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STABILIZATION OF STATIONARY SOLUTIONS OF EVOLUTION EQUATIONS BY NOISE

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Abstract. We investigate the existence, uniqueness and exponential stability of non-constant stationary solutions of stochastic semilinear evolution equations. Our main result shows, in particular, that noise can have a stabilization effect on deterministic equations. Moreover, we do not require any commutative condition on the noise terms.

1. **Introduction.** It has long been known that the introduction of noise to a deterministic system can give rise to new dynamical behaviour. Horsthemke and Lefever provide an extensive discussion of such noise-induced transition in their monograph [14], while Kirupaharan and Allen [16] consider similar effects in epidemiological models, and Caraballo *et al.* [5] show how noise may lead to a strengthened synchronization effect in reaction-diffusion problems on thin domains separated by a membrane.

The stabilization of equilibria in mechanical systems has important applications in engineering (see, for instance [21] and the references therein). Indeed, as we will show in this paper, the presence of noise may even introduce a stable stationary solution which has no counterpart in the noise free model.

Mao considers in [21] Chapter 4 the following ordinary differential equation on $\mathbb{R}^d$:

$$\frac{dx}{dt} = f(x), \qquad |f(x)| \le K|x| \tag{1}$$

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where f is sufficiently regular. Hence $x^*(t) \equiv 0$ is a stationary solution for this equation. This stationary solution (i.e. steady state) may not be exponentially stable. (For instance, consider the scalar function $f(x) = Kx$ with $K > 0$). If we now add a noise to this differential equation, namely one given by the Ito differential $\sigma x \dot{W}$, where $W(t)$ is a one dimensional Wiener process, we obtain the one dimensional Ito equation

$$dx = f(x)dt + \sigma x dW.$$

Assume now that $\sigma^2 - 2K > 0$, then (see, e.g., [21] or [19]) this equation has the almost surely exponentially stable stationary solution $x^*(t) \equiv 0$. In other words,

$$\lim_{t\to\infty} x(t) = 0 \quad \text{exponentially fast}, \quad \text{almost surely},$$

where $x(t)$ is any solution of (1) which is defined for all future time.

In what follows, we consider an evolution equation of the form

$$dX = AXdt + F(X)dt + B(X)dW, \tag{2}$$

on some separable Hilbert space H, where A is the generator of a C_0-semigroup, F is assumed to be a Lipschitz continuous nonlinear operator, and $B(X)$ is a diffusion coefficient which is also assumed to be Lipschitz continuous with respect to some Hilbert-Schmidt norm. $W(t)$ is an appropriate Wiener process. For a precise formulation of the assumptions on the coefficients see Section 2. Under particular assumptions, it has been proved in [4] that Eq. (2) possesses a unique *non-trivial* stochastic stationary solution which is exponentially stable in the mean square sense or almost surely. This stochastic stationary solution is generated by a random variable X^* with values in H and the Wiener shift $\{\theta_t\}_{t\in\mathbb{R}}$ (see below) such that

$$\mathbb{R}^+ \ni t \mapsto X^*(\theta_t\omega)$$

is the solution to (2) with the initial datum $X(0,\omega) = X^*(\omega)$. To obtain almost sure or, more precisely, ω-wise stability we used the theory of random dynamical systems in [4], which differs from the method to be used in the present work.
To our knowledge the existing literature concerning the stabilization of differential equations by noise only deals with stabilizing steady state (constant) solutions of the deterministic problem, i.e. Eq. (2) with $B = 0$, by using suitable noisy terms which ensure that this steady solution is also a solution of the stochastic perturbed model (2). In most cases, it is assumed that $F(0) = B(0) = 0$, so the investigation refers to the trivial solution (see, for instance, Arnold et al. [2], Hasminskii [13], Scheutzow [23], Mao [21], Caraballo et al. [7], Caraballo and Robinson [6], Kwiecinska [18], Leha et al. [19]...).
The goal of this article is to find diffusion coefficients B ensuring the existence of an almost surely exponentially stable solution for the stochastic evolution equation (2), *even if* this equation without noise does not have any exponentially stable stationary solution. The main idea to construct a stationary solution is to apply the pullback technique together with the exponential martingale inequality.

