

Accepted Manuscript

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PII: S0362-546X(09)00342-3

DOI: [10.1016/j.na.2009.02.065](https://doi.org/10.1016/j.na.2009.02.065)

Reference: NA 7398

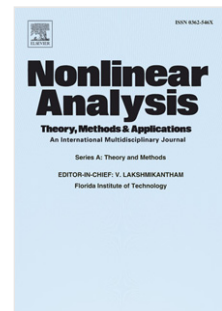
To appear in: *Nonlinear Analysis*

Received date: 4 March 2008

Accepted date: 13 February 2009

Please cite this article as: P. Marín-Rubio, J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Analysis* (2009), doi:[10.1016/j.na.2009.02.065](https://doi.org/10.1016/j.na.2009.02.065)

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On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems

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Abstract

For an abstract dynamical system, we establish under minimal assumptions the existence of \mathcal{D} -attractor, i.e. a pullback attractor for a given class \mathcal{D} of families of time varying subsets of the phase space. We relate this concept with the usual attractor of fixed bounded sets, pointing out its usefulness in order to ensure the existence of this last attractor in particular situations. Moreover, we prove that under a simple assumption these two notions of attractors generate in fact the same object. This is then applied to a Navier-Stokes model, improving some previous results on attractor theory.

Key words: Pullback attractors, non-autonomous dynamical systems, tempered sets.

1 Introduction

Much attention has been paid in the last few decades to attractor theory with the aim of going further in the study of complex dynamical systems. Although much information can be obtained with it, such as finite fractal and Hausdorff dimensions, determining modes, inertial manifolds and finite-dimensionality behaviour among others (see [13] and the references therein), a general modelization involves additional difficulties, even in the starting concepts.

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Namely, this is the case when the model is non-autonomous. Stability, bifurcation, and even the concept of attractor need to be revised. Indeed, after the uniform attractor of Chepyzhov and Vishik, only valid for some situations, the notions of pullback and forward attractors seem to be general ways to extend results on this direction (see [5] for a comparison of these two last concepts).

Several branches are then developed. On the one hand, there exists the pullback attractor of “fixed” bounded sets as the most usual option. See for instance [9, Section 1] for a brief description on deterministic pullback attractors for non-autonomous systems; for this reason we will denote this attractor by \mathcal{A}_{CDF} (the attractor of Crauel, Debussche, and Flandoli).

On the other hand, several authors use the concept of attraction in a universe \mathcal{D} not only composed by “fixed” sets, but also moving in time, which usually appears in applications and is defined in terms of a tempered condition. We will denote $\mathcal{A}_{\mathcal{D}}$ the attractor in this case.

Nevertheless, some features of attractors can be circumvented for any of the two options, notwithstanding the time dependence, such as the invariance (adapting the concept in a suitable sense). A distinguishing matter is why the attractor (a family of time dependent sets) has to be unique. It is obvious, whatever of the above two cited definitions one chooses, that it is minimal, and that is the only way to talk about uniqueness when dealing with a universe of autonomous bounded sets, since the attractor is not an object of the universe and cannot be attracted by itself. On the other hand, for a universe of time-dependent families, the uniqueness holds easily as for the global attractor of an autonomous dynamical system, being the attractor a member of this universe (and therefore attracted by itself). So it is not absolutely clear which of the two options is better.

A first attempt to consider the topic of whether or not to use autonomous bounded sets as universe is the paper by Crauel and Flandoli [8]. There they develop a theory for random attractors and deal with some problems where the deterministic equations have added terms that represent additive or multiplicative noise. In principle, they develop (in Section 3) concepts for a universe that may change with the fiber ω of the realization of the noise, and define and prove nice properties for the omega-limit sets. However, they announce in Remark 3.10 (ii) that they are unable to obtain more than attraction of fixed bounded sets, so they quit this framework and continue with the *standard* universe of autonomous bounded sets. They did not remark that the estimates with exponential decay w.r.t. initial data allow to extend the universe to a tempered one with the same properties (the tempered condition is related to time and ω through the family of measure preserving transformations $\{\theta_t\}$).

