Attractors for differential equations with unbounded delays

T. Caraballo [a], P. Marín–Rubio [a,*], and J. Valero [b]

[a] *Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080 Sevilla, Spain*

[b] *Centro de Investigación Operativa, Universidad Miguel Hernández, Avda. Universidad s/n 03202, Elche, Alicante, Spain*

Abstract

We prove the existence of attractors for some types of differential problems containing infinite delays. Applications and examples are provided to illustrate the theory, which is valid for both cases with and without explicit dependence of time, and with or without uniqueness of solutions, as well.

Key words: autonomous and non-autonomous (pullback) attractors, delay differential equations, infinite delays.
Mathematics Subject Classifications (2000): 34D45, 34K20, 34K25, 37L30

1 Introduction

Retarded differential equations are an important area of applied mathematics due to physical reasons with non instant transmission phenomena as high velocity fields in wind tunnel experiments, or other memory processes, and specially biological motivations (e.g. [13,23,27]) like species' growth or incubating time on disease models among many others.

On the other hand, the asymptotic behaviour of such models has meaningful interpretations like permanence of species on a given domain, with or without competition, their possible extinction, instability and sometimes chaotic developments, being therefore of obvious interest. There exists a wide literature devoted to the stability of fixed points, and also to the study of global attractors. This is another useful tool but still valid with more general conditions than those for stability, and the equations for which the existence of an attractor (and so both stable and unstable regions) can be ensured is therefore an interesting subject which is receiving much attention.

* Corresponding author
Email addresses: caraball@us.es (T. Caraballo), pmr@us.es (P. Marín–Rubio), jvalero@umh.es (J. Valero).

The theory of global attractors for autonomous systems as developed by Hale in [15] owes much to examples arising in the study of (finite and infinite) retarded functional differential equations [17] (for slightly different approaches see Babin and Vishik [2], Ladyzhenskaya [24], or Temam [29]). Although the classical theory has been extended in a relatively straightforward manner to deal with time-periodic equations, general non-autonomous equations such as

$$x'(t) = F(t, x(t), x(t - \rho(t))),$$

with variable delay, or

$$x'(t) = \int_{-h}^{0} b(t, s, x(t + s))\, \mathrm{d}s,$$

for distributed delay terms, including the possibility of being $h = +\infty$, fall outside this scope.

Recently, a theory of 'pullback attractors' has been developed for stochastic and non-autonomous systems in which the trajectories can be unbounded when time increases to infinity, allowing many of the ideas for the autonomous theory to be extended to deal with such examples (cf. [11,22])). In this case, the global attractor is defined as a parameterized family of sets $A(t)$ depending on the final time, such that attracts solutions of the system 'from $-\infty$', i.e. initial time goes to $-\infty$ while the final time remains fixed.

The cases in which the hereditary characteristics in the models involve bounded (also termed finite) delays has already been analysed for instance in [7] and [9]. In the latter, also the situations in which uniqueness of solutions cannot be ensured (or it is not known) are considered thanks to the concept of multi-valued semigroup or semiprocess.

However, there are reasons that make sensible the appearance of unbounded delays, for instance when a problem has different delay intervals (possibly unknown) where may be applied, and a unified model is required, as in economic situations or the pantograph equation (physics), or properly a complete influence of the whole past of the state (e.g. versions of the logistic model, see below).

As far as we know, the existence of attractors in the case of differential equations with infinite (or unbounded) delays has only been analysed in the autonomous case (e.g. see Hale & Lunel [17]). This means that the existence of attractors for very simple equations as, for example,

$$x'(t) = F(x(t), x(qt)), \quad q \in (0, 1),$$

(which includes the interesting pantograph equations, e.g. [21,14,28]), has not been studied yet.

Several technical reasons must be taken into account. On the one hand, for some of these problems, it is not possible to use the autonomous form $x'(t) = f(x_t)$, since f depends explicitly on time, which motivates the necessity of using the theory of non-autonomous dynamical systems.

On the other hand, being infinite in most cases the time interval influence, the choice of the phase space for these problems is delicate (see [16] for a discussion on this problem). Even this is an important difference for stability results (see [19]). This fact also implies that the compactness techniques to ensure existence of attractor for finite delay equations used in [9] may not be applied directly here.

Additionally, we point out that uniqueness is now a more rare condition to obtain, which leads us to state our study in a (more) general multi-valued framework.

We aim to show, jointly with classical results on global attractors, that the theory of pullback attractors for non-autonomous dynamical systems (with or without uniqueness) can be very useful in order to prove the existence of attracting sets for differential equations with infinite delay.

The content of the paper is as follows. Section 2 is devoted to preliminaries on infinite delay differential equations and their associated dynamical systems. In Section 3 we recover and state some new results on attractors which are suitable for the considered equations. Finally, Section 4 is devoted to several applications of the theory, some of them with biologic motivation, as Logistic or Lotka-Volterra models.

2 Preliminaries. Dynamical systems

In this section we aim to establish briefly some preliminaries on existence and uniqueness of integro-differential equations with infinite delays, the definitions of (generally multi-valued, if uniqueness does not hold) semiflows and processes associated to the autonomous and non-autonomous problems. For a more detailed exposition we refer to [1,16–18].

Let us first introduce some notation: we will consider $\mathbb{R}^m$ with its usual Euclidean topology and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ its inner product and norm respectively. The delay functions will be denoted as usual by x_t, that is, $x_t(s) = x(t+s)$ for every s such that it has sense. In this paper it will be $s \in (-\infty, 0]$, so that $x_t : (-\infty, 0] \to \mathbb{R}^m$. Also, it will be useful to denote $\mathbb{R}_d = \{(t,s) \in \mathbb{R}^2,\ t \geq s\}$, $B_X(y,r)$ the open ball in a metric space X with center y and radius r, and $P(X)$ the non-empty subsets of X.

Solutions for delay differential equations

Delay differential and integro-differential equations have been intensively studied for a long time, and deeply developed since Volterra's works (see [13,17,23] and the references therein among others).

Consider the canonical model

$$x'(t) = f(t, x_t), \tag{1}$$

with f a function regular enough, for example continuous (of course, this may be weakened from the mathematical point of view, but the biological motivations lead us to consider so), on suitable spaces to be specified below. This functional may contain, for instance, terms of the form

$$F(t, x(t), x(t-\rho(t))) + \int_{-h}^{0} b(t,s,x(t+s))\mathrm{d}s + \int_{-\infty}^{0} c(t,s,x(t+s))\mathrm{d}s,$$

though for simplicity in the exposition we will restrict ourselves in the distributed term to the case without the integral over $[-h, 0]$, since it does not contribute significantly to our study, but only the improper integral.

There are many results concerning existence (and uniqueness) of solutions using for instance iterative methods, contraction arguments, and other infinite-dimensional fixed point techniques, among others (see for instance [17, Ch. 12], [16,18–20]). Let us only comment that unlike the finite delay case, the initial data is always part of the solution. So, there is not a time with immediate regularization, and some kind of regularity must be imposed from the beginning (cf. [1,16–18]). This leads us to work with a canonical phase space:

$$C_\gamma = \{x \in C((-\infty, 0]; \mathbb{R}^m) \ : \ \sup_{\theta \in (-\infty,0]} \mathrm{e}^{\gamma\theta}|x(\theta)| < \infty \text{ and } \exists \lim_{\theta \to -\infty} e^{\gamma\theta} x(\theta)\}, \tag{2}$$

where the parameter $\gamma > 0$ will be determined later on.

The space C_γ is Banach with the norm $\|\psi\|_{C_\gamma} := \sup_{\theta \in (-\infty,0]} \mathrm{e}^{\gamma\theta}|\psi(\theta)|$. Standard results on existence can be posed here naturally (as we recall below). Due to realistic situations as biological models, it will also be used the Banach space C_γ^+, that is, the positive cone of C_γ.

Other choices are also valid, but we will restrict our attention only to this situation just for clarity in the presentation (for more comments see Remark 1 below).

Nevertheless, we would like to mention that the infinite delay case may be quite more suitable from the existence than from the uniqueness point of view, since the right hand side should now incorporate additional assumptions for uniqueness over more particular functional spaces. This will make the multi-valued framework more suitable.

As usual, a priori bounds for possible solutions and bounded map in the right hand side of (1), i.e. that maps bounded sets onto bounded sets, lead to non-explosion of solutions (cf. [18, Ch. 2]) (These are all our considered situations). This implies that solutions are continuable and well defined for all times, and the study of its asymptotic behaviour is sensible. Some essential definitions and results in this sense are described now.

Semiflows and processes for delay differential equations

In order to avoid unnecessary repetitions, we shall first state the results for the non-autonomous case and will particularize later on for the autonomous framework, i.e. without explicit dependence on time.

To construct the dynamical system associated to (1), we need a suitable phase space, for instance C_γ, and a smooth enough right hand side, for example $f \in C(\mathbb{R} \times C_\gamma; \mathbb{R}^m)$. [If f is only Caratheodory we would deal with solutions that ar absolutely continuous; however all through the paper f will be continuous functionals and solutions will be classical]. At least, we can obtain some local results on the existence of solutions to the initial value problem

$$\begin{cases} x'(t) = f(t, x_t), \quad t > \tau \\ x_\tau = \phi, \end{cases} \tag{3}$$

for each $\tau \in \mathbb{R}$ and $\phi \in C_\gamma$; i.e. there exist an interval $[\tau, \tau + \delta)$ and a function $x \in C([\tau, \tau + \delta); \mathbb{R}^m)$ that satisfies the integral equality $x(t) = \phi(0) + \int_\tau^t f(s, x_s)\mathrm{d}s$ for $t \in [\tau, \tau + \delta)$ (e.g. cf. [18, Th.1.1,p.36]). In the sequel we shall use the notations $u(t, \tau, \phi)$ or simply $u(t)$ for the solutions.

Now, if uniqueness of solutions holds, we can construct a (local, i.e. defined for $t \in (\tau, \tau + \delta)$)

two-parameter process $U(t,\tau,\cdot): C_\gamma \to C_\gamma$ as

$$U(t,\tau,\phi) = u_t(\cdot,\tau,\phi),$$

where $u(\cdot,\tau,\phi)$ denotes the unique solution to (3).

However, since we are interested in the study of the long-time behaviour for the problem, we restrict ourselves to deal with solutions that exist for all times (see Remark 4 below).

Remark 1 *Observe that the choice of C_γ as state space makes that any element $x \in C_\gamma$ satisfies that the map $t \mapsto x_t$ is continuous. This fact is not necessary in order to construct our two-parameter semigroup (see Definition 2 below), and to prove the existence of attractors [for that we will use essentially the squeezing weight of the exponential in the queue]. But it is an important point in the existence of solutions.*

Another difficulty that arises in many cases is the fact that we cannot ensure the uniqueness of solutions of problem (3), so in order to construct the most general associated multi-valued dynamical system, we have now to consider all the solutions which are globally defined for positive times associated to each initial datum. If we assume that for every $\tau \in \mathbb{R}$ and $\phi \in C_\gamma$ there exists at least one solution $u(t,\tau,\phi)$ defined for any $t \geq \tau$, then a multi-valued process U can be defined correctly. Namely, let $D(\tau,\phi)$ be the set of all solutions $u(t,\tau,\phi)$ which are defined for $t \geq \tau$. Then we put

$$U(t,\tau,\phi) = \{u_t \ : \ u(\cdot,\tau,\phi) \in D(\tau,\phi)\}.$$

This fits precisely into the following definition (here X denotes an abstract complete metric space; we put this since it will be used not only with C_γ but, for instance, with the positive cone C_γ^+):

Definition 2 *The map $U : \mathbb{R}_d \times X \to P(X)$ is said to be a multi-valued dynamical process (MDP) on X if*
(1) $U(t,t,\cdot) = Id$ (identity map);
(2) $U(t,s,x) \subset U(t,\tau,U(\tau,s,x))$, for all $x \in X$, $s \leq \tau \leq t$,
where $U(t,\tau,U(\tau,s,x)) = \cup_{y \in U(\tau,s,x)} U(t,\tau,y)$.
The MDP U is said to be strict if

$$U(t,s,x) = U(t,\tau,U(\tau,s,x)), \textit{ for all } x \in X, \ s \leq \tau \leq t.$$

It can be proved easily (cf. [9]), just using concatenation and translation of solutions to (3), that U is a strict MDP.

