

ATTRACTORS FOR PARAMETRIC DELAY DIFFERENTIAL EQUATIONS AND THEIR CONTINUOUS BEHAVIOUR

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Dedicated to the memory of Valery S. Melnik

The problem of the continuity of global attractors under minimal assumptions for a general class of parameterized delay differential equations is considered. The theory of equi-attraction developed by Li and Kloeden in [Li & Kloeden, 2004a] is adapted to this framework, so it is proved that the continuity of the attractors with respect to the parameter is equivalent to this equi-attraction property.

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1. Introduction

In many fields of science, Biology, Economics, Physics, Chemistry, etc, delay differential equations (DDE for short) are of major interest. Besides the well-posedness of these problems, the qualitative behaviour of DDE includes many features as for general differential equations, although most attention has been paid to stability properties. In any case, even when such results do not hold, it is still useful the study of their long-time behaviour (leading to meaningful situations as permanence, extinction, chaotic phenomena, etc), in particular the study of attractors. There exists a wide literature on this topic, see for instance the seminal monographs [Hale, 1988, Hale & Lunel, 1993] and the references therein, or some recent developments in this field as multi-valued (semi)flows and/or

non-autonomous terms in the involved models (e.g. [Caraballo *et al.*, 2005, Caraballo *et al.*, 2007b]).

On the other hand, it is a complex matter to understand the structure of the attractor for a general problem and its continuous behaviour, for instance when changes in the model happen. This also indicates somehow robustness of the model (cf. [Hale *et al.*, 1988, Hale & Raugel, 1989]. Many papers are devoted to study continuity properties, being the upper semi continuity one half of the answer, and sometimes the most one can obtain (e.g. cf. [Caraballo & Langa, 2003]). Indeed, the continuity w.r.t. to variations in the original model and the structure of attractors remain as open problems (e.g. see [Carvalho & Langa, 2007, Carvalho *et al.*, 2007, Langa *et al.*, 2007, Efendiev *et al.*, 2005, Gatti *et al.*, 2006] for some recent developments in different situations).

The theory for dynamical systems associated to DDE obtains similar results too, for instance in the study of approximation of attractors (e.g. [Hines, 1995, Caraballo *et al.*, 2007a]), or when considering changes in the delay length (cf. [Kloeden, 2006]). An early paper on upper semi continuity for a retarded nonlinear PDE is [Boutet de Monvel *et al.*, 1997]. See also [Boutet de Monvel *et al.*, 1998, Chueshov *et al.*, 2005] for the same question but focused on inertial manifolds to deterministic and stochastic problems.

Since a middle step between autonomous and non-autonomous models is the appearance of parameters (also as approximation technique), it is natural the study of these properties in parametric systems. In [Marín-Rubio, 2008] the upper semi-continuity of the attractors for dynamical systems associated to DDE with a parametric dependence and without uniqueness was studied.

Very recently, Li and Kloeden [Li & Kloeden, 2004a] proved some continuity results for attractors of problems corresponding to different values of a parameter, using the new concept of equi-attraction. This approach translates the problem of the continuity of attractors to another property; in any case, it seems an interesting option to throw light on the question. That work was also adapted to some different situations as a non-autonomous framework with the cocycle formulation (cf. [Li & Kloeden, 2004b]) or even in a multi-valued context (cf. [Li & Kloeden, 2005]). The idea was adapted in [Kloeden & Marín-Rubio, 2008] to an abstract dynamical system related to DDE problems, and applied to a simple DDE just to illustrate its validity.

In this paper we consider a general class of parameterized DDE which includes fixed and distributed delays. Each equation is stated in a different state space, and the existence of the global attractor is ensured. Our goal is to study the behaviour of these attractors when varying the parameter. More exactly, we aim to establish continuity results without knowledge of the structure of the attractors, but in the same terms as Li and Kloeden's theory, relating to the concept of equi-attraction. So, it may be considered the main example to justify and complement

our contributions [Kloeden & Marín-Rubio, 2008, Marín-Rubio, 2008].

The structure of the paper is as follows. In Sec. 2 the parametric family of DDEs and main hypotheses are stated. In Sec. 3 some useful estimates and main concepts of dynamical systems and attractor theory are given. Existence of attractors for our family of problems is proved here. Finally, in Sec. 4 the Li and Kloeden's theory of equi-attraction is recalled, adapted and applied to our model.

2. Statement of the parametric family of DDEs

Let us introduce some notation which will be used all through the paper.

For a given metric space (X, d) , $B_X(a, r)$ will denote the open ball of X with center a and radius r . In addition, denote the Hausdorff semi distance and Hausdorff distance on X , respectively, by

$$H_X^*(A, B) = \sup_{x \in A} d(x, B), \quad \text{and}$$

$$H_X(A, B) = \max \{H_X^*(A, B), H_X^*(B, A)\}$$

for any nonempty subsets A and B of X .

In \mathbb{R}^m ($m \in \mathbb{N}$), we denote $|\cdot|$ the Euclidean norm; and for any $T > 0$ we will denote $(C_T, \|\cdot\|_T)$ the Banach space $C([-T, 0]; \mathbb{R}^m)$ endowed with the norm $\|\varphi\|_T = \sup_{t \in [-T, 0]} |\varphi(t)|$. The usual notation for delay function will be a sub-script: $x_t(s) = x(t + s)$ where it has sense.

We consider a similar model to that in [Marín-Rubio, 2008]. Concretely, let us state the following assumptions.

