial data in the corresponding phase-space) possesses a converging subsequence in $C([\tau, t]; H)$, whence all claims follows. □

The following two results finally show the existence of attractors for the universe of fixed bounded sets and for $\mathcal{D}^\delta_{C_H}$ and $\mathcal{D}^\delta_{M_H^{2p'}}$ (for clarity we have treated separately the cases of $M_H^{2p'}$ and C_H as phase-spaces).

Theorem 4. *[Attractors in the C_H framework] Assume that (c'1)-(c'3) and (27)-(29) hold. Then, there exist global pullback attractors* $\mathcal{A}_{C_H} = \{\mathcal{A}_{C_H}(t) \ : \ t \in \mathbb{R}\}$ *and* $\mathcal{A}_{\mathcal{D}^\delta_{C_H}} = \{\mathcal{A}_{\mathcal{D}^\delta_{C_H}}(t) \ : \ t \in \mathbb{R}\}$ *for the process* (C_H, U) *in the universes of fixed bounded sets and in* $\mathcal{D}^\delta_{C_H}$ *respectively. Moreover, they are unique (in the sense of Theorem 3) and negatively invariant for* U*, and the following relation holds:*

$$\mathcal{A}_{C_H}(t) \subset \mathcal{A}_{\mathcal{D}^\delta_{C_H}}(t) \ \forall t \in \mathbb{R}.$$

Proof. The results follow from applying Theorem 3 and Corollary 2, in view of propositions 6 and 8. □

Theorem 5 (Attractors in the $M_H^{2p'}$ framework)**.** *Under the assumptions of Theorem 4, there exist global pullback attractors*

$$\mathcal{A}_{M_H^{2p'}} = \{\mathcal{A}_{M_H^{2p'}}(t) \ : \ t \in \mathbb{R}\} \qquad \textit{and} \qquad \mathcal{A}_{\mathcal{D}^\delta_{M_H^{2p'}}} = \{\mathcal{A}_{\mathcal{D}^\delta_{M_H^{2p'}}}(t) \ : \ t \in \mathbb{R}\}$$

for the process $(M_H^{2p'}, U)$ *in the universes of fixed bounded sets and in* $\mathcal{D}^\delta_{M_H^{2p'}}$ *respectively. They are unique (in the sense of Theorem 3) and negatively invariant for* U, *and the following relation holds:*

$$\mathcal{A}_{M_H^{2p'}}(t) \subset \mathcal{A}_{\mathcal{D}^\delta_{M_H^{2p'}}}(t) \ \forall t \in \mathbb{R}.$$

Moreover, they are related with the attractors obtained in Theorem 4 for (C_H, U) *in the following way:*

$$\mathcal{A}_{M_H^{2p'}}(t) = j(\mathcal{A}_{C_H}(t)) \quad \textit{and} \quad \mathcal{A}_{\mathcal{D}^\delta_{M_H^{2p'}}}(t) = j(\mathcal{A}_{\mathcal{D}^\delta_{C_H}}(t)), \tag{36}$$

where $j : C_H \to M_H^{2p'}$ *is the continuous mapping defined by* $j(\phi) = (\phi(0), \phi)$.

Proof. Indeed all the claims but (36) follow exactly as in the proof of Theorem 4.

In order to prove the first identification in (36), observe that $U(t,\tau)$ maps $M_H^{2p'}$ into C_H if $t \geq \tau + h$, and bounded sets from $M_H^{2p'}$ goes to bounded sets of C_H (these claims are both consequences of Theorem 1).

The inclusion $\mathcal{A}_{M_H^{2p'}}(t) \subset j(\mathcal{A}_{C_H}(t))$ follows since $\mathcal{A}_{M_H^{2p'}}(t)$ is the minimal closed set with the property of attracting in pullback sense to bounded sets in $M_H^{2p'}$, and from the above arguments we have that $j(\mathcal{A}_{C_H}(t))$ also attracts in pullback sense to bounded sets in $M_H^{2p'}$.

The opposite inclusion,

$$\mathcal{A}_{M_H^{2p'}}(t) \supset j(\mathcal{A}_{C_H}(t)) \tag{37}$$

follows from the continuous injection $j(C_H) \subset M_H^{2p'}$ and the construction of the attractor

$$\mathcal{A}_{C_H}(t) = \overline{\bigcup_{\substack{B \subset C_H \\ \text{bounded}}} \Lambda_{C_H}(B,t)}^{C_H},$$

(where we have denoted $\Lambda_{C_H}(\cdot,\cdot)$ the omega-limit construction in the space C_H to distinguish from the case of phase space $M_H^{2p'}$, which we will denote $\Lambda_{M_H^{2p'}}(\cdot,\cdot)$) and so

$$j(\mathcal{A}_{C_H}(t)) \subset \overline{\bigcup_{\substack{B \subset C_H \\ \text{bounded}}} j(\Lambda_{C_H}(B,t))}^{M_H^{2p'}}.$$

And finally it is obvious that for each bounded set $B \subset C_H$, one has that

$$j(\Lambda_{C_H}(B,t)) = \Lambda_{M_H^{2p'}}(j(B),t),$$

whence (37) follows.

The second identification in (36) can be proved analogously. □

Acknowledgments. We would like to thank the referees for their interesting and helpful suggestions and comments.

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