

## EQUIVALENCE OF INVARIANT MEASURES AND STATIONARY STATISTICAL SOLUTIONS FOR THE AUTONOMOUS GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS

PETER E. KLOEDEN, PEDRO MARÍN-RUBIO, AND JOSÉ REAL

ABSTRACT. A new proof of existence of solutions for the three dimensional system of globally modified Navier-Stokes equations introduced in [3] by Caraballo, Kloeden and Real is obtained using a smoother Galerkin scheme. This is then used to investigate the relationship between invariant measures and statistical solutions of this system in the case of temporally independent forcing term. Indeed, we are able to prove that a stationary statistical solution is also an invariant probability measure under suitable assumptions.

**1. Introduction.** Let  $\Omega \subset \mathbb{R}^3$  be an open bounded set with regular boundary  $\Gamma$ , and consider the following system of Navier-Stokes equations (NSE) on  $\Omega$  with a homogeneous Dirichlet boundary condition

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(t) & \text{in } (0, +\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \Gamma, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

where  $\nu > 0$  is the kinematic viscosity,  $u$  is the velocity field of the fluid,  $p$  the pressure,  $u_0$  the initial velocity field, and  $f(t)$  a given external force field.

There exist many modified versions of Navier-Stokes equations due to Leray and others with mollification (and/or cut off) of the nonlinear term as a way to approximate the original problem, see for instance the review paper of Constantin [1]. We also mention the paper [5] by Flandoli and Maslowski with a global cut off function used in a 2-dimensional stochastic context.

In 2006, Caraballo, Kloeden and Real (cf. [3]) proposed a 3-dimensional model where the nonlinear term included a cut off factor  $F_N(\|u\|)$  based on the norm of the gradient of the solution in the whole domain. Namely, for  $N \in (0, +\infty)$  the

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function  $F_N : [0, +\infty) \rightarrow (0, 1]$  is defined by

$$F_N(r) := \min \left\{ 1, \frac{N}{r} \right\}, \quad r \in [0, +\infty).$$

They called the resulting system

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|) [(u \cdot \nabla)u] + \nabla p = f(t) & \text{in } (0, +\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \Gamma, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

the globally modified Navier-Stokes equations (GMNSE) and established the well-posedness of the model, in particular the absence of blow-up of solutions, as well as the existence of global  $V$ -attractors and certain relationships with the original NSE model. See also [7, 8] for other studies and applications of the GMNSE as well as the review paper [2].

Foias *et al.* [6] (see also the papers cited therein) have made a very systematic study of invariant measures and statistical solutions of the NSE models in 2 and 3 dimensions. This investigation was continued in [4] by Caraballo, Kloeden and Real for the GMNSE, who proved the existence of time-averaged measures, then of invariant measures and finally that any invariant measure is a stationary statistical solution. We note that a relation between a family of time-average probability measures and the pullback attractor of a non-autonomous version of the NSE was established in [10].

Our aim in this paper is to establish the inverse of the last result in [4], i.e. that a stationary statistical solution of the GMNSE is an invariant probability measure, which we prove under suitable assumptions.

The structure of the paper is as follows. We recall some preliminaries on the used functional spaces, definitions and existence results in Section 2. In Section 3 we establish a new proof of the existence of strong solutions for the GMNSE with a different approach from that used in [3] since we need more regularity on the Galerkin approximations. Estimates in  $D(A)$  are obtained for these solutions of the new Galerkin scheme in Section 4. Then, in Section 5 we establish our main result, Theorem 15, proving under additional assumptions that a statistical solution is an invariant probability measure. Finally, to improve the clarity of the paper, proofs of some technical results used earlier are given in the Appendix.

**2. Preliminaries.** To set our problem in the abstract framework, we consider the following usual abstract spaces (see Lions [9] and Temam [12, 13]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0 \right\},$$

$H$  = the closure of  $\mathcal{V}$  in  $(L^2(\Omega))^3$  with inner product  $(\cdot, \cdot)$  and associated norm  $|\cdot|$ , where for  $u, v \in (L^2(\Omega))^3$ ,

$$(u, v) = \sum_{j=1}^3 \int_{\Omega} u_j(x) v_j(x) dx,$$

$V$  = the closure of  $\mathcal{V}$  in  $(H_0^1(\Omega))^3$  with inner product  $((\cdot, \cdot))$  and associated norm  $\|\cdot\|$ , where for  $u, v \in (H_0^1(\Omega))^3$ ,

$$((u, v)) = \sum_{i,j=1}^3 \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that  $V \subset H \equiv H' \subset V'$ , where the injections are dense and compact. Finally, we will use  $\|\cdot\|_*$  for the norm in  $V'$  and  $\langle \cdot, \cdot \rangle$  for the duality pairing between  $V$  and  $V'$ .

Now we define the trilinear form  $b$  on  $V \times V \times V$  by

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V,$$

and we denote

$$b_N(u, v, w) = F_N(\|v\|)b(u, v, w), \quad \forall u, v, w \in V.$$

The form  $b_N$  is linear in  $u$  and  $w$ , but it is nonlinear in  $v$ . Evidently we have  $b_N(u, v, v) = 0$ , for all  $u, v \in V$ . Moreover, by the properties of  $b$  (see [11] or [12]), there exists a constant  $C_1 > 0$  only dependent on  $\Omega$  such that

$$|b(u, v, w)| \leq C_1 \|u\| \|v\| \|w\|^{1/4} \|w\|^{3/4}, \quad \forall u, v, w \in V, \quad (2)$$

$$|b(u, v, w)| \leq C_1 |u|^{1/4} \|u\|^{3/4} \|v\| \|w\|^{1/4} \|w\|^{3/4}, \quad \forall u, v, w \in V, \quad (3)$$

$$|b(u, v, w)| \leq C_1 \|u\| \|v\| \|w\|, \quad \forall u, v, w \in V. \quad (4)$$

Thus, if we denote

$$\langle B_N(u, v), w \rangle = b_N(u, v, w), \quad \forall u, v, w \in V,$$

we have for example

$$\|B_N(u, v)\|_* \leq NC_1 \|u\|, \quad \forall u, v \in V. \quad (5)$$

We also consider the operator  $A : V \rightarrow V'$  defined by  $\langle Au, v \rangle = ((u, v))$ . Denoting  $D(A) = (H^2(\Omega))^3 \cap V$ , then  $Au = -P\Delta u, \forall u \in D(A)$ , is the Stokes operator ( $P$  is the ortho-projector from  $(L^2(\Omega))^3$  onto  $H$ ).

We recall (see [12] and [11]) that there exists a constant  $C_2 > 0$  depending only on  $\Omega$  such that

$$|b(u, v, w)| \leq C_2 |Au| \|v\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H, \quad (6)$$

$$|b(u, v, w)| \leq C_2 |u|^{1/4} |Au|^{3/4} \|v\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H, \quad (7)$$

$$|b(u, v, w)| \leq C_2 \|u\|^{1/2} |Au|^{1/2} \|v\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H, \quad (8)$$

$$|b(u, v, w)| \leq C_2 \|u\| \|v\|^{1/2} |Av|^{1/2} \|w\|, \quad \forall u \in V, v \in D(A), w \in H, \quad (9)$$

$$|b(u, v, w)| \leq C_2 |u| \|v\|^{1/2} |Av|^{1/2} \|w\|, \quad \forall u \in H, v \in D(A), w \in V. \quad (10)$$

**Definition 1.** Let  $u_0 \in H$  and  $f \in L^2(0, T; (L^2(\Omega))^3)$  for all  $T > 0$  be given. A weak solution of (1) is any  $u \in L^2(0, T; V)$  for all  $T > 0$  such that

$$\begin{cases} u'(t) + \nu Au(t) + B_N(u(t), u(t)) = f(t) \text{ in } \mathcal{D}'(0, +\infty; V'), \\ u(0) = u_0, \end{cases}$$

or equivalently

$$\begin{aligned} & (u(t), w) + \nu \int_0^t ((u(s), w)) ds + \int_0^t b_N(u(s), u(s), w) ds \\ &= (u_0, w) + \int_0^t (f(s), w) ds, \quad \text{for all } t \geq 0 \text{ and all } w \in V. \end{aligned}$$

**Remark 2.** Observe that if  $u \in L^2(0, T; V)$  for all  $T > 0$  and satisfies the equation

$$u'(t) + \nu Au(t) + B_N(u(t), u(t)) = f(t) \quad \text{in } \mathcal{D}'(0, +\infty; V'),$$

then, as a consequence of (5),  $u'(t) \in L^2(0, T; V')$ , and hence (see [13])  $u \in C([0, +\infty); H)$  and satisfies the energy equality

$$|u(t)|^2 - |u(s)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr = 2 \int_s^t (f(r), u(r)) dr \quad \text{for all } 0 \leq s \leq t.$$

The following result was proved in [3].