The natural continuation of the above-mentioned paper was published three

years later. In [9], Crauel, Debussche, and Flandoli kept the same option of a universe of autonomous bounded sets. For the other option in the random case, see for instance [10, p.385] for an early comment.

The question in this random framework goes much further with the very nice paper [7] by Crauel. There, the author continues with the concept of random attractor w.r.t. the universe of autonomous bounded sets (or the universe of autonomous compact sets). Even without ergodicity of the flow, he is able to prove that random attractors (i.e. compact random sets, strictly invariant, and attracting autonomous bounded sets) that attract compact non-random sets are almost surely the same object, and one can choose a non-random compact set K such that $\mathbb{P}(\mathcal{A} \subset \Omega_K) \geq 1 - \varepsilon$, where Ω_K denotes the omega-limit set of K , and with ε as small as desired. When the flow is ergodic the result can be improved and the random attractor satisfies $\mathcal{A} = \Omega_D$, both for attractors of non-random bounded or non-random compact sets.

This seems to give a definitive answer to the random case, since an attractor with nice properties in any universe bigger than that of the autonomous bounded sets is uniquely determined by the attraction of non-random compact (even more restrictive than bounded) sets. However, there are some inconveniences even in this setting. The results use a version of Poincaré's recurrence theorem, so in the end they make use of the probability structure behind the problem. The second obstacle is that the existence of a random compact attracting set is an assumption in the result, and sometimes the result itself of existence of a random compact set that attracts is the open question in applications. So, to circumvent these matters, some authors are directly focused on the tempered framework (e.g. cf. [1,6]).

The goal of this note is two-fold: on the one hand, we aim to point out that there exist situations where real difficulties appear in the framework related to the universe of fixed bounded sets, and this requires, to be solved, to apply the tempered tools. Examples in this sense are provided. So, this bigger framework is not for the sake of generality. On the other hand, we aim to establish and answer the natural question that arises now: can the resulting families corresponding to these two notions of attraction be the same? We show that in fact they are the same under a suitable assumption.

2 Existence of pullback attractor for fixed bounded sets

We combine in this section some results in the literature, see for instance [14], about existence of attractors for fixed bounded sets, leading to a sufficient condition to apply one result of [9], passing through a condition stated in terms of a family of time-dependent sets.

To avoid confusion among the two frameworks of this note, since conditions are similar, we establish statements with explicit references to the suitable definitions on each case.

Consider a metric space (X, d) . As usual, let us denote by $\text{dist}(C_1, C_2)$ the Hausdorff semidistance between C_1 and C_2 , i.e.

$$\text{dist}(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} d(x, y) \quad \text{for } C_1, C_2 \subset X.$$

Suppose given a process in X , i.e. a family of maps $U(t, \tau) : X \rightarrow X$ for $t \geq \tau \in \mathbb{R}$, such that

$$U(t, \tau) = U(t, r)U(r, \tau) \quad \forall \tau \leq r \leq t.$$

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X , and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ [observe that we do not require any additional condition on these sets as compactness or boundedness].

Definition 1 *We say that the process U is \widehat{D}_0 -asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\tau_n \rightarrow -\infty$ and $x_n \in D_0(\tau_n)$, the sequence $U(t, \tau_n)x_n$, with $\tau_n \leq t$, is precompact in X .*

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D_0(\tau)}^X \quad \forall t \in \mathbb{R}. \quad (1)$$

Proposition 2 *If the process $U(t, \tau)$ is \widehat{D}_0 -asymptotically compact, then the set $\Lambda(\widehat{D}_0, t)$ given by (1) is a nonempty compact subset of X for all $t \in \mathbb{R}$.*

Proof. Consider fixed a value $t \in \mathbb{R}$.

