THE EXPONENTIAL BEHAVIOUR AND STABILIZABILITY OF STOCHASTIC 2D-NAVIER-STOKES EQUATIONS

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Abstract. Some results on the pathwise exponential stability of the weak solutions to a stochastic 2D-Navier-Stokes equation are established. The first ones are proved as a consequence of the exponential mean square stability of the solutions. However, some of them are improved by avoiding the previous mean square stability in some more particular and restrictive situations. Also, some results and comments concerning the stabilizability and stabilization of these equations are stated.

1. **Introduction**

The long-time behaviour of flows is a very interesting and important problem in the theory of fluid dynamics, as the vast literature shows (see Temam [19], Hale [13], Ladyzhenskaya [14], among others, and the references therein), and has been receiving very much attention over the last three decades.

One of the most studied models is the Navier-Stokes one (and its variants) since it provides a suitable model which covers several important fluids (see Temam [17]-[19] and the references inside these).

On the other hand, another interesting question is to analyze the effects produced on a deterministic system by some stochastic or random disturbances appeared in the problem. These facts have motivated the present work whose main objective is to show some aspects of the effects produced

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in the long-time behaviour of the solution to a two dimensional Navier-Stokes equation under the presence of stochastic perturbations.

In the deterministic case, it is well known for a long time that, for small enough Reynolds number (or, equivalently, large viscosity), the solutions of 2D-Navier-Stokes equations tend to a stationary one (unique, in fact) when time goes to infinite and, as this number increases, the dynamics of the system turns more and more complex (see, e.g. Temam [18] for a detailed description of the Couette-Taylor experiment). The problem of detecting the critical value where the instability appears is a difficult challenging one. Thus, in a general framework, one can only ensure that for small values of the Reynolds number the stationary solution is stable but we do not know when it becomes unstable. This motivates that people working in this kind of problems use to consider particular examples in order to obtain sharper results.

Our first aim in this work is to provide some light in some aspects concerning the stability of the stationary solutions of the following stochastic 2D-Navier-Stokes:

$$\begin{cases} dX = [\nu \Delta X - \langle X, \nabla \rangle X + f(X) + \nabla p]dt + g(t,X)dW(t) \\ divX \ = 0 \ \text{ in } \ [0,\infty) \times D, \\ X = 0 \ \text{ on } \ [0,\infty) \times \Gamma, \\ X(0,x) = X_0(x), \ \ x \in D, \end{cases}$$

where D is a regular open bounded domain of $\mathbb{R}^2$ with boundary Γ, u is the velocity field of the fluid, p the pressure, $\nu > 0$ the kinematic viscosity, X_0 the initial velocity field, f the external force field and $g(t,x)dW(t)$ the random field where $W(t)$ is an infinite dimensional Wiener process.

Concerning the effects produced by random perturbations in deterministic systems, it is worth mentioning that this is a very difficult task which is being investigated actually by many authors within the framework of the theory of random attractors recently introduced by Crauel and Flandoli [10]. On the one hand, existence of random attractors is only known for specific random terms (see, for instance, Crauel and Flandoli [10], Capinski and Cutland [6]). On the other hand, almost nothing is known on the structure of these random sets, so that many challenging open problems, as those related to stability and instability, are still open.

Also, it is very interesting to investigate if a fluid subjected to random influences is asymptotically more or less stable than the deterministic unperturbed one. In the finite dimensional case, there exits a wide literature on this topic (see Arnold [1] and the references therein) which proves that some kind of multiplicative noise may produce a stabilization effect on deterministic unstable systems. However, for the infinite dimensional case, a similar result has not been proved yet, mainly due to the fact that the technique developed in the finite dimensional framework cannot be extended to this case or, at least, it is not known how to do that. The main result proved in [1] ensures that an unstable linear differential system in $\mathbb{R}^n$, namely $\dot{x}(t) = Ax(t)$ with *trace* $A < 0$, can be stabilized by adding a multiplicative noise in the Stratonovich sense containing a suitable skew-symmetric matrix. One interesting remark is that when the stochastic multiplicative perturbation is considered in the Ito sense, this uses to imply a general stabilization effect on the system. In a limit sense, the Ito equations with multiplicative noise correspond to deterministic equations with a mean-zero fluctuating control plus a stabilizing systematic control (see Section 4 for more details and comments). This would mean that only the stabilization produced by Stratonovich terms could be considered as proper stabilization produced by random noise, since the Stratonovich multiplicative noise acts like a periodic zero-mean feedback control, and consequently, its stabilizing effect is unexpected and therefore very interesting. In this paper, we consider the stochastic disturbances in Ito sense, so the stabilization results proved should be interpreted in a suitable sense (see also Caraballo and Langa [7] for an analysis on the different long-time behaviour of Ito and Stratonovich equations in the linear case).

The content of this paper is as follows. In Section 2, we include some preliminaries. In Section 3, we shall prove some results on pathwise exponential stability by extending to this case the stability theory previously developed for semilinear stochastic partial differential equations (see Caraballo and Liu [8], Taniguchi [16]). Finally, in Section 4, we deal with the interesting stabilizability problem, that is, we shall analyze the possible reasons implying a stabilizing effect on the deterministic problem by the appearance of a random disturbance.

2. **Preliminaries**

Firstly, we introduce the following Hilbert spaces:

H = the closure of the set $\left\{u \in C_0^\infty(D,\mathbb{R}^2) : div\ u = 0\right\}$ in $L^2(D,\mathbb{R}^2)$ with the norm $|u| = (u,u)^{\frac{1}{2}}$, where for $u, v \in L^2(D,\mathbb{R}^2)$,

$$(u,v) = \sum_{j=1}^{2} \int_D u^j(x) v^j(x) dx,$$

V = the closure of the set $\left\{u \in C_0^\infty(D,\mathbb{R}^2) : div\ u = 0\right\}$ in $H_0^1(D,\mathbb{R}^2)$ with the norm $\|u\| = ((u,v))^{\frac{1}{2}}$, where for $u, v \in H_0^1(D,\mathbb{R}^2)$,

$$((u,v)) = \sum_{j=1}^{2} \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right).$$

Then, it follows that H and V are separable Hilbert spaces with associated inner products $(\cdot,\cdot)$ and $((\cdot,\cdot))$ and the following is safisfied:

$$V \subset H \equiv H' \subset V',$$

where injections are dense, continuous and compact. Now, we can set $A = -P\triangle$ where P is the orthogonal projector from $L^2(D,\mathbb{R}^2)$ onto H, and define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_D u^i(x) \frac{\partial v^j}{\partial x_i}(x) w^j(x) dx.$$