**Theorem 3.** [cf. [3, Th.7(a)]] Suppose that  $f \in L^2(0, T; (L^2(\Omega))^3)$  for all  $T > 0$ , and let  $u_0 \in V$  be given. Then, there exists a unique weak solution  $u$  of (1), which is in fact a strong solution, i.e., it satisfies

$$u \in C([0, T]; V) \cap L^2(0, T; D(A)) \quad \text{for all } T > 0.$$

In the next section, for technical reasons, we obtain a similar result on existence of solution (see Proposition 5 and Remark 6 below), using a different Galerkin approximation from that used in [3].

**3. A smooth Galerkin approximation for GMNSE.** By regularity requirements for the manipulations in the proof of our main result (cf. Theorem 15), we need to introduce a new Galerkin approximation scheme for the GMNSE.

Our first step is to consider a family of smoother functions that approximates the truncated term  $F_N$  appearing in the GMNSE. This can be carried out in the classical way, by mollification.

Let  $\{\rho_m, m \geq 1\} \subset C^\infty(\mathbb{R})$  be a regularizing sequence in  $\mathbb{R}$ , i.e., such that  $0 \leq \rho_m(r) \leq 1$ , the support of  $\rho_m$  is included in the interval  $[-1/m, 1/m]$ , and  $\int_{\mathbb{R}} \rho_m(r) dr = 1$ , for all  $m \geq 1$ .

Let us consider  $F_N$  prolonged as  $F_N(r) = 1$  if  $r \leq 0$ , and denote  $F_{N,m}$  the convolution  $\rho_m * F_N$ , i.e.,

$$F_{N,m}(r) = \int_{\mathbb{R}} \rho_m(r-s) F_N(s) ds \quad \forall r \in \mathbb{R}. \quad (11)$$

It is well known that  $F_{N,m} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , and also  $F_{N,m}(r) \rightarrow F_N(r)$ , as  $m \rightarrow +\infty$ , uniformly on bounded subsets of  $\mathbb{R}$ . Moreover,

$$0 \leq F_{N,m}(r) \leq \|F_N\|_{L^\infty(\mathbb{R})} = 1 \quad \forall r \in \mathbb{R}, \quad \forall m \geq 1.$$

In fact, this last estimate can be improved. The following result concerning the functions  $F_{N,m}$  holds.

**Lemma 4.** *The functions  $F_{N,m}$  defined by (11) satisfy the following inequalities:*

$$0 \leq rF_{N,m}(r) \leq N+1 \quad \forall r \geq 0, \quad \forall m \geq 1. \quad (12)$$

$$|F'_{N,m}(r)| \leq 1/N \quad \forall r \in \mathbb{R}, \quad \forall m \geq 1. \quad (13)$$

$$r|F'_{N,m}(r)| \leq \frac{N+1}{N}, \quad r^2|F'_{N,m}(r)| \leq \frac{(N+1)^2}{N} \quad \forall r \geq 0, \quad \forall m \geq 1. \quad (14)$$

$$F'_{N,m}(r) = 0 \quad \forall r \leq 1/m, \quad \forall m > 2/N. \quad (15)$$

Its proof is given in the Appendix to avoid distracting from the main flow of ideas here.

Now, let us denote  $B_{N,m}(v, v)$  the element of  $V'$  defined by

$$\langle B_{N,m}(v, v), w \rangle = F_{N,m}(\|v\|)b(v, v, w) \quad \forall u, w \in V.$$

Consider the Galerkin approximations for the GMNSE given by

$$u'_m + \nu Au_m + P_m B_{N,m}(u_m, u_m) = P_m f, \quad u_m(0) = P_m u_0, \quad (16)$$

where  $u_m = \sum_{j=1}^m u_{m,j} \phi_j$ ,  $Au_m = \sum_{j=1}^m \lambda_j u_{m,j} \phi_j$ . Here the  $\lambda_j$  and  $\phi_j$  are the corresponding eigenvalues and eigenfunctions (orthonormal in  $H$ , orthogonal in  $V$ ) of the operator  $A$  and  $P_m$  is the projection onto the subspace of  $H$  spanned by  $\{\phi_1, \dots, \phi_m\}$ .

Now, we establish the main result of this section, which provides a new proof of the existence of solution given in Theorem 3.

**Proposition 5.** *Under the above notation, if  $f \in L^\infty(0, T; H)$  for all  $T > 0$ , the scheme (16) is well defined, i.e. for each  $u_0 \in V$ , there exists a smooth family ( $C^1$  in time) of functions  $\{u_m\}$ , solutions of (16), that are defined in  $(0, +\infty)$ . Moreover, there exists a subsequence of  $\{u_m\}_m$  that converges (in several senses) to the unique solution of (1).*

**Remark 6.** *The assumption  $f \in L^\infty(0, T; H)$  instead of  $L^2(0, T; (L^2(\Omega))^3)$  as in Theorem 3 is not essential, but only for simplicity in the notation. Indeed, since projector operators are involved in the test-functions space, there is no restriction in considering  $f$  taking values in  $H$  instead of  $(L^2(\Omega))^3$ . On the other hand, the stronger requirement in time,  $L^\infty$  instead of  $L^2$ , is also for the sake of convenience in the calculus. Observe that our final goal in Section 5 is with an autonomous term  $f$ .*

*Proof.* [of Proposition 5] The existence and uniqueness of local solution of (16) is a consequence of the Picard Theorem, and the fact that the local solution is a global one defined in the same interval of time as  $f$  is in view of the estimates (19) and (24) below.

After that, we will proceed by the compactness method: firstly proving uniform estimates for  $u_m$  in  $H$  and then in  $V$ . Observe that for the estimates in  $H$  we only need  $u_0 \in H$ .

Fix a value  $T > 0$  and let us denote

$$\|f\|_\infty = \|f\|_{L^\infty(0, T; H)}.$$

It is standard that if we take the inner product of the Galerkin ODE (16) with  $u_m$  and use  $b(u_m, u_m, u_m) = 0$ , we have

$$\frac{d}{dt}|u_m|^2 + 2\nu\|u_m\|^2 = 2(f, u_m), \quad (17)$$

and then, using  $\lambda_1|u_m|^2 \leq \|u_m\|^2$ , we obtain

$$\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 \leq \frac{|f|_\infty^2}{\nu\lambda_1}, \quad (18)$$

and consequently

$$|u_m(t)|^2 + \nu \int_0^t \|u_m(s)\|^2 ds \leq |u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1} t \quad (19)$$

for all  $t \geq 0$  in the interval of definition of  $u_m$ . The well posedness of  $u_m$  in  $(0, T)$  follows from Gronwall's lemma.

Now we obtain uniform estimates in  $V$  for all  $u_m$  in  $(0, T)$  using  $u_0 \in V$ . We take the inner product of the Galerkin ODE (16) with  $Au_m$  and obtain

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \nu|Au_m|^2 + b_{N,m}(u_m, u_m, Au_m) = (f, Au_m). \quad (20)$$

Evidently,

$$|(f, Au_m)| \leq \frac{\nu}{4}|Au_m|^2 + \frac{|f|_\infty^2}{\nu}$$

In addition, by (8), (12) and Young's inequality, one obtains

$$\begin{aligned} |b_{N,m}(u_m, u_m, Au_m)| &\leq C_2(N+1)\|u_m\|^{1/2}|Au_m|^{3/2} \\ &\leq \frac{\nu}{4}|Au_m|^2 + C'_N\|u_m\|^2, \end{aligned}$$

with  $C'_N = \frac{27(N+1)^4 C_2^4}{4\nu^3}$ .