In the next section we present some basic results which will be needed to prove our stability results. The main results are given in Section 3 and in Section 4, we will apply our results to a situation which, in particular, improve on those of Kwiecinska [18].

2. **Stochastic evolution equations.** In this section we will state and prove some basic properties for stochastic evolution equations.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a filtered probability space, for instance, with the set of elementary events given by $\Omega = C_0(\mathbb{R}, U)$, the set of continuous functions on $\mathbb{R}$ with values in some separable Hilbert space U, equipped with the compact open topology. Then for $\mathcal{F}$, we choose the Borel σ-algebra of $C_0(\mathbb{R}, U)$, whereas $\mathcal{F}_t$ is generated by the events

$$\sigma\{\omega(u) - \omega(v) : u,\, v \leq t,\, \omega \in \Omega\}, \qquad \omega \in \Omega.$$

The probability measure $\mathbb{P}$ is defined to be the Wiener measure on $\mathcal{F}$ with some covariance Q which is the distribution of a *two-sided* Wiener process, see Arnold [1] Appendix A3. The completion of the above probability space is denoted by $(\Omega, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, where we suppose that $\bar{\mathcal{F}}_t$ contains all null sets of $\bar{\mathcal{F}}$. This filtration $\{\bar{\mathcal{F}}_t\}_{t\in\mathbb{R}}$ is right continuous.
We now consider the measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on Ω:

$$\theta : (\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R})) \to (\Omega, \mathcal{F}), \quad \theta_{t+\tau} = \theta_t \circ \theta_\tau, \quad \theta_0 = \mathrm{id}_\Omega, \tag{3}$$

where id_Ω denotes the identical map on Ω. This flow is given by the *Wiener shift* operators

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},\, \omega \in \Omega.$$

Note that $\mathbb{P}$ is ergodic (hence invariant) with respect to θ. For the above filtration we have that

$$\theta_u^{-1}\mathcal{F}_t = \mathcal{F}_{t+u} \tag{4}$$

for any $t,\, u \in \mathbb{R}$, see Arnold [1] Page 72. To guarantee the measurability of θ it is not allowed to replace the σ-algebras $\mathcal{F}_t$ by their completions, see also Arnold [1] Appendix A3. However, for a fixed $t \in \mathbb{R}$, we have the measurability of the mapping $\theta_t : (\Omega, \bar{\mathcal{F}}) \to (\Omega, \bar{\mathcal{F}})$. Since our probability space is a canonical one, we have $W(t,\omega) = \omega(t)$ and

$$W(t, \theta_s\omega) = \omega(t+s) - \omega(s) = W(t+s, \omega) - W(s,\omega) =: \theta_s W(t,\omega).$$

We will investigate the stability behaviour of the stochastic evolution equation

$$dX = AX\,dt + F(X)dt + B(X)dW, \qquad X(0) = X_0. \tag{5}$$

This evolution equation is defined with respect the rigged spaces $V \subset H \subset V'$ consisting of separable Hilbert spaces, where V' denotes the dual of V. The norms of the spaces H, V are denoted by $|\cdot|$, $\|\cdot\|$ respectively with embedding constant a_1:

$$a_1|u|^2 \leq \|u\|^2, \qquad \text{for } v \in V.$$

The inner product in H is denoted by $(\cdot,\cdot)$, and for the duality map between V' and V we write $\langle\cdot,\cdot\rangle$. The initial condition in (5) is assumed to be $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable. For $-A$ we choose a linear operator from V into V' so that, for any $v \in V$,

$$\langle -Av, v\rangle \geq -a_2\|v\|^2 + a_3|v|^2,$$

where $a_2 < 0$, $a_3 \in \mathbb{R}$. It is well known (see e.g. Dautray and Lions [10]) that A is the generator of a strongly continuous semigroup $\{S(t)\}_{t\in\mathbb{R}^+}$ with the operator norm $\|S(t)\|_{\mathcal{L}(H)} \leq e^{a\,t}$ for $a := a_1a_2 - a_3$. $\mathcal{L}(H)$ denotes the space of linear bounded