Take an arbitrary sequence $\tau_n \rightarrow -\infty$, and for each $\tau_n \leq t$ choose $x_n \in D_0(\tau_n)$, then, there exist two subsequences τ_μ and x_μ , and one point $y \in X$ such that $U(t, \tau_\mu)x_\mu$ converges to y in X . Therefore, y belongs to $\Lambda(\widehat{D}_0, t)$, which is therefore nonempty.

On the other hand, it is clear by definition that $\Lambda(\widehat{D}_0, t)$ is closed, so in order to show that it is compact, it is enough to prove that it is relatively compact in X . Consider a sequence $\{y_n\} \subset \Lambda(\widehat{D}_0, t)$, we have to prove that it is possible to extract a subsequence converging in X .

By definition of $\Lambda(\widehat{D}_0, t)$, for each integer n there exist $\tau_n \leq t - n$ and $x_n \in D_0(\tau_n)$ such that $d(y_n, U(t, \tau_n)x_n) \leq 1/n$. As long as U is \widehat{D}_0 -asymptotically compact, we can extract from $\{U(t, \tau_n)x_n\}$ a subsequence converging in X .

Therefore, it is clear that the corresponding subsequence of $\{y_n\}$ will also converge in X to the same point. ■

The following definition contains what seem to be minimal requirements on continuity for a process in order to study its asymptotic behaviour (cf. [12,14]).

Definition 3 *We say that the process U is strong-weak continuous if for any pair $t \geq \tau \in \mathbb{R}$, the map $U(t, \tau)$ is continuous from X with the strong topology into X with the weak topology.*

Proposition 4 *Suppose that the process U is \widehat{D}_0 -asymptotically compact and strong-weak continuous, then the family of sets $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$, defined by (1), is invariant for U , i.e.*

$$\Lambda(\widehat{D}_0, t) = U(t, \tau)\Lambda(\widehat{D}_0, \tau) \quad \forall \tau \leq t.$$

Proof. Fix a pair of values $\tau \leq t$.

If $y \in \Lambda(\widehat{D}_0, \tau)$, then there exist sequences $\tau_n \rightarrow -\infty$ and $x_n \in D_0(\tau_n)$, such that $U(\tau, \tau_n)x_n \rightarrow y$ in X when $n \rightarrow \infty$. Then,

$$U(t, \tau_n)x_n = U(t, \tau)U(\tau, \tau_n)x_n \rightharpoonup U(t, \tau)y \quad \text{weakly in } X.$$

On the other hand, from the sequence $\{U(t, \tau_n)x_n\}$ we can extract a subsequence converging strongly in X , so the limit has to be $U(t, \tau)y$. Thus, $U(t, \tau)y \in \Lambda(\widehat{D}_0, t)$. We have proved the inclusion $U(t, \tau)\Lambda(\widehat{D}_0, \tau) \subset \Lambda(\widehat{D}_0, t)$.

For the other inclusion, consider an element $z \in \Lambda(\widehat{D}_0, t)$. Then there exist two sequences $\tau_n \rightarrow -\infty$ and $x_n \in D_0(\tau_n)$, such that $U(t, \tau_n)x_n \rightarrow z$ in X when $n \rightarrow \infty$. For each $\tau_n \leq \tau$ it holds that

$$U(t, \tau_n)x_n = U(t, \tau)U(\tau, \tau_n)x_n. \quad (2)$$

Taking into account that U is \widehat{D}_0 -asymptotically compact, we can extract a subsequence $\{U(\tau, \tau_\mu)x_\mu\}$ converging in X to one point y , and by construction it belongs to $\Lambda(\widehat{D}_0, \tau)$. By (2) and being U strong-weak continuous, we have that $U(t, \tau_\mu)x_\mu \rightharpoonup U(t, \tau)y$, so $z = U(t, \tau)y$. This proves the required inclusion $\Lambda(\widehat{D}_0, t) \subset U(t, \tau)\Lambda(\widehat{D}_0, \tau)$. ■