Thus (20) simplifies to

$$\frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 \leq \frac{2}{\nu}|f|_\infty^2 + 2C'_N\|u_m\|^2. \quad (21)$$

From (21) and the fact that

$$\|u_m(0)\| = \|P_m u_0\| \leq \|u_0\|$$

by the choice of the basis  $\{\phi_j\}$  of  $H$ , one easily concludes that the sequence  $\{u_m\}$  is bounded in  $C([0, T]; V)$  and in  $L^2(0, T; D(A))$  for all  $T > 0$ .

Then, observe that  $|b_{N,m}(u_m, u_m, w)| \leq (N+1)C_2\|u_m\|^{1/2}|Au_m|^{1/2}|w|$ , for any  $w \in H$ , and thus, the sequence  $\{P_m B_{N,m}(u_m, u_m)\}$  is bounded in  $L^2(0, T; H)$  for all  $T > 0$ .

Therefore, from the equation  $u'_m = -\nu Au_m - P_m B_{N,m}(u_m, u_m) + P_m f$ , one sees that the sequence  $\{u'_m\}$  is also bounded in  $L^2(0, T; H)$ .

Consequently, as  $D(A) \subset V \subset H$  with compact injection, by Theorem 5.1 in Chapter 1 of [9] there exists an element  $u \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$  for all

$T > 0$ , and a subsequence  $\{u_\mu\}$  of  $\{u_m\}$ , such that

$$\left\{ \begin{array}{l} u_\mu \rightarrow u \text{ strong in } L^2(0, T; V), \\ u_\mu \rightarrow u \text{ a.e. in } (0, T) \times \Omega, \\ u_\mu \rightharpoonup u \text{ weak in } L^2(0, T; D(A)), \\ u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(0, T; V), \\ u'_\mu \rightharpoonup u' \text{ weak in } L^2(0, T; H), \end{array} \right. \quad (22)$$

for all  $T > 0$ .

Also, as  $u_\mu$  converges to  $v$  in  $L^2(0, T; V)$  for all  $T > 0$ , we can assume, possibly extracting a subsequence, that

$$\|u_\mu(t)\| \rightarrow \|u(t)\| \quad \text{a.e. in } (0, +\infty).$$

Then, by the boundedness of  $\{u_\mu\}$  in  $C([0, T]; V)$  and the fact that  $F_N$  is continuous and  $F_{N,m}$  converges to  $F_N$  uniformly on bounded subsets of  $\mathbb{R}$  as  $m \rightarrow +\infty$ , one obtains

$$F_{N,m}(\|u_\mu(t)\|) \rightarrow F_N(\|u(t)\|) \quad \text{a.e. in } (0, +\infty). \quad (23)$$

From (22) and (23) we can take limits in (16) exactly as in [3], and we obtain that  $u$  is a solution of (1).  $\square$

**Remark 7.** *If we assume that  $f$  is also defined for negative time, for instance,  $f \in L^\infty(-T, T; H)$ , the above solutions  $u_m$  are also well defined for all  $(-T, T)$ . This will be used below -in the proof of Theorem 15- to ensure that we can go backwards in time on each solution  $u_m$  of the Galerkin scheme.*

*This is due to (17), whence we have*

$$\frac{d}{dt}|u_m|^2 + 2\nu\lambda_m|u_m|^2 \geq -|f|_\infty^2 - |u_m|^2,$$

where  $|f|_\infty$  denotes now  $\|f\|_{L^\infty(-T, T; H)}$ . Therefore, integrating between  $t \in (-T, 0)$  and 0 we deduce

$$|u_m(t)|^2 \leq |u_0|^2 + |t||f|_\infty^2 + (1 + 2\nu\lambda_m) \int_t^0 |u_m(s)|^2 ds \quad (24)$$

for all  $t < 0$  in the interval of definition of  $u_m$ . Thus, by (19), (24) and again Gronwall's lemma,  $u_m$  is defined in all  $(-T, T)$ .

**Remark 8.** *Observe that, since the sequence  $\{u'_m\}$  is bounded in  $L^2(0, T; H)$  for any  $T > 0$ , the sequence  $u_m$  as a sequence of functions from  $[0, T]$  into  $H$  is equicontinuous. Then, since  $\{u_m\}$  is bounded in  $C([0, T]; V)$  and  $V \subset H$  with compact injection, we can assert that from any subsequence of  $\{u_m\}$  we can extract a subsequence that converges in  $C([0, T]; H)$ , and taking into account the previous arguments for the existence of solution  $u$  of (1), and the uniqueness of such a solution (cf. Theorem 3), one has that  $u_m \rightarrow u$  in  $C([0, T]; H)$ , as  $m \rightarrow +\infty$ , for all  $T > 0$ .*

**4. Estimates in  $D(A)$  for the Galerkin approximations.** In this section we go further in developing new estimates for the solutions of (16), but now with initial data  $u_0 \in D(A)$ , and assuming that  $f \in W^{1,\infty}(\mathbb{R}; H)$ . [The more restrictive hypothesis of  $f \in W^{1,\infty}(\mathbb{R}; H)$ , which we will use from here on instead of  $W^{1,\infty}((-T, T); H)$  for all  $T$ , is only for the sake of brevity in the statements.]

**Proposition 9.** *With the notation of Section 3, if  $f \in W^{1,\infty}(\mathbb{R}; H)$ , there exists a positive constant  $\tilde{M}_f^{(N)}$ , independent of  $u_0$ ,  $t$  and  $m$ , such that*

$$|Au_m(t)| \leq \tilde{M}_f^{(N)}(1 + |Au_0|) \quad \forall t \geq 0, \text{ for all } m > 2/N, u_0 \in D(A). \quad (25)$$

*Proof.* Now let us denote  $|\cdot|_\infty = \|\cdot\|_{L^\infty(\mathbb{R}; H)}$ . From (18) we have

$$\frac{d}{dt} (e^{\nu\lambda_1 t} |u_m(t)|^2) \leq \frac{|f|_\infty^2}{\nu\lambda_1} e^{\nu\lambda_1 t}, \quad t \geq 0,$$

and integrating one obtains

$$|u_m(t)|^2 \leq |u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu^2 \lambda_1^2} \quad \text{for all } t \geq 0. \quad (26)$$

Now, if we use (20) and take into account that  $\lambda_1 \|u_m(t)\|^2 \leq |Au_m(t)|^2$  and that, by (7) and (12),  $|b_{N,m}(u_m(t), u_m(t), Au_m(t))| \leq (N+1)C_2 |u_m(t)|^{1/4} |Au_m(t)|^{7/4}$ , then we can obtain the inequality

$$\frac{d}{dt} \|u_m(t)\|^2 + \nu\lambda_1 \|u_m(t)\|^2 \leq \frac{2}{\nu} |f|_\infty^2 + C_1^{(N)} |u_m(t)|^2, \quad (27)$$

with  $C_1^{(N)} = \frac{(N+1)^8 C_2^8 7^7}{2^9 \nu^7}$ .