Due to realistic reasons related to the particular models under study (biological, physical, etc.), we may be interested only in the solutions which remain non-negative for all $t \geq \tau$. In such a case we define $D^+(\tau,\phi)$ as the set of all solutions $u(t,\tau,\phi)$ which are defined for $t \geq \tau$ and such that $u_t \in C_\gamma^+$, for all i and $t \geq \tau$. Assuming that for all τ and $\phi \in C_\gamma^+$ such a solution exists, then we can define the map

$$U^+(t,\tau,\phi) = \{u_t : u(\cdot,\tau,\phi) \in D^+(\tau,\phi)\}.$$

When the problem is autonomous, there is no need to mark both initial and final time, but only the elapsed time. This will be usually denoted by $G(t,\psi)$ and called multi-valued semiflow. If the solution of the Cauchy problem is unique, it defines a semigroup in the usual sense.

Although consistency in the definition of a process implies having the same space as initial and final, let us introduce, for convenience of notation in the assumptions, another map:

$$\bar{U}(t,\tau,\psi) = \{u(t,\tau,\psi) \,:\, u(\cdot,\tau,\psi) \in D(\tau,\phi)\}.$$

We also use the analogous notation $\bar{U}^+$ for $D^+(\tau,\phi)$ instead of $D(\tau,\phi)$ above.

Observe that we have no necessity of introducing the auxiliary process

$$\tilde{U}(t,s,(u_0,\psi)) = (u(t,s,(u_0,\psi),u_t(\cdot,s,(u_0,\psi))))$$

as in [10] since we are dealing all the time with continuous functions.

The structure of these processes and semiflows comes at last by the solutions and is stated in the following result. A similar result with finite delay can be found in [9, Prop. 10].

Let us firstly observe that the continuity notion for multi-valued maps is not unique, and the upper semicontinuity is the suitable notion for results on attractors (see below). A multi-valued map $F : X \to P(X)$ is upper semicontinuous if for every $x \in X$ and every neighbourhood M of $F(x)$, there exists a neighbourhood N of x such that $F(y) \subset M$ for any $y \in N$. When the process is single valued, we recover the usual notion of continuity.

We are again back to the space C_γ as phase space of our problem, instead of X.

Proposition 3 *Suppose $f \in C(\mathbb{R} \times C_\gamma; \mathbb{R}^m)$ is bounded and that the differential equation $x'(\tau) = f(\tau, x_\tau)$ generates a MDP U. Assume that $\bar{U}$ is uniformly bounded in the following sense: for every pair $(t,s) \in \mathbb{R}_d$ and $R > 0$, there exists a constant $M(R,s,t) > 0$ such that $\bar{U}(\theta,s,B_{C_\gamma}(0,R)) \subset B_{\mathbb{R}^m}(0,M(R,s,t))$ for all (s,θ) such that $s \le \theta \le t$. Then, $U(t,s,\cdot) : C_\gamma \to P(C_\gamma)$ has compact values and is upper semicontinuous.*

Remark 4 *We point out what may seem to be a duplicity in the hypothesis or an "abuse of notation" in the above statement. As we announced before, we are only concerned with solutions defined globally in time. In order to obtain that in differential problems, it is usual to proceed by a priori estimates on possible solutions. This is represented in the above statement "formally" by the bound for $\bar{U}$ (formal since we have written it with $\bar{U}$ which is composed of solutions). Local existence and continuation results already cited (cf. [16,18]; see also [9, Corollary 6] for the case with finite delay) allow to construct correctly global solutions and therefore to define the MDP U.*

In the applications in which we will restrict ourselves to positive cone of solutions, we will have to do something more than simple a priori estimates, and to prove properly the existence, at least, of one globally defined positive solution.

Proof. Let $\psi \in C_\gamma$ and $t \ge s$ be given. We will see that $U(t,s,\psi)$ is compact. Suppose we have a sequence $\varphi^n \in U(t,s,\psi)$. Let us check we can extract a convergent subsequence.

Indeed, the solutions to the differential problem are $x^n(\tau) = \varphi^n(\tau - t)$ for $\tau \in [s,t]$, and $\psi(\theta) = \varphi^n(\theta + s - t)$ for $\theta \le 0$.

The uniform bound of all x^n on $[s,t]$ and the fact that f is bounded gives that $\{x^n\}$ is an equicontinuous family. Therefore, by the Ascoli-Arzelà theorem we can extract a convergent subsequence $x^{n'} \to x$ in $C([s,t];\mathbb{R}^m)$. And so, extending x suitably by ψ till $-\infty$, the convergence of these elements also holds in C_γ.

Using the continuity and boundedness of f to obtain an upper bound, we can apply the Lebesgue Theorem to

$$x^n(t) - x^n(s) = \int_s^t f(\tau, \varphi_\tau^n)\mathrm{d}\tau$$

to obtain, passing through the limit, an equality for x which is proved to be a solution, as desired.

The upper semicontinuity follows analogously. Indeed, by contradiction, for every M neighbourhood of $U(t,s,x)$ there would exist an element y (close enough to x) such that $U(t,s,y)$ is not contained in M. Consider such a sequence $y^n \to x$ and elements $z^n \in U(t,s,y^n)$ with $z^n \notin M$. We will see that there exists a convergent subsequence $z^{n'} \to z$ which belongs to $U(t,s,x)$, a contradiction. Actually, the arguments are the same that in the first part: the Ascoli-Arzelà Theorem allows to extract the convergent subsequence, and from the equality $z^n(t) - z^n(s) = \int_s^t f(\tau, y_\tau^n)\mathrm{d}\tau$ we can pass to the limit using the Lebesgue Theorem and conclude: $z(t) - z(s) = \int_s^t f(\tau, x_\tau)\mathrm{d}\tau$. ■

It is straightforward to obtain the autonomous version.

Proposition 5 *Suppose $f \in C(C_\gamma;\mathbb{R}^m)$ is bounded and that $x'(\tau) = f(x_\tau)$ generates a semiflow G. Assume that $\bar{U}(\cdot,0,\cdot)$ is uniformly bounded in the following sense: for every $t > 0$ and $R > 0$ there exists a constant $M(R,t) > 0$ such that $\bar{U}(s,0,B_{C_\gamma}(0,R)) \subset B_{\mathbb{R}^m}(0,M(R,t))$ for all $0 \le s \le t$. Then, $G(t,\cdot) : C_\gamma \to P(C_\gamma)$ has compact values and is upper semicontinuous.*

Remark 6 *The above results remain true for U^+, supposed that it is well defined, and the same type of bounds holds for $\bar{U}^+$.*

Remark 7 *It is also useful for the theoretical results exposed below (cf. Theorem 14 and its original version) to observe that a multi-valued map F which is upper semicontinuous and has closed values has closed graph.*

3 Attractors, general results, and infinite delays

Our aim in this section is to expose briefly some of the main results on existence of attractors, forward and pullback, for multi valued semiflows and processes, which generalize and extend the stability studies for dynamical systems. As long as semiflows and processes are not compact, we will only be concerned with asymptotically compact properties and associated results.

Denote by d the metric over X. Let us also denote by $dist\,(A,B)$ the Hausdorff semi-metric, i.e., for given subsets A and B we have

$$dist(A,B) = \sup_{x\in A}\inf_{y\in B} d\,(x,y)\,.$$

Definition 8 *It is said that the set $\mathcal{A} \subset X$ is a global attractor of the multi-valued semiflow G if:*

(1) It is attracting, i.e,

$$dist\left(G\left(t,B\right),\mathcal{A}\right)\rightarrow 0 \text{ as } t\rightarrow +\infty, \text{ for all bounded } B\subset X;$$

(2) $\mathcal{A}$ is negatively semi-invariant, i.e., $\mathcal{A}\subset G\left(t,\mathcal{A}\right)$, for all $t\geq 0$;
(3) It is minimal, that is, for any closed attracting set Y, we have $\mathcal{A}\subset Y$.

In applications it is desirable for the global attractor to be compact and invariant (i.e. $\mathcal{A}=G\left(t,\mathcal{A}\right)$, for all $t\geq 0$), which is usually obtained if the semiflow is strict. This will be the case in this paper.

When the differential equation is non-autonomous and we wish to study its asymptotic behaviour, the above concept is a bit restrictive, and a new formulation, like kernel sections, skew-product flows, cocycle attractors or pullback attractors may be more suitable. We will be concerned with the last one, according to the following definition:

Definition 9 *The family $\{A(t)\}_{t\in\mathbb{R}}$ is said to be a non-autonomous or pullback attractor of the MDP U if:*

(1) $A(t)$ is pullback attracting at time t for all $t\in\mathbb{R}$:

$$dist\left(U\left(t,s,B\right),A(t)\right)\rightarrow 0 \text{ as } s\rightarrow -\infty, \text{ for all bounded } B\subset X;$$

(2) It is negatively invariant, that is,

$$A(t)\subset U(t,s,A(s)), \text{ for any } (t,s)\in\mathbb{R}_d;$$

(3) It is minimal, that is, for any closed set Y attracting at time t, we have $A(t)\subset Y$.

The pullback attracting property considers the state of the system at time t when the initial time s goes to $-\infty$.

In the applications it is also desirable for every $A\left(t\right)$ to be compact (if so, we shall say that the attractor is compact). It would be also of interest to obtain the invariance of $A\left(t\right)$ (i.e. $A(t)=U(t,s,A(s))$). However, in order to prove this we need to assume that the map $U\left(t,s,\cdot\right)$ is lower semicontinuous (cf. [5,6]), which is a strong assumption (although this may be circumvent in a probability framework, cf. [25]).

The main idea behind the attractors rely on two facts: an attraction of each bounded set by another one (the ω-limit set) with "good properties", and, when possible, some kind of absorption towards a unique set that makes every ω-limit may be "reduced" to this last one, as we will see below in the theoretical results.

Naturally, autonomous and non-autonomous cases have different formulations, but the autonomous one can be derived from the non-autonomous case in the standard way: omitting a final time and going to ∞ instead of coming from $-\infty$, and for the sake of brevity some autonomous definitions will be omitted here.

The concepts of (shift) orbit until s and ω-limit set at time t are formulated respectively by:

$$\gamma^s(t,B)=\bigcup_{\tau\leq s}U(t,\tau,B),\quad \omega(t,B)=\bigcap_{s\leq t}\overline{\gamma^s(t,B)}. \tag{4}$$

We have the following result concerning the ω-limit sets:

Theorem 10 *(cf. [6, Th. 6]) Let X be a complete metric space. Let U be a multi-valued process, and suppose that for every $t \in \mathbb{R}$ and every bounded set $B \subset X$ there exists a compact set $D(t, B) \subset X$ such that*

$$\lim_{s \to -\infty} dist(U(t, s, B), D(t, B)) = 0. \tag{5}$$

Then, $\omega(t, B)$ defined by (4) is nonempty, compact and the minimal set attracting B at time t.

The following property is actually an equivalent condition to have a compact set $D(t, B)$ satisfying (5) (see Lemma 8 in [6]):

Definition 11 *The MDP U is called (pullback) asymptotically upper semicompact if for any bounded set $B \subset X$ and for each $t \in \mathbb{R}$, any sequence $\xi^m \in U(t, s^n, B)$, where $s^n \to -\infty$, is precompact.*

The next property says that all the dynamic starting in one element accumulates near one only set (parameterized in t, of course; though the useful concept is for the autonomous version as we will see):

Definition 12 *The MDP U is called (pullback) point dissipative if for any $t \in \mathbb{R}$ there exists a bounded set $B_0(t) \subset X$ such that*

$$\mathrm{dist}(U(t, s, \psi), B_0(t)) \to 0, \ \ \textit{as } s \to -\infty, \ \ \textit{for all}\, \psi \in X.$$

With the above definitions, the main results for existence of attractors which will be valid in our context are as follows (observe that the definition of asymptotic upper semicompactness and statement of the theorems may be slightly different in the cited papers).

The autonomous version (adapting in the usual manner the above definitions to a semiflow) is given by the following:

Theorem 13 *[cf. [26, Th. 3 & Remark 8]] Let G be a pointwise dissipative and asymptotically upper semicompact multi-valued semiflow. Suppose that $G(t, \cdot) : X \to P(X)$ has closed values and is upper semicontinuous for any $t \in \mathbb{R}_+$. Then G has the compact global attractor $\mathcal{A}$. It is minimal among all closed sets attracting each bounded set, and it is invariant: $G(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$.*

It is worth pointing out that there are stronger conditions to ensure existence of attractors, but they are not valid here. Precisely, if one obtains the existence of a bounded absorbing set (or a family of bounded absorbing sets in the non-autonomous case) and has some compactness property of the process, this implies the existence of an attractor. The compactness of the semiflow or process is easy to obtain, for instance, for finite delay differential equations applying the Ascoli-Arzelà Theorem, cf. [9], under bounded maps assumption and after the delay time period, which has no sense here obviously.