Hypothesis 2.1. *Let $\Lambda \subset \mathbb{R}$ be a closed interval, and suppose that positive numbers $0 < T_* < T^*$, and functions $\tau, \rho \in C(\Lambda; [T_*, T^*])$ are given.*

Consider also the functions $F_0, F_1 \in C(\mathbb{R}^m; \mathbb{R}^m)$, and $b : [-\max_{\Lambda} \tau, 0] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, measurable w.r.t. its first variable and continuous w.r.t. the second variable, $m_0, m_1 \in L^1((-\max_{\Lambda} \tau, 0); \mathbb{R}_+)$, and $\alpha, \beta > 0$, and $k_1, k_2 \geq 0$, such that

$$|b(s, x)| \leq m_1(s)|x| + m_0(s),$$

$$\forall x \in \mathbb{R}^m, \text{ a.e. } s \in (-\max_{\Lambda} \tau, 0),$$

$$\langle x, F_0(x) \rangle \leq -\alpha|x|^2 + \beta, \quad \forall x \in \mathbb{R}^m,$$

$$|F_1(x)|^2 \leq k_1^2 + k_2^2|x|^2, \quad \forall x \in \mathbb{R}^m.$$

Hypothesis 2.2 (Lipschitz character). *There exist positive constants L_{F_0} , L_{F_1} , and L_b such that $|F_j(x) - F_j(x')| \leq L_{F_j}|x - x'|$ for all $x, x' \in \mathbb{R}^m$, $j = 0, 1$, and $|b(s, x) - b(s, x')| \leq L_b|x - x'|$ for all $x, x' \in \mathbb{R}^m$, a. e. $s \in (-\max_{\Lambda} \tau, 0)$.*

For convenience we introduce the following notation

$$M_\lambda = \max\{\rho(\lambda), \tau(\lambda)\},$$

$$m_i = \max_{\Lambda} \int_{-\tau(\lambda)}^0 m_i(s) ds \quad \text{for } i = 0, 1.$$

Under the above assumptions, consider (for each $\lambda \in \Lambda$) the functional

$$f(\lambda, \cdot) : C_{M_\lambda} \rightarrow \mathbb{R}^m$$

given by

$$\begin{aligned} f(\lambda, \varphi) &= F_0(\varphi(0)) + F_1(\varphi(-\rho(\lambda))) \\ &\quad + \int_{-\tau(\lambda)}^0 b(s, \varphi(s)) ds, \end{aligned} \quad (1)$$

and the family of DDE

$$\begin{aligned} x'(t) &= f(\lambda, x_t) \\ &= F_0(x(t)) + F_1(x(t - \rho(\lambda))) \\ &\quad + \int_{-\tau(\lambda)}^0 b(s, x(t + s)) ds. \end{aligned} \quad (2)$$

Remark 2.3.

(a) More general functionals depending on the parameter and/or more different delay terms can be considered. However, for clarity in the presentation, we prefer to restrict to this case.

(b) The parametric dependence of the problem will be denoted by upper script (λ) when necessary, i.e. if no confusion is possible, we will just use the notation x for any solution instead of $x^{(\lambda)}$.

3. Parameterized semi dynamical systems and their attractors

Local existence of solutions for (2) is a well-known result (cf. [Hale & Lunel, 1993]) thanks to the continuity assumptions given in Hypothesis 2.1. With a priori estimates from the following result (proved in [Marín-Rubio, 2008; Lem.4], see also [Caraballo *et al.*, 2005; Lem.34]) we obtain global (and not only local) solutions but no uniqueness.

Lemma 3.1. *Under the Hypothesis 2.1, consider a local solution x to (2) with initial data in C_{M_λ} defined on an interval $[0, T_x)$. Then, there exist positive values A , B , and δ such that x satisfies for all $t < T_x$:*

$$e^{\delta t} |x(t)|^2 \leq |x(0)|^2 + \int_0^t e^{\delta s} (A + B \|x_s\|_{M_\lambda}^2) ds. \quad (3)$$

Proof. By the Hypothesis 2.1 and the Young inequality with suitable constants ε and $\bar{\varepsilon}$, it is not difficult to obtain

$$\frac{d}{dt} |x(t)|^2 \leq -\delta |x(t)|^2 + A + B \|x_t\|_{M_\lambda}^2,$$

with

$$A = 2\beta + \frac{k_1^2}{\varepsilon} + \frac{m_0^2}{\bar{\varepsilon}}, \quad B = \frac{k_2^2}{\varepsilon} + 2m_1, \quad (4)$$

and δ given by $\varepsilon + \bar{\varepsilon} = 2\alpha - \delta$, whence (3) follows. ■

Although for the results in [Marín-Rubio, 2008] (multi-valued) it was enough to fulfill model (2) with the Hypothesis 2.1, here we will need a more restrictive structure (semi dynamical systems in a single-valued sense), and uniqueness is required. It is well-known that Hypothesis 2.2 provides uniqueness of solution for the DDE.

Definition 3.2. A map $\mathcal{S} : \mathbb{R}_+ \times X \rightarrow X$ is called a semi dynamical system (SDS for short) if for each fixed $t \geq 0$, $\mathcal{S}(t, \cdot) : X \rightarrow X$ is continuous, and it has a semi-group structure in time, i.e.

- (a) $\mathcal{S}(0, \cdot) = \text{Id}$ (identity map)
- (b) For any pair $t_1, t_2 \geq 0$ and for all $x \in X$,

$$\mathcal{S}(t_1 + t_2, x) = \mathcal{S}(t_1, \mathcal{S}(t_2, x)).$$

Next proposition follows from standard continuation results (cf. [Hale & Lunel, 1993; Ch.2]).