Definition 5 *We say that the family of sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ is pullback-absorbing for U if for every $t \in \mathbb{R}$ and every bounded subset B of X , there exists $\tau(t, B) \leq t$ such that*

$$U(t, \tau)B \subset D_0(t) \quad \forall \tau \leq \tau(t, B).$$

Proposition 6 *If the family of sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ is pullback-absorbing for U and the process U is \widehat{D}_0 -asymptotically compact, then for every $t \in \mathbb{R}$ and every bounded set B of X , it holds*

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)B, \Lambda(\widehat{D}_0, t)) = 0. \quad (3)$$

Proof. We proceed by a contradiction argument. Suppose that there exist $t \in \mathbb{R}$ and a bounded set B of X such that (3) fails. Then, there exist $\varepsilon > 0$ and two sequences $\tau_n \rightarrow -\infty$ and $\{x_n\} \subset B$ such that

$$d(U(t, \tau_n)x_n, \Lambda(\widehat{D}_0, t)) \geq \varepsilon \quad \forall n \geq 1. \quad (4)$$

For each integer $k \geq 1$, define $t_k = t - k$. As \widehat{D}_0 is pullback-absorbing, for each $k \geq 1$ there exists a value τ_{n_k} in the sequence $\{\tau_n\}$ such that

$$\tau_{n_k} \leq t_k \quad \text{and} \quad U(t_k, \tau_{n_k})B \subset D_0(t_k).$$

In particular,

$$\lim_{k \rightarrow \infty} \tau_{n_k} = -\infty \quad \text{and} \quad y_k := U(t_k, \tau_{n_k})x_{n_k} \in D_0(t_k).$$

Therefore, the sequence

$$U(t, t_k)y_k = U(t, \tau_{n_k})x_{n_k}$$

is precompact, and we can conclude the existence of subsequences $(\tau_{n_\mu}, x_{n_\mu})$ from (τ_{n_k}, x_{n_k}) and of a point $z \in X$ such that

$$\lim_{\mu \rightarrow \infty} U(t, t_\mu)y_\mu = \lim_{\mu \rightarrow \infty} U(t, \tau_{n_\mu})x_{n_\mu} = z.$$

Since $U(t, t_\mu)y_\mu$ converges to z , $z \in \Lambda(\widehat{D}_0, t)$, and so $U(t, \tau_{n_\mu})x_{n_\mu}$ converges to z , which is a contradiction with (4). ■

As a consequence of the above results, we can establish easily a sufficient condition in order to ensure the existence of the global pullback attractor $\{\mathcal{A}_{\text{CDF}}(t)\}$. But firstly, for clarity in our comparison, let us recall the original result as was formulated in [9, Th.1.1, p.311].

Theorem 7 *Assume that $U(t, s) : X \rightarrow X$ is continuous for all $s \leq t$. Given $t \in \mathbb{R}$, assume that there exists a compact attracting set $K(t)$. Then the set*

$$\mathcal{A}_{\text{CDF}}(t) = \overline{\bigcup_{B \text{ bounded}} \Lambda(B, t)}^X, \quad (5)$$

is a nonempty compact subset of $K(t)$. It attracts all bounded sets from $-\infty$: for all $B \subset X$

$$\lim_{s \rightarrow -\infty} \text{dist}(U(t, s)B, \mathcal{A}_{\text{CDF}}(t)) = 0.$$

Moreover, it is the minimal closed set with this property: if $\tilde{A}(t)$ is a closed set that attracts all bounded sets from $-\infty$, then $\mathcal{A}_{\text{CDF}}(t) \subset \tilde{A}(t)$. Finally, $\mathcal{A}_{\text{CDF}}(\tau)$ is also well defined for all $\tau > t$ and satisfies the invariance property

$$U(\tau, r)\mathcal{A}_{\text{CDF}}(r) = \mathcal{A}_{\text{CDF}}(\tau), \quad \forall \tau \geq r \geq t.$$

For these reasons, we say that $\mathcal{A}_{\text{CDF}}(t)$ is the global attractor of the dynamical system $U(t, s)$ at time t .