The non-autonomous results we will apply are the following:

Theorem 14 *[cf. [6, Th. 11]] Let X be a complete metric space, and U a multi-valued dynamical process with closed values, such that for all $s \leq t$, $U(t, s, \cdot)$ is upper semicontinuous and* *asymptotically upper semicompact process. Then, there exists a pullback attractor given by*

$$A(t) = \bigcup_{B \text{ bounded}} \omega(t, B).$$

It is desirable to have good properties for $\{A(t)\}$, for instance compactness although this uses stronger assumptions. The next result uses condition (5) from Theorem 10 uniformly for every bounded set B :

Theorem 15 *[cf. [6, Th. 18]] Under the same assumptions of Theorem 14, if there exists a compact set $D(t)$ which satisfies for any bounded set $B \subset X$*

$$\lim_{s \to -\infty} dist(U(t, s, B), D(t)) = 0,$$

then the closure of the attractor $A(t)$ obtained in Theorem 14, is a compact attractor.

The following theorem does not use the strong assumption of compactness of the process, which would imply the compactness of the attractor, but still fits to our situation and the above result giving the desired compactness. It is based in the paper [8], although there it is stated in another framework of dynamical systems: tempered sets.

We need previously the following definition:

Definition 16 *A family $B(t)$ is said pullback absorbing for the process U if for every bounded set $B \subset X$, there exists a time $\tau(t, B)$ such that*

$$U(t, s, B) \subset B(t) \quad \forall s \leq \tau(t, B).$$

Theorem 17 *Under the same assumptions of Theorem 14, if there exists a family of absorbing bounded sets $\{B(t)\}_{t \in \mathbb{R}}$ such that*

$$B(s) \subset B(t) \quad \forall\ s \leq t, \tag{6}$$

then the extra assumption in Theorem 15 holds. Indeed, one can take $D(t) = \omega(t, B(t))$, and the attractor from Theorem 14 becomes $A(t) = \omega(t, B(t))$.

Proof. The construction of the attractor is standard:

$$A(t) = \overline{\bigcup_{B \text{ bounded}} \omega(t, B)}.$$

By (4), using the absorbing family $\{B(t)\}$ and the inclusion relation between these sets, one has for any fixed bounded set B, and times $t \geq r$, that there exists a time $\tau(r, B)$ such that for any $s \leq \tau(r, B)$ one has

$$U(t, s, B) \subset U(t, r, U(r, s, B)) \subset U(t, r, B(r)) \subset U(t, r, B(t))$$

and, indeed, $\omega(t, B) \subset \omega(t, B(t))$ for any bounded set B. As $B(t)$ is another bounded set, this proves that $A(t) = \omega(t, B(t))$. ■

Remark 18 (i) *The assumption of increasing family of absorbing bounded sets can be weakened. Indeed, by the proof, one can see that it is enough to have, for each $B(t)$, a sequence of* *positive values $\{r_k(t)\}_{k\in\mathbb{N}}$ increasing to $+\infty$, such that for every $r_k(t)$ the following inclusion holds $B(t-r_k(t))\subset B(t)$.*

(ii) *The compact pullback attractor $\{A(t)\}_t$ has also the following interesting boundedness property: $\bigcup_{t\leq\tau} A(t)\subset B(\tau)$.*

(iii) *Given a family $\{B(t)\}$ of bounded absorbing sets, one could be tempted (to apply the result) to construct the following family: $\tilde{B}(t)=\bigcup_{s\leq t} B(s)$. Of course, this is an increasing family of absorbing sets, as required in the statement of the theorem, but each $\tilde{B}(t)$ does not have to be bounded.*

Asymptotic compactness on delay differential equations

Now let us expose, similarly to the continuous properties of semiflows and processes, one of the properties that will be essential for the construction of attractors. This theoretical presentation, based on some kind of uniform estimates, will be proved particularly for each of the applications.

Proposition 19 *Suppose that f is continuous, bounded and such that U is well defined globally in time. Let $t\in\mathbb{R}$ be given, and assume that $\bar{U}(t,\cdot,\cdot)$ is uniformly bounded in the following sense: for every $t\in\mathbb{R}$ and $R>0$, there exists a constant $M(R,t)>0$ such that $\bar{U}(\theta,s,B_{C_\gamma}(0,R))\subset B_{\mathbb{R}^m}(0,M(R,t))$ for all (s,θ) such that $s\leq\theta\leq t$. Then, U is asymptotically upper semi-compact.*

Proof. Consider the sequences $s^n\to-\infty$ and $\xi^n\in U(t,s^n,\varphi^n)$ with $\varphi^n\in B_{C_\gamma}(0,R)$. We will check that $\{\xi^n\}$ is precompact.

By the assumptions on boundedness of f and $\bar{U}(t,\cdot,\cdot)$, we can apply the Ascoli-Arzelà Theorem on solutions to ensure precompactness on compact intervals of time for $\xi^n|_{[-T,0]}$ for every $T>0$. So, we can obtain a continuous function $\psi:(-\infty,0]\to\mathbb{R}^m$ such that $|\psi(\theta)|\leq M_R$ for all $\theta\leq 0$, and such that a subsequence, relabelled the same, converges uniformly to ψ on $\mathbb{R}^m$ on every interval $[-T,0]$.

Actually, we claim that ξ^n converges to ψ in C_γ. Indeed, we have to see that for every $\epsilon>0$ there exists n_ϵ such that

$$\sup_{\theta\in(-\infty,0]}|\xi^n(\theta)-\psi(\theta)|\mathrm{e}^{\gamma\theta}\leq\epsilon \qquad \forall n\geq n_\epsilon. \tag{7}$$

Fix T_ϵ such that $M_R\mathrm{e}^{-\gamma T_\epsilon}\leq\epsilon/2$, and take n_ϵ such that $t-s^{n_\epsilon}\geq\max\left(T_\epsilon,-\frac{1}{\gamma}\ln\frac{\epsilon}{2R}\right)$. Since the convergence of ξ^n to ψ holds in compact intervals of time, in order to prove (7) we only have to check

$$\sup_{\theta\in(-\infty,-T_\epsilon]}|\xi^n(\theta)-\psi(\theta)|\mathrm{e}^{\gamma\theta}\leq\epsilon \qquad \forall n\geq n_\epsilon.$$

By the uniform bound on ψ and the choice of T_ϵ, it suffices to prove the following:

$$\sup_{\theta\in(-\infty,-T_\epsilon]}|\xi^n(\theta)|\mathrm{e}^{\gamma\theta}\leq\epsilon/2 \qquad \forall n\geq n_\epsilon.$$

We remind that an element ξ^n of the (possibly multi-valued) process U has two parts:

$$\xi^n(\theta) = \begin{cases} \varphi^n(\theta + t - s^n), & \text{if } \theta \in (-\infty, s^n - t), \\ u^n(\theta + t, s^n, \varphi^n), & \text{if } \theta \in [s^n - t, 0], \end{cases}$$

where $u^n \in D(s^n, \varphi^n)$. Thus, the proof is finished if we prove that

$$\max\left\{ \sup_{\theta \in (-\infty, s^n - t)} |\varphi^n(\theta + t_n)| \mathrm{e}^{\gamma\theta}, \sup_{\theta \in [s^n - t, -T_\epsilon]} |u^n(\theta + t, s^n, \varphi^n)| \mathrm{e}^{\gamma\theta} \right\} \leq \epsilon/2.$$

The first term can be bounded as follows:

$$\begin{aligned} \sup_{\theta \leq s^n - t} |\varphi^n(\theta + t - s^n)| \mathrm{e}^{\gamma\theta} &= \sup_{\theta \leq s^n - t} |\varphi^n(\theta + t - s^n)| \mathrm{e}^{\gamma(\theta + t - s^n)} \mathrm{e}^{\gamma(s^n - t)} \\ &\leq R \mathrm{e}^{\gamma(s^n - t)} \leq \frac{\epsilon}{2}, \end{aligned}$$

thanks to the choice of n_ϵ. And finally, the second term is, again by the choice of T_ϵ, less than $\epsilon/2$. ■

The autonomous case follows similarly to the above proof. Actually, it consists essentially of putting $t^n = t - s^n$. For clarity, we give the statement here.

Proposition 20 *Suppose that f is continuous, bounded and has no explicit dependence on time, and is such that G is well defined, and uniformly bounded, that is, for every $R > 0$, there exists a constant $M_R > 0$ such that $\bar{U}(t, 0, B_{C_\gamma}(0, R)) \subset B_{\mathbb{R}^m}(0, M_R)$ for all $t \geq 0$. Then, G is asymptotically upper semi-compact.*

Remark 21 *Again we remind that the above results are true for U^+, supposed that it is well defined, and the same type of bounds holds for $\bar{U}^+$.*

4 Applications

Next, we consider several situations where the theoretical results in Section 3 can be applied. Our first examples will be devoted to autonomous equations where the terms are multiplying themselves as the logistic and Lotka-Volterra models.

A second subsection will be concerned with additive and non-autonomous terms, considering successive weaker situations in some way, which show some restrictions that have to be taken into account on parameters to ensure the existence of an attractor.

In many physical and biological applications the variables x_i have to be non-negative. Hence, we need to define a multi-valued semiflow in a more restrictive phase space that C_γ, namely, as announced before, we will consider

$$C_\gamma^+ = \{\psi \in C_\gamma : \psi_i(s) \geq 0, \text{ for all } i \text{ and } s \leq 0\}.$$

Firstly, we proceed with a result that shows, for a continuous and "positive" function f, that a semiflow composed only of positive solutions defined globally in time can be constructed.

Lemma 22 *Let f be continuous and bounded. Suppose that*

$$f_i(t,\psi) \geq 0, \text{ for all } i,t \text{ and } \psi \in C_\gamma^+ \text{ such that } \psi_i(0) = 0. \tag{8}$$

Then, for any $\psi \in C_\gamma^+$ there exist $A > t_0$ and a solution x to $x'(t) = f(t, x_t)$ with $x_{t_0} = \psi$, such that $x_t \in C_\gamma^+$, for any $t \in [t_0, A]$.

If for any $T_0 > t_0$ there exists $B = B(\psi, T_0, t_0)$ such that for every solution $x \in D^+(t_0,\psi)$ satisfying $x_t \in C_\gamma^+$, $t \in [t_0, T_x)$, $T_x \leq T_0$, one has

$$|x(t)| \leq B, \text{ for all } t \in [t_0, T_x), \tag{9}$$

then there exists at least one global solution such that $x_t \in C_\gamma^+$, for any $t \geq t_0$.

Proof. Define the approximate functions $f^\varepsilon(t,\psi) = f(t,\psi) + \varepsilon d$, $\varepsilon > 0$, where $d = (1, ..., 1)$, which satisfy $f_i^\varepsilon(t,\psi) \geq \varepsilon$, for all i and $\psi \in C_\gamma^+$ such that $\psi_i(0) = 0$. Consider an arbitrary solution $x^\varepsilon(t)$ of the equation $x'(t) = f^\varepsilon(t, x_t)$ corresponding to $x_{t_0} = \psi \in C_\gamma^+$, and defined in $[t_0, A_\varepsilon]$. Suppose that $x^\varepsilon(t)$ is not positive for some t. Let the i component of this solution be the first one such that the respective solution component becomes negative in some interval (t_1, t_2), $x_i(t) \geq 0$, for $t \leq t_1$. Thus, by the continuity of f^ε in C_γ and (8) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i^\varepsilon(t) = f_i^\varepsilon(t, x_t^\varepsilon) > 0, \text{ for } t \in (t_1, t_1 + \delta),$$

which is a contradiction. Hence, $x^\varepsilon(t) \geq 0$, for all $t \in [t_0, A_\varepsilon]$.

We note that following the same lines of the proof of the theorem of existence of solutions [16, Theorem 2.1] one can choose an interval $[t_0, A]$ such that $x^\varepsilon(t)$ are defined for all $\varepsilon > 0$, $t \in [t_0, A]$. Also, there exists $\eta > 0$ such that

$$|x^\varepsilon(t)| \leq \eta, \text{ for all } \varepsilon > 0, t \in [t_0, A].$$

The boundedness of f implies that the functions $\frac{d}{\mathrm{d}t} x^\varepsilon(t)$ are uniformly bounded in $C([t_0, A]; \mathbb{R}^m)$, so that by the Ascoli-Arzelà theorem there exists a converging subsequence x^{ε_n}.The limit function $x(t)$ is a solution of $x'(t) = f(t, x_t)$ defined on $[t_0, A]$ (see [16, Lemma 2.3]). Clearly, $x(t) \geq 0$ for all $t \in [t_0, A]$, so that $x_t \in C_\gamma^+$ as we wanted to prove.