Proposition 3.3. *Under Hypotheses 2.1 and 2.2, the following map defines a semi dynamical system:*

$$\begin{aligned} \mathbb{S}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda} &\rightarrow C_{M_\lambda} \\ (t, \psi) &\mapsto \mathbb{S}(t, \psi) = x_t \end{aligned}$$

with x the global solution of (2) with $x_0 = \psi$.

The goal of our study is the following object.

Definition 3.4. A nonempty compact subset \mathcal{A} of X is called a global attractor of a semi dynamical system \mathcal{S} if it is invariant, i.e. $\mathcal{S}(t, \mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{R}_+$, and attracts bounded subsets B of X , i.e.

$$\lim_{t \rightarrow +\infty} H^*(\mathcal{S}(t, B), \mathcal{A}) = 0,$$

where $\mathcal{S}(t, B) = \bigcup_{b \in B} \mathcal{S}(t, b)$.

The following two notions suffice in order to have a global attractor (e.g. cf. [Hale, 1988; Ch.3, Sec.4] or [Robinson, 2001; Ch.10]).

Definition 3.5. A semi dynamical system $\mathcal{S} : \mathbb{R}_+ \times X \rightarrow X$ is called pointwise dissipative if there exists a bounded set $B \subset X$ such that it attracts the dynamics starting at all single points, i.e.

$$\lim_{t \rightarrow +\infty} H^*(\mathcal{S}(t, x), B) = 0 \quad \forall x \in X.$$

It is called asymptotically compact if for any bounded set $B \subset X$ and any sequences $\{t_n\}$ with $t_n \rightarrow +\infty$ and $\{\psi^{(n)}\} \subset B$, the set $\{\mathcal{S}(t_n, \psi)\}$ is relatively compact in X .

Pointwise dissipativity can be obtained using ideas from Wang & Xu [Wang & Xu, 2003] and Ball [Ball, 2004] already used in [Caraballo *et al.*, 2007b], but here for finite delay (see also Theorem 8 in [Marín-Rubio, 2008]).

Theorem 3.6. *Assume that Hypotheses 2.1 and 2.2 hold, and*

$$\alpha > k_2 + m_1. \quad (5)$$

Then there exist values A , B , and δ as in Lemma 3.1 satisfying $\delta > B$. Moreover, the semi dynamical system $\mathbb{S}^{(\lambda)}$ is pointwise attracted by the set

$$B_0^{(\lambda)} = \{\psi \in C_{M_\lambda} : \|\psi\|_{M_\lambda}^2 \leq K := \frac{A}{\delta - B}\}, \quad (6)$$

that is, $\lim_{t \rightarrow +\infty} H_{C_{M_\lambda}}^(\mathbb{S}^{(\lambda)}(t, \varphi), B_0^{(\lambda)}) = 0$ for all $\varphi \in C_{M_\lambda}$.*

Although the proof is very similar to that of [Marín-Rubio, 2008; Th.8], it is included here for the sake of clarity, since it will be used below in other results.

Proof. The function $g : (0, 2\alpha) \rightarrow \mathbb{R}$ given by $g(\delta) = \delta - 2m_1 - \frac{k_2^2}{2\alpha - \delta}$, which represents $\delta - B$ from Lemma

3.1, has its maximum value $g(2\alpha - k_2) = 2(\alpha - m_1 - k_2) > 0$. This proves the first claim.

For the second part, we proceed in two steps.

Step 1: For any $R \geq 1$, $B_{C_{M_\lambda}}(0, \sqrt{RK})$, is positively invariant for the semi dynamical system $\mathbb{S}^{(\lambda)}$ associated to Eq. (2).

If not, there must be an initial data ψ with $\|\psi\|_{M_\lambda}^2 < RK$ and a solution x of (2) with $x_0 = \psi$ and a first time t_1 such that $\|x_{t_1}\|_{M_\lambda}^2 = RK$, i.e. $|x(t_1)|^2 = RK$.

But from (3) we deduce that

$$\begin{aligned} |x(t_1)|^2 &< e^{-\delta t_1} RK + \int_0^{t_1} e^{-\delta(t_1-s)} (A + BRK) ds \\ &= e^{-\delta t_1} RK + \frac{A + BRK}{\delta} (1 - e^{-\delta t_1}). \end{aligned}$$

Observe that

$$\frac{A + BRK}{\delta} \leq \frac{R(A + BK)}{\delta} = RK,$$

which is a contradiction with $|x(t_1)|^2 = RK$.

Step 2: The closed ball $B_0^{(\lambda)} = \bar{B}_{C_{M_\lambda}}(0, \sqrt{K})$ attracts any solution of (2).

Consider a solution $x(\cdot)$ with initial data $\psi \in C_{M_\lambda}$ with $\|\psi\|_{M_\lambda}^2 = \zeta \geq K$ (otherwise, the claim holds by Step 1).

Thanks to Step 1 we have that $|x(t)| \leq \zeta$ for all $t \geq 0$. Therefore, there exists $\limsup_{t \rightarrow +\infty} |x(t)|^2 = \sigma$.