operators from H into H.
The mapping F is assumed to be Lipschitz continuous with a Lipschitz constant L:

$$|F(x_1) - F(x_2)| \leq L\,|x_1 - x_2| \quad \text{for } x_1,\, x_2 \in H.$$

Let W be a continuous two-sided Wiener process with values in the separable Hilbert space U given above. For the diffusion operator we suppose Lipschitz continuity with respect to the Hilbert-Schmidt norm $\mathcal{L}_2^Q(U,H)$ of linear operators from U to H:

$$\mathrm{tr}_H((B(u_1) - B(u_2))Q(B(u_1) - B(u_2))^*) =: \|B(u_1) - B(u_2)\|^2_{\mathcal{L}_2^Q} \leq L_B|u_1 - u_2|^2$$

for $u_1,\, u_2 \in H$.
We interpret the solution of (5) as a *mild solution* on $[0,\infty)$. A mild solution to (5) is an $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$–adapted process $X(t)$ with values in H such that for every $t \in [0,\infty)$

$$X(t) = S(t)X_0 + \int_0^t S(t-\tau)F(X(\tau))d\tau + \int_0^t S(t-\tau)B(X(\tau))dW(\tau)$$

holds, see [12].
Let $L_{2,s} := L_2(\Omega, \bar{\mathcal{F}}_s, \mathbb{P}; H)$ be the space of square integrable random variables which are $(\bar{\mathcal{F}}_s, \mathcal{B}(H))$-measurable. We then have the following fundamental results on the solution of (5):

Theorem 1. *Assume that $X_0 \in L_{2,0}$. Then, (5) possesses a unique (up to equivalence) mild solution $X(\cdot)$ on $[0,\infty)$ which has a continuous version. In addition,*

$$\mathbb{E}\int_0^T \|X(t)\|^2 dt < \infty$$

and

$$\mathbb{E}\sup_{t\in[0,T]} |X(t)|^2 < \infty.$$

for any $T \geq 0$.

For the existence of a mild solution see Da Prato and Zabczyk [11], Theorem 7.4. The regularity assertion can be found, for instance, in Krylov and Rozovskii [17] Chapter 2.

Lemma 1. *Let $X_1,\, X_2$ be two solution versions with continuous paths of (5) where*

$$X_1(0) = X_0^1 \in L_{2,0}, \qquad X_2(0) = X_0^2 \in L_{2,0}.$$

Assume that $X_0^1(\omega) = X_0^2(\omega)$ for $\omega \in \Omega' \in \bar{\mathcal{F}}_0$. Then, we have the equality $X_1(\cdot) = X_2(\cdot)$, for almost all $\omega \in \Omega'$.

For the proof see Da Prato and Zabczyk [11] Pages 190-191.
The following Lemmata are simple conclusions of Theorem 1 if we do assume that $X_0 \in L_{2,0}$.

Lemma 2. *Assume that X_0 is an $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable random variable. Then there exists a unique solution $X(\cdot)$ (up to equivalence) of (5) which has continuous paths in H. In addition, we have*

$$\int_0^T \|X(\tau)\|^2 d\tau < \infty,$$

for any $T > 0$ almost surely.

Note that

$$X_{0,N}(\omega) = \begin{cases} X_0, & \text{if} \quad |X_0| \leq N \\ 0, & \text{if} \quad |X_0| > N \end{cases} \tag{6}$$

is contained in $L_{2,0}$. The associated solution to this initial condition is denoted by X^N. Hence X^N satisfies all assertions of Theorem 1 and, in particular, the conclusions of this lemma. By the Lemma 1 we have $X^N(\omega) = X^{N+1}(\omega)$ for almost all $\omega \in \{|X_{0,N}|\} \leq N\}$. Hence

$$X(\cdot) := \lim_{N\to\infty} X^N(\cdot) \qquad \text{almost surely}$$

inherits all properties of X^N except square integrability.

Corollary 1. *Lemma 1 remains true if we assume only that X_0^1, X_0^2 are $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable.*

In the sequel, for $s \in \mathbb{R}$, we denote by $\Phi(\cdot, s, X_0)$ a continuous solution version of (5) with an $(\bar{\mathcal{F}}_s, \mathcal{B}(H))$-measurable initial condition X_0 and driven by the Wiener process $\theta_s W$.
We also have the following continuity result.