The nice thing here is that our conditions on the family \widehat{D}_0 do not require compactness nor boundedness as in [9] is asked for the family $\{K(t)\}$.

Remark 8 *Theorem 7 can also be obtained under the assumption of a process strong-weak continuous instead of continuous, provided the existence of a family $\{K(t)\}$ with each $K(t)$ a compact (pullback) attracting set at time t . Indeed, this generalization can be proved similarly to propositions 2, 4 and 6.*

The only difference is that the assumption in [9] of existence of a compact attracting set $K(t)$ at one single time t is enough –jointly with the continuity of the process– to ensure (cf. [9, Remark 1.1]) the existence of compact attracting sets for all future: namely take $K(r) = U(r, t)K(t)$ for $r \geq t$. For a strong-weak continuous process this is not so. But in fact we have a family of compact attracting sets already built: $\{\Lambda(\widehat{D}_0, r) : r \in \mathbb{R}\}$.

Corollary 9 *Consider a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ of nonempty subsets of X and a process U that is \widehat{D}_0 -asymptotically compact and strong-weak continuous, and assume also that \widehat{D}_0 is pullback-absorbing for U . Then, it exists the attractor of Crauel, Debussche and Flandoli $\mathcal{A}_{\text{CDF}} = \{\mathcal{A}_{\text{CDF}}(t) : t \in \mathbb{R}\}$.*

Moreover, the following relation holds:

$$\mathcal{A}_{\text{CDF}}(t) \subset \Lambda(\widehat{D}_0, t) \quad \forall t \in \mathbb{R}. \quad (6)$$

Proof. Firstly, observe that Theorem 7 is also valid for a strong-weak continuous process U (see Remark 8). In order to rearrange that proof and obtain the results, we only need a family $\{K(t) : t \in \mathbb{R}\}$ of compact pullback attracting sets. It is enough to take as $K(t) = \Lambda(\widehat{D}_0, t)$ thanks to Proposition 2 and Proposition 6.

Finally, the relation (6) is another consequence of Theorem 7 since \mathcal{A}_{CDF} is minimal among all families of closed sets attracting bounded sets. ■

Remark 10 Properties of $\Lambda(\widehat{D}_0, t)$ proved in propositions 2, 4 and 6 point out that it is an attractor. However, since the property of pullback absorption is only for bounded sets, the most we can expect for $\{\Lambda(\widehat{D}_0, t)\}$ is to be an attractor of fixed bounded sets. But \mathcal{A}_{CDF} is already the smallest family with such property. Therefore, the best we can expect in (6) is to have not only an inclusion but an equality.

The expectation of obtaining in all cases an equality in (6) instead of an strict inclusion is vain. Consider the following example.

Example 11 Assume that $f = f(t, x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function, globally lipschitz w.r.t. x , and such that $f(t, x) = -x$ if $|x| \leq e^{-t}$. Then, for the process $U(t, \tau)$ defined on \mathbb{R} by

$$U(t, \tau)x_0 = x(t; \tau, x_0), \quad x_0 \in \mathbb{R}, \quad t \geq \tau,$$

where $x(\cdot; \tau, x_0)$ is the unique solution of the (PC) $\dot{x} = f(t, x)$ with $x(\tau) = x_0$, one has that $\mathcal{A}_{\text{CDF}}(t) = \{0\}$ for all $t \in \mathbb{R}$.

On the other hand, if we take $\widetilde{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ with $D_0(t) = [-e^{-t}, e^{-t}]$ for all $t \in \mathbb{R}$, one can check that $U(t, \tau)D_0(\tau) = D_0(t)$. So,

$$\Lambda(\widetilde{D}_0, t) = D_0(t) = [-e^{-t}, e^{-t}], \quad \forall t \in \mathbb{R}. \quad (7)$$

Proposition 12 Under the assumptions of Corollary 9, if there exists a value $T \in \mathbb{R}$ such that $\bigcup_{t \leq T} D_0(t)$ is bounded, then $\mathcal{A}_{\text{CDF}}(t) = \Lambda(\widehat{D}_0, t)$ for all $t \leq T$.