As we shall need some properties on this trilinear form b, we list here the ones we will use later on (see Temam [19]):

$$\begin{aligned} &|b(u,v,w)| \leq c_1 \mid u \mid^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| \mid w \mid^{\frac{1}{2}} \|w\|^{\frac{1}{2}}, \quad \forall u,v,w \in V, \\ &b(u,v,v) = 0, \quad \forall u,v \in V, \\ &b(u,u,v-u) - b(v,v,v-u) = -b(v-u,u,v-u), \quad \forall u,v \in V, \end{aligned} \tag{2.1}$$

where $c_1 > 0$ is an appropriate constant which depends on the regular open domain D (see Constantin and Foias [9, (6.9), p.50]) . Furthermore, we can define the operator $B : V \times V \to V'$ by

$$\langle B(u,v), w \rangle = b(u,v,w), \quad \forall u,v,w \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality $\langle V', V \rangle$. We also set

$$B(u) = B(u,u), \ \ \forall u \in V.$$

Let $(\Omega, P, \Im)$ be a probability space on which an increasing and right continuous family $\{\Im_t\}_{t\in[0,\infty)}$ of complete sub-σ-algebra of $\Im$ is defined. Let

$\beta_n(t)$ $(n = 1, 2, 3, \cdots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, P, \Im)$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n'} \beta_n(t) e_n, t \geq 0$$

where $\lambda_n' \geq 0$ $(n = 1, 2, 3 \cdots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n' < +\infty$, and $\{e_n\}$ $(n = 1, 2, 3, \cdots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Let $Q \in L(K, K)$ be the operator defined by $Qe_n = \lambda_n' e_n$. The above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

Thus the stochastic 2D-Navier-Stokes equation can be rewritten as follows in the abstract mathematical setting:

$$dX(t) = [-\nu AX(t) - B(X(t)) + f(X(t))] \, dt + g(t, X(t)) dW(t), \tag{2.2}$$

where $f : V \to V'$, $g : [0, \infty) \times V \to L(K, H)$ are continuous functions satisfying some additional assumptions (see conditions below). Also we consider the deterministic version of this equation, namely,

$$dX(t) = [-\nu AX(t) - B(X(t)) + f(X(t))] \, dt. \tag{2.3}$$

First, we give the definition of the weak solutions to stochastic 2D-Navier-Stokes equation (2.2)

Definition 2.1. *A stochastic process $X(t)$, $t \geq 0$, is said to be a weak solution of (2.2) if*

(1a) $X(t)$ *is* $\Im_t-$*adapted,*

(1b) $X(t) \in L^\infty(0, T; H) \cap L^2(0, T; V)$ *almost surely for all* $T > 0$,

(1c) *the following equation holds as an identity in* V' *almost surely, for* $t \in [0, \infty)$

$$\begin{aligned} X(t) &= X(0) + \int_0^t [-\nu AX(s) - B(X(s)) + f(X(s))] \, ds \\ &\quad + \int_0^t g(s, X(s)) dW(s). \end{aligned}$$

As we are mainly interested in the analysis of the exponential stability of the weak solutions to the problem (2.2), we will assume the existence of such weak solutions (see, for instance, Bensoussan [2] or Capinski and Gatarek [4] for results on the existence and uniqueness of solutions).

Now we are going to establish an Ito's formula which is going to be necessary for our purposes (see Pardoux [15])

Let $C^{(1,2)}([0,\infty)\times H,\mathbb{R}^+)$ denote the space of all $\mathbb{R}^+-$valued functions Ψ defined on $[0,\infty)\times H$ with the following properties:

(1) $\Psi(t,x)$ is differentiable in $t\in[0,\infty)$ and twice Frechet differentiable in x with $\Psi_t(t,\cdot),\Psi_x(t,\cdot)$ and $\Psi_{xx}(t,\cdot)$ locally bounded on H

(2) $\Psi(t,\cdot)$, $\Psi_t(t,\cdot)$ and $\Psi_x(t,\cdot)$ are continuous on H,

(3) for all trace class operators R, $tr(\Psi_{xx}(t,\cdot)R)$ is continuous from H into $\mathbb{R}$.

(4). if $v\in V$ then $\Psi_x(t,v)\in V$, and $u\to\langle\Psi_x(t,u),v^*\rangle$ is continuous for each $v^*\in V'$,

(5). $\|\Psi_x(t,v)\|\leq C_0(t)(1+\|v\|)$, $C_0(t)>0$, for all $v\in V$.

Theorem 2.1. *(Ito's formula) Let $\Psi\in C^{(1,2)}([0,\infty)\times H,\mathbb{R}^+)$. If stochastic process $X(t)$ is a weak solution to (2.2), then, it holds that*

$$\begin{aligned}\Psi(t,X(t)) &= \Psi(0,X(0))+\int_0^t L\Psi(s,X(s))ds\\ &\quad+\int_0^t\left(\Psi_x(s,X(s)),g(s,X(s))dW(s)\right),\end{aligned}$$

where

$$\begin{aligned}L\Psi(s,X(s)) &= \Psi_t(s,X(s))\\ &\quad+\langle-\nu AX(s)-B(X(s))+f(X(s)),\Psi_x(s,X(s))\rangle\\ &\quad+\tfrac{1}{2}tr\left(\Psi_{xx}(s,X(s))g(s,X(s))Qg(s,X(s))^*\right).\end{aligned}$$

Definition 2.2. *We say that a weak solution $X(t)$ to (2.2) converges to $x_\infty\in H$ exponentially in mean square if there exist $a>0$ and $M_0=M_0(X(0))>0$ (which may depend on $X(0)$) such that*

$$E\left|X(t)-x_\infty\right|^2\leq M_0e^{-at},t\geq0,$$

In particular, if x_∞ is a solution to (2.2), then it is said that x_∞ is exponentially stable in mean square provided that every weak solution to (2.2) converges to x_∞ exponentially in mean square with the same exponential order $a>0$.

Definition 2.3. *We say that a weak solution $X(t)$ to (2.2) converges to $x_\infty\in H$ almost surely exponentially if there exists $\gamma>0$ such that*

$$\lim_{t\to\infty}\frac{1}{t}\log|X(t)-x_\infty|\leq-\gamma,\ \textit{almost surely.}$$

In particular, if x_∞ is a solution to (2.2), then it is said that x_∞ is almost surely exponentially stable provided that every weak solution to (2.2) converges to x_∞ almost surely exponentially with the same constant γ.

3. **The exponential stability of solutions**

In this section we discuss the moment exponential stability and almost sure exponential stability of weak solutions to stochastic NSE (2.2). Let $\lambda_1 > 0$ be the first eigenvalue of A. We remark that $\|v\|^2 \geq \lambda_1 \mid v \mid^2, \forall v \in V$. We also denote by

$$\| g(t,u) \|_{L_2^0}^2 = tr(g(t,u)Qg(t,u)^*).$$

Throughout this section we will use the following condition:

Condition A. *There exists $\beta > 0$ such that*

$$\| f(u) - f(v) \|_{V'} \leq \beta \| u - v \|, \beta > 0, u, v \in V.$$

In this paper, we first consider the existence of the stationary solution to the next equation

$$\nu Au + B(u) = f(u) \quad (\text{equality in } V'). \tag{3.1}$$

Then we have the following lemma. The proof is similar to the one of Theorem 10.1 in Temam [18]. But, since the proof depends on the conditions of the function f, we give the proof for the convenience of the reader.