Lemma 3. *Assume that for a sequence (X_0^k) of $(\bar{\mathcal{F}}_s, \mathbb{B}(H))$-measurable initial conditions*

$$(\mathbb{P}) \lim_{k\to\infty} X_0^k = X_0.$$

Then,

$$(\mathbb{P}) \lim_{k\to\infty} \Phi(t, 0, X_0^k) = \Phi(t, 0, X_0).$$

Proof. By Prokhorov's theorem, for every $\varepsilon > 0$ there exists an $N > 0$ such that

$$\sup_k \mathbb{P}\{\omega : |X_0^k(\omega)| > N\} < \frac{\varepsilon}{2}.$$

Thus $(X_{0,N}^k)$ tends to $X_{0,N} \in L_{2,0}$, see (6). Consequently, by Lemma 1, we have for $t \geq 0$

$$(L_2) \lim_{k\to\infty} \Phi(t, 0, X_{0,N}^k) = \Phi(t, 0, X_{0,N}).$$

Hence, we can conclude for any $\delta > 0$

$$\begin{aligned} &\mathbb{P}(\{\omega : |\Phi(t, 0, X_0^k)(\omega) - \Phi(t, 0, X_0)(\omega)| > \delta\}) \\ &\quad \leq \mathbb{P}(\{\omega : |\Phi(t, 0, X_{0,N}^k)(\omega) - \Phi(t, 0, X_{0,N})(\omega)| > \delta\}) + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

for $k > k_0(\varepsilon, \delta)$. □

Since the shift operator θ_s does not change the Wiener measure $\mathbb{P}$, then $\theta_s W(\cdot, \omega)$ is also a Wiener process with the same covariance Q. For $t \geq 0$ this process is adapted to the filtration $\{\bar{\mathcal{F}}_{s+t}\}_{t\geq 0}$, what can be seen from (4). Let us denote by $\Phi(t, s, X_0)(\omega)$, or for short $\Phi(\cdot, s, X_0)$, the solution of (5), corresponding to the initial value $X_0 \in L_{2,s}$, which is driven by $\theta_s W$, and satisfying the assertions of Theorem 1.
The following equality holds for $X_0 \in L_{2,0}$:

$$\Phi(\cdot, 0, X_0)(\theta_s \cdot) = \Phi(\cdot, s, X_s)(\cdot), \qquad \text{almost surely}, \tag{7}$$

where $X_s(\cdot) := X_0(\theta_s \cdot)$. It is easily seen that both sides of (7) are driven by the same Wiener process and the same initial condition and the fact that solutions of (5) are unique. Indeed, by (4) $X_0(\theta_s \cdot)$ is $\bar{\mathcal{F}}_s$-measurable.

Let $T\geq 0$ be a stopping time with respect to $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$. Then the process $\theta_T W$ defined as

$$(\theta_T W)(t,\omega)=\left(\theta_{T(\omega)}W\right)(t,\omega)=W(t,\theta_{T(\omega)}(\omega))=W(t+T(\omega),\omega)-W(T(\omega),\omega),$$

is a continuous Wiener process with the same distribution as W. This Wiener process is adapted to the optional filtration $\{\bar{\mathcal{F}}_{T+t}\}_{t\geq 0}$, see Karatzas and Shreve [15] Chapter 2, Theorem 6.16, page 86. Suppose that Eq. (5) is driven by $\theta_T W$ with an $(\bar{\mathcal{F}}_T,\mathcal{B}(H))$-measurable initial condition. It is straightforward that this equation has a solution satisfying the conclusion of Theorem 1 and the following Lemma. A continuous version of this solution will be denoted by $\Phi(\cdot,T,X_0)$.