For the second statement, let for any $T_0 > t_0$ there exists $B(\psi, T_0)$ such that (9) holds. Suppose that a global solution such that $x_t \in C_\gamma^+$, for any $t \geq t_0$, does not exist. Then for any solution $x(t)$ one can choose the maximal interval $[t_0, T_x)$ where $x_t \in C_\gamma^+$ and $T_x < +\infty$. Observe that condition (9) implies that $x(t)$ cannot blow up and cannot oscillate at T_x (by the boundedness of f). Then, the limit of x in T_x exists. By continuation of solutions (cf. [16, Theorem 2.3]), $x(t)$ exists on $[t_0, T_x + \delta]$ but for some i we have $x_i(T_x) = 0$ and $x_i(t) < 0$, for $t \in (T_x, T_x + \delta)$. Let

$$T_0 = \sup \{T_x : x(\cdot) \text{ is solution with } x_{t_0} = \psi\}.$$

We state that $T_0 = +\infty$. If not, we can choose an increasing sequence T_{x_n} such that $T_{x_n} \to T_0$. Consider any $T_m \in (t_0, T_0)$. Using (9), the boundedness of f and the Ascoli-Arzelà theorem, one can prove that up to a subsequence $x_n(\cdot)$ converges to some function $x(\cdot)$ in $C([t_0, T_m], \mathbb{R}^m)$ and $x(\cdot)$ is a solution [16, Lemma 2.3]. Taking $T_m \to T_0$ and using a diagonal argument we obtain a solution $x(t)$ such that $x_t \in C_\gamma^+$ for all $t \in [t_0, T_0)$.

We use again a continuation argument as before, so we can extend the solution to $x(T_0)$. Then, we have that $x(t)$ can be continued to a solution defined on $[t_0, T_0+\delta)$, $\delta > 0$, with $x_t \in C_\gamma^+$ (by the first claim), a contradiction.

Let now $T_{x_n} \to +\infty$. Repeating the previous argument with $T_m \to +\infty$ we obtain a global solution $x(t)$ such that $x_t \in C_\gamma^+$, for any $t \geq t_0$. ■

Remark 23 *In the next paragraph, the natural phase space for the problem will be C_γ^+. Bearing in mind Remark 6, Proposition 5, Proposition 20 and Remark 21, the rest of theoretical results will be valid for U^+, the MDP of globally defined solutions which are positive in all components, and $\bar{U}^+$, instead of U and $\bar{U}$.*

4.1 Logistic equation

Consider the delayed logistic equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = rx(t)\left(1 - K^{-1}\displaystyle\int_{-\infty}^{0} w(s)\, P(x(s+t))\,\mathrm{d}s\right), \\ \qquad\qquad x_0 = \psi, \end{cases} \tag{10}$$

where $x(t) \geq 0$, $r > 0$, $P \in C(\mathbb{R};\mathbb{R})$, $P(x) \geq 0$ if $x \geq 0$, $w \in C(\mathbb{R};\mathbb{R}^+)$ and there exist $C_i, L > 0$, $\zeta \geq 1$, such that for all $x \in \mathbb{R}$

$$\begin{aligned} |P(x)| &\leq C_1 |x|^\zeta + C_2, \\ P(x) &\geq L|x|. \end{aligned} \tag{11}$$

Moreover, we assume that

$$\int_{-\infty}^{0} w(s)\,\mathrm{e}^{-\eta s}\mathrm{d}s < \infty, \tag{12}$$

for some $\eta > 0$. In particular, this implies that $\int_{-\infty}^{0} w(s)\,\mathrm{d}s < \infty$.

Remark 24 *This model contains as a particular case the standard logistic equation, where $P(x) = x$ (cf. [27, Sec.3.1]).*

We state that the function $M : C_\gamma \to \mathbb{R}$,

$$\psi \mapsto M(\psi) = \int_{-\infty}^{0} w(s)\, P(\psi(s))\,\mathrm{d}s$$

is continuous on C_γ if $\gamma = \frac{\eta}{\zeta}$. Indeed, let $\varepsilon > 0$ be arbitrary. Then in view of (12) for any $D > 0$ there exists $K = K(D, \varepsilon) > 0$ such that for all $\psi \in C_\gamma$ with $\|\psi\|_{C_\gamma} \leq D$ one has

$$\begin{aligned} \int_{-\infty}^{-K} w(s)\,|P(\psi(s))|\,\mathrm{d}s &= \int_{-\infty}^{-K} w(s)\,\mathrm{e}^{-\eta s}\,|P(\psi(s))|\,\mathrm{e}^{\eta s}\mathrm{d}s \\ &\leq \int_{-\infty}^{-K} w(s)\,\mathrm{e}^{-\eta s}\left(C_1\left|\psi(s)\,\mathrm{e}^{\frac{\eta}{\zeta}s}\right|^\zeta + C_2\mathrm{e}^{\eta s}\right)\mathrm{d}s \\ &\leq \left(C_1 D^\zeta + C_2\right)\int_{-\infty}^{-K} w(s)\,\mathrm{e}^{-\eta s}\mathrm{d}s < \frac{\varepsilon}{3}. \end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_{-\infty}^{0} w(s)\, P(\psi_1(s))\, \mathrm{d}s - \int_{-\infty}^{0} w(s)\, P(\psi_2(s))\, \mathrm{d}s\right| \\
&\leq \int_{-K}^{0} w(s)\, |P(\psi_1(s)) - P(\psi_2(s))|\, \mathrm{d}s \\
&\quad + \int_{-\infty}^{-K} w(s)\, (|P(\psi_1(s))| + |P(\psi_2(s))|)\, \mathrm{d}s < \varepsilon,
\end{aligned}$$

if $\|\psi_1 - \psi_2\|_{C_\gamma} < \delta(\varepsilon, D)$ and $\|\psi_i\|_{C_\gamma} \leq D$.

Therefore, the map $f : C_\gamma \to \mathbb{R}$ defined by

$$f(\varphi) = r\varphi(0)\left(1 - K^{-1}\int_{-\infty}^{0} w(s)\, P(\varphi(s))\, \mathrm{d}s\right)$$

is continuous and the existence of a local solution to (10) is guaranteed (cf. [18, Th.1.1,p.36]). Boundedness of f can be proved analogously. This leads us to the following result.

Lemma 25 *For any $\psi \in C_\gamma^+$ there exists at least one global solution such that $x_t \in C_\gamma^+$, for any $t \geq 0$.*

Proof. First, multiplying (10) by an arbitrary solution $x(t)$ such that $x_t \in C_\gamma^+$ in $[0, T_x)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \leq 2r\,|x(t)|^2\,,$$

so that, by the Gronwall lemma,

$$|x(t)|^2 \leq |\psi(0)|^2 \exp(2rT_0)\,, \text{ for all } 0 \leq t < T_x \leq T_0, \tag{13}$$

where $\psi = x_0$, so that condition (9) holds. It is evident that condition (8) is satisfied. Hence, the result follows from Lemma 22. ■

Lemma 25 implies that a multi-valued semiflow $G^+ : C_\gamma^+ \to P(C_\gamma^+)$ can be well defined in the way we have explained before.

Lemma 26 *For any bounded set B in C_γ^+ we have that*

$$\bigcup_{t\geq 0} G^+(t, B)$$

is bounded. Moreover, there exists a bounded absorbing set for G^+.

Proof. Let $x(t)$ be an arbitrary global solution such that $x_t \in C_\gamma^+$ for $t \geq 0$. Denote $c = \int_{-\infty}^{0} w(s)\, \mathrm{d}s$ and $b = c - \varepsilon$, $\varepsilon > 0$. Let $\psi(0) \leq \frac{K}{bL}$. Then (13) gives

$$|x(t)| \leq R(T_0) = \frac{K}{bL}\exp(rT_0)\,, \text{ for all } 0 \leq t \leq T_0. \tag{14}$$

We shall prove that for some $T_0 > 0$ the set $B_0 = \left\{x : \|x\|_{C_\gamma} \leq R(T_0)\right\}$ is absorbing.

Observe that T_0 can be chosen such that

$$\int_{-T_0}^{0} w(s)\,\mathrm{d}s \geq b.$$

We do so, and claim that for such T_0 we have that $|x(t)| \leq R(T_0)$, for all $t \geq 0$, provided that $|\psi(0)| \leq \frac{K}{bL}$.

Suppose the opposite. Then, since the problem is autonomous, using (14), and a shift if necessary, there exist a solution $x(t)$, $T_1(\psi) \geq 0$, and times T_2 and T_3 with $T_3 > T_2 \geq T_1 + T_0$ such that

$$\begin{aligned} |x(T_1)| &= \frac{K}{bL}, \qquad |x(t)| > \frac{K}{bL} \quad \text{for } t \in (T_1, T_2], \\ |x(T_2)| &= R(T_0), \qquad |x(t)| > R(T_0) \quad \text{for } t \in (T_2, T_3]. \end{aligned}$$

We note that if $t \geq T_2$, then $s + t \geq T_1$, for all $s \in [-T_0, 0]$, so that using (10)-(11) we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 &\leq r|x(t)|^2\left(1 - K^{-1}L\int_{-\infty}^{0} w(s)|x(s+t)|\,\mathrm{d}s\right) \\ &< r|x(t)|^2\left(1 - \frac{1}{b}\int_{-T_0}^{0} w(s)\,\mathrm{d}s\right), \end{aligned}$$

for $t \in (T_2, T_3)$. Now, from the choice of T_0, we deduce that the derivative of $|x(t)|^2$ is negative on (T_2, T_3) and we have a contradiction. This implies that the set B_0 is absorbing for any bounded B set such that $|\psi(0)| \leq \frac{K}{bL}$, for all $\psi \in B$. Indeed, using that

$$\sup_{\theta \in (-\infty, -t]} \mathrm{e}^{\gamma(t+\theta)}|x(t+\theta)| = \|\psi\|_{C_\gamma}$$

for any solution $x(t)$, we have

$$\begin{aligned} \|x_t\|_{C_\gamma} &= \sup_{\theta \in (-\infty, 0]} \mathrm{e}^{\gamma\theta}|x(t+\theta)| \\ &= \max\left\{\sup_{\theta \in (-\infty, -t]} \mathrm{e}^{\gamma\theta}|x(t+\theta)|, \sup_{\theta \in (-t, 0]} \mathrm{e}^{\gamma\theta}|x(t+\theta)|\right\} \\ &\leq \max\left\{\mathrm{e}^{-\gamma t}\|\psi\|_{C_\gamma}, R(T_0)\right\} \leq R(T_0), \end{aligned} \tag{15}$$

if $t \geq T(B)$.

Consider now an arbitrary bounded set B. Let us prove the existence of $T(B)$ such that any solution starting in B satisfies $|x(t)| \leq R(T_0)$, for all $t \geq T(B)$. We have to consider only the case $|\psi(0)| > \frac{K}{bL}$. If we suppose the opposite, then there exists a sequence $x_k(t_k)$, $t_k \to +\infty$, such that $|x_k(t_k)| > R(T_0)$. We note that T_0 can be chosen such that $R(T_0) > \frac{K}{bL}$, so that it follows from the previous results that $|x_k(t)| > \frac{K}{bL}$, for all $t \in [0, t_k]$ (if not, we could apply again the above argument in such a point). Since $t_k > T_0$, for $k \geq k_0$, arguing as before we obtain that the solutions are decreasing on $[T_0, t_k]$, so that $|x_k(t)| > R(T_0)$, for all $t \in [T_0, t_k]$. We note that $s + t \geq T_0$ if $t \geq 2T_0$ and $s \geq -T_0$. Hence,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x_k(t)|^2 &\leq r|x_k(t)|^2\left(1 - K^{-1}L\int_{-T_0}^{0} w(s)|x_k(s+t)|\,\mathrm{d}s\right) \\ &\leq r|x_k(t)|^2\left(1 - K^{-1}bLR(T_0)\right) = -\alpha|x_k(t)|^2, \ \alpha > 0, \end{aligned}$$

for $t \in [2T_0, t_k]$, so that $|x_k(t_k)| \leq |x_k(2T_0)| \mathrm{e}^{-\alpha(t_k - 2T_0)}$ and then in view of (13) there exists k such that $|x_k(t_k)| < R(T_0)$, a contradiction. Arguing in a similar way as in (15), we obtain that the set B_0 is absorbing.