So,

$$\forall \varepsilon > 0 \exists T_1(\varepsilon, \lambda) > 0 : |x(t)|^2 \leq \sigma + \varepsilon \quad \forall t \geq T_1(\varepsilon, \lambda),$$

$$\text{and } \|x_t\|_{M_\lambda}^2 \leq \sigma + \varepsilon \quad \forall t \geq T_1(\varepsilon, \lambda) + M_\lambda. \quad (7)$$

Take now $T_2(\varepsilon)$ such that

$$e^{-\delta t} \zeta + \frac{A + B\zeta}{\delta} (e^{-\delta T_2(\varepsilon)} - e^{-\delta t}) \leq \varepsilon \quad \forall t \geq T_2(\varepsilon). \quad (8)$$

So, for any $t \geq T_2(\varepsilon) + T_1(\varepsilon, \lambda) + M_\lambda$, from (3), splitting the integral in two parts,

$$\begin{aligned} |x(t)|^2 &\leq e^{-\delta t} |x(0)|^2 \\ &\quad + \int_0^{t-T_2(\varepsilon)} e^{-\delta(t-s)} (A + B\|x_s\|_{M_\lambda}^2) ds \\ &\quad + \int_{t-T_2(\varepsilon)}^t e^{-\delta(t-s)} (A + B\|x_s\|_{M_\lambda}^2) ds, \end{aligned}$$

applying (8) to the first two addends (thanks to Step 1), and (7) to the last addend, we obtain for all $t \geq T_2(\varepsilon) + T_1(\varepsilon, \lambda) + M_\lambda$:

$$|x(t)|^2 \leq \varepsilon + \frac{A + B(\sigma + \varepsilon)}{\delta} \left(1 - e^{-\delta T_2(\varepsilon)}\right).$$

Passing through the limit when ε goes to zero, we deduce that

$$\sigma = \limsup_{t \rightarrow +\infty} |x(t)|^2 \leq \frac{A + B\sigma}{\delta},$$

in other words, $\sigma \leq \frac{A}{\delta - B} = K$, which finishes the proof. \blacksquare

Remark 3.7. In Step 2, T_1 depends on λ . In principle they are not uniformly bounded in λ , which will have importance in Proposition 4.13 below.

The following result is an immediate consequence of the Ascoli-Arzelà theorem, and its proof is similar to [Caraballo *et al.*, 2005; Prop.10] or [Caraballo *et al.*, 2007b; Prop.2].

Proposition 3.8. *Let be given $T > 0$ and a continuous functional $h : C_T \rightarrow \mathbb{R}^n$ which is bounded (i.e. the image of a bounded set is also bounded), and such that the DDE $x'(t) = h(x_t)$ generates a semi dynamical system \mathcal{S} . If \mathcal{S} satisfies the following boundedness condition,*

$$\begin{aligned} \forall R > 0 \exists M(R) > 0 \text{ such that} \\ \mathcal{S}(t, B_{C_T}(0, R)) \subset B_{C_T}(0, M(R)), \end{aligned}$$

then \mathcal{S} is asymptotically compact.

We can combine the above results to conclude the existence of attractors for the semi dynamical systems $\{\mathbb{S}^{(\lambda)}\}_{\lambda \in \Lambda}$.

Theorem 3.9. *Assume that Hypotheses 2.1 and 2.2 and (5) hold. Then, for each $\lambda \in \Lambda$, (2) generates a semi dynamical system $\mathbb{S}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda} \rightarrow C_{M_\lambda}$, which possesses a global attractor $\mathbb{A}^{(\lambda)}$, which satisfies a uniform bound (for all λ) on the Euclidean projected space \mathbb{R}^m :*

$$\|\psi\|_{M_\lambda}^2 \leq K, \quad \forall \psi \in \mathbb{A}^{(\lambda)},$$

where the constant K is given in Theorem 3.6.

Proof. The pointwise dissipativity of each $\mathbb{S}^{(\lambda)}$ already proved in Theorem 3.6 is fulfilled with the asymptotic compactness since the uniform boundedness condition required in Proposition 3.8 holds from Step 1 in the same theorem.

Finally, the value K of the radius of the pointwise attracting ball, that contains the attractor, is independent of λ as (6) and (4) show. \blacksquare

4. Continuous dependence of the attractors on the parameter

Our aim now is to show a result of continuous dependence for the obtained attractors $\mathbb{A}^{(\lambda)}$ in Theorem 3.9. Recently, Li and Kloeden [Li & Kloeden, 2004a, Li & Kloeden, 2005] have developed a theory to show equivalent conditions to continuity of parametric attractors (for single valued and multi-valued frameworks respectively). However, it is necessary to adapt these concepts to our framework since the delay affects the phase-space for each λ . We will recover some results in this sense from [Kloeden & Marín-Rubio, 2008].

4.1. On the Li and Kloeden's equi-attraction theory

Definition 4.1. Let $\{\mathcal{S}^{(\lambda)}\}_{\lambda \in \Lambda}$ be a family of semi dynamical systems on X . We say that

(a) $\{\mathcal{S}^{(\lambda)}\}$ is **equi-dissipative** on X if there exists a bounded subset \mathcal{U} of X so that for any bounded subset $B \subset X$, there exists a $T_B \in \mathbb{R}_+$ independent of $\lambda \in \Lambda$ such that

$$\mathcal{S}^{(\lambda)}(t, B) \subset \mathcal{U}, \quad t \geq T_B;$$

(b) $\{\mathcal{S}^{(\lambda)}\}$ is **eventually equi-compact** (or uniformly compact for large t in [Li & Kloeden, 2004a]) if for any bounded subset B of X , there exists a $T_B \in \mathbb{R}_+$ independent of $\lambda \in \Lambda$, such that $\cup_{\lambda \in \Lambda} \mathcal{S}^{(\lambda)}(t, B)$ is relatively compact for any $t \geq T_B$.

The following theorem was proved by Li and Kloeden [Li & Kloeden, 2004a; Th.2.9].