Substituting the bound (26) for  $|u_m(t)|^2$  in the differential inequality (27) gives

$$\frac{d}{dt} \|u_m(t)\|^2 + \nu\lambda_1 \|u_m(t)\|^2 \leq C_1^{(N)} |u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu} \left( 2 + \frac{C_1^{(N)}}{\nu\lambda_1^2} \right).$$

Integrating this inequality we deduce that

$$\|u_m(t)\|^2 \leq (\|u_0\|^2 + C_1^{(N)} t |u_0|^2) e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu^2 \lambda_1} \left( 2 + \frac{C_1^{(N)}}{\nu\lambda_1^2} \right), \quad \forall t \geq 0. \quad (28)$$

On the other hand, integrating (21) between  $t$  and  $t+1$ , we obtain in particular

$$\nu \int_t^{t+1} |Au_m(s)|^2 ds \leq \frac{2}{\nu} |f|_\infty^2 + 2C'_N \int_t^{t+1} \|u_m(s)\|^2 ds + \|u_m(t)\|^2 \quad \forall t \geq 0,$$

and then, by (28), one obtains

$$\begin{aligned} \int_t^{t+1} |Au_m(s)|^2 ds &\leq \frac{1 + 2C'_N}{\nu} (\|u_0\|^2 + C_1^{(N)}(t+1)|u_0|^2) e^{-\nu\lambda_1 t} \\ &\quad + \frac{|f|_\infty^2}{\nu^2} \left[ 2 + \frac{1 + 2C'_N}{\nu\lambda_1} \left( 2 + \frac{C_1^{(N)}}{\nu\lambda_1^2} \right) \right] \quad \forall t \geq 0 \quad \forall m \geq 1. \end{aligned} \quad (29)$$

Suppose now that  $m > 2/N$ . From (16) and (15), we obtain for the second derivative of  $u_m$ ,

$$\begin{aligned} u_m''(t) &= -\nu Au_m'(t) - F'_{N,m}(\|u_m(t)\|) \frac{((u_m'(t), u_m(t)))}{\|u_m(t)\|} \chi_{\mathcal{O}_m}(t) P_m B(u_m(t), u_m(t)) \\ &\quad - F_{N,m}(\|u_m(t)\|) P_m B(u_m'(t), u_m(t)) \\ &\quad - F_{N,m}(\|u_m(t)\|) P_m B(u_m(t), u_m'(t)) + P_m f'(t), \end{aligned} \quad (30)$$

where

$$\mathcal{O}_m = \{t \in [0, +\infty) : \|u_m(t)\| > 1/m\}$$

and  $\chi_{\mathcal{O}_m}(t)$  is the indicator function of the set  $\mathcal{O}_m$ .



Multiplying in (30) by  $u'_m$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u'_m(t)|^2 + \nu \|u'_m(t)\|^2 \\ &= -F'_{N,m}(\|u_m(t)\|) \frac{((u'_m(t), u_m(t)))}{\|u_m(t)\|} \chi_{\mathcal{O}_m}(t) b(u_m(t), u_m(t), u'_m(t)) \\ & \quad - F_{N,m}(\|u_m(t)\|) b(u'_m(t), u_m(t), u'_m(t)) + (f'(t), u'_m(t)) \quad t \geq 0. \end{aligned} \quad (31)$$

From (2), (14) and Young's inequality, we have

$$\begin{aligned} & |F'_{N,m}(\|u_m(t)\|) \frac{((u'_m(t), u_m(t)))}{\|u_m(t)\|} \chi_{\mathcal{O}_m}(t) b(u_m(t), u_m(t), u'_m(t))| \\ & \leq \frac{2C_1(N+1)^2}{N} \|u'_m(t)\| |u'_m(t)|^{1/4} \|u'_m(t)\|^{3/4} \\ & = \frac{2C_1(N+1)^2}{N} C_1 |u'_m(t)|^{1/4} \|u'_m(t)\|^{7/4} \\ & \leq \frac{\nu}{2} \|u'_m(t)\|^2 + \left(\frac{7}{4\nu}\right)^7 2^5 \left(\frac{C_1^2(N+1)^2}{N}\right)^8 |u'_m(t)|^2. \end{aligned} \quad (32)$$

By (3), (12) and Young's inequality again

$$\begin{aligned} & |2F_{N,m}(\|u_m(t)\|) b(u'_m(t), u_m(t), u'_m(t))| \\ & \leq 2(N+1)C_1 |u'_m(t)|^{1/2} \|u'_m(t)\|^{3/2} \\ & \leq \frac{\nu}{2} \|u'_m(t)\|^2 + \frac{27}{2\nu^3} (N+1)^4 C_1^4 |u'_m(t)|^2. \end{aligned} \quad (33)$$

Thus, if we denote

$$L_1^{(N)} = 1 + \frac{1}{2} \left(\frac{7}{2\nu}\right)^7 \left(\frac{C_1^2(N+1)^2}{N}\right)^8 + \frac{27}{\nu^3} (N+1)^4 C_1^4,$$

from (31), (32) and (33) we easily obtain

$$\frac{d}{dt} |u'_m(t)|^2 \leq L_1^{(N)} |u'_m(t)|^2 + |f'|_\infty^2 \quad \forall t \geq 0 \quad \forall m > 2/N. \quad (34)$$

If we integrate this inequality between  $s \in [t, t+1]$  and  $t+1$ , we have

$$\begin{aligned} |u'_m(t+1)|^2 & \leq |u'_m(s)|^2 \\ & \quad + L_1^{(N)} \int_s^{t+1} |u'_m(r)|^2 dr + |f'|_\infty^2 \quad \forall 0 \leq t \leq s \leq t+1, \quad \forall m > 2/N. \end{aligned}$$

Integrating now this last inequality for  $s$  between  $t$  and  $t+1$ , we obtain

$$|u'_m(t+1)|^2 \leq (1 + L_1^{(N)}) \int_t^{t+1} |u'_m(s)|^2 ds + |f'|_\infty^2 \quad \forall t \geq 0, \quad \forall m > 2/N. \quad (35)$$

Now, observe that by (16), (6) and (12),

$$\begin{aligned} |u'_m(t)| & \leq \nu |Au_m(t)| + |F_{N,m}(\|u_m(t)\|) B(u_m(t), u_m(t))| + |f(t)| \\ & \leq [\nu + (N+1)C_2] |Au_m(t)| + |f|_\infty, \quad t \geq 0, \end{aligned}$$

and therefore

$$\begin{aligned} \int_t^{t+1} |u'_m(s)|^2 ds & \leq 2|f|_\infty^2 + 2[\nu + (N+1)C_2]^2 \\ & \quad \times \int_t^{t+1} |Au_m(s)|^2 ds \quad \forall t \geq 0, \quad \forall m \geq 2/N. \end{aligned} \quad (36)$$

From (29), (35) and (36), it is clear that there exist two positive constants  $\tilde{C}_f^{(N)}$  and  $\tilde{D}_f^{(N)}$ , independents of  $u_0$ ,  $t$  and  $m$ , such that

$$|u'_m(t+1)|^2 \leq \tilde{C}_f^{(N)} + \tilde{D}_f^{(N)}(\|u_0\|^2 + (t+1)|u_0|^2)e^{-\nu\lambda_1 t} \quad \forall t \geq 0, \quad (37)$$

for all  $m > 2/N$ ,  $u_0 \in V$ .

Again, by (16), (8) and (12),

$$\begin{aligned} \nu|Au_m(t)| &\leq |u'_m(t)| + |F_{N,m}(\|u_m(t)\|)B(u_m(t), u_m(t))| + |f(t)| \\ &\leq |u'_m(t)| + (N+1)C_2\|u_m(t)\|^{1/2}|Au_m(t)|^{1/2} + |f|_\infty \\ &\leq |u'_m(t)| + \frac{\nu}{2}|Au_m(t)| + \frac{(N+1)^2C_2^2}{2\nu}\|u_m(t)\| + |f|_\infty, \quad t \geq 0, \end{aligned} \quad (38)$$

and therefore

$$|Au_m(t)|^2 \leq \frac{12}{\nu^2}|u'_m(t)|^2 + \frac{3(N+1)^4C_2^4}{\nu^4}\|u_m(t)\|^2 + \frac{12}{\nu^2}|f|_\infty^2, \quad \forall t \geq 0, \quad \forall m \geq 1. \quad (39)$$

From (28), (37) and (39), one finds that there exist two positive constants  $\tilde{K}_f^{(N)}$  and  $\tilde{R}_f^{(N)}$ , independent of  $u_0$ ,  $t$  and  $m$ , such that

$$|Au_m(t)|^2 \leq \tilde{R}_f^{(N)} + \tilde{K}_f^{(N)}(1+t)\|u_0\|^2e^{-\nu\lambda_1 t} \quad \forall t \geq 1, \quad (40)$$

for all  $m > 2/N$ ,  $u_0 \in D(A)$ .

Observe that with (40) we are almost done. However, in (40) the valid time is  $t \geq 1$ , forced by (37). In order to obtain (25), we need to treat separately the time interval  $[0, 1]$ .