Furthermore, using an analogous expression to (13) in the time interval $[0, T_B]$, we have a uniform bound. Thus, we have obtained also that $\bigcup_{t \geq 0} G^+(t, B)$ is bounded. ∎

As a consequence of Lemma 26, Remark 21, Theorem 13 and Remark 6, we have:

Theorem 27 *The semiflow G^+ has a compact global invariant attractor.*

4.2 Lotka-Volterra equations

Consider the following predator-prey system with a possibly saturating predator [23, p.283]:

$$\begin{cases} x_1'(t) = x_1(t)\left(a - bx_1(t) - cx_2(t)\right), \\ x_2'(t) = x_2(t)\left(-d + \int_{-\infty}^{0} K(s) \dfrac{x_1(t+s)}{\lambda + vx_1(t+s)} \mathrm{d}s\right), \end{cases}$$

where $x_i(t) \geq 0$, a, b, c, d, λ are positive constants, v is a nonnegative constant, and $K : (-\infty, 0] \to [0, \infty)$ is also nonnegative, continuous and such that for some $\gamma > 0$ we have

$$\int_{-\infty}^{0} \mathrm{e}^{-\gamma\tau} K(\tau)\, \mathrm{d}\tau < \infty, \tag{16}$$

$$\int_{-\infty}^{0} \mathrm{e}^{-\gamma s} \int_{-\infty}^{s} K(\tau)\, \mathrm{d}\tau \mathrm{d}s < \infty.$$

Arguing as in the logistic equation one can prove that the function $M : C_\gamma \to \mathbb{R}$ defined by

$$\psi \mapsto M(\psi) = \int_{-\infty}^{0} K(s) \frac{\psi(s)}{\lambda + v\psi(s)} \mathrm{d}s$$

is continuous.

Moreover, since the function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\rho(\theta) = \theta/(\lambda + v\theta)$ is Lipschitz (its derivative is bounded by 1), multiplying by $\mathrm{e}^{\gamma s}\mathrm{e}^{-\gamma s}$ and using (16), we deduce that M is also Lipschitz.

As in the logistic case, one can see that M is also bounded. Then, the Cauchy problem has a solution, unique by the Lipschitz character (cf. [17, Ch. 2, Th. 2.3] or [16,18]).

In this case, the phase space is C_γ^+, the positive cone of C_γ defined in (2) with $m = 2$. Denote $x = (x_1, x_2)$. Then it can be proved following exactly the same lines as in the proof of [4, Lemma 5.1] that for any $B > 0$ there exists B_1 such that an estimate for the initial data $\|\psi\| \leq B$ gives a uniform estimate for any solution $|x(t)| \leq B_1$, for all $t \geq 0$. Hence, the solutions are globally bounded and, since Lemma 22 can be applied, the semigroup $G^+ : C_\gamma^+ \to P(C_\gamma^+)$ is well defined.

Also, there is a constant $R > 0$ such that for any $B > 0$ there exists $T(B)$ for which $\|\psi\| \leq B$ implies $|x(t)| \leq R$, for all $t \geq T$. Hence, arguing as in (15), we obtain that the set $B_0 = \left\{x : \|x\|_{C_\gamma} \leq R\right\}$ is absorbing.

Again, as a consequence of Remark 21, Theorem 13 and Remark 6, we have:

Theorem 28 *The semigroup G^+ has a compact global invariant attractor.*

4.3 Strong dissipative conditions with and without sublinear terms

In this section we are concerned with two different results. Indeed we first consider a strong dissipative equation without sublinear terms. This will provide an easy proof of boundedness without restrictions in the choice of the phase space. A second result is concerned with sublinear terms added to the dissipativity conditions. The way used to obtain estimates will imply stronger assumptions on the parameter conditions, which will be weakened in the next section.

Case 1: Strong dissipativity without sublinear terms

Observe that the following result is very restrictive, since we are imposing a condition which points out a predominant importance of the final state over the rest of the delay.

Proposition 29 *If $f : \mathbb{R} \times C_\gamma \to \mathbb{R}^m$ is such that the MDP U associated to the equation $x'(\tau) = f(\tau, x_\tau)$ is well defined, and satisfies*

$$\langle f(t,\varphi), \varphi(0)\rangle \leq -\alpha|\varphi(0)|^2 + \beta \qquad \forall \varphi \in C_\gamma \tag{17}$$

for some $\alpha > 0$ and $\beta \geq 0$, then U is eventually bounded in both forward and pullback senses, that is, there exists a bounded set $B_0 \subset C_\gamma$ such that for any bounded set $B \subset C_\gamma$ if s is fixed, there exists $t_0(s,B)$ such that for any $t \geq t_0(s,B)$ [or analogously, if t is fixed, there exists $s_0(t,B)$ such that for any $s \leq s_0(t,B)$] the following inclusion holds:

$$U(t,s,B) \subset B_0.$$

Proof. Consider an arbitrary solution to the Cauchy problem $x'(t) = f(t, x_t)$ with $x_s = \varphi \in B = B_{C_\gamma}(0,d)$. Let $\tau \in (s,t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|x(\tau)|^2 \leq -2\alpha|x(\tau)|^2 + 2\beta,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{2\alpha\tau}|x(\tau)|^2\right) \leq 2\mathrm{e}^{2\alpha\tau}\beta.$$

Therefore, integrating over $[s,\tau]$ with $\tau \in [s,t]$:

$$\begin{aligned}
e^{2\alpha\tau}|x(\tau)|^2 &\leq e^{2\alpha s}|x(s)|^2 + 2\beta\int_s^\tau e^{2\alpha\rho}d\rho \\
&= e^{2\alpha s}|\varphi(0)|^2 + \frac{\beta}{\alpha}\left(e^{2\alpha\tau} - e^{2\alpha s}\right).
\end{aligned}$$

Notice that we are interested in checking the norm of the arbitrary solution, x, in the phase space C_γ. Hence:

$$\sup_{\theta\in(-\infty,0]} e^{2\gamma\theta}|x(t+\theta,s,\varphi)|^2 = \max\left(\sup_{\theta\in(-\infty,s-t]} e^{2\gamma\theta}|\varphi(\theta+t-s)|^2, \sup_{\theta\in(s-t,0]} e^{2\gamma\theta}|x(\theta+t)|^2\right).$$

Observe that the first term on the right hand side is bounded:

$$e^{2\gamma\theta}|\varphi(\theta+t-s)|^2 = e^{2\gamma(s-t)}e^{2\gamma(\theta+t-s)}|\varphi(\theta+t-s)|^2 \leq d^2 e^{2\gamma(s-t)}.$$

For the second term, denote again $\tau = \theta + t$. Now we use the estimation on $x(\tau)$: depending wether $\gamma \leq \alpha$ or not, one gets different expressions for the supreme, but both can be written as follows:

$$\sup_{\tau\in[s,t]} e^{-2\gamma t}e^{2\gamma\tau}|x(\tau)|^2 \leq e^{2(s-t)\min(\gamma,\alpha)}d^2 + \frac{\beta}{\alpha}\left(1 - e^{2(s-t)\max(\gamma,\alpha)}\right).$$

Thus, the ball $B_{C_\gamma}(0, \frac{\beta}{\alpha})$ is absorbing in both senses: forward ($t \to +\infty$ and s fixed) and pullback ($s \to -\infty$ and t fixed). ■

As a direct consequence of Proposition 29, Proposition 3, Proposition 19, and Theorem 17, we obtain

Theorem 30 *Let $f \in C(\mathbb{R}\times C_\gamma;\mathbb{R}^m)$ be bounded and satisfies the condition (17). Then, $x'(\tau) = f(\tau, x_\tau)$ defines correctly a MDP U and there exists a compact pullback attractor.*

Remark 31

(i) Actually, the above case is rather restrictive in the sense that the classical notion of global attractor for non-autonomous dynamical systems is also suitable (as considered by Chepyzhov and Vishik [12]). [For the sake of brevity we do not extend more here.]

(ii) The above situation also admits slight modifications, as to allow non-autonomous growth, for example using a function $\beta = \beta(t)$ with suitable growth [a sufficient condition is $\beta \in L^1(\mathbb{R})$; see also Remark 33 (ii)]. Nevertheless, this framework is essentially restrictive, and says that the effect of the delay is not very significant in comparison with the present time. For instance, an example is given by the following:

$$f : \mathbb{R}\times C_\gamma \to \mathbb{R}^m : (t,\varphi) \mapsto \bar{f}(t,\varphi(0)) + \int_{-\infty}^0 b(s,\varphi(s))\mathrm{d}s$$

where the dissipativity effect is given by $\bar{f} \in C(\mathbb{R}\times\mathbb{R}^m;\mathbb{R}^m)$, which satisfies $\langle \bar{f}(t,x), x\rangle \leq -\alpha|x|^2 + \bar{\beta}$, and, for example, $b : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ Carathéodory (measurable in s, continuous in x) and satisfying $|b(s,x)| \leq g(s)$ for all $x \in \mathbb{R}^m$ being $g \in L^1((-\infty,0))$.

However, simple but important examples as the pantograph equation, $x'(t) = ax(t) + bx(qt)$ for $t \geq 0$ with $0 < q < 1$, which is linear, do not fall within and then are not allowed in this case. So we extend the above result to deal with a sublinear term in a similar way to [9, Theorem 35]. In any case, it is remarkable that, even with this extension, the pantograph example cannot be handled; actually it will be stated separately in a forthcoming paper with other general situations.

Case 2: Nonlinearities with sublinear and non-autonomous terms

Consider the equation

$$x'(t) = F_0(t, x(t)) + F_1(t, x(t - \rho(t))) + \int_{-\infty}^{0} b(t, s, x(t + s))\mathrm{d}s \tag{18}$$

with $F_0, F_1 \in C(\mathbb{R} \times \mathbb{R}^m; \mathbb{R}^m)$, $h > 0$, $\rho \in C^1(\mathbb{R}; [0, h])$, and $b : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ continuous on its first and third variables, measurable w.r.t. the second variable, and satisfying the following conditions:

(A1) There exist positive functions $y = m_0(r)$ and $y = m_1(r)$ such that $m_0, \mathrm{e}^{-\gamma r} m_1(r) \in L^1((-\infty, 0))$ such that

$$|b(t, s, x)| \leq m_0(s) + m_1(s)|x|, \qquad \forall t \in \mathbb{R}.$$

We will denote

$$m_0 = \int_{-\infty}^{0} m_0(s)\mathrm{d}s \quad \text{and} \quad m_1 = \int_{-\infty}^{0} \mathrm{e}^{-\gamma s} m_1(s)\mathrm{d}s.$$

(A2) There exist positive constants k_1, k_2, α and a positive function β such that

$$\langle x, F_0(t, x)\rangle \leq -\alpha|x|^2 + \beta(t), \qquad \forall t \in \mathbb{R},\ x \in \mathbb{R}^m,$$

$$|F_1(t, x)|^2 \leq k_1^2 + k_2^2|x|^2, \qquad \forall t \in \mathbb{R},\ x \in \mathbb{R}^m.$$

Additionally, we suppose that $|\rho'(t)| \leq \rho_* < 1$ for all $t \in \mathbb{R}$.

Equation (18) is a unified way to treat at the same time fixed, variable and distributed delays such that our model is valid for each of above situations separately and mixed (e.g. cf. [23] and the reference therein or [9] for some examples with finite delays, and [18, Ch.8,Sec.5] and the references therein for the infinite delay case).

Analogously to the above cases, one can see that $f(t, x_t)$, the right hand side of (18), is continuous and bounded. Local existence of solutions is guaranteed (cf. [18, Th.1.1,p.36]).