Proof. Fix any $t \leq T$. By Corollary 9 it remains to prove the inclusion $\mathcal{A}_{\text{CDF}}(t) \supset \Lambda(\widehat{D}_0, t)$.

By definition (1)

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D_0(\tau)}^X \subset \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B_0}^X = \Lambda(B_0, t),$$

where we have denoted $B_0 = \bigcup_{\tau \leq T} D_0(\tau)$, that is bounded by assumption. Now, by (5) the result is clear. ■

3 Pullback attractors for families of sets depending on time

In this section we establish the basic result about the existence of pullback attractor, close to that in [3] (see also [4]), for the case where the attracted universe is not composed of fixed bounded sets but of families of sets depending on time.

Let be given \mathcal{D} a nonempty class of sets parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 13 *The process U is said to be pullback \mathcal{D} -asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \rightarrow -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .*

Remark 14 *Observe that the pullback \mathcal{D} -asymptotic compactness above in Definition 13 is w.r.t. a class of families (\mathcal{D}), while the \widehat{D}_0 -asymptotic compactness in the sense of Definition 1 corresponds to a single family (\widehat{D}_0).*

The property given in Definition 13 is enough to prove the following result, whose proof is analogous to [3, Prop.12] combined with some ideas used in Section 2:

Proposition 15 *Let us assume that the process U is strong-weak continuous, pullback \mathcal{D} -asymptotically compact. For each $\widehat{D} \in \mathcal{D}$, the set $\Lambda(\widehat{D}, t)$ defined by (1) is a non-empty compact subset of X , invariant for U , that attracts \widehat{D} in the pullback sense, i.e.:*

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)D(\tau), \Lambda(\widehat{D}, t)) = 0.$$

Definition 16 *It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is pullback \mathcal{D} -absorbing for the process U if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that*

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Remark 17 *Observe again that by the notation no possible confusion arises with the property given in Definition 3.*

Joining the concepts given in Definitions 13 and 16 we have the following result (whose proof is analogous to [3, Th.7] combined with some ideas used in Section 2):

Theorem 18 *Let us suppose that U is a strong-weak continuous process and it is pullback \mathcal{D} -asymptotically compact, and that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is a family of pullback \mathcal{D} -absorbing sets for U . Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t), \quad t \in \mathbb{R}, \quad (8)$$

has the following properties:

- (1) *the set $\mathcal{A}_{\mathcal{D}}(t)$ is compact for any $t \in \mathbb{R}$,*
- (2) *$\mathcal{A}_{\mathcal{D}}$ is pullback \mathcal{D} -attracting, i.e.*

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D},$$

- (3) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,
 (4) and

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad \text{for } t \in \mathbb{R}.$$

The family $\mathcal{A}_{\mathcal{D}}$, called the global pullback \mathcal{D} -attractor for the process U , is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)B(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 19 If we assume that $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family \mathcal{D} is inclusion-closed (i.e. if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t , then $\widehat{D}' \in \mathcal{D}$), then the pullback \mathcal{D} -attractor $\mathcal{A}_{\mathcal{D}}$ belongs to \mathcal{D} , and it is the only family in \mathcal{D} satisfying properties (1)–(3) above.

The following results give a comparison of the two introduced attractors.

Corollary 20 Under the assumptions of Theorem 18, if the universe \mathcal{D} contains all bounded sets of X , then both attractors, \mathcal{A}_{CDF} and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:

$$\mathcal{A}_{\text{CDF}}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \in \mathbb{R}.$$

Proof. Apply Theorem 18 and Corollary 9. ■

Corollary 21 Under the assumptions of Theorem 18, if the universe \mathcal{D} contains all bounded sets of X , and the family \widehat{D}_0 satisfies for some $T \in \mathbb{R}$ that $\bigcup_{t \leq T} D_0(t)$ is bounded, then $\mathcal{A}_{\text{CDF}}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.