From the inequality (34), we obtain (remember that  $f$  is continuous with values in  $H$ , and therefore  $u_m$  is  $C^1$ )

$$|u'_m(t)|^2 \leq (|u'_m(0)|^2 + |f'|_\infty^2)e^{L_1^{(N)}t} \quad \forall 0 \leq t \leq 1 \quad \forall m > 2/N. \quad (41)$$

But

$$\begin{aligned} |u'_m(0)| &\leq \nu|AP_m u_0| + |F_{N,m}(\|P_m u_0\|)B(P_m u_0, P_m u_0)| + |f(0)| \\ &\leq \nu|AP_m u_0| + C_2(N+1)\|P_m u_0\|^{1/2}|AP_m u_0|^{1/2} + |f(0)| \\ &\leq \nu|Au_0| + C_2(N+1)\|u_0\|^{1/2}|Au_0|^{1/2} + |f(0)| \quad \forall m \geq 1. \end{aligned} \quad (42)$$

From (41) and (42) it is clear that

$$|u'_m(t)| \leq \left( \nu|Au_0| + C_2(N+1)\|u_0\|^{1/2}|Au_0|^{1/2} + |f(0)| + |f'|_\infty \right) e^{L_1^{(N)}t/2} \quad (43)$$

for all  $0 \leq t \leq 1$  and any  $m > 2/N$ .

On the other hand, from (38) we have that

$$\nu|Au_m(t)| \leq 2|u'_m(t)| + \frac{C_2^2(N+1)^2}{\nu}\|u_m(t)\| + 2|f|_\infty, \quad \forall t \geq 0 \quad \forall m \geq 1. \quad (44)$$

From the estimates (28), (40), (43) and (44), the inequality (25) follows.  $\square$

5.  **$S_N$ -invariant measures in the autonomous case.** From now on we suppose that the right hand side in (1) is independent of time, i.e. simply  $f \in H$ , and for any  $u_0 \in V$  we denote  $S_N(\cdot)u_0$  the corresponding solution of (1).

By a probability measure on  $H$  we will understand a probability measure on the  $\sigma$ -algebra of Borel subsets of  $H$ .

We recall that any Borel set in  $V$  is a Borel set in  $H$ , and a set  $E \subset V$  is a Borel subset of  $V$  if and only if there exists a Borel subset  $F$  of  $H$  such that  $E = F \cap V$ . The same properties follow with  $D(A)$  instead of  $V$  (see [6]).

**Definition 10.** Let  $\mu$  be a probability measure on  $H$ . We will say that  $\mu$  is  $S_N$ -invariant if

$$\mu(V) = 1 \quad \text{and} \quad \mu(E) = \mu(S_N(t)^{-1}E) \quad \forall t \geq 0,$$

for every Borel subset  $E$  of  $V$ .

**Remark 11.** By the continuity of  $S_N(t)(\cdot)$  as operator of  $V$  (cf. [3, Th.8]),  $S_N(t)^{-1}E$  is Borel, so the expression  $\mu(S_N(t)^{-1}E)$  above makes sense.

**Definition 12.** We define  $\mathcal{T}$  as the set of real valued functionals  $\Phi = \Phi(v)$  on  $H$  such that

$$(i) \quad c_r := \sup_{|v| \leq r} |\Phi(v)| < +\infty \quad \text{for all } r > 0;$$

(ii) for any  $v \in V$  there exists  $\Phi'(v) \in V$  such that

$$\frac{|\Phi(v+w) - \Phi(v) - (\Phi'(v), w)|}{|w|} \rightarrow 0 \quad \text{as } |w| \rightarrow 0 \text{ with } w \in V;$$

(iii) the mapping  $v \mapsto \Phi'(v)$  is continuous and bounded as function from  $V$  into  $V$ .

Let us now extend the notation of  $\|\cdot\|$  such that  $\|v\| = +\infty$  if  $v \in H \setminus V$ . With this convention, if  $\mu$  is a probability measure on  $H$  and  $\int_H \|v\|^2 d\mu(v) < +\infty$ , then  $\mu(H \setminus V) = 0$ .

We define

$$G_N(v) = -\nu Av - B_N(v, v) + f \quad \forall v \in V.$$

Observe that the mapping  $G_N : V \rightarrow V'$  is continuous. Also, by (5),

$$\|G_N(v)\|_* \leq (\nu + NC_1)\|v\| + \lambda_1^{-1/2}|f| \quad \forall v \in V.$$

Thus, if  $\Phi \in \mathcal{T}$ ,

$$|\langle G_N(v), \Phi'(v) \rangle| \leq [(\nu + NC_1)\|v\| + \lambda_1^{-1/2}|f|] \sup_{w \in V} \|\Phi'(w)\| \quad \forall v \in V,$$

and consequently, if  $\mu$  is a probability measure on  $H$  with  $\mu(H \setminus V) = 0$  and satisfying (i) below, then the integral  $\int_H \langle G_N(v), \Phi'(v) \rangle d\mu(v)$  is finite.

**Definition 13.** A stationary statistical solution of GMNSE is a probability measure  $\mu$  on  $H$  such that

$$(i) \quad \int_H \|v\|^2 d\mu(v) < +\infty;$$

$$(ii) \quad \int_H \langle G_N(v), \Phi'(v) \rangle d\mu(v) = 0 \quad \text{for any } \Phi \in \mathcal{T};$$

$$(iii) \quad \int_{\{a \leq |v|^2 < b\}} \{\nu\|v\|^2 - (f, v)\} d\mu(v) \leq 0 \quad \text{for any } 0 \leq a < b \leq +\infty.$$

The following result was proved in [4].

**Theorem 14.** *Every  $S_N$ -invariant probability measure on  $H$  is a stationary statistical solution of GMNSE.*

Now, as a partial counterpart of the above, we can prove our main result:

**Theorem 15.** *Let  $\mu$  be a stationary statistical solution of GMNSE such that there exists a bounded and measurable subset  $\mathcal{B}_N$  of  $D(A)$  satisfying  $\mu(H \setminus \mathcal{B}_N) = 0$ . Then  $\mu$  is a  $S_N$ -invariant probability measure on  $H$ .*

*Proof.* Suppose that we already proved that for any  $\Phi \in \mathcal{T}$ ,

$$\int_V \Phi(S_N(t)v) d\mu(v) = \int_V \Phi(v) d\mu(v) \quad \forall t \geq 0. \quad (45)$$

Taking into account that  $\mathcal{T}$  is dense in  $L^1(H, \mu)$ , we can also take in (45) the characteristic function of any measurable subset  $F$  of  $V$ , and we find that  $\mu(S_N(t)^{-1}F) = \mu(F)$  for all  $t \geq 0$ , and therefore, by Definition 10,  $\mu$  is a  $S_N$ -invariant probability measure on  $H$ .

Thus, we must prove (45) for any  $\Phi \in \mathcal{T}$ . To do this, we split the proof in two steps. The final manipulations for  $\Phi(S_N(t)v)$  leading to (45) requires the correct definition and passing to the limit of a suitable Fréchet derivative acting on the approximations  $\Phi(S_N^{(m)}(t)v)$ , this is the step 1 with which we start.

**Step 1: Extension of Fréchet derivative w.r.t.  $v$  of  $S_N^{(m)}(t)v$  as an operator in  $\mathcal{L}(V')$ .**

For each  $m \geq 1$ , any  $u_0 \in H$  and any  $t \in \mathbb{R}$ , let us denote  $S_N^{(m)}(t)u_0 = u_m(t)$ , where  $u_m$  is the solution of (16). In this way, for each  $m \geq 1$  we have defined in particular a group  $\{S_N^{(m)}(t)\}_{t \in \mathbb{R}}$  of nonlinear operators in  $P_m H$ .

Taking into account that the mapping  $v \in H \rightarrow ((v, \phi_1), \dots, (v, \phi_m)) \in \mathbb{R}^m$  is Fréchet differentiable, from the regularity of the  $F_{N,m}$  and the results of differentiability of the solutions of an ordinary differential systems with respect to the initial conditions, we obtain that for each  $m \geq 1$  and any  $t \in \mathbb{R}$ , the mapping  $v \in H \rightarrow S_N^{(m)}(t)v \in H$  is Fréchet differentiable. We will denote  $D_v S_N^{(m)}(t)v$  its Fréchet differential at  $v \in H$ , thus  $D_v S_N^{(m)}(t)v \in \mathcal{L}(H)$ . We now study some additional properties of the operator  $D_v S_N^{(m)}(t)v \in \mathcal{L}(H)$ .