Lemma 3.1. *Suppose that condition A is satisfied and the function f satisfies that $f(v_m)$ converges to $f(v)$ weakly in V' whenever $\{v_m\} \subset V$ converges to $v \in V$ weakly in V and strongly in H. Then,*

(a) if $\nu > \beta$, there exists a stationary solution $u_\infty \in V$ to (3.1);

(b) furthermore, if $\nu > \frac{c_1 \|f(0)\|_{V'}}{\sqrt{\lambda_1}(\nu-\beta)} + \beta$, then the stationary solution to (3.1) is unique.

Proof. (a) Let $v_1, v_2, v_3, \cdots, v_m, \cdots$ be the orthonormal basis of V. Consider the finite dimensional Hilbert space V_m spanned by $\{v_1, \cdots, v_m\}$ with the scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ induced by the corresponding ones in V. Now we define a mapping $R_m : V_m \to V_m$ as follows

$$[R_m u, v] = ((R_m u, v)) := \nu ((u,v)) + b(u,u,v) - \langle f(u), v \rangle, \forall u, v \in V_m. \tag{3.2}$$

If we prove that this mapping is continuous in V_m with respect to the norm $[\cdot]$, and that $[R_m u, u] > 0$ for some $u \in V_m$ with $[u] = k > 0$, then Lemma 1.4 in Temam [17, p. 164] guarantees that there exists $u_m \in V_m$ such that $[u_m] \leq k$ and $R_m u_m = 0$.

The continuity of R_m follows easily from the properties of b and the assumptions on f. Now, from (3.2) it holds for $u \in V_m$

$$\begin{aligned}
[R_m u, u] &= \nu((u,u)) + b(u,u,u) - \langle f(u), u\rangle \\
&\geq \nu((u,u)) - \|f(u)\|_{V'} \|u\| \\
&\geq \nu \|u\|^2 - (\|f(0)\|_{V'} + \beta \|u\|) \|u\|
\end{aligned}$$

Since $\nu > \beta$, we can choose a positive real number $k > 0$ such that $(\nu-\beta)k^2 - \|f(0)\|_{V'} k > 0$, and for $u \in V_m$ such that $\|u\| = k$, we have $[R_m u, u] > 0$. Then, there exists an element $u_m \in V_m \subset V$ which is a solution of (3.2) with $\|u_m\| \leq k$. Furthermore, we can easily deduce (see estimation (3.3) below) that

$$\|u_m\| \leq \frac{\|f(0)\|_{V'}}{(\nu - \beta)},$$

and, consequently, we have that a suitable subsequence of $\{u_m\}$ converges weakly in V to some limit u_∞ and, thanks to the compact injection, strongly in H. Now, the properties of b and assumptions on f enable us to prove that this u_∞ is a solution of (3.1).

(b) As for the uniqueness statement, let us assume that u_1 and u_2 are two solutions, then

$$\begin{aligned}
\nu((u_1,v)) + b(u_1,u_1,v) &= \langle f(u_1), v\rangle, \forall v \in V, \\
\nu((u_2,v)) + b(u_2,u_2,v) &= \langle f(u_2), v\rangle, \forall v \in V.
\end{aligned}$$

Setting $v = u_1 - u_2$, by substracting the second relation from the first one, and taking into account the properties of the trilinear form b and condition A we obtain that

$$\begin{aligned}
\nu \|u_1 - u_2\|^2 &= -b(u_1,u_1,u_1-u_2) + b(u_2,u_2,u_1-u_2) \\
&\quad + \langle f(u_1) - f(u_2), u_1 - u_2\rangle \\
&= -b(u_1-u_2,u_2,u_1-u_2) + \langle f(u_1) - f(u_2), u_1 - u_2\rangle \\
&\leq \frac{c_1}{\sqrt{\lambda_1}} \|u_1 - u_2\|^2 \|u_2\| + \beta \|u_1 - u_2\|^2 .
\end{aligned}$$

Observing that

$$\begin{aligned}
\nu \|u_2\|^2 &= \langle f(u_2), u_2\rangle \\
&\leq \|f(u_2)\|_{V'} \|u_2\| \\
&\leq \beta \|u_2\|^2 + \|f(0)\|_{V'} \|u_2\|,
\end{aligned} \tag{3.3}$$

if follows that

$$\|u_2\| \leq \frac{\|f(0)\|_{V'}}{\nu - \beta}.$$

Consequently,

$$\nu \|u_1 - u_2\|^2 \leq \left(\frac{c_1 \|f(0)\|_{V'}}{\sqrt{\lambda_1}(\nu - \beta)} + \beta \right) \|u_1 - u_2\|^2 ,$$

and as $\nu > \frac{c_1 \|f(0)\|_{V'}}{\sqrt{\lambda_1}(\nu - \beta)} + \beta$, uniqueness follows immediately. This completes the proof of the lemma.

Now, using this lemma, we discuss the long-time behaviour of weak solutions $X(t)$ to the stochastic Navier-Stokes equation (2.2) under some conditions including that the kinematic viscosity ν is sufficiently large. Hence throughout this paper we assume that there exists a unique stationary solution $u_\infty \in V$ to (3.1). In this section, we use the following condition.

Condition B. $\| g(t, u) \|^2_{L_2^0} \leq \gamma(t) + (\xi + \delta(t)) \mid u - u_\infty \mid^2,$
where $\xi > 0$ *is a constant and* $\gamma(t), \delta(t)$ *are nonnegative integrable functions such that there exist real numbers* $\theta > 0$, M_γ, $M_\delta \geq 1$ *with*

$$\gamma(t) \leq M_\gamma e^{-\theta t}, \delta(t) \leq M_\delta e^{-\theta t}, t \geq 0.$$