Proof. This is a particular case of Proposition 12. ■

Remark 22 (i) When \mathcal{D} is the universe of all constant bounded subsets of X , the existence of $\mathcal{A}_{\mathcal{D}}$ implies the existence of \mathcal{A}_{CDF} , and both are equal: $\mathcal{A}_{\text{CDF}}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.

However, the existence of \mathcal{A}_{CDF} does not necessarily imply the existence of $\mathcal{A}_{\mathcal{D}}$ defined by (8), because it may happen that $\widehat{D}_0 \notin \mathcal{D}$. This is in fact the situation in many cases in applications where the sets $D_0(t)$ do depend effectively on t , as we can see in the example in Section 4.

(ii) As observed in the Introduction, in a random framework an analogous question was addressed and solved by Crauel (cf. [7, Cor.5.5 and Cor.5.8]), but without any special additional requirement on the sets or the universe. The

proofs rely on the use of some probabilistic arguments, such as the Poincaré's recurrence theorem.

In the deterministic non-autonomous framework the situation is different. In Example 11, if one takes as \mathcal{D} the universe of all constant bounded subsets of \mathbb{R} and the family \widehat{D}_0 defined by (7), one has that

$$\mathcal{A}_{\text{CDF}}(t) = \{0\}, \quad \mathcal{A}_{\mathcal{D}}(t) = [-e^{-t}, e^{-t}], \quad \forall t \in \mathbb{R}.$$

4 An application: non-autonomous 2D-Navier-Stokes equations

Besides the abstract setting, the applications involve an election of the universe for the dynamical system on each case. This is usually chosen through the estimates for the problem, and finally leads to a tempered condition –using a suitable function– on the radius of the absorbing balls or whatever other object relevant for the global dynamic.

We state an example to illustrate how the simplest universe of fixed bounded sets can be amplified in these situations to a tempered universe, but still fulfilling the conditions in Corollary 21. Thus, the global attractor of fixed bounded sets in fact will attract more objects: those of the new tempered universe.

The dynamical system associated to a non-autonomous 2D-Navier-Stokes problem in bounded and even in some unbounded domains was proved to have a global pullback attractor in [2] and [3] respectively. Let us recall briefly the main points we are interested in (we refer for a more detailed exposition to the papers cited above).

Consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i} = f(t) - \nabla p & \text{in } (\tau, +\infty) \times \mathcal{O}, \\ \operatorname{div} u = 0 & \text{in } (\tau, +\infty) \times \mathcal{O}, \\ u = 0 & \text{on } (\tau, +\infty) \times \partial \mathcal{O}, \\ u(\tau, x) = u^0(x), & x \in \mathcal{O}, \end{cases} \quad (9)$$

with the standard notation, and where $\mathcal{O} \subset \mathbb{R}^2$ is an open set with boundary $\partial \mathcal{O}$. We do not suppose that Γ is regular, and \mathcal{O} is not necessarily bounded, but satisfies a Poincaré inequality, i.e., there exists $\lambda_1 > 0$ such that

$$\int_{\mathcal{O}} |\phi|^2 dx \leq \frac{1}{\lambda_1} \int_{\mathcal{O}} |\nabla \phi|^2 dx, \quad \text{for all } \phi \in H_0^1(\mathcal{O}).$$

Assume also the standard notation for the spaces \mathcal{V} , H and V (see [13]).

In [3] the authors prove that the problem above generates a strong-weak continuous process in H (even more than this, see [3, Prop.16]), and in [3, Th.17] they prove that if

$$\int_{-\infty}^t e^{\sigma s} \|f(s)\|_*^2 ds < +\infty \text{ for all } t \in \mathbb{R},$$

(where $\sigma = \nu\lambda_1$) then there exists the global pullback \mathcal{D} -attractor, where \mathcal{D} is the universe (the class) of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$ for some function $r_{\widehat{D}} : \mathbb{R} \rightarrow (0, +\infty)$ satisfying the tempered condition $\lim_{t \rightarrow -\infty} e^{\sigma t} r^2(t) = 0$.