Theorem 4.2. *Suppose that $\{\mathcal{S}^{(\lambda)}\}_{\lambda \in \Lambda}$ is equi-dissipative and eventually equi-compact and that $\mathbb{A}^{(\lambda)}$ is the global attractor of $\mathcal{S}^{(\lambda)}$ for $\lambda \in \Lambda$. In addition, suppose that the following conditions are satisfied:*

(A1) For any $t \in \mathbb{R}_+$ fixed, $\mathcal{S}^{(\lambda)}(t, x)$ is jointly continuous in (x, λ) on $X \times \Lambda$.

(A2) $\mathcal{S}^{(\lambda)}(t, x)$ is equi-continuous in λ for (t, x) in any bounded subset of $\mathbb{R}_+ \times X$.

Then $\mathcal{A}^{(\lambda)}$ is continuous in λ with respect to the Hausdorff distance **if and only if** $\{\mathcal{A}^{(\lambda)}\}$ is equi-attracting, i.e. for any bounded subset B of X and any $\varepsilon > 0$, there is a $\tau = \tau(B, \varepsilon) > 0$ independent of λ such that $H_X^*(\mathcal{S}^{(\lambda)}(t, B), \mathcal{A}^{(\lambda)}) < \varepsilon$ for all $t \geq \tau$ and for all $\lambda \in \Lambda$.

Remark 4.3. The above equivalence also holds if (A2) is replaced by:

(A2') $\mathcal{S}^{(\lambda)}(t, x)$ is equi-continuous in λ for t in any bounded subset of \mathbb{R}_+ and x in any bounded subset of $\bigcup_{\lambda \in \Lambda} \mathcal{A}^{(\lambda)}$.

Finally, we recall an additional continuity notion, which will be useful for the following result (cf. [Li & Kloeden, 2004a; Th.2.7]) and below:

(A3) For any bounded subset B of X and $T > 0$, $\mathcal{S}^{(\lambda)}(t, x)$ is uniformly continuous in $x \in B$ uniformly w.r.t. $\lambda \in \Lambda$ and $t \leq T$, i.e.

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta > 0 : x, y \in B, d(x, y) < \delta \\ \Rightarrow d(\mathcal{S}^{(\lambda)}(t, x), \mathcal{S}^{(\lambda)}(t, y)) < \varepsilon, \forall t \leq T, \lambda \in \Lambda. \end{aligned}$$

Theorem 4.4. Suppose that $\{\mathcal{S}^{(\lambda)}\}_{\lambda \in \Lambda}$ is equi-dissipative, eventually equi-compact, and the assumptions (A1) and (A3) given above hold.

Then, if $\{\mathcal{A}^{(\lambda)}\}$ is continuous in λ , it is also uniformly Lyapunov stable, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ (independent of λ) such that for all $\lambda \in \Lambda$, if $d(x, \mathcal{A}^{(\lambda)}) < \delta$, then $d(\mathcal{S}^{(\lambda)}(t, x), \mathcal{A}^{(\lambda)}) < \varepsilon$ for all $t \in \mathbb{R}_+$.

4.2. Embedding on a common state space for DDEs and equi-concepts

In order to apply Li and Kloeden's results to our parametric problem (2) and their attractors $\mathbb{A}^{(\lambda)}$ we need to do some adaptations. We reproduce here main required results from [Kloeden & Marín-Rubio, 2008].

Firstly, it is required a common state space for all the problems independently of the parameter. This means to extend the semi dynamical systems $\mathbb{S}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda} \rightarrow C_{M_\lambda}$ to another SDS.

Theorem 4.5. [cf. [Kloeden & Marín-Rubio, 2008; Th.6]] Let $\mathcal{S}^{(\tau)} : \mathbb{R}_+ \times C_\tau \rightarrow C_\tau$ be a family of semi dynamical system, with $\tau \in [T_*, T^*]$. Then, $\widehat{\mathcal{S}}^{(\tau)}$ defines a semi dynamical system on C_{T^*} , where $\widehat{\mathcal{S}}^{(\tau)}(t, \phi)(s) := x(t + s, \phi)$, $s \in [-T^*, 0]$, being

$$x(t, \phi) := \begin{cases} \phi(t) & t \in [-T^*, 0], \\ \mathcal{S}^{(\tau)}(t, \phi|_{[-\tau, 0]})(0) & t > 0, \end{cases}$$

Moreover, if $\mathcal{S}^{(\tau)}$ is jointly continuous w.r.t. $(t, \phi) \in \mathbb{R}_+ \times C_\tau$, so it is $\widehat{\mathcal{S}}^{(\tau)}$.

It is clear that we can apply the above result to our family of SDS $\mathbb{S}^{(\lambda)}$. We will denote the extended by SDS $\widehat{\mathbb{S}}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda} \rightarrow C_{M_\lambda}$.

Remark 4.6. When there is a DDE generating the SDS, as it is the case here, it is easy to check that $\widehat{\mathbb{S}}^{(\lambda)}$ is the SDS associated to the DDE with right hand side $\widehat{F} \in C(C_{T^*}; \mathbb{R}^m)$ defined as

$$\widehat{F}(\widehat{\phi}) = F(\widehat{\phi}|_{M_\lambda}).$$

The existence of attractors $\mathbb{A}^{(\lambda)}$ for $\mathbb{S}^{(\lambda)}$ also guaranties the existence of attractors $\widehat{\mathbb{A}}^{(\lambda)}$ for the extended SDS $\widehat{\mathbb{S}}^{(\lambda)}$ as an application of the following abstract result (cf. [Kloeden & Marín-Rubio, 2008; Th.7]).