Theorem 3.2. *Let* $u_\infty \in V$ *be the unique stationary solution to (3.1) and let* $2\nu > \lambda_1^{-1}\xi + 2\beta + \frac{2c_1}{\sqrt{\lambda_1}} \| u_\infty \|$ *. Suppose that conditions A and B are satisfied. Then, any weak solution* $X(t)$ *to (2.2) converges to the stationary solution* u_∞ *to (3.1) exponentially in mean square. That is, there exist real numbers* $a \in (0, \theta), M_0 = M_0(X(0)) > 0$ *such that*

$$E \mid X(t) - u_\infty \mid^2 \leq M_0 e^{-at}, t \geq 0.$$

Proof. Since $2\nu > \lambda_1^{-1}\xi + 2\beta + \frac{2c_1}{\sqrt{\lambda_1}} \| u_\infty \|$, we can take a positive real number $a \in (0,\, \theta)$ such that $2\nu > \lambda_1^{-1}(\xi + a) + 2\beta + \frac{2c_1}{\sqrt{\lambda_1}} \| u_\infty \|$. Then, by applying the Ito formula to the function $e^{at} \mid X(t) - u_\infty \mid^2$, we have that

$$\begin{aligned} e^{at} E \mid X(t) - u_\infty \mid^2 \;&=\; E \mid X(0) - u_\infty \mid^2 + \int_0^t a e^{as} E \mid X(s) - u_\infty \mid^2 ds \\ &\quad -2 \int_0^t e^{as} E \left\langle \nu A X(s), X(s) - u_\infty \right\rangle ds \\ &\quad -2 \int_0^t e^{as} E \left\langle B(X(s)), X(s) - u_\infty \right\rangle ds \\ &\quad +2 \int_0^t e^{as} E \left\langle f(X(s)), X(s) - u_\infty \right\rangle ds \\ &\quad + \int_0^t e^{as} E \left\| g(s, X(s)) \right\|^2_{L_2^0} ds. \end{aligned}$$

Since u_∞ satisfies the identity (3.1),

$$\begin{aligned} \int_0^t e^{as} E \left\langle \nu A u_\infty, X(s) - u_\infty \right\rangle ds \;&+\; \int_0^t e^{as} E \left\langle B(u_\infty), X(s) - u_\infty \right\rangle ds \\ &=\; \int_0^t e^{as} E \left\langle f(u_\infty), X(s) - u_\infty \right\rangle ds. \end{aligned}$$

Therefore, noting the next identity:

$$\langle B(X(s)) - B(u_\infty), X(s) - u_\infty\rangle = b(X(s) - u_\infty, u_\infty, X(s) - u_\infty),$$

we obtain that

$$\begin{aligned}
&e^{at}E \mid X(t) - u_\infty \mid^2 \\
&\quad \leq E \mid X(0) - u_\infty \mid^2 + \int_0^t ae^{as}E \mid X(s) - u_\infty \mid^2 ds \\
&\qquad - 2\int_0^t \nu e^{as}E \parallel X(s) - u_\infty \parallel^2 ds \\
&\qquad + 2 \int_0^t e^{as}E \langle f(X(s)) - f(u_\infty), X(s) - u_\infty\rangle ds \\
&\qquad - 2 \int_0^t e^{as}Eb(X(s) - u_\infty, u_\infty, X(s) - u_\infty)ds \\
&\qquad + \int_0^t e^{as}E \|g(s, X(s))\|^2_{L_2^0} ds \\
&\quad \leq E \mid X(0) - u_\infty \mid^2 \\
&\qquad + \int_0^t (\lambda_1^{-1}a + 2\beta + 2\tfrac{c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel - 2\nu)e^{as}E \parallel X(s) - u_\infty \parallel^2 ds \\
&\qquad + \int_0^t e^{as}(\gamma(s) + (\xi + \delta(s))E \mid X(s) - u_\infty \mid^2)ds.
\end{aligned}$$

Here we used that $\lambda_1 \mid X(s) - u_\infty \mid^2 \leq \parallel X(s) - u_\infty \parallel^2$ and the esimate of the function b as follows:

$$\begin{aligned}
&\mid b(X(s) - u_\infty, u_\infty, X(s) - u_\infty) \mid \\
&\leq c_1 \mid X(s) - u_\infty \mid^{\frac{1}{2}} \parallel X(s) - u_\infty \parallel^{\frac{1}{2}} \parallel u_\infty \parallel \mid X(s) - u_\infty \mid^{\frac{1}{2}} \parallel X(s) - u_\infty \parallel^{\frac{1}{2}} \\
&= c_1 \mid X(s) - u_\infty \mid \parallel X(s) - u_\infty \parallel \parallel u_\infty \parallel \\
&\leq \tfrac{c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel \parallel X(s) - u_\infty \parallel^2.
\end{aligned}$$

Therefore, we obtain that

$$e^{at}E \mid X(t) - u_\infty \mid^2 \leq E \mid X(0) - u_\infty \mid^2 + \int_0^t e^{as}(\gamma(s) + \delta(s)E \mid X(s) - u_\infty \mid^2)ds.$$

Noticing that $\theta > a$, we have by applying the Gronwall lemma that

$$\begin{aligned}
e^{at}E \mid X(t) - u_\infty \mid^2 \quad \leq \quad & (E \mid X(0) - u_\infty \mid^2) \exp(\int_0^t 2\delta(s)ds) \\
& + \exp(\int_0^t 2\delta(s)ds)(\int_0^t e^{as}(\gamma(s) + 2\delta(s) \mid u_\infty \mid^2)ds.
\end{aligned}$$

Thus, there exists a positive real number $M_0 = M_0(X(0)) > 0$ such that

$$E \mid X(t) - u_\infty \mid^2 \leq M_0 e^{-at} \text{ for all } t > 0.$$

This completes the proof.

Theorem 3.3. *Suppose that all the conditions in Theorem 3.2 are satisfied. Then, any weak solution $X(t)$ to (2.2) converges to the stationary solution u_∞ of (3.1) almost surely exponentially.*

Proof. Let N be a natural number. By the Ito formula, it follows for any $t \geq N$,

$$\begin{aligned} \mid X(t) - u_\infty \mid^2 \; = \; & \mid X(N) - u_\infty \mid^2 \\ & -2 \int_N^t \langle \nu A X(s), X(s) - u_\infty \rangle \, ds \\ & -2 \int_N^t \langle B(X(s)), X(s) - u_\infty \rangle \, ds \\ & +2 \int_N^t \langle f(X(s)), X(s) - u_\infty \rangle \, ds \\ & + \int_N^t \|g(s, X(s))\|_{L_2^0}^2 \, ds \\ & +2 \int_N^t (X(s) - u_\infty, g(s, X(s)) dW(s)) \, . \end{aligned}$$