Consider  $v, w \in H$  and  $m > 2/N$  given, and denote

$$w_m(t) = (D_v S_N^{(m)}(t)v)w, \quad t \in \mathbb{R},$$

which satisfies the initial condition

$$w_m(0) = P_m v,$$

and the equation

$$\begin{aligned} & \frac{d}{dt} w_m(t) + \nu A w_m(t) \\ = & -F_{N,m}(\|v_m(t)\|) P_m B(v_m(t), w_m(t)) \\ & + F_{N,m}(\|v_m(t)\|) P_m B(w_m(t), v_m(t)) \\ & - F'_{N,m}(\|v_m(t)\|) \frac{((v_m(t), w_m(t)))}{\|v_m(t)\|} \chi \tilde{\mathcal{C}}_m(v_m(t)) P_m B(v_m(t), v_m(t)), \quad (46) \end{aligned}$$

for  $t \in \mathbb{R}$ , where

$$\tilde{\mathcal{O}}_m = \{z \in V : \|z\| > 1/m\},$$

and  $v_m(t) = S_N^{(m)}(t)v$ .

Taking the inner product in  $H$  with  $A^{-1}w_m(t)$  in equation (46), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |A^{-1/2}w_m(t)|^2 + \nu |w_m(t)|^2 \\ = & -F_{N,m}(\|v_m(t)\|)b(v_m(t), w_m(t), P_m A^{-1}w_m(t)) \\ & + F_{N,m}(\|v_m(t)\|)b(w_m(t), v_m(t), P_m A^{-1}w_m(t)) \\ & - F'_{N,m}(\|v_m(t)\|) \frac{((v_m(t), w_m(t)))}{\|v_m(t)\|} \chi_{\tilde{\mathcal{O}}_m}(v_m(t))b(v_m(t), v_m(t), P_m A^{-1}w_m(t)), \end{aligned} \quad (47)$$

for  $t \in \mathbb{R}$ .

Now, observe that by (9), (12) and Young's inequality,

$$\begin{aligned} & |F_{N,m}(\|v_m(t)\|)b(v_m(t), w_m(t), P_m A^{-1}w_m(t))| \\ = & |F_{N,m}(\|v_m(t)\|)b(v_m(t), P_m A^{-1}w_m(t), w_m(t))| \\ \leq & C_2(N+1)|A^{1/2}P_m A^{-1}w_m(t)|^{1/2}|AP_m A^{-1}w_m(t)|^{1/2}|w_m(t)| \\ \leq & C_2(N+1)|A^{-1/2}w_m(t)|^{1/2}|w_m(t)|^{3/2} \\ \leq & \frac{\nu}{3}|w_m(t)|^2 + \frac{1}{4} \left(\frac{9}{4\nu}\right)^3 C_2^4(N+1)^4 |A^{-1/2}w_m(t)|^2, \end{aligned} \quad (48)$$

and analogously, by (10) and (12),

$$\begin{aligned} & |F_{N,m}(\|v_m(t)\|)b(w_m(t), v_m(t), P_m A^{-1}w_m(t))| \\ = & |F_{N,m}(\|v_m(t)\|)b(w_m(t), P_m A^{-1}w_m(t), v_m(t))| \\ \leq & \frac{\nu}{3}|w_m(t)|^2 + \frac{1}{4} \left(\frac{9}{4\nu}\right)^3 C_2^4(N+1)^4 |A^{-1/2}w_m(t)|^2. \end{aligned} \quad (49)$$

On the other hand, as  $v_m(t)$  belongs to  $D(A)$ , we have that

$$((v_m(t), w_m(t))) = (Av_m(t), w_m(t)),$$

and therefore, by (9), (14) and Young's inequality,

$$\begin{aligned} & |F'_{N,m}(\|v_m(t)\|) \frac{((v_m(t), w_m(t)))}{\|v_m(t)\|} \chi_{\tilde{\mathcal{O}}_m}(v_m(t))b(v_m(t), v_m(t), P_m A^{-1}w_m(t))| \\ = & |F'_{N,m}(\|v_m(t)\|) \frac{(Av_m(t), w_m(t))}{\|v_m(t)\|} \chi_{\tilde{\mathcal{O}}_m}(v_m(t))b(v_m(t), P_m A^{-1}w_m(t), v_m(t))| \\ \leq & |F'_{N,m}(\|v_m(t)\|)| |Av_m(t)| |w_m(t)| C_2 |A^{1/2}P_m A^{-1}w_m(t)|^{1/2} \\ & \times |AP_m A^{-1}w_m(t)|^{1/2} |v_m(t)| \\ \leq & \lambda_1^{-1/2} C_2 |Av_m(t)| |F'_{N,m}(\|v_m(t)\|)| \|v_m(t)\| |w_m(t)|^{3/2} |A^{-1/2}w_m(t)|^{1/2} \\ \leq & \lambda_1^{-1/2} C_2 \frac{N+1}{N} |Av_m(t)| |w_m(t)|^{3/2} |A^{-1/2}w_m(t)|^{1/2} \\ \leq & \frac{\nu}{3}|w_m(t)|^2 + \frac{1}{4} \left(\frac{9}{4\nu}\right)^3 \lambda_1^{-2} C_2^4 \frac{(N+1)^4}{N^4} |Av_m(t)|^4 |A^{-1/2}w_m(t)|^2. \end{aligned}$$

From this inequality and (47)-(49), denoting

$$\tilde{C}^{(N)} = \left(\frac{9}{4\nu}\right)^3 C_2^4 (N+1)^4 \max\left\{1, \frac{1}{2\lambda_1^2 N^4}\right\},$$

we have that

$$\frac{d}{dt} |A^{-1/2} w_m(t)|^2 \leq \tilde{C}^{(N)} (1 + |Av_m(t)|^4) |A^{-1/2} w_m(t)|^2 \quad \forall t \in \mathbb{R} \quad \forall m > 2/N,$$

and therefore

$$|A^{-1/2} w_m(t)|^2 \leq |A^{-1/2} w_m(0)|^2 e^{\tilde{C}^{(N)} \int_0^t (1 + |Av_m(s)|^4) ds} \quad \forall t \geq 0, \quad \forall m > 2/N,$$

i.e.,

$$|A^{-1/2} (D_v S_N^{(m)}(t)v)w|^2 \leq |A^{-1/2} w|^2 e^{\tilde{C}^{(N)} \int_0^t (1 + |Av_m(s)|^4) ds} \quad \forall t \geq 0, \quad (50)$$

for all  $v, w \in H$  and any  $m > 2/N$ .

Observe that in the above expression, the integral of  $|Av_m(s)|^4$  makes sense by (25). So, inequality (50) implies that for each  $v \in H$  and any  $t \geq 0$  and  $m > 2/N$ , the operator  $D_v S_N^{(m)}(t)v$  can be extended in a unique way to an operator in  $\mathcal{L}(V')$ , and therefore the adjoint  $(D_v S_N^{(m)}(t)v)^*$  belongs to  $\mathcal{L}(V)$ .

### Step 2: Manipulations on $\Phi(S_N^{(m)}(t)v)$ and passing to the limit.

Let  $\Phi \in \mathcal{T}$  be given. Since  $S_N^{(m)}(t)v \in V$  for all  $(t, v) \in \mathbb{R} \times H$ , and since  $S_N^{(m)}(\cdot)v \in C^1(\mathbb{R}; H)$  for any  $v \in H$ , we obtain

$$\frac{d}{dt} \Phi(S_N^{(m)}(t)v) = (P_m G_{N,m}(S_N^{(m)}(t)v), \Phi'(S_N^{(m)}(t)v)) \quad \forall (t, v) \in \mathbb{R} \times H,$$

where

$$G_{N,m}(w) = -\nu Aw - B_{N,m}(w, w) + f \quad \forall w \in V.$$

Integrating between 0 and  $T$  we have

$$\Phi(S_N^{(m)}(T)v) - \Phi(P_m v) = \int_0^T (P_m G_{N,m}(S_N^{(m)}(t)v), \Phi'(S_N^{(m)}(t)v)) dt \quad (51)$$

for all  $v \in H$  and any  $T > 0$ .