Theorem 4.7. Suppose that a semi dynamical system $\mathcal{S}^{(\tau)} : \mathbb{R}_+ \times C_\tau \rightarrow C_\tau$ has a global attractor $\mathcal{A}^{(\tau)}$. Then, the extended semi dynamical system $\widehat{\mathcal{S}}^{(\tau)}$ given in Theorem 4.5 possesses a global attractor, denoted $\widehat{\mathcal{A}}^{(\tau)}$, and it has the following characterization:

$$\widehat{\mathcal{A}}^{(\tau)} := \left\{ \psi \in C_{T^*} : \exists \text{ entire trajectory } \bar{\Phi}_t^{(\tau)} \text{ of } \mathcal{S}^{(\tau)} \right. \\ \left. \text{in } \mathcal{A}^{(\tau)} \text{ with } \psi(s) = \bar{\phi}(s) \forall s \in [-T^*, 0] \right\},$$

where $\bar{\phi}(t)$ is the projection in \mathbb{R}^m of the entire solution $\bar{\Phi}_t^{(\tau)}$ defined by $\bar{\phi}(t) := \bar{\Phi}_t^{(\tau)}(0)$ for all $t \in \mathbb{R}$.

On the other hand, to apply Theorem 4.2, we need to know how equi-concepts are translated from the extended SDS to the original ones.

The equi-dissipative notion is not difficult to relate between the original and the extended SDS, passing through the projected trajectories in \mathbb{R}^m .

Lemma 4.8. [cf. [Kloeden & Marín-Rubio, 2008; Lem.8]] *A family of SDS $\{\widehat{\mathcal{S}}^{(\tau)}, \tau \in [T_*, T^*]\}$ is equi-dissipative if and only if there exists a bounded subset U of \mathbb{R}^m such that for every bounded subset B of \mathbb{R}^m there exists a $T_B \in \mathbb{R}_+$, which is independent of τ , such that*

$$\mathcal{S}^{(\tau)}(t, \mathcal{B}|_{[-\tau, 0]})(0) \subset U \quad \forall t \geq T_B \text{ and } \tau \in [T_*, T^*],$$

where

$$\mathcal{B} := \{\phi \in C_{T^*} : \phi(s) \in B \quad \forall s \in [-T^*, 0]\}.$$

Remark 4.9. As a consequence of the Ascoli-Arzelà theorem, coming from a DDE, the equi-dissipativity and the compactness of the SDS after an elapsed time T^* implies the eventual equi-compactness.

So, it only rests to know how to translate the equi-attraction property. One sensible attempt to translate in an abstract situation the equi-attraction property to the original SDS is the following:

For every $\varepsilon > 0$ and bounded subset \mathcal{B} of C_{T^*} there exists $T_{\varepsilon, \mathcal{B}} \in \mathbb{R}_+$ independent of $\tau \in [T_*, T^*]$ such that

$$H_{C_\tau}^* \left(\mathcal{S}^{(\tau)}(t, \phi|_{[-\tau, 0]}), \mathcal{A}^{(\tau)} \right) < \varepsilon$$

for all $t \geq T_{\varepsilon, \mathcal{B}}, \phi \in \mathcal{B}, \tau \in [T_*, T^*]$. (9)

The next result was proved in [Kloeden & Marín-Rubio, 2008; Th.11].

Theorem 4.10. *Consider $\mathcal{S}^{(\tau)} : \mathbb{R}_+ \times C_\tau \rightarrow C_\tau$ for $\tau \in [T_*, T^*]$, a family of semi dynamical systems, with attractors $\mathcal{A}^{(\tau)}$, equi-dissipative and equi-attracting in the sense of (9), and satisfying the following condition:*

(A3') *For any bounded subset \mathcal{B} of C_{T^*} and $T > 0$, $\mathcal{S}^{(\tau)}(t, x)$ is uniformly continuous in $x \in \mathcal{B}|_{[-\tau, 0]}$, uniformly w.r.t. τ and $t \leq T$, i.e.*

$$\begin{aligned} & \forall \varepsilon > 0, \exists \delta > 0 : \\ & x, y \in \mathcal{B}, \|x|_{[-\tau, 0]} - y|_{[-\tau, 0]}\|_\tau < \delta \\ \Rightarrow & \|\mathcal{S}^\tau(t, x|_{[-\tau, 0]}) - \mathcal{S}^{(\tau)}(t, y|_{[-\tau, 0]})\|_\tau < \varepsilon, \\ & \forall t \leq T, \tau \in [T_*, T^*]. \end{aligned} \quad (10)$$

Then $\widehat{\mathcal{A}}^{(\tau)}$ is equi-attracting.

4.3. Equi-properties for the DDE model

The following result gives the required continuity notions to apply the above results.

Let us introduce for commodity the following notation:

$$\begin{aligned} m_{\lambda, \lambda'}^\tau &= \min\{\tau(\lambda), \tau(\lambda')\}, \\ M_{\lambda, \lambda'}^\tau &= \max\{\tau(\lambda), \tau(\lambda')\}. \end{aligned}$$

Lemma 4.11. *Assume Hypotheses 2.1 and 2.2 and (5) hold. Then,*

(a) *For any $\varphi \in C_{M_\lambda}$ and $\psi \in C_{M_{\lambda'}}$, the functional f given in (1) satisfies:*

$$\begin{aligned} & |f(\lambda, \varphi) - f(\lambda', \psi)| \quad (11) \\ & \leq L_{F_0}|\varphi(0) - \psi(0)| + L_{F_1}|\varphi(-\rho(\lambda)) - \psi(-\rho(\lambda'))| \\ & \quad + L_b m_{\lambda, \lambda'}^\tau \|\varphi - \psi\|_{m_{\lambda, \lambda'}^\tau} + \int_{-M_{\lambda, \lambda'}^\tau}^{-m_{\lambda, \lambda'}^\tau} |b(s, \xi(s))| ds, \end{aligned}$$

where ξ is φ or ψ depending on which is the maximum and minimum between $\tau(\lambda)$ and $\tau(\lambda')$.