Theorem 32 *Consider equation (18) with the above assumptions (A1) and (A2), and suppose that there exists $\lambda > 0$ such that*

$$\lambda - 2\alpha + 2k_2 \left(\frac{\mathrm{e}^{\lambda h}}{1 - \rho_*}\right)^{1/2} < 0 \qquad \textit{and} \tag{19}$$

$$2m_1 < \lambda \leq 2\gamma. \tag{20}$$

Then, the process $U : \mathbb{R}_d \times C_\gamma \to C_\gamma$ is well defined and it is eventually bounded in the following sense:
a) forward, that is, there exists a family of bounded sets $\{B(t_0)\}_{t_0 \in \mathbb{R}}$ such that for every bounded set $B \subset C_\gamma$ and $t_0 \in \mathbb{R}$, there exists $\tau(t_0, B)$ such that $U(t, t_0, B) \subset B(t_0)$ if $t \geq \tau(t_0, B)$, provided the following condition holds:

$$\sup_{t \in [t_0, +\infty)} \int_{t_0}^{t} \mathrm{e}^{(2m_1 - \lambda)(t - r)} \beta(r)\mathrm{d}r < \infty \qquad \forall t_0 \in \mathbb{R}. \tag{21}$$

b) pullback, that is, there exists a family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ such that for every bounded set $B \subset C_\gamma$ and $t \in \mathbb{R}$, there exists $\tau(t, B)$ such that $U(t, t_0, B) \subset B(t)$ if $t_0 \leq \tau(t, B)$, if the following condition holds:

$$\int_{-\infty}^{t} \mathrm{e}^{(2m_1-\lambda)(t-r)}\beta(r)\mathrm{d}r < \infty \qquad \forall t \in \mathbb{R}. \tag{22}$$

Remark 33 (i) *The gap condition involving γ and m_1 in the statement of the theorem is satisfied, for instance, by $m_1(r) = C(\gamma)\frac{\mathrm{e}^{\gamma r}}{1+(-r)^a}$ with $a > 1$ and $C(\gamma) = \frac{\gamma}{2}\left(\int_{-\infty}^{0} \frac{1}{1+(-s)^a}\mathrm{d}s\right)^{-1}$.*

(ii) *Since $\mathrm{e}^{(2m_1-\lambda)(t-s)} \leq 1$ in the integral, stronger but simpler conditions than (21) and (22) are, respectively, that the integrals of β in $[t, \infty)$ and $(-\infty, t]$ are finite for any t, or more generally: $\beta \in L^1(\mathbb{R})$.*

Proof. We start by proving a priori bounds of possible solutions (so in the end we will obtain that U is well defined): consider a time t_0 and a bounded data $\varphi \in B = B_{C_\gamma}(0, d)$. We will prove that $U(t, t_0, B)$ comes into a bounded set of C_γ according to the conditions of the theorem.

For an arbitrary solution of (18), using Young's inequality and (A2), we have the following estimate (here λ and ε are positive constants to be determined below):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}|x(t)|^2\right) &= \lambda \mathrm{e}^{\lambda t}|x(t)|^2 + 2\mathrm{e}^{\lambda t}\langle x(t), F_0(t, x(t)) + F_1(t, x(t-\rho(t)))\rangle \\ &\quad +2\mathrm{e}^{\lambda t}\langle x(t), \int_{-\infty}^{0} b(t, s, x(t+s))\mathrm{d}s\rangle \\ &\leq (\lambda - 2\alpha + \varepsilon)\mathrm{e}^{\lambda t}|x(t)|^2 + 2\mathrm{e}^{\lambda t}\beta(t) + \frac{1}{\varepsilon}\mathrm{e}^{\lambda t}\left(k_1^2 + k_2^2|x(t-\rho(t))|^2\right) \\ &\quad +2\mathrm{e}^{\lambda t}\langle x(t), \int_{-\infty}^{0} b(t, s, x(t+s))\mathrm{d}s\rangle.\end{aligned}$$

The term with finite delay will be treated by using a Gronwall inequality in integral form. So, first we integrate between t_0 and t :

$$\begin{aligned}\mathrm{e}^{\lambda t}|x(t)|^2 &\leq \mathrm{e}^{\lambda t_0}|x(t_0)|^2 + (\lambda - 2\alpha + \varepsilon)\int_{t_0}^{t} \mathrm{e}^{\lambda r}|x(r)|^2\mathrm{d}r + 2\int_{t_0}^{t} \mathrm{e}^{\lambda r}\beta(r)\mathrm{d}r \\ &\quad +\frac{1}{\varepsilon}\int_{t_0}^{t} \mathrm{e}^{\lambda r}\left(k_1^2 + k_2^2|x(r-\rho(r))|^2\right)\mathrm{d}r \\ &\quad +2\int_{t_0}^{t} \mathrm{e}^{\lambda r}\langle x(r), \int_{-\infty}^{0} b(r, s, x(r+s))\mathrm{d}s\rangle \mathrm{d}r.\end{aligned} \tag{23}$$

We apply a change of variables, $r - \rho(r) = u$, in the integral with the finite variable delay. Separating the part with the initial condition (introducing appropriate exponential terms), we have

$$\begin{aligned}\int_{t_0}^t \mathrm{e}^{\lambda r}|x(r-\rho(r))|^2\mathrm{d}r \le& \frac{\mathrm{e}^{\lambda h}}{1-\rho_*}\int_{t_0-h}^t \mathrm{e}^{\lambda u}|x(u)|^2du\\ =& \frac{\mathrm{e}^{\lambda h}}{1-\rho_*}\left(\int_{t_0-h}^{t_0}\mathrm{e}^{\lambda u\pm 2\gamma(t-t_0)}|x(u)|^2du+\int_{t_0}^t \mathrm{e}^{\lambda u}|x(u)|^2du\right)\\ \le& \frac{\mathrm{e}^{\lambda h}d^2}{1-\rho_*}\int_{t_0-h}^{t_0}\mathrm{e}^{\lambda u-2\gamma(u-t_0)}du+\frac{\mathrm{e}^{\lambda h}}{1-\rho_*}\int_{t_0}^t \mathrm{e}^{\lambda u}|x(u)|^2du.\end{aligned}$$

Before combining this with (23), we estimate the infinite delay term similarly:

$$\begin{aligned}&\int_{t_0}^t \mathrm{e}^{\lambda r}\langle x(r),\int_{-\infty}^0 b(r,s,x(r+s))\mathrm{d}s\rangle\mathrm{d}r\\ \le& \int_{t_0}^t \mathrm{e}^{\lambda r}|x(r)|\int_{-\infty}^0 (m_0(s)+m_1(s)|x(r+s)|)\mathrm{d}s\mathrm{d}r\\ \le& m_0\int_{t_0}^t \mathrm{e}^{\lambda r}|x(r)|\mathrm{d}r+m_1\int_{t_0}^t \mathrm{e}^{\lambda r}|x(r)|\|x_r\|_{C_\gamma}\mathrm{d}r.\end{aligned}$$

Using Young's inequality (with another arbitrary positive constant $\bar{\varepsilon}$) for the first term in the right hand side, and the trivial bound $|x(s)|\le\|x_s\|_{C_\gamma}$ on the second term, and joining with the last estimate, we deduce from (23) that

$$\begin{aligned}\mathrm{e}^{\lambda t}|x(t)|^2 \le& \mathrm{e}^{\lambda t_0}|x(t_0)|^2+\left(\lambda-2\alpha+\varepsilon+\frac{\mathrm{e}^{\lambda h}k_2^2}{\varepsilon(1-\rho_*)}+\bar{\varepsilon}\right)\int_{t_0}^t \mathrm{e}^{\lambda r}|x(r)|^2\mathrm{d}r\\ &+2\int_{t_0}^t \mathrm{e}^{\lambda r}\beta(r)\mathrm{d}r+\left(\frac{k_1^2}{\varepsilon\lambda}+\frac{m_0^2}{\bar{\varepsilon}\lambda}\right)\left(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda t_0}\right)\\ &+\frac{k_2^2d^2\mathrm{e}^{\lambda t_0}(\mathrm{e}^{\lambda h}-\mathrm{e}^{2\gamma h})}{\varepsilon(1-\rho_*)(\lambda-2\gamma)}+2m_1\int_{t_0}^t \mathrm{e}^{\lambda r}\|x_r\|_{C_\gamma}^2\mathrm{d}r.\end{aligned}$$

Taking the less restrictive (minimal) choice $\varepsilon=k_2\left(\frac{\mathrm{e}^{\lambda h}}{1-\rho_*}\right)^{1/2}$ (for convenience we will keep denoting it by ε) we can neglect one term since by (19)

$$\lambda-2\alpha+\varepsilon+\frac{\mathrm{e}^{\lambda h}k_2^2}{\varepsilon(1-\rho_*)}+\bar{\varepsilon}<0$$

if $\bar{\varepsilon}$ is chosen small enough. Thus, we conclude that

$$\begin{aligned}\mathrm{e}^{\lambda t}|x(t)|^2 \le& \mathrm{e}^{\lambda t_0}|x(t_0)|^2+2\int_{t_0}^t \mathrm{e}^{\lambda r}\beta(r)\mathrm{d}r+\left(\frac{k_1^2}{\varepsilon\lambda}+\frac{m_0^2}{\bar{\varepsilon}\lambda}\right)\left(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda t_0}\right)\\ &+\frac{k_2^2d^2\mathrm{e}^{\lambda t_0}(\mathrm{e}^{\lambda h}-\mathrm{e}^{2\gamma h})}{\varepsilon(1-\rho_*)(\lambda-2\gamma)}+2m_1\int_{t_0}^t \mathrm{e}^{\lambda r}\|x_r\|_{C_\gamma}^2\mathrm{d}r. \qquad (24)\end{aligned}$$

We would like to observe specially that the natural way, that is, to substitute now t by $t+\theta$ to obtain a useful bound for $\|x_t\|$, has to be done carefully. Similar estimates to [9] are misleading here since $\theta\in(-\infty,0]$ does not allow easy estimates (in comparison with the case $\theta\in[-h,0]$).

The additional assumption (20) has been imposed on the phase space to overcome the cited difficulty. We use the extra assumption $\lambda\le 2\gamma$ and so $\mathrm{e}^{(2\gamma-\lambda)\theta}\le 1$ for $\theta\le 0$. Multiplying (24) by $e^{2\gamma\theta}e^{-2\gamma\theta}$ and replacing t by $t+\theta$, it leads to

$$\begin{aligned}&\sup_{\theta\in[t_0-t,0]}|x(t+\theta)|^2e^{2\gamma\theta}\\&\le \mathrm{e}^{\lambda(t_0-t)}d^2+2\mathrm{e}^{-\lambda t}\int_{t_0}^t \mathrm{e}^{\lambda r}\beta(r)\mathrm{d}r+\left(\frac{k_1^2}{\varepsilon\lambda}+\frac{m_0^2}{\bar\varepsilon\lambda}\right)\left(1-\mathrm{e}^{\lambda(t_0-t)}\right)\\&+\frac{k_2d^2\left(\mathrm{e}^{2\gamma h}-\mathrm{e}^{\lambda h}\right)}{[(1-\rho_*)\mathrm{e}^{\lambda h}]^{1/2}(2\gamma-\lambda)}\mathrm{e}^{\lambda(t_0-t)}+2m_1\mathrm{e}^{-\lambda t}\int_{t_0}^t \mathrm{e}^{\lambda r}\|x_r\|^2_{C_\gamma}\mathrm{d}r.\end{aligned}$$

In order to treat the whole norm of $\|x_t\|^2_{C_\gamma}$ we need to include the initial data, that is, the values $\theta\in(-\infty,t_0-t]$. This gives:

$$\begin{aligned}e^{2\gamma\theta}|x(t+\theta)|^2&\le e^{-2\gamma(t-t_0)}\|\phi\|^2_{C_\gamma}\\&\le e^{\lambda(t_0-t)}d^2,\quad \forall\theta\in(-\infty,t_0-t].\end{aligned}$$

So, we conclude as wanted

$$\begin{aligned}\|x_t\|^2_{C_\gamma}\le{}& \mathrm{e}^{\lambda(t_0-t)}d^2+2\mathrm{e}^{-\lambda t}\int_{t_0}^t \mathrm{e}^{\lambda r}\beta(r)\mathrm{d}r+\left(\frac{k_1^2}{\varepsilon\lambda}+\frac{m_0^2}{\bar\varepsilon\lambda}\right)\left(1-\mathrm{e}^{\lambda(t_0-t)}\right)\\&+\frac{k_2d^2\left(\mathrm{e}^{2\gamma h}-\mathrm{e}^{\lambda h}\right)}{[(1-\rho_*)\mathrm{e}^{\lambda h}]^{1/2}(2\gamma-\lambda)}\mathrm{e}^{\lambda(t_0-t)}+2m_1\mathrm{e}^{-\lambda t}\int_{t_0}^t \mathrm{e}^{\lambda r}\|x_r\|^2_{C_\gamma}\mathrm{d}r.\end{aligned}$$

Multiplying both terms by $\mathrm{e}^{\lambda t}$, Fubini's theorem and Gronwall's lemma yield

$$\begin{aligned}\|x_t\|^2_{C_\gamma}\le{}&\frac{\lambda}{\lambda-2m_1}\left(\frac{k_1^2}{\varepsilon\lambda}+\frac{m_0^2}{\bar\varepsilon\lambda}\right)\left(1-\mathrm{e}^{(2m_1-\lambda)(t-t_0)}\right)\\&+d^2\left(1+\frac{k_2\left(\mathrm{e}^{2\gamma h}-\mathrm{e}^{\lambda h}\right)}{[\mathrm{e}^{\lambda h}(1-\rho_*)]^{1/2}(2\gamma-\lambda)}\right)\mathrm{e}^{(2m_1-\lambda)(t-t_0)}\\&+2\int_{t_0}^t \mathrm{e}^{(2m_1-\lambda)(t-r)}\beta(r)\mathrm{d}r.\end{aligned}$$

Now, we use the condition $2m_1-\lambda<0$ and (21) or (22) to finish the proof. ■

Now, it is immediate to obtain the following result

Corollary 34 *Under the assumptions of Theorem 32, there exists a pullback attractor* $\{A(t)\}_{t\in\mathbb{R}}$ *for the process* U.