Now observe that by (19),

$$|S_N^{(m)}(t)v|^2 \leq \frac{\|v\|^2}{\lambda_1} + \frac{|f|^2}{\nu\lambda_1} T \quad \forall (t, v) \in [0, T] \times H, \quad (52)$$

and therefore  $|G_{N,m}(S_N^{(m)}(t)v)|$  is bounded in  $[0, T] \times H$ . Thus, by the properties of  $\Phi$  and  $\mu$ , we can integrate equality (51) with respect to  $\mu$  and apply the Fubini theorem, and we obtain

$$\begin{aligned} & \int_H \Phi(S_N^{(m)}(T)v) d\mu(v) \\ &= \int_H \Phi(P_m v) d\mu(v) \\ &+ \int_0^T \int_H (P_m G_{N,m}(S_N^{(m)}(t)v), \Phi'(S_N^{(m)}(t)v)) d\mu(v) dt \quad \forall T > 0. \end{aligned} \quad (53)$$

Let us define

$$\Phi_m(t, v) = \Phi(S_N^{(m)}(t)v) \quad \forall (t, v) \in \mathbb{R} \times H, \quad m > 2/N.$$

It is not difficult to see that  $\Phi_m(t, \cdot) \in \mathcal{T}$  for any  $t \in \mathbb{R}$ , with Fréchet derivative

$$D_v(\Phi_m)(t, v) = (D_v S_N^{(m)}(t)v)^* \Phi'(S_N^{(m)}(t)v) \quad \forall (t, v) \in \mathbb{R} \times H.$$

Using the group property of  $\{S_N^{(m)}(t)\}_{t \in \mathbb{R}}$  in  $P_m H$ , it is immediate that

$$\Phi_m(t, S_N^{(m)}(-t)w) = \Phi(P_m w) \quad \forall (t, w) \in \mathbb{R} \times H,$$

and therefore

$$\begin{aligned} 0 &= \frac{d}{dt} \Phi_m(t, S_N^{(m)}(-t)w) \\ &= (\Phi_m)'_t(t, S_N^{(m)}(-t)w) - (P_m G_{N,m}(S_N^{(m)}(-t)v), D_v(\Phi_m)(t, S_N^{(m)}(-t)w)) \end{aligned}$$

for all  $(t, w) \in \mathbb{R} \times H$ , where  $(\Phi_m)'_t(\cdot, \cdot)$  denotes the derivative of  $\Phi_m$  with respect to the first variable.

Thus, taking  $w = S_N^{(m)}(t)v = S_N^{(m)}(t)P_m v$ , we have

$$(\Phi_m)'_t(t, P_m v) = (P_m G_{N,m}(P_m v), D_v(\Phi_m)(t, P_m v)) \quad \forall (t, v) \in \mathbb{R} \times H. \quad (54)$$

But

$$(\Phi_m)'_t(t, P_m v) = \frac{d}{dt} \Phi(S_N^{(m)}(t)v) = (P_m G_{N,m}(S_N^{(m)}(t)v), \Phi'(S_N^{(m)}(t)v)),$$

and then, by (54),

$$\begin{aligned} &(P_m G_{N,m}(S_N^{(m)}(t)v), \Phi'(S_N^{(m)}(t)v)) \\ &= (P_m G_{N,m}(P_m v), D_v(\Phi_m)(t, P_m v)) \quad \forall (t, v) \in \mathbb{R} \times H. \end{aligned} \quad (55)$$

From (53) and (55) it follows that

$$\begin{aligned} &\int_H \Phi(S_N^{(m)}(T)v) d\mu(v) \\ &= \int_H \Phi(P_m v) d\mu(v) \\ &\quad + \int_0^T \int_H (P_m G_{N,m}(P_m v), D_v(\Phi_m)(t, P_m v)) d\mu(v) dt \quad \forall T > 0 \end{aligned} \quad (56)$$

for any  $v \in H$ .

Taking into account that  $\Phi_m(t, \cdot) \in \mathcal{T}$  for any  $t \in \mathbb{R}$ , and  $\mu$  is a stationary statistical solution of GMNSE,

$$\int_H (G_N(v), D_v(\Phi_m)(t, v)) d\mu(v) = 0 \quad \forall t \in \mathbb{R},$$

and then, by (56),

$$\begin{aligned} \int_H \Phi(S_N^{(m)}(T)v) d\mu(v) &= \int_H \Phi(P_m v) d\mu(v) \\ &\quad + \int_0^T \int_H [(P_m G_{N,m}(P_m v), D_v(\Phi_m)(t, P_m v)) \\ &\quad \quad - (G_N(v), D_v(\Phi_m)(t, v))] d\mu(v) dt \quad \forall T > 0. \end{aligned} \quad (57)$$

Since  $\Phi_m(t, v) = \Phi_m(t, P_m v)$ , one can deduce that

$$D_v(\Phi_m)(t, v) = D_v[(\Phi_m)(t, P_m v)] = P_m D_v(\Phi_m)(t, P_m v) = P_m D_v(\Phi_m)(t, v). \quad (58)$$

Then, since  $P_m \in \mathcal{L}(H)$  is self-adjoint, from (57) and (58) we obtain

$$\begin{aligned} & \int_H \Phi(S_N^{(m)}(T)v) d\mu(v) \\ = & \int_H \Phi(P_m v) d\mu(v) \\ & + \int_0^T \int_H (P_m G_{N,m}(P_m v) - P_m G_N(v), D_v(\Phi_m)(t, v)) d\mu(v) dt \quad \forall T > 0. \end{aligned} \quad (59)$$

Now, observe that  $P_m G_{N,m}(P_m v) - P_m G_N(v) = P_m B_N(v, v) - P_m B_{N,m}(P_m v, P_m v)$ , and then, by (58) and since  $P_m^* = P_m$ , we obtain from (59)

$$\begin{aligned} & \int_H \Phi(S_N^{(m)}(T)v) d\mu(v) \\ = & \int_H \Phi(P_m v) d\mu(v) \\ & + \int_0^T \int_H (P_m B_N(v, v) - P_m B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v)) d\mu(v) dt \\ = & \int_H \Phi(P_m v) d\mu(v) \\ & + \int_0^T \int_H (B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v)) d\mu(v) dt \quad \forall T > 0. \end{aligned} \quad (60)$$

Now, we prove that as  $m \rightarrow +\infty$ , the last integral term goes to zero for all  $T > 0$ . Observe that

$$\begin{aligned} & |(B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v))| \\ = & |F_N(\|v\|)b(v, v, D_v(\Phi_m)(t, v)) - F_{N,m}(\|P_m v\|)b(P_m v, P_m v, D_v(\Phi_m)(t, v))| \\ \leq & |F_N(\|v\|)b(v - P_m v, v, D_v(\Phi_m)(t, v))| \\ & + |F_N(\|v\|) - F_{N,m}(\|v\|)|b(P_m v, v, D_v(\Phi_m)(t, v))| \\ & + |F_{N,m}(\|v\|)b(P_m v, v - P_m v, D_v(\Phi_m)(t, v))| \\ & + |F_{N,m}(\|v\|) - F_{N,m}(\|P_m v\|)|b(P_m v, P_m v, D_v(\Phi_m)(t, v))|, \end{aligned}$$

and therefore, by (4), (12) and (13),

$$\begin{aligned} & |(B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v))| \\ \leq & NC_1 \|v - P_m v\| \|D_v(\Phi_m)(t, v)\| \\ & + |F_N(\|v\|) - F_{N,m}(\|v\|)| C_1 \|v\|^2 \|D_v(\Phi_m)(t, v)\| \\ & + (N+1)C_1 \|v - P_m v\| \|D_v(\Phi_m)(t, v)\| \\ & + \frac{C_1}{N} \|v - P_m v\| \|v\|^2 \|D_v(\Phi_m)(t, v)\|. \end{aligned} \quad (61)$$

On the other hand, reasoning exactly as on page 232 in [6], one obtains

$$\begin{aligned} & \|D_v(\Phi_m)(t, v)\| \\ = & \|\Phi'(S_N^{(m)}(t)v)\| \sup_{\substack{w \in H, \\ |A^{-1/2}w| \leq 1}} |A^{-1/2}(D_v S_N^{(m)}(t)v)w| \quad \forall t \geq 0, \quad \forall v, w \in H, \end{aligned}$$



for any  $m \geq 1$ , and then, taking into account that  $\Phi \in \mathcal{T}$  has derivative  $\Phi'(\cdot)$  bounded on  $V$ , there exists a constant  $C_\Phi > 0$  such that

$$\|D_v(\Phi_m)(t, v)\| \leq C_\Phi \sup_{\substack{w \in H, \\ |A^{-1/2}w| \leq 1}} |A^{-1/2}(D_v S_N^{(m)}(t)v)w| \quad (62)$$

$\forall t \geq 0, \quad \forall v, w \in H$ , for any  $m \geq 1$ .