Furthermore, by the Burkholder-Davis-Gundy lemma,

$$\begin{aligned} & 2E\left[\sup_{N \leq t \leq N+1} \int_N^t (X(s) - u_\infty, g(s, X(s)) dW(s))\right] \\ & \leq n_1 E\left[\int_N^{N+1} \mid X(N) - u_\infty \mid^2 \| \, g(s, X(s)) \, \|_{L_2^0}^2 \, ds\right]^{1/2} \\ & \leq n_1 E\left[\sup_{N \leq s \leq N+1} \mid X(N) - u_\infty \mid^2 \int_N^{N+1} \| \, g(s, X(s)) \, \|_{L_2^0}^2 \, ds\right]^{1/2} \\ & \leq n_2 \int_N^{N+1} E \, \| \, g(s, X(s)) \, \|_{L_2^0}^2 \, ds \\ & \quad + \frac{1}{2} E\left[\sup_{N \leq t \leq N+1} \mid X(t) - u_\infty \mid^2\right] , \end{aligned}$$

where $n_1, n_2 > 0$. Therefore, we obtain a positive real number $n_0 > 0$ such that

$$\begin{aligned} E\left[\sup_{N \leq t \leq N+1} \mid X(t) - u_\infty \mid^2\right] \; \leq \; & E \mid X(N) - u_\infty \mid^2 \\ & -2\nu \int_N^{N+1} E \, \| \, X(s) - u_\infty \, \|^2 \, ds \\ & + \frac{2c_1}{\sqrt{\lambda_1}} \, \| \, u_\infty \, \| \, \int_N^{N+1} E \, \| \, X(s) - u_\infty \, \|^2 \, ds \\ & +2\beta \int_N^{N+1} E \, \| \, X(s) - u_\infty \, \|^2 \, ds \\ & +n_0 \int_N^{N+1} E \, \| \, g(s, X(s)) \, \|_{L_2^0}^2 \, ds \\ & + \frac{1}{2} E \sup_{N \leq t \leq N+1} \mid X(t) - u_\infty \mid^2 \, . \end{aligned}$$

Thus, since $2\nu > \lambda_1^{-1}\xi + 2\beta + \frac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel$, by simple computations,

$$\begin{aligned}
&\tfrac{1}{2}E\sup\nolimits_{N\le t\le N+1} \mid X(t) - u_\infty \mid^2 \\
&\quad \le E \mid X(N) - u_\infty \mid^2 \\
&\qquad + n_0 \int_N^{N+1} (\gamma(s) + (\xi + \delta(s))E \mid X(s) - u_\infty \mid^2) ds.
\end{aligned}$$

Since $\gamma(t) \le M_\gamma e^{-\theta t}$ and $\delta(t) \le M_\delta e^{-\theta t}$, $a \in (0, \theta), M_\gamma \ge 1, M_\delta \ge 1$, we obtain, thanks to Theorem 3.2, that there exists $M_1 = M_1(X(0)) \ge 1$ such that

$$E\left[\sup_{N\le t\le N+1} \mid X(t) - u_\infty \mid^2\right] \le M_1 e^{-aN}.$$

Finally, using the Borel-Cantelli lemma one can easily finish the proof.

Theorem 3.4. *Let $u_\infty \in V$ be the unique stationary solution to (3.1). Assume that condition A and the following ones hold:*

(3.4a) $\quad g(t, u_\infty) \equiv 0, t \ge 0,$

(3.4b) $\quad \parallel g(t, u) - g(t, v) \parallel_{L_2^0} \le c_g \parallel u - v \parallel, \ c_g > 0, \ u, v \in V.$

If $2\nu > 2\beta + c_g^2 + \frac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel$, then any weak solution $X(t)$ to (2.2) converges to u_∞ exponentially in mean square and so u_∞ is exponentially stable in mean square. That is, there exists a real number $\gamma > 0$ such that

$$E \mid X(t) - u_\infty \mid^2 \le E \mid X(0) - u_\infty \mid^2 e^{-\gamma t}, \ t \ge 0.$$

Furthermore, pathwise exponential stability with probability one of u_∞ also holds.

Proof . We have that the following equality is satisfied:

$$\begin{aligned}
X(t) - u_\infty \quad = \quad & X(0) - u_\infty + \int_0^t -(\nu AX(s) - \nu Au_\infty) \\
& + \int_0^t \{-(B(X(s)) - B(u_\infty)) + [f(X(s)) - f(u_\infty)]\} \, dt \\
& + \int_0^t [g(t, X(s)) - g(s, u_\infty)] \, dW(t).
\end{aligned}$$

Now, we can take $\gamma > 0$ small enough (fixed later) and, applying the Ito formula and taking expectation,

$$\begin{aligned}
e^{\gamma t} E \mid X(t) - u_\infty \mid^2 &= E \mid X(0) - u_\infty \mid^2 \\
&\quad + \gamma \int_0^t e^{\gamma s} E \mid X(s) - u_\infty \mid^2 ds \\
&\quad -2 \int_0^t \nu e^{\gamma s} E \parallel X(s) - u_\infty \parallel^2 ds \\
&\quad -2 \int_0^t e^{\gamma s} E \left\langle B(X(s)) - B(u_\infty), X(s) - u_\infty \right\rangle ds \\
&\quad +2 \int_0^t e^{\gamma s} E \left\langle f(X(s)) - f(u_\infty), X(s) - u_\infty \right\rangle ds \\
&\quad + \int_0^t e^{\gamma s} E \parallel g(s, X(s)) - g(s, u_\infty) \parallel^2_{L_2^0} ds \\
&\leq E \mid X(0) - u_\infty \mid^2 + \gamma \int_0^t e^{\gamma s} E \mid X(s) - u_\infty \mid^2 ds \\
&\quad -2 \int_0^t \nu e^{\gamma s} E \parallel X(s) - u_\infty \parallel^2 ds \\
&\quad +2 \int_0^t e^{\gamma s} E \left[\parallel f(X(s)) - f(u_\infty) \parallel_{V'} \parallel X(s) - u_\infty \parallel \right] ds \\
&\quad + \int_0^t e^{\gamma s} c_g^2 E \parallel X(s) - u_\infty \parallel^2 ds \\
&\quad -2 \int_0^t e^{\gamma s} E \left\langle B(X(s)) - B(u_\infty), X(s) - u_\infty \right\rangle ds.
\end{aligned}$$

Then, arguing as before, it follows that

$$\begin{aligned}
\left\langle B(X(s)) - B(u_\infty), X(s) - u_\infty \right\rangle &= b(X(s) - u_\infty, u_\infty, X(s) - u_\infty) \\
&\leq \tfrac{c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel \parallel X(s) - u_\infty \parallel^2 .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&e^{\gamma t} E \mid X(t) - u_\infty \mid^2 \\
&\quad \leq E \mid X(0) - u_\infty \mid^2 \\
&\qquad + 2 \int_0^t \left[\gamma \lambda_1^{-1} - 2\nu + 2\beta + c_g^2 + \tfrac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel \right] e^{\gamma s} E \parallel X(s) - u_\infty \parallel^2 ds.
\end{aligned}$$

As $-2\nu + 2\beta + c_g^2 + \frac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel < 0$, we can choose a real number $\gamma > 0$ such that

$$\gamma \lambda_1^{-1} - 2\nu + 2\beta + c_g^2 + \frac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel < 0,$$

which completes the proof of the first part of the theorem. As the rest of the theorem is proved by a similar method to the one in the proof of Theorem 3.3, we omit it.