(b) *Properties (A1) and (A2') hold for the semi dynamical systems $\widehat{\mathcal{S}}^{(\lambda)}$.*

Proof. Taking into account the definition of f in (1), one deduces

$$\begin{aligned} & |f(\lambda, \varphi) - f(\lambda', \psi)| \\ & = |F_0(\varphi(0)) - F_0(\psi(0)) \\ & \quad + F_1(\varphi(-\rho(\lambda))) - F_1(\psi(-\rho(\lambda')))| \\ & \quad + \left| \int_{-\tau(\lambda)}^0 b(s, \varphi(s)) ds - \int_{-\tau(\lambda')}^0 b(s, \psi(s)) ds \right| \\ & \leq L_{F_0}|\varphi(0) - \psi(0)| \\ & \quad + L_{F_1}|\varphi(-\rho(\lambda)) - \psi(-\rho(\lambda'))| \\ & \quad + \int_{-m_{\lambda, \lambda'}^\tau}^0 |b(s, \varphi(s)) - b(s, \psi(s))| ds \\ & \quad + \int_{-M_{\lambda, \lambda'}^\tau}^{-m_{\lambda, \lambda'}^\tau} |b(s, \xi(s))| ds. \end{aligned} \quad (12)$$

So, one obtains (11).

Now, we proceed to check (A1) and (A2').

Consider two functions $\varphi, \psi \in C_{T^*}$ and the respective solutions $x(t) = \widehat{\mathcal{S}}^{(\lambda)}(t, \varphi)(0)$ and $y(t) = \widehat{\mathcal{S}}^{(\lambda')}(t, \psi)(0)$.

Denote $z(t) = x(t) - y(t)$ for $t \geq -T^*$. Using (12) and the uniform estimate obtained in Step 1

in Theorem 3.6, we know that there exists a constant C , depending on $\max(\|\varphi\|_{T^*}, \|\psi\|_{T^*})$ and b such that

$$\begin{aligned}
& |z(t)| \\
\leq & |z(0)| + \int_0^t \left\{ L_{F_0}|x(s) - y(s)| \right. \\
& \quad + L_{F_1}|x(s - \rho(\lambda)) - y(s - \rho(\lambda'))| \\
& \quad + L_b m_{\lambda, \lambda'}^\tau \|x_s - y_s\|_{m_{\lambda, \lambda'}^\tau} \\
& \quad \left. + C|\tau(\lambda) - \tau(\lambda')| \right\} ds \\
\leq & |z(0)| + \int_0^t \left\{ L_{F_0}|x(s) - y(s)| \right. \\
& \quad + L_{F_1} \left[|x(s - \rho(\lambda)) - y(s - \rho(\lambda))| \right. \\
& \quad \left. + |y(s - \rho(\lambda)) - y(s - \rho(\lambda'))| \right] \\
& \quad + L_b m_{\lambda, \lambda'}^\tau \|x_s - y_s\|_{m_{\lambda, \lambda'}^\tau} \\
& \quad \left. + C|\tau(\lambda) - \tau(\lambda')| \right\} ds \\
\leq & |z(0)| \\
& + (L_{F_0} + L_{F_1} + L_b T^*) \int_0^t \|x_s - y_s\|_{T^*} ds \\
& + L_{F_1} \int_0^t |y(s - \rho(\lambda)) - y(s - \rho(\lambda'))| ds \\
& + C|\tau(\lambda) - \tau(\lambda')| t.
\end{aligned}$$

If we consider a fixed interval of time $[0, T]$, thanks to the continuity of ρ , τ , and the solution y (therefore, it is uniformly continuous on $[0, T]$), it is possible to take a value $\varepsilon = \varepsilon(\lambda, \lambda', \rho, \tau, T, y) > 0$, as small as desired, such that

$$\begin{aligned}
|z(t)| & \leq |z(0)| + (L_{F_1} + C) \varepsilon t \quad (13) \\
& + (L_{F_0} + L_{F_1} + L_b T^*) \int_0^t \|z_s\|_{T^*} ds.
\end{aligned}$$

Now, property (A1) is a consequence of the Gronwall lemma manipulating (13) to change $|z(t)|$ by $\|z(t)\|_{T^*}$.

In order to obtain (A2) we have to proceed more carefully. We cannot obtain an equi-continuous bound for the term $|y(s - \rho(\lambda)) - y(s - \rho(\lambda'))|$ if we deal now not only with a fixed solution and a unique initial data but with a bounded set of initial data. However, we observe (cf. Remark 4.3) that it is enough to prove (A2) for the bounded set composed by the union of all attractors, which is, by the uniform bound, the differential equation, and

the Ascoli-Arzelà theorem an equi-continuous set of functions. This circumvents the difficulty, and one obtains (A2'). \blacksquare

Remark 4.12. Following the proof of (A1) above, one deduces that property (A3') given in Theorem 4.10 (see (10)) also holds.