Lemma 4. *For $s\in\mathbb{R}$, let $T\geq 0$ be an $\{\bar{\mathcal{F}}_{s+t}\}_{t\geq 0}$ adapted stopping time and X_0 an $(\bar{\mathcal{F}}_s,\mathcal{B}(H))$-measurable random variable. Then we have*

$$\Phi(\cdot,s+T,\Phi(T,s,X_0))=\Phi(\cdot+T,s,X_0),\qquad \textit{almost surely.}$$

Proof. Let $0=q_0^\delta<q_1^\delta<\cdots$ be a partition of $\mathbb{R}^+$ such that the maximal mash size tends to zero for $\delta\to 0$ and define

$$\mathbf{1}_s(q)=\begin{cases}1 & \text{if} \quad q\leq s\\ 0 & \text{if} \quad q>s\end{cases}.$$

The expression of $\Phi(\cdot+T,s,X_0))$, as a mild solution to Eq. (5), contains in its right hand side a stochastic integral which can be written as

$$\int_0^{T+\tau}S(T+\tau-q)B(\Phi(q,s,X_0))d\theta_sW(q)=S(\tau)\int_0^T S(T-q)B(\Phi(q,s,X_0))d\theta_sW(q)$$
$$+\int_T^{T+\tau}S(T+\tau-q)B(\Phi(q,s,X_0))d\theta_sW(q).$$

The second integral on the right hand side can be approximated by random sums

$$\sum_i(\mathbf{1}_{T+\tau}-\mathbf{1}_T)(q_i)S(T+\tau-q_i)B(\Phi(q_i,s,X_0))(\theta_sW(q_{i+1})-\theta_sW(q_i)).$$

Introducing the random partition $\tilde{q}_i=q_i-T$ for $q_i>T$, this term can be rewritten as

$$\sum_i\mathbf{1}_\tau(\tilde{q}_i)S(\tau-\tilde{q}_i)B(\Phi(\tilde{q}_i+T,s,X_0))(\theta_{T+s}W(\tilde{q}_{i+1})-\theta_{T+s}W(\tilde{q}_i)).$$

For $\delta\to 0$ this sum tends in probability to

$$\int_0^\tau S(\tau-q)B(q+T,s,X_0))d\theta_{T+s}W(q).$$

(Indeed, the stochastic integral can be defined using random adapted partitions, see Protter [22], Chapter 2). Similarly, we can treat the non-stochastic integral in the formula defining the mild solution

$$\int_0^{T+\tau}S(T+\tau-q)F(\Phi(q,s,X_0))dq=S(\tau)\int_0^T S(T-q)F(\Phi(q,s,X_0))dq$$
$$+\int_0^\tau S(\tau-q)F(\Phi(q+T,s,X_0))dq$$

which shows that

$$\Phi(\tau + T, s, X_0) = \Phi(\tau, s + T, \Phi(T, s, X_0)) \quad \text{almost surely.}$$

□

Remark 1. Note that the assertion of the last Lemma is also true if we replace the stopping time T by a deterministic time $t \geq 0$.
We also note that the composition property of the last Lemma does not express any flow property (as required in the theory of random dynamical systems). Indeed, the exceptional sets for $\Phi(\cdot)$ may depend explicitly on the initial condition X_0.

3. **Existence of exponentially stable stationary solutions.** As we have already mentioned, the intention of this article is to find exponentially stable stationary solutions to (5), which means to find a solution process for which the finite dimensional distributions do not depend on time shifts. Assume, we can find an $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable random variable X^* such that $t \to X^*(\theta_t\omega)$ solves (5) (or more precisely, that this process has a version solving (5)), then we have for $0 \leq t_1 < t_2 < \cdots < t_n$

$$\mathbb{P}(X^*(\theta_{t_1}\omega) \in D_1, \cdots, X^*(\theta_{t_n}\omega) \in D_n) = \mathbb{P}(X^*(\theta_{t_1+t}\omega) \in D_1, \cdots, X^*(\theta_{t_n+t}\omega) \in D_n)$$

for any $t \geq 0$ and Borel sets $D_1, \cdots, D_n$ from H which follows directly from θ_t-invariance of $\mathbb{P}$. Hence X^* generates a stationary solution.
We are also interested in almost sure exponential stability which means that for any initial condition X_0 which is an $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable random variable

$$\lim_{t\to\infty} |\Phi(t, 0, X_0) - \Phi(t, 0, X^*)| = 0 \quad \text{exponentially fast,} \qquad \text{almost surely.} \tag{8}$$

The process $t \to \Phi(t, 0, X^*)$ is a continuous version of $t \to X^*(\theta_t\omega)$. The following theorem was proved in Caraballo et al. [4].