Proof. Combine Proposition 3, Theorem 14, Proposition 19, and Theorem 32. ■

Corollary 35 *Under the assumptions of Theorem 32, there exists a pullback attractor* $\{A(t)\}_{t\in\mathbb{R}}$ *for the process* U *which, in addition, is compact if any of the following conditions hold:*

(a) *the function* $r(t)=\int_{-\infty}^t \mathrm{e}^{(2m_1-\lambda)(t-s)}\beta(s)\mathrm{d}s$ *is increasing;*
(b) $\beta\in L^1(-\infty,t),\ \forall t\in\mathbb{R}$.

Proof. Observe that the radii of the absorbing bounded sets in the proof of Theorem 32 are precisely

$$\frac{\lambda}{\lambda - 2m_1}\left(\frac{k_1^2}{\varepsilon\lambda} + \frac{m_0^2}{\bar{\varepsilon}\lambda}\right) + r(t).$$

This gives condition (6) from (a).

The result follows from Proposition 3, Theorem 15, Theorem 17 and Proposition 19.

The second condition is similar, although stronger, since the exponential in $r(t)$ can be bounded by 1 and the function $\int_{-\infty}^{t} \beta(s)\mathrm{d}s$ generates a bigger radius function for a family of absorbing balls. ■

4.4 Sharp use of the dissipativity for the autonomous case

Opposite to the second case in the above paragraph, in this section we will prove in a different way with less restrictive conditions that it is possible to obtain boundedness for the semigroup associated to an autonomous infinite delay differential equation, combining the ideas from Wang & Xu [30] and Ball [3].

In order to state the main result, we need a preliminary lemma for the estimates on the solutions.

Consider, as before, the non-autonomous equation (18) and assume conditions (A1) and (A2). We will state firstly an estimate valid for this general equation.

Lemma 36 *Under the above conditions, there exist positive values A, B, and δ such that for any solution $x(\cdot)$ and $t \in [t_0, T_x)$ the following inequality holds*

$$|x(t)|^2 \le \mathrm{e}^{-\delta(t-t_0)}|x(t_0)|^2 + \int_{t_0}^{t} \mathrm{e}^{-\delta(t-s)}\left(A + 2\beta(s) + B\|x_s\|_{C_\gamma}^2\right)\mathrm{d}s \tag{25}$$

Proof. We multiply (18) by $x(t)$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le& -\alpha|x(t)|^2 + \beta(t) + \left(k_1^2 + k_2^2|x(t-\rho(t))|^2\right)^{1/2}|x(t)| \\ &+\langle x(t), \int_{-\infty}^{0} b(t,s,x(t+s))\mathrm{d}s\rangle \\ \le& -\alpha|x(t)|^2 + \beta(t) + \frac{1}{2\varepsilon}\left(k_1^2 + k_2^2|x(t-\rho(t))|^2\right) + \frac{\varepsilon}{2}|x(t)|^2 \\ &+\frac{\bar{\varepsilon}}{2}|x(t)|^2 + \frac{m_0^2}{2\bar{\varepsilon}} + \left(\int_{-\infty}^{0} m_1(s)|x(t+s)|\mathrm{d}s\right)|x(t)|.\end{aligned}$$

Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le -\left(\alpha - \frac{\varepsilon}{2} - \frac{\bar{\varepsilon}}{2}\right)|x(t)|^2 + \beta(t) + \frac{k_1^2}{2\varepsilon} + \frac{1}{2\varepsilon}k_2^2\mathrm{e}^{2\gamma h}\|x_t\|_{C_\gamma}^2 + \frac{m_0}{2\bar{\varepsilon}} + m_1\|x_t\|_{C_\gamma}^2.$$

Introducing another parameter $\delta \in (0, 2\alpha)$ (choosing $\varepsilon, \bar{\varepsilon}$ small) such that

$$\varepsilon + \bar{\varepsilon} = 2\alpha - \delta, \tag{26}$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \leq -\delta|x(t)|^2 + 2\beta(t) + \frac{k_1^2}{\varepsilon} + \frac{1}{\varepsilon}k_2^2 \mathrm{e}^{2\gamma h}\|x_t\|_{C_\gamma}^2 + \frac{m_0}{\bar{\varepsilon}} + 2m_1\|x_t\|_{C_\gamma}^2.$$

Multiplying by $\mathrm{e}^{\delta t}$ and integrating in $[t_0, t]$ we have

$$\mathrm{e}^{\delta t}|x(t)|^2 \leq \mathrm{e}^{\delta t_0}|x(t_0)|^2 + \int_{t_0}^t (A + 2\beta(s))\mathrm{e}^{\delta s}\mathrm{d}s + B\int_{t_0}^t \mathrm{e}^{\delta s}\|x_s\|_{C_\gamma}^2 \mathrm{d}s,$$

where we have denoted

$$A = \frac{k_1^2}{\varepsilon} + \frac{m_0}{\bar{\varepsilon}} \qquad \text{and} \qquad B = \frac{1}{\varepsilon}k_2^2\mathrm{e}^{2\gamma h} + 2m_1. \tag{27}$$

■

For the main result of this section, we will restrict ourselves to the autonomous case. Here on we suppose, jointly with assumptions (A1) and (A2), that

(A3) $\beta(t) \equiv \beta$ and $\rho(t) \equiv h$ for the equation

$$x'(t) = F_0(x(t)) + F_1(x(t-h)) + \int_{-\infty}^0 b(s, x(t+s))\mathrm{d}s \tag{28}$$

Before establishing the main theorem, we give an auxiliary result which will be used below.

Proposition 37 *The following condition*

$$\alpha > m_1 + k_2\mathrm{e}^{\gamma h} \tag{29}$$

is optimal to obtain a pair of values B and δ as in Lemma 36 satisfying in addition that $B < \delta$.

Proof. Due to expressions (26) and (27) in Lemma 36, if we wish $B < \delta$, we put all weight of (26) in ε. So, we only have to show that the function $g(\delta) = \delta - 2m_1 - \frac{k_2^2}{2\alpha-\delta}\mathrm{e}^{2\gamma h}$, which represents "*grosso modo*" $\delta - B$, admits a positive value for some $\delta \in (0, 2\alpha)$. We proceed to calculate its maximum and to impose to be positive its value there, making the condition on the dissipativity optimal.

A simple analysis shows that if (and only if) $\alpha \leq k_2\mathrm{e}^{\gamma h}/2$, then it is not possible to obtain positive values of $g|_{(0,2\alpha)}$. But if $2\alpha > k_2\mathrm{e}^{\gamma h}$, the function g may have positive values if we add the extra condition (29): more precisely, the maximum of g is achieved in δ_* such that $2\alpha - \delta_* = k_2\mathrm{e}^{\gamma h}$ and $g(\delta_*) = 2(\alpha - m_1 - k_2\mathrm{e}^{\gamma h})$. ■

Remark 38 *Observe that the above result claims an optimal condition for the dissipativity, not for the values ε and $\bar{\varepsilon}$, and so neither for B nor A. This is because ε should be taken close to $2\alpha - \delta_*$ but not exactly equal, since $\bar{\varepsilon}$ must be positive and satisfy (26).*

In other words, although the dissipativity condition is optimal (and so is δ_*), the choice we can do so for A and B is not. While we choose a value B closer to B_* (this is obviously possible by continuity, taking ε close to $\varepsilon_* = 2\alpha - \delta_*$), we obtain a value A which grows to infinity as long as $\bar{\varepsilon}$ goes to zero.

Theorem 39 *Consider the equation (28) with conditions (A1), (A2) and (A3), and that inequality (29) holds. Then the generated multi-valued semiflow G is well defined and pointwise dissipative.*

More exactly, denoting B and δ the values from Proposition 37 and the associate value A by (26)-(27), the following set attracts $G(t,\psi)$ for every $\psi \in C_\gamma$ when $t \to +\infty$:

$$B_0 = \left\{ \psi \ : \ \|\psi\|^2_{C_\gamma} \leq \frac{A+2\beta}{\delta - B} \right\}.$$

Proof. Since here we are concerned with the autonomous case, denote for convenience $\bar{A} = A + 2\beta$. Let K be such that $\delta K = \bar{A} + BK$, i.e. $K = \frac{\bar{A}}{\delta - B}$ (w.l.o.g. we can assume that $K > 0$).

Step 1: We will see that for any $R \geq 1$, the open ball $B_{C_\gamma}(0, (\bar{K})^{1/2})$ is positively invariant for $\bar{K} = RK$, that is, for every $\psi \in C_\gamma$ with $\|\psi\|^2_{C_\gamma} < \bar{K}$, any solution $x(\cdot) \in D(0,\psi)$ satisfies $\|x_t\|^2_{C_\gamma} < \bar{K}$ for all $t \geq 0$.

By a contradiction argument, if not, there exists a time $t_1 > 0$ such that $\|x_t\|^2_{C_\gamma} < \bar{K}$ (in particular, $|x(t)| < \bar{K}$ too) for all $t < t_1$ and $\|x_{t_1}\|^2_{C_\gamma} = \bar{K}$. With the above strict inequalities, this equality means that $|x(t_1)|^2 = \bar{K}$.

Now, writing (25) for $t_0 = 0$ and $t = t_1$,

$$\begin{aligned}
|x(t_1)| &< \mathrm{e}^{-\delta t_1} \bar{K} + \int_0^{t_1} \mathrm{e}^{-\delta(t_1 - s)} (\bar{A} + B\bar{K}) \mathrm{d}s \\
&= \mathrm{e}^{-\delta t_1} \bar{K} + \frac{\bar{A} + B\bar{K}}{\delta} \left(1 - \mathrm{e}^{-\delta t_1}\right).
\end{aligned}$$

As long as

$$\frac{\bar{A} + B\bar{K}}{\delta} = \frac{\bar{A} + BRK}{\delta} \leq \frac{R(\bar{A} + BK)}{\delta} = RK = \bar{K},$$

we obtain $|x(t_1)| < \bar{K}$, a contradiction.

Therefore, we deduce that G is well defined.

Step 2: We prove now the statement of the theorem for a single solution: it is attracted by the ball $B_0 = \{\psi \ : \ \|\psi\|^2_{C_\gamma} \leq K\}$.