Remark 3.1. *Assume that $\nu Au + B(u) = f(u)$ has a unique stationary solution u_∞. If the stochastic NSE (2.2) has a time-independent solution $u_1 \in V$, then $u_1 = u_\infty$, almost surely. Indeed, assume $u_1 \in V$ is a solution* *to (2.2). Then we have*

$$\begin{aligned} \int_0^t (-\nu A u_1 - B(u_1) + f(u_1))dt + \int_0^t g(t,u_1)dW(t) &= 0, \\ (-\nu A u_1 - B(u_1) + f(u_1))t + \int_0^t g(t,u_1)dW(t) &= 0, \\ -\nu A u_1 - B(u_1) + f(u_1) + \frac{1}{t}\int_0^t g(t,u_1)dW(t) &= 0. \end{aligned}$$

Letting $t \to \infty$, we obtain that $-\nu A u_1 - B(u_1) + f(u_1) = 0$ P-almost surely, which implies that $u_1 = u_\infty$.

Remark 3.2. *It is worth pointing out that if g satisfies an additional assumption, namely,*

$$(u-v,(g(u)-g(v))h) = 0,\ \ u,v \in V, h \in K,$$

(this condition is used by Capinski and Cutland [6]*), then the almost sure exponential stability can be obtained directly by the computations in the theorem, since it is not necessary to take expectation in order to eliminate the stochastic integral. Also, the above condition is implied by the following one, $(u,g(v)) = -(g(u),v)$, which is fulfilled, for example, for some particular examples of first order partial differential operators (e.g. solenoidal ones) in the case of a one dimensional Wiener process.*

In the final of this section we consider the case where the external force f can depend on time, that is, $f : [0,\infty) \times V \to V'$. In this case, we assume

Condition C. $\langle f(t,x),x\rangle \le \alpha(t) + (c+\beta(t)) \mid x \mid^2, c > 0,$
where $\alpha(t), \beta(t)$ are integrable functions such that there exist real numbers $\theta > 0$, M_α, $M_\beta \ge 1$ with

$$\alpha(t) \le M_\alpha e^{-\theta t},\ \beta(t) \le M_\beta e^{-\theta t},\ t \ge 0.$$

Then, we prove the following result.

Theorem 3.5. *Suppose that condition C is satisfied and there exists a constant $\zeta > 0$ such that $\| g(t,u) \|^2_{L_2^0} \le \gamma(t) + (\zeta + \delta(t)) \mid u \mid^2$ where the functions $\gamma(t)$, $\delta(t)$ satisfy the same condition as the ones in Condition B. Furthermore, let $2\nu\lambda_1 > \zeta + 2c$. Then any weak solution $X(t)$ to (2.2) converges to zero almost surely exponentially.*

Proof. We can take a positive number $a \in (0,\ \theta)$ such that $2\nu\lambda_1 > a + 2c + \zeta$. First we have

$$\begin{aligned} e^{at}E \mid X(t) \mid^2 \ &= \ E \mid X(0) \mid^2 + \textstyle\int_0^t ae^{as}E \mid X(s) \mid^2 ds \\ &\quad -2 \ \textstyle\int_0^t e^{as}E \left\langle \nu AX(s), X(s) \right\rangle ds \\ &\quad -2 \ \textstyle\int_0^t e^{as}E \left\langle B(X(s)), X(s) \right\rangle ds \\ &\quad +2 \ \textstyle\int_0^t e^{as}E \left\langle f(s, X(s)), X(s) \right\rangle ds \\ &\quad + \ \textstyle\int_0^t e^{as}E \parallel g(s, X(s)) \parallel_{L_2^0}^2 ds. \end{aligned}$$

Therefore, since $(-2\nu\lambda_1 + a + 2c + \zeta) < 0$, we have that

$$\begin{aligned} e^{at}E \mid X(t) \mid^2 \ &\leq \ E \mid X(0) \mid^2 \\ &\quad + \ \textstyle\int_0^t e^{as}(\gamma(s) + 2\alpha(s) + (2\beta(s) + \delta(s))E \mid X(s) \mid^2)ds. \end{aligned}$$

By the Gronwall lemma, we get that any weak solution $X(t)$ to (2.2) converges to zero exponentially in mean square.

Now, the proof can be finished by the same method as the one in the proof of Theorem 3.3.

4. **Stabilizability and stabilization of solutions**

In this Section, we shall analyze some aspects related to the problem of stabilizability and stabilization of our Navier-Stokes model. Firstly, notice that the pathwise stability in the previous Section has been deduced as a by product of the mean square stability. However, it may happen that a solution of a stochastic equation can be pathwise exponentially stable and not exponentially stable in mean square.

Indeed, let us consider the following scalar ordinary differential equation to illustrate this fact,

$$dx(t) = ax(t)dt + bx(t)dW(t),$$

where a, b are real numbers and W is a one dimensional Wiener process. As this equation can be solved directly, we can easily check that the solution is given by

$$x(t) = x(0) \exp \left\{ \left(a - \frac{b^2}{2} \right) t + bW(t) \right\}.$$

Thus, it is easy to see that the zero solution is pathwise exponentially stable with probability one if and only if $a - \frac{b^2}{2} < 0$. Also, one can prove that

$$E \left| x(t) \right|^2 = E \left| x(0) \right|^2 \exp \left\{ \left(2a + b^2 \right) t \right\},$$

and therefore, the zero solution is exponentially stable in mean square if and only if $a + \frac{b^2}{2} < 0$. So, we observe that there exist many possibilities of being the zero solution pathwise exponentially stable and, at the same time, exponentially unstable in mean square.

Consequently, it would be very interesting to obtain pathwise exponential stability results by avoiding the method of using mean square stability as a previous step. This will be one of the aims of this section. However, it is worth pointing out that to get some results in this direction, we will need to assume some additional hypotheses on the stochastic perturbation so that we can obtain better stability criteria but for more specific situations. In particular, in some of our situations, the noise is so special that one can perform a time change, a substitution that transform the stochastic equation into a deterministic one. For example, the Ito formula for the logarithm in the proof of Theorem 4.2 in this section is one way to perform this transformation; another is to multiply by the exponential of the noise (see Crauel and Flandoli [10, p. 382]).

To this end let us firstly state the following condition

Condition D. $f : H \to H$, and satisfies

$$|f(u) - f(v)| \leq c\,|u - v|\,,\ c > 0,\ u, v \in H,$$

$$g(t, \cdot) : H \to L(K, H), \text{ and satisfies}$$

$$\|g(t,u) - g(t,v)\|_{L(K,H)} \leq C_g\,|u - v|\,,\ \forall t \in [0, \infty), \forall u, v \in H.$$

Observe that if $\nu\lambda_1 > c$ and $f(0) = 0$, then the zero solution to (2.3) is exponentially stable. But when $\nu\lambda_1 \leq c$ and $f(0) = 0$ we do not know, in general, if the zero solution is exponentially stable or not. The following theorem is going to state that, under some particular conditions, any weak solution of the stochastic Navier-Stokes equation converges to zero almost surely exponentially stable. So, in a sense, we can interpret that a kind of stabilization could have taken place in the system.