Since  $\mathcal{B}_N$  is a bounded subset of  $D(A)$ , from (50), Proposition 9 and (62) we deduce that there exists a constant  $C_{f, \mathcal{B}_N}$  such that

$$\|D_v(\Phi_m)(t, v)\| \leq C_\Phi e^{TC_{f, \mathcal{B}_N}} \quad \forall 0 \leq t \leq T, \quad \forall v \in \mathcal{B}_N, \quad (63)$$

for any  $m > 2/N$ .

From (61) and (63) we deduce that for each  $T > 0$  there exists a constant  $C_T > 0$ , such that

$$\begin{aligned} & |(B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v))| \\ & \leq C_T (\|v - P_m v\| + |F_N(\|v\|) - F_{N,m}(\|v\|)|) \end{aligned}$$

for all  $(t, v) \in [0, T] \times \mathcal{B}_N$ , for any  $m > 2/N$ .

Then, as  $\mu(H \setminus \mathcal{B}_N) = 0$ , we obtain that for each  $T > 0$ ,

$$\begin{aligned} & \int_0^T \int_H |(B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v))| d\mu(v) dt \quad (64) \\ & \leq TC_T \int_{\mathcal{B}_N} (\|v - P_m v\| + |F_N(\|v\|) - F_{N,m}(\|v\|)|) d\mu(v) \quad \forall m > 2/N. \end{aligned}$$

As  $\|v - P_m v\| \rightarrow 0$  as  $m \rightarrow +\infty$  and  $\|v - P_m v\| \leq 2\|v\|$  for any  $v \in \mathcal{B}_N$ , and  $F_{N,m}$  converges to  $F_N$  as  $m \rightarrow +\infty$  uniformly on bounded subsets of  $\mathbb{R}$ , it is immediate form (64) that

$$\int_0^T \int_H |(B_N(v, v) - B_{N,m}(P_m v, P_m v), D_v(\Phi_m)(t, v))| d\mu(v) dt \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

and then, from (60) we conclude that

$$\int_H \Phi(S_N^{(m)}(T)v) d\mu(v) - \int_H \Phi(P_m v) d\mu(v) \rightarrow 0 \quad \text{as } m \rightarrow +\infty, \quad (65)$$

for all  $T > 0$ .

But, taking again into account that  $\mu(H \setminus \mathcal{B}_N) = 0$ , that  $\mathcal{B}_N$  is a bounded subset of  $D(A)$ , and that  $\Phi$  is bounded on bounded subsets of  $H$ , we deduce from Remark 8 and (52) that

$$\int_H \Phi(S_N^{(m)}(T)v) d\mu(v) \rightarrow \int_V \Phi(S_N(T)v) d\mu(v) \quad \text{as } m \rightarrow +\infty, \text{ for all } T > 0,$$

and

$$\int_H \Phi(P_m v) d\mu(v) \rightarrow \int_V \Phi(v) d\mu(v) \quad \text{as } m \rightarrow +\infty,$$

and then, by (65) we have (45), as desired.  $\square$

**Appendix: Estimates on the truncated function.** This section is devoted to prove some technical estimates concerning the functions  $F_{N,m}$ , defined by (11), namely those of Lemma 4.

To start, observe that

$$F_{N,m}(r) = \int_{r-1/m}^{r+1/m} \rho_m(r-\tau) F_N(\tau) d\tau.$$

Consequently, if  $N < r - 1/m$ , then

$$\begin{aligned} F_{N,m}(r) &= \int_{r-1/m}^{r+1/m} \rho_m(r-\tau) \frac{N}{\tau} d\tau \\ &\leq \frac{N}{r-1/m}. \end{aligned}$$

Thus,

- if  $N \geq r - 1/m$ , then  $rF_{N,m}(r) \leq r \leq N + 1/m$ , and
- if  $N < r - 1/m$ , then

$$\begin{aligned} rF_{N,m}(r) &\leq \frac{rN}{r-1/m} \\ &= N \left( 1 + \frac{1}{m(r-1/m)} \right) \\ &\leq N \left( 1 + \frac{1}{mN} \right) = N + 1/m. \end{aligned}$$

Hence, (12) is proved.

Moreover, it is straightforward to check that  $F'_{N,m}(r) = (\rho_m * F'_N)(r)$ , where  $F'_N(r) = -\frac{N}{r^2} \chi_{(N,+\infty)}(r)$ , and therefore (13) also holds.

For (14), related to  $r|F'_{N,m}(r)|$  and  $r^2|F'_{N,m}(r)|$ , we have to proceed more carefully.

Observe that

$$\begin{aligned} |F'_{N,m}(r)| &\leq \int_{r-1/m}^{r+1/m} \rho_m(r-\tau) |F'_N(\tau)| d\tau \\ &\leq \int_{r-1/m}^{r+1/m} \rho_m(r-\tau) \frac{N}{\tau^2} d\tau, \end{aligned}$$

and consequently, if  $r > N + 1$ , then

$$\begin{aligned} |F'_{N,m}(r)| &\leq \frac{N}{(r-1/m)^2} \\ &\leq \frac{N}{(r-1)^2}. \end{aligned}$$

Thus,

- if  $r > N + 1$ ,

$$\begin{aligned} r|F'_{N,m}(r)| &\leq \frac{Nr}{(r-1)^2} \\ &= \frac{N}{r-1} \left(1 + \frac{1}{r-1}\right) \\ &< 1 + \frac{1}{N} \\ &= \frac{N+1}{N}, \end{aligned}$$

and

$$\begin{aligned} r^2|F'_{N,m}(r)| &\leq \frac{Nr^2}{(r-1)^2} \\ &= N \left(1 + \frac{1}{r-1}\right)^2 \\ &< N \left(1 + \frac{1}{N}\right)^2 \\ &= \frac{(N+1)^2}{N}. \end{aligned}$$

- if  $0 \leq r \leq N + 1$ , then, by (13), one also has

$$r|F'_{N,m}(r)| \leq \frac{N+1}{N}, \quad r^2|F'_{N,m}(r)| \leq \frac{(N+1)^2}{N}.$$

Hence, combining all situations one recovers the inequality in (14).

Finally, in order to prove (15) observe that if  $m > 2/N$  and  $r \leq 1/m$ , then

$$F_{N,m}(r) = \int_{r-1/m}^{r+1/m} \rho_m(r-\tau)F_N(\tau) d\tau = 1,$$

whence the last result follows.

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*E-mail address*, Peter E. Kloeden: `kloeden@math.uni-frankfurt.de`

*E-mail address*, Pedro Marín-Rubio: `pmr@us.es`

*E-mail address*, José Real: `jreal@us.es`

(Peter E. Kloeden) INSTITUT FÜR MATHEMATIK, JOHANN WOLFGANG GOETHE-UNIVERSITÄT,  
D-60054 FRANKFURT AM MAIN, GERMANY

(Pedro Marín-Rubio and José Real) DPTO. ECUACIONES DIFERENCIALES Y ANÁLISIS NUMÉRICO,  
UNIVERSIDAD DE SEVILLA, APDO. DE CORREOS 1160, 41080-SEVILLA (SPAIN)