Take the bounded open ball $B_{C_\gamma}(0, d^{1/2})$ with $d \geq K$ (otherwise it is trivial by Step 1). Observe that the norm in C_γ of a solution with initial datum in the above set is given by

$$\begin{aligned}
\|x_t\|_{C_\gamma} &= \sup_{s \leq 0} \mathrm{e}^{\gamma s} |x(t+s)| \\
&= \max \left(\sup_{s \leq -t} \mathrm{e}^{\gamma s} |x(t+s)|, \sup_{s \in [-t,0]} \mathrm{e}^{\gamma s} |x(t+s)| \right) \\
&\leq \max \left(d\mathrm{e}^{-\gamma t}, \sup_{s \in [-t,0]} \mathrm{e}^{\gamma s} |x(t+s)| \right).
\end{aligned} \tag{30}$$

Consider an arbitrary solution $x(\cdot)$. By the first step it is possible, putting $\bar{K} = d$, to show that it is globally bounded (actually, it holds that $|x(t)| \le d \ \forall t \ge 0$), and therefore there exists

$$\limsup_{t \to +\infty} |x(t)|^2 = \sigma.$$

This means that

$$\forall \epsilon > 0, \ \exists T_1(\epsilon) > 0 \ s.t. \ |x(t)|^2 \le \sigma + \epsilon \quad \forall t \ge T_1(\epsilon). \tag{31}$$

Before joining (30) and (31) to obtain a bound on $\|x_t\|_{C_\gamma}$ we must care about the interval time $[0, T_1]$. Note that (for $t \ge T_1$)

$$\sup_{s \in [-t,0]} \mathrm{e}^{\gamma s} |x(t+s)| = \max \left\{ \sup_{s \in [-t, T_1 - t]} \mathrm{e}^{\gamma s} |x(t+s)|, \sup_{s \in [T_1 - t, 0]} \mathrm{e}^{\gamma s} |x(t+s)| \right\}.$$

Let now $T_2(\epsilon) \ge T_1(\epsilon)$ be such that $d\mathrm{e}^{\gamma(T_1 - T_2)} \le \sigma + \epsilon$. Using this choice in the first term of the maximum and (31) in the second term, putting this in (30), we conclude that

$$\exists T_2(\epsilon) \ge T_1(\epsilon) \text{ such that } \|x_t\|^2_{C_\gamma} \le \sigma + \epsilon \quad \forall t \ge T_2(\epsilon). \tag{32}$$

On other hand, since we will use the bound obtained in the first step, we fix now a time $T_3(\epsilon)$ such that

$$\mathrm{e}^{-\delta t} d + \frac{\bar{A} + Bd}{\delta} \left(\mathrm{e}^{-\delta T_3} - \mathrm{e}^{-\delta t} \right) \le \epsilon \quad \forall t \ge T_3(\epsilon). \tag{33}$$

Now, recovering (25) for $t_0 = 0$ and splitting the integral into two parts, $[0, t - T_3]$ and $[t - T_3, t]$ we have

$$|x(t)|^2 \le \mathrm{e}^{-\delta t} |x(0)|^2 + \int_0^{t - T_3} \mathrm{e}^{-\delta(t-s)} \left(\bar{A} + B\|x_s\|^2_{C_\gamma} \right) \mathrm{d}s + \int_{t - T_3}^{T_3} \mathrm{e}^{-\delta(t-s)} \left(\bar{A} + B\|x_s\|^2_{C_\gamma} \right) \mathrm{d}s.$$

If we assume that $t - T_3 \ge T_2$, using the bound (32) for the second integral and (33) for the remaining, we conclude that

$$|x(t)|^2 \le \epsilon + \frac{\bar{A} + B(\sigma + \epsilon)}{\delta} \left(1 - \mathrm{e}^{-\delta T_3} \right) \qquad \forall t \ge T_2 + T_3,$$

whence passing to the limit if ϵ goes to zero, we have

$$\sigma = \limsup |x(t)|^2 \le \frac{\bar{A} + B\sigma}{\delta}.$$

In other words, we have deduced that $\sigma(\delta - B) \le \bar{A}$, or equivalently, $\sigma \le \frac{\bar{A}}{\delta - B} = K$. This inequality and (32) imply (since ϵ is arbitrary) the statement of the theorem for a single solution.

Step 3: We prove now the general result: the semiflow is pointwise dissipative, i.e. for any fixed initial data ψ, the set $G(t, \psi)$ (possibly not a singleton) is attracted by B_0.

Firstly let us denote (for an arbitrary $\eta > 0$)

$$B_{0,\eta} = \{ \psi \ : \ \|\psi\|^2_{C_\gamma} \le K + \eta \}.$$

We claim that $B_{0,\eta}$ is absorbing for $G(t, \psi)$ (since this will be proved being $\eta > 0$ arbitrarily small, we will obtain the main statement of this step).

We proceed by a contradiction argument. Assume that there exist a sequence of times $t_n \to +\infty$, and solutions $x^n_{t_n}$ with the same initial data $x^n_0 = \psi$ such that $x^n_{t_n} \notin B_{0,\eta}$.

Therefore, by the first step, we deduce that $x^n_t \notin B_0$ for all $0 \le t \le t_n$. Besides this, we know that solutions are uniformly bounded since it is so for the (unique) initial datum. So, by Ascoli-Arzelà Theorem and a diagonal procedure argument, we obtain the existence of a function $y \in C([0,+\infty);\mathbb{R}^m)$ and a subsequence (relabelled the same) such that

$$x^n|_{[0,T]} \to y|_{[0,T]} \quad \text{in } C([0,T];\mathbb{R}^m), \quad \forall T > 0.$$

In particular, extending y to $\mathbb{R}_-$ in the natural way, concatenating with the same datum ψ (denote this function again by y), we have that $x^n_t \to y_t$ for all $t \ge 0$. By standard arguments (cf. [16]) we deduce that y is solution of the problem, but on the other hand it satisfies

$$\|y_t\|_{C_\gamma} \ge K + \eta, \quad \forall t \ge 0.$$

This is a contradiction with the result of the second step since B_0 attracts any solution, in particular y. ∎

Remark 40 (i) *Condition (29) is optimal to ensure existence of a pair of values δ and B as in Lemma 36 and satisfying, in addition, the condition $\delta < B$ in Step 1 of the proof. This allows to obtain a bounded absorbing set and, at last, the existence of attractor under minimal dissipativity assumptions.*

It also says that dissipativity must increase with values m_1, k_2, γ and h.

(ii) *However, it does not imply the smallest radius for the bounded absorbing set as can be seen in the following proof (see also Remark 38).*

Taking into account relations (25), (26) and (27), observe that since the parameter $\bar{\varepsilon}$ has no influence on the integral with the $\|x_s\|$, we will play essentially with the parameter ε, making $\bar{\varepsilon} \sim 0$. This means that the absorbing set proved in the theorem is bigger as far as $\bar{\varepsilon}$ becomes smaller. To obtain the optimal radius of the absorbing set one should optimize the function

$$r(\varepsilon) = \frac{\frac{k_1^2}{\varepsilon} + \frac{m_0}{2\alpha - \delta_* - \varepsilon}}{\delta_* - \frac{k_2^2}{\varepsilon} e^{2\gamma h} - 2m_1}$$

with $\varepsilon \in (\nu, \varepsilon_*)$ being ν the value where $B(\nu)$ collapse to δ_*, and $\varepsilon_* = 2\alpha - \delta_*$.

(iii) *δ positive is necessary to involve an exponential, which is essential in this proof, making a stronger use of the dissipativity than in [9] and the above section. However, we have only been able to apply it to the autonomous case.*

Corollary 41 *Under the assumptions of Theorem 39, there exists a global attractor for the multi-valued semiflow G associated to the differential equation (28).*

Proof. We have only to apply Theorem 13. Observe that the asymptotic compactness follows from Proposition 20, which can be applied by Step 1 in Theorem 39, and the condition on the map $\bar{U}$ used in Proposition 5 is satisfied by the same reason, giving the upper semicontinuity of G (cf. Proposition 5). ∎

Conclusions

The theory of attractors can be extended with some care to the case of infinite delay differential equations, even if uniqueness does not hold. We have been able to apply an asymptotically compact property in a suitable space (although it can be done in more general abstract spaces) and checked the eventual bounded character or the pointwise dissipativity of the associated (single or multi-valued) process in some general situations depending on suitable relations of the parameters.

However, some unbounded delay equations have not been treated in this framework, as for instance the pantograph equation, which needs to be handled more carefully since the function ρ containing the delay is now not bounded by any quantity h, and the asymptotic behaviour depends on the way we see the equation (it has a proper physical meaning only forward in time). On the other hand, the non-autonomous results admit a comparison on the assumptions with the corresponding *tempered* framework (see [8]), where a *tempered* attractor with better properties, but bigger in principle, can be obtained. Extensions on these directions will be object of a forthcoming paper.

Acknowledgments. P.M.-R. thanks sincerely the staff of the Departamento de Matemáticas of the University of Huelva (Spain), for the kind treatment during the beginning of this work. The authors thank Professor José Real for discussions concerning paper [8] which led to Theorem 17, and to an anonymous referee for pointing out a difficulty in a previous version of the paper concerning the state space of these problems (cf. Remark 1).

This work has been partially supported by Ministerio de Educación y Ciencia (MEC, Spain) and FEDER (Fondo Europeo de Desarrollo Regional), grants MTM2005-01412 and MTM2005-03868, by Generalitat Valenciana (Spain), grant GV05/064, and Consejería de Cultura y Educación (Comunidad Autónoma de Murcia), grant PI-8/00807/FS/01.

References

[1] F. V. Atkinson and J. R. Haddock, On determining phase spaces for functional differential equations, *Funkcial. Ekvac.* **31** (1988), 331-347.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations,* vol. 25 of *Studies in Mathematics and its Applications.* Amsterdam: North-Holland Publishing Co., 1992.

[3] J. M. Ball, Global attractors for damped semilinear wave equations, *Discrete Contin. Dyn. Syst.* **10**(1-2) (2004), 31–52.

[4] T. Burton, V. Hutson, Repellers in systems with infinite delay, *J. Math. Anal. Appl.*, **137** (1989), 240-263.

[5] T. Caraballo, J. A. Langa and J. Valero, Global attractors for multi-valued random dynamical systems, *Nonlinear Anal.* **48** (2002), 805-829.

[6] T. Caraballo, J. A. Langa, V. S. Melnik, and J. Valero, Pullback attractors of non-autonomous and stochastic multi-valued dynamical systems, *Set-Valued Anal.* **11**(2) (2003), 153-201.

[7] T. Caraballo, J. A. Langa, and J. Robinson, Attractors for differential equations with variable delays, *J. Math. Anal. Appl.* **260**(2) (2001), 421-438.

[8] T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, *Nonlinear Analysis, TMA*, to appear.

[9] T. Caraballo, P. Marín-Rubio, and J. Valero, Autonomous and non-autonomous attractors for differential equations with delays, *J. Diff. Eqns.* 208 (2005), no. 1, 9-41.

[10] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays. *J. Differential Equations* **205** (2004), 271-297.

[11] D. N. Cheban, P. E. Kloeden and B. Schmalfuß, The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems, *Nonlinear Dynamics & Systems Theory,* vol. 2, no. 2, pp. 9–28, 2002.

[12] V. V. Chepyzhov and M. I. Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.* **76** (1997), 913-964.

[13] J. M. Cushing, *Integrodifferential Equations and Delay Models in Population Dynamics,* Lecture Notes in Biomathematics 20, Heidelberg: Springer-Verlag, 1979.

[14] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, On a functional differential equation, *J. Inst. Math. Appl.*, **8** (1971) 271–307.

[15] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, vol. 25 of *Mathematical Surveys and Monographs*, Providence, RI: American Mathematical Society, 1988.

[16] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11-41.

[17] J. K. Hale and S. M. V. Lunel, *Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, New York: Springer-Verlag, 1993.

[18] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, 1473, Berlin: Springer-Verlag, 1991.

[19] J. Kato, Stability problem in functional differential equations with infinite delay. *Funkcial. Ekvac.* **21** (1978), 63-80.

[20] S. Kato, Existence, uniqueness, and continuous dependence of solutions of delay-differential equations with infinite delays in a Banach space. *J. Math. Anal. Appl.* **195** (1995), 82-91.

[21] T. Kato and J. B. McLeod, The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$, *Bull. Amer. Math. Soc.*, **77** (1971) 891–937.

[22] P. E. Kloeden and B. Schmalfuß, Non-autonomous systems, cocycle attractors and variable time–step discretization, *Numer. Algorithms*, vol. 14, no. 1-3, pp. 141–152, 1997.

[23] Y. Kuang, *Delay differential equations with applications in population dynamics.*Boston: Academic Press, 1993.

[24] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, vol. 25 of *Lincei Lectures.* Cambridge: Cambridge University Press, 1991.

[25] P. Marín-Rubio and J. Robinson, Attractors for the stochastic 3D Navier-Stokes equations, *Stoch. Dyn.* **3** (2003), 279-297.

[26] V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Anal.* **6** (1998), 83-111.

[27] J. D. Murray, *Mathematical Biology,* Berlin: Springer-Verlag, 1993.

[28] H. Péics, On the asymptotic behaviour of a pantograph-type difference equation, *J. Differ. Equations Appl.*, **6**(3) (2000) 257–273.

[29] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 of *Applied Mathematical Sciences*. New York: Springer-Verlag, 1997.

[30] L. Wang and D. Xu, Asymptotic behavior of a class of reaction-diffusion equations with delays, *J. Math. Anal. Appl.* **281**(2) (2003), 439–453.