Theorem 4.1. *In addition to condition D, assume that $f(0) = 0$ and $g(t, 0) = 0$ for all $t \geq 0$, and that there exists $\rho > 0$ such that*

$$\widetilde{Q}\psi(s,x) := tr\left[(\psi_x(x) \otimes \psi_x(x))(g(s,x)Qg(s,x)^*)\right] \geq \rho^2\,|x|^4\,,$$

where $\psi(x) = |x|^2$ (recall that $(\psi_x(x) \otimes \psi_x(x))(h) = \psi_x(x)\,(\psi_x(x), h)\,,$ for $x, h \in H$). Then, there exists $\Omega_0 \subset \Omega, P(\Omega_0) = 0$, such that for $\omega \notin \Omega_0$ there exists $T(\omega) > 0$ such that any weak solution $X(t)$ to (2.2) satisfies

$$|X(t)|^2 \leq |X(0)|^2\, e^{-\gamma t} \quad \textit{for any} \quad t \geq T(\omega),$$

where $\gamma := \frac{1}{2}(\lambda_1\nu - c - \frac{C_g^2}{2} + \frac{\rho^2}{2})$. *In particular, exponential stability of sample paths with probability one holds if* $\gamma > 0$.

Proof. Let us apply Ito's formula for our solution $X(t)$.Then, it follows

$$\begin{aligned} |X(t)|^2 &= |X(0)|^2 + 2\int_0^t \langle -\nu AX(s) - B(X(s)) + f(X(s)), X(s)\rangle \, ds \\ &\quad + \int_0^t \|g(s,X(s))\|^2_{L_2^0} \, ds \\ &\quad +2\int_0^t (X(s), g(s,X(s))dW(s)) \\ &= |X(0)|^2 + 2\int_0^t \left[-\nu \|X(s)\|^2 + \langle f(X(s)), X(s)\rangle \right] ds \\ &\quad + \int_0^t \|g(s,X(s))\|^2_{L_2^0} \, ds \\ &\quad +2\int_0^t (X(s), g(s,X(s))dW(s)), \end{aligned}$$

and, applying once again Ito's formula to the function $\log |X(t)|^2$, and taking into account the hypotheses, it follows

$$\begin{aligned} \log|X(t)|^2 &= \log|X(0)|^2 + \frac{1}{2}\int_0^t \frac{1}{|X(s)|^2}\left[-2\nu\|X(s)\|^2 + 2\langle X(s), f(X(s))\rangle \right] ds \\ &\quad + \frac{1}{2}\int_0^t \frac{1}{|X(s)|^2}\|g(s,X(s))\|^2_{L_2^0} \, ds \\ &\quad +2\int_0^t \frac{1}{|X(s)|^2}(X(s), g(s,X(s))dW(s)) - \frac{1}{2}\int_0^t \frac{\widetilde{Q}\psi(s,X(s))}{|X(s)|^4} ds \\ &\leq \log|X(0)|^2 + \int_0^t \frac{1}{|X(s)|^2}\left[-\nu\lambda_1 + c + \frac{C_g^2}{2}\right] |X(s)|^2 \, ds \\ &\quad +2\int_0^t \frac{1}{|X(s)|^2}(X(s), g(s,X(s))dW(s)) - \frac{\rho^2}{2}t. \end{aligned}$$

Now, due to our assumptions, the term $M(t) = \int_0^t \frac{2}{|X(s)|^2}(X(s), g(s,X(s))dW(s))$ is a real martingale and it is not difficult to prove, by means of the law of iterated logarithm,

$$\lim_{t\to+\infty} \frac{M(t)}{t} = 0, \; P - \text{almost surely.}$$

Thus, we can assure that there exists a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$, such that for every $\omega \notin \Omega_0$ there exists $T(\omega) > 0$ such that for all $t \geq T(\omega)$

$$\frac{M(t)}{t} \leq \frac{1}{2}(\lambda_1\nu - c - \frac{C_g^2}{2} + \frac{\rho^2}{2}).$$

Therefore, it easily follows that for any $t \geq T(\omega)$

$$\log|X(t)|^2 \leq \log|X(0)|^2 + \frac{1}{2}(-\lambda_1\nu + c + \frac{C_g^2}{2} - \frac{\rho^2}{2})t.$$

The proof is now complete.

Remark 4.1. *Observe that although we do not know whether the stationary solution to the deterministic problem is stable or not, it is possible to ensure sample exponential stability of the stochastic equation provided that the* *lipschitz constant and the lower bound on the stochastic term (namely, C_g and ρ) imply that $\gamma > 0$. For instance, in the particular case of a linear term, i.e., when g is given for example as*

$$g(t,x)k = \frac{\sigma}{\sqrt{\lambda_1'}} x(k,e_1)_K, \quad t > 0, x \in H, k \in K,$$

the constants appearing in the previous theorem are:

$$C_g = \sigma, \quad \rho = 2\sigma, \quad \gamma = \frac{1}{2}(\lambda_1\nu - c - \frac{C_g^2}{2} + \frac{\rho^2}{2}) = \frac{1}{2}(\lambda_1\nu - c - \frac{\sigma^2}{2} + 2\sigma^2).$$

Consequently, although $\lambda_1\nu - c < 0$, one can always choose σ large enough so that $\gamma > 0$.

Remark 4.2. *For the finite dimensional case, there exists a wide literature on stabilization by noise (see Arnold [1] and the references therein), but for the infinite dimensional case, as far as we know, this question remains open, mainly due to the fact that the technique used in the finite dimensional case seems very difficult to extend to this situation. However, we have to point out that, in general, when one considers a deterministic system and a perturbed version of it by adding a stochastic Ito term, for instance a linear multiplicative one of the form $\sigma u dW(t)$ (being u the solution), in a limit sense, the stochastic equation corresponds to a deterministic equation with a mean-zero fluctuation feedback control plus a stabilizing systematic control, in fact, one can say that an Ito multiplicative noise with intesity σ, acts like a feedback stabilizing control of the form $-\frac{\sigma^2}{2}u$, so maybe not the noise is responsible of the stabilizing effect but this additional damping one. However, the most interesting results in the literature concerning stabilization deals with the one produced by considering the stochastic term in the Stratonovich sense. In this case, as this term is like a periodic zero-mean feedback control, its stabilizing effect is unexpected and very interesting since there is no new damping terms in the equations and when the stabilization is produced, one can properly say that the noise has stabilized the system.*

Remark 4.3. *Noticing that, in order to produce a stabilization effect, it is sufficient to consider a one dimensional Wiener process, in the rest of this section we assume that $K = \mathbb{R}, Q = 1$ and $W(t)$ is a one dimensional Wiener process.*