Proposition 4.13. *Under the assumptions of Lemma 4.11, $\{\widehat{\mathbb{S}}^{(\lambda)}\}_{\lambda \in \Lambda}$ is equi-dissipative.*

Proof. We can complete the uniform bound obtained in Theorem 3.9 in order to gain the equi-dissipativity.

We have to prove that there exists a bounded set $\mathcal{U} \subset C_{T^*}$, such that for any closed ball $\mathcal{B} = \bar{B}_{C_{T^*}}(0, R) \subset C_{T^*}$, there exists a time $T_{\mathcal{B}} > 0$, independent of λ , such that for all $t \geq T_{\mathcal{B}}$, one has $\widehat{\mathbb{S}}^{(\lambda)}(t, \mathcal{B}) \subset \mathcal{U}$ for all $\lambda \in \Lambda$.

Just consider $\varepsilon > 0$ arbitrary, and the value K given in Theorem 3.6.

For any $\lambda \in \Lambda$, thanks to the existence of the attractor $\mathbb{A}^{(\lambda)}$, and therefore, that of $\widehat{\mathbb{A}}^{(\lambda)}$, there exists a time $T_1(\varepsilon, \lambda, \mathcal{B})$ (w.l.o.g. bigger than $T^* = \max_{\Lambda} M_{\lambda}$) such that

$$\begin{aligned}
\widehat{\mathbb{S}}^{(\lambda)}(t, \mathcal{B}) & \subset B_{C_{T^*}}(0, (K + \varepsilon)^{1/2}) \\
& \quad \forall t \geq T_1(\varepsilon, \lambda, \mathcal{B}). \quad (14)
\end{aligned}$$

We have to prove that $T_1(\varepsilon, \cdot, \mathcal{B})$, as function of λ , is bounded in Λ . Suppose the opposite, then there exist sequences $\{\lambda_n\}$ and $\{\psi^{(n)}\} \subset \mathcal{B}$ such that

$$\lim_{n \rightarrow +\infty} T_1(\varepsilon, \lambda_n, \mathcal{B}) = +\infty, \quad \text{and} \quad (15)$$

$$\exists \theta_n < T_1(\varepsilon, \lambda_n, \mathcal{B}) : \|\widehat{\mathbb{S}}^{(\lambda_n)}(\theta_n, \psi^{(n)})\|_{T^*}^2 \geq K + \varepsilon.$$

By the compactness of Λ , there exist a value λ and a subsequence $\{\lambda_{n'}\} \subset \{\lambda_n\}$ such that $\lim_{n' \rightarrow +\infty} \lambda_{n'} = \lambda$. We will see that (15) comes in contradiction with (14) using this particular value λ .

Observe that the uniform bound (Step 1 in Theorem 3.6) of solutions for the DDE's, the finite delay of all the involved DDE, and the fact that all $f(\lambda, \cdot)$ are bounded maps, provide compactness: after the time $T^* = \max_{\Lambda} M_{\lambda}$, the set $\{\varphi^{(n)}\}$ with $\varphi^{(n)} = \widehat{\mathbb{S}}^{(\lambda_n)}(T^*, \psi^{(n)})$, is relatively compact. There exists a subsequence $\{\varphi^{(n')}\} \subset \{\varphi^{(n)}\}$ with $\varphi^{(n')} \rightarrow \varphi$ in C_{T^*} .

Now we make the most of property (A1) in a fixed finite interval since we have $\lambda_{n'} \rightarrow \lambda$ and $\varphi^{(n')} \rightarrow \varphi$. We know by (14) that the solution with initial data φ satisfies $\|x_{T_1}^{(\lambda)}\|_{T^*}^2 \leq K + \varepsilon'$ (for certain value $0 < \varepsilon' < \varepsilon$).

Consider the finite time interval $[0, T]$ with $T = T_1(\varepsilon, \lambda, \mathcal{B}) - T^*$ (since we have let an elapsed time T^* to get $\varphi^{(n')}$). By (A1), the solutions with initial data $\varphi^{(n')}$, $x_{T_1}^{(\lambda_{n'})}$ stay close of $x_{T_1}^{(\lambda)}$ if n' is big enough. In other words,

$$\begin{aligned} \exists n'(\varepsilon - \varepsilon') \quad : \quad & \|x_{T_1}^{(\lambda_{n'})}\|_{T^*}^2 \leq K + (\varepsilon + \varepsilon')/2 \\ & \forall n' \geq n'(\varepsilon - \varepsilon'). \end{aligned}$$

Once again, the positive invariance of any ball proved in Step 1 in Theorem 3.6 points out that this contradicts (15). ■

The main result of this section is the following.

Theorem 4.14. *Assume that Hypotheses 2.1 and 2.2 and (5) hold. Then,*

(a) *Equi-attraction of the attractors $\{\mathbb{A}^{(\lambda)}\}$ in the sense of (9) is equivalent to the continuity of the attractors $\{\widehat{\mathbb{A}}^{(\lambda)}\}$.*

(b) *If $\{\widehat{\mathbb{A}}^{(\lambda)}\}$ is continuous in λ , then this family is also uniformly Lyapunov stable in the sense given in Theorem 4.4.*

Proof. Firstly, we have to check the required conditions for Theorem 4.2. These were proved in Lemma 4.11 (recall Remark 4.3 to substitute (A2) by (A2')), Proposition 4.13, and by the uniform bound in Theorem 3.9, which provides equi-compactness (cf. Remark 4.9).

The second statement is a consequence of Theorem 4.4 and Theorem 4.10. ■

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