Indeed, the condition to apply Theorem 18 here is to consider the absorbing family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$, where $D_0(t) = \{v \in H : |v| \leq R_\sigma(t)\}$ with the positive value $R_\sigma(t)$ given by $(R_\sigma(t))^2 = \frac{\varepsilon - \sigma t}{\nu} \int_{-\infty}^t e^{\sigma s} \|f(s)\|_*^2 ds + 1$.

Actually the definition of a single family or of a class of families such that the process has “good” asymptotic properties is obtained from the estimates of the decay of the solutions. The point is that if one aims to construct the attractor \mathcal{A}_{CDF} , it is unclear in this situation how to proceed keeping only with fixed bounded sets, but clearer using Corollary 9. Moreover, once we are involved in this framework, *with the same effort* (applying Theorem 18) we may obtain the attractor $\mathcal{A}_{\mathcal{D}}$, which in principle is a bigger family. But the interesting point is that if these two families are the same, what one really obtains is that the usual concept of attractor, i.e. \mathcal{A}_{CDF} , in fact attracts in a richer universe.

Proposition 23 *Under the notation and assumptions above, if there exists a value $T \in \mathbb{R}$ such that*

$$\sup_{t \leq T} R_\sigma(t) < +\infty, \quad (10)$$

then the following equality holds:

$$\mathcal{A}_{\text{CDF}}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \in \mathbb{R},$$

where \mathcal{A}_{CDF} is the attractor of fixed bounded sets for the dynamic generated by (9) and $\mathcal{A}_{\mathcal{D}}$ is the global pullback \mathcal{D} -attractor.

Remark 24 (i) *It is not difficult to check that a sufficient condition for (10) is that $f \in L^\infty(-\infty, T; V^*)$.*

(ii) *An analogous result can be obtained for the delay cases stated in [2] and [11].*

Conclusions

It is clear from applications that in order to establish the existence of (pull-back) attractors of bounded sets for the dynamical system associated to one problem, it is sometimes useful to go up to a bigger framework, replacing the universe of autonomous bounded sets by a universe of families of time-depending sets. This more general concept of \mathcal{D} -attractor is better adapted to different situations, and easier to obtain, even when the existence of the usual attractor is unclear.

In examples, this \mathcal{D} -attractor is usually related to a *tempered universe*, i.e. where the families of time-depending sets are given by a tempered condition on their growth in time. Moreover, it usually happens that the universe of autonomous bounded sets is a subset of the *tempered universe*.

Therefore, the two concepts of attractor, respectively \mathcal{A}_{CDF} and $\mathcal{A}_{\mathcal{D}}$, are usually related through the inclusion $\mathcal{A}_{\text{CDF}} \subset \mathcal{A}_{\mathcal{D}}$. More precisely, the existence of $\mathcal{A}_{\mathcal{D}}$ provides a sufficient condition that ensures the existence of \mathcal{A}_{CDF} . Although the cases of an ODE or a PDE in a bounded domain do not usually require this way of proceeding, in other situations where compactness of the (semi-)process does not hold or is unknown, this approach is essential, a fact that we aimed to point out.

Even in the random case, where the relation between both objects is well known (they coincide, or at least in a probability sense, cf. [7]), it is sometimes useful to study previously the existence of a random \mathcal{D} -attractor in order to obtain a sufficient condition that ensures the existence of the random attractor in the sense of Crauel, Debussche, and Flandoli.

In the non-random case, we have obtained sufficient conditions that guarantee that these two objects are in fact the same. Therefore, we can claim that \mathcal{A}_{CDF} , previously considered only the attractor of bounded sets, in fact attracts more objects.

Acknowledgements

The authors would like to thank one referee for comments that helped to improve the presentation of the paper, and discussions that led to Example 11 making the exposition clearer.

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