Lastly, consider the case where $f(0) \neq 0$. If $\nu > \beta$, $\nu > \frac{c_1\|f(0)\|_{V'}}{\sqrt{\lambda_1}(\nu-\beta)} + \beta$ and all the conditions of Lemma 3.1 are satisfied, we have the existence of a unique stationary solution $u_\infty \in V$ to (3.1). Here we note that this u_∞

is also the stationary solution to (2.3). Then, we shall show that it can be chosen $g(t,x) = \sigma(x - u_\infty)$ so that the stationary solution u_∞ to the deterministic equation, becomes an almost sure exponentially stable solution to the stochastic 2D-Navier-Stokes equation (2.2), when the kinematic viscosity ν is large enough and, for simplicity, when W is a one dimensional Wiener process. By the following lemma we get that if the Lipschitz constant $c > 0$ of the external force field f is sufficiently small, that is, if $\lambda_1\nu > c_1\sqrt{\lambda_1} \parallel u_\infty \parallel +c$, then the stationary solution u_∞ to (2.3) is exponentially stable. This lemma can be proved by the similar method as in the proof of Theorem 10.2 (Temam[18, p.69]).

Lemma 4.2. *Let $u_\infty \in V$ be the unique stationary solution to (3.1). If the function f satisfies condition D and $\lambda_1\nu > c_1\sqrt{\lambda_1} \parallel u_\infty \parallel +c$, then the stationary solution u_∞ to (2.3) is exponentially stable.*

But if the Lipschitz constant $c > 0$ is sufficiently large, that is, if $\lambda_1\nu \leq c_1\sqrt{\lambda_1} \parallel u_\infty \parallel +c$, then we do not know if u_∞ is exponentially stable or not. However, we can prove the following result:

Theorem 4.3. *Let $u_\infty \in V$ be the unique stationary solution to (3.1). Let $c_0 := \lambda_1\nu - c_1\sqrt{\lambda_1} \parallel u_\infty \parallel > 0$ and let $\lambda_1\nu \leq c_1\sqrt{\lambda_1} \parallel u_\infty \parallel +c$. Assume that σ is a real number such that $2\lambda_1\nu - 2c_1\sqrt{\lambda_1} \parallel u_\infty \parallel +\sigma^2 > 2c$. If the function f satisfies condition D, then there exists $\Omega_0 \subset \Omega, P(\Omega_0) = 0$, such that for $\omega \notin \Omega_0$ there exists $T(\omega) > 0$ such that*

$$|X(t) - u_\infty|^2 \leq |X(0) - u_\infty|^2 e^{-\gamma t} \text{ for all } t \geq T(\omega),$$

where $\gamma := \frac{1}{2}(\ \sigma^2 - 2c + 2c_0) > 0$, and $X(t)$ is any weak solution to (2.2) where the function g is given by $g(t,x) = \sigma(x - u_\infty)$.

Proof. Applying Ito's formula to the function $|X(t) - u_\infty|^2$, we have that

$$\begin{aligned} |X(t) - u_\infty|^2 &= |X(0) - u_\infty|^2 \\ &\quad -2 \int_0^t \langle \nu AX(s), X(s) - u_\infty \rangle \, ds \\ &\quad -2 \int_0^t \langle B(X(s)), X(s) - u_\infty \rangle \, ds \\ &\quad +2 \int_0^t (f(X(s)), X(s) - u_\infty) \, ds \\ &\quad + \int_0^t \|g(s, X(s))\|^2_{L_2^0} \, ds \\ &\quad +2 \int_0^t (X(s) - u_\infty, g(s, X(s)) dW(s)) \, . \end{aligned}$$

And so

$$\begin{aligned} \mid X(t)-u_\infty \mid^2 &= \mid X(0)-u_\infty \mid^2 \\ &\quad -2\int_0^t \nu \parallel X(s)-u_\infty \parallel^2 ds \\ &\quad -2\int_0^t b(X(s)-u_\infty, u_\infty, X(s)-u_\infty)ds \\ &\quad +2\int_0^t (X(s)-u_\infty, f(X(s))-f(u_\infty))ds \\ &\quad +\int_0^t \parallel g(s,X(s)) \parallel^2_{L_2^0} ds \\ &\quad +2\int_0^t (X(s)-u_\infty, g(s,X(s))dW(s)) . \end{aligned}$$

Hence, since $c_0 := \lambda_1\nu - c_1\sqrt{\lambda_1} \parallel u_\infty \parallel > 0$, using the inequality $\mid b(X(s)-u_\infty, u_\infty, X(s)-u_\infty) \mid \le \frac{c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel\parallel X(s)-u_\infty \parallel^2$, we obtain that

$$\begin{aligned} &-2\nu \|X(s)-u_\infty\|^2 + 2 \mid b(X(s)-u_\infty, u_\infty, X(s)-u_\infty) \mid \\ &\qquad \le (-2\nu + \tfrac{2c_1}{\sqrt{\lambda_1}} \parallel u_\infty \parallel) \parallel X(s)-u_\infty \parallel^2 \\ &\qquad \le (-2\lambda_1\nu + 2c_1\sqrt{\lambda_1} \parallel u_\infty \parallel) \mid X(s)-u_\infty \mid^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} \log|X(t)-u_\infty|^2 &= \log|X(0)-u_\infty|^2 \\ &\quad +\int_0^t \tfrac{1}{|X(s)-u_\infty|^2}(-2\nu \|X(s)-u_\infty\|^2 \\ &\quad +\sigma^2 |X(s)-u_\infty|^2 \\ &\quad -2b(X(s)-u_\infty, u_\infty, X(s)-u_\infty) \\ &\quad +2(f(X(s))-f(u_\infty), X(s)-u_\infty)ds \\ &\quad +2\int_0^t \tfrac{\sigma|X(s)-u_\infty|^2}{|X(s)-u_\infty|^2} dW(s) \\ &\quad -\tfrac{1}{2}\int_0^t \tfrac{4\sigma^2|X(s)-u_\infty|^4}{|X(s)-u_\infty|^4} ds \\ &\le \log \mid X(0)-u_\infty \mid^2 + (2c-2c_0-\sigma^2)t + 2\sigma W(t). \end{aligned}$$

As $\lim_{t\to\infty} \frac{W(t)}{t} = 0$, almost surely, we can find a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$, such that, for each $\omega \notin \Omega_0$, there exists $T(\omega)$ such that for all $t \ge T(\omega)$

$$\frac{2\sigma W(t)}{t} \le \frac{1}{2}(-2c + 2c_0 + \sigma^2).$$

Thus, we obtain that for any $t \ge T(\omega)$

$$\log|X(t)-u_\infty|^2 \le \log|X(0)-u_\infty|^2 + \frac{1}{2}(2c-2c_0-\sigma^2)t.$$

This completes the proof of the theorem.

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