Theorem 2. *Assume that* $\mathbb{E}|X_0|^2 < \infty$ *and*

$$\mu := 2a + 2L + L_B < 0. \tag{9}$$

Then Eq. (5) has a unique stationary solution which is pathwise exponentially stable with probability one. Moreover, (8) also holds in the mean square sense.

Observe that as soon as condition (9) is satisfied, then necessarily the constant a has to be negative. We now consider the diffusion part of the stochastic partial differential equation

$$B(X)dW = B_1 X dw_1 + \cdots + B_N X dw_N,$$

where $w_1, \cdots, w_N$ are one dimensional mutually independent standard Wiener processes and $W = (w_1, \cdots, w_N)$ so that the phase space U for the Wiener process is given by $\mathbb{R}^N$, and $B_i \in \mathcal{L}(H)$ for $i = 1, \cdots, N$.
We will denote $b_i = \|B_i\|_{\mathcal{L}(H)}$.
We study the stochastic evolution equation

$$dX = (AX + F(X))dt + \sum_{i=1}^{N} B_i X dw_i. \tag{10}$$

Notice that, in this situation, the constant L_B is given by $L_B = \sum_{i=1}^N b_i^2$, so it may happen that $\mu = 2a + 2L + L_B > 0$, in which case we cannot apply Theorem 2. However, supposing that the linear operators B_i have a particular form, we will be able to prove that (10) possesses an exponentially stable stationary solution.

The following lemma will be crucial

Lemma 5. *Let $X_i(\cdot) = \Phi(t, 0, X^i)$, $i = 1, 2$ be two solutions of (5) with $(\bar{\mathcal{F}}_0, \mathcal{B}(H))$-measurable initial conditions X^i. Consider the stopping time*

$$T_0(\omega) = \inf\{t \geq 0 : |X_1(t,\omega) - X_2(t,\omega)| = 0\}$$

where $T_0 = \infty$ if $|X_1(t,\omega) - X_2(t,\omega)| > 0$ for any $t \geq 0$.
i) Let $R_K := T_0 \wedge K$, $K \in \mathbb{N}$, then

$$|X_1(R_K(\omega) + t, \omega) - X_2(R_K(\omega) + t, \omega)| = 0, \qquad t \geq 0$$

for almost all $\omega \in \{R_K < K\}$.
ii) For almost all $\omega \in \{T_0 < \infty\}$ we have for $t \geq 0$

$$|X_1(T_0(\omega) + t, \omega) - X_2(T_0(\omega) + t, \omega)| = 0.$$

Proof. By Lemma 4 we have almost surely

$$X_i(t + R_K) = \Phi(t + R_K, 0, X^i) = \Phi(t, R_K, \Phi(R_K, 0, X^i)).$$

The process $Y_i(\cdot) = \Phi(\cdot, R_K, \Phi(R_K, 0, X^i))$ solves (5) driven by the Wiener process $\theta_{R_K}W$ with initial condition $\Phi(R_K, 0, X^i)$. For $\omega \in \{R_K < K\}$ we have

$$\Phi(R_K, 0, X^1) = \Phi(R_K, 0, X^2).$$

Corollary 1 gives us i).
ii) follows easily from i) because

$$\lim_{K\to\infty} R_K = T_0.$$

□

To prove our stability result, we need the following martingale inequality (see, e.g. Liu and Mao [20])

Lemma 6. *Let $M(t)$ be a continuous local martingale. Then for any positive constants k, γ and δ we have*

$$\mathbb{P}\left(\sup_{t\in[0,k]} (M(t) - \frac{\gamma}{2}\langle M\rangle_t) > \delta\right) < e^{-\gamma\delta},$$

where $\langle M\rangle_t$ denotes the quadratic variation process associated to $M(t)$.

The following result is the main theorem in this article. It is worth mentioning that, in particular, this result also completes and improves similar ones from Liu and Mao [20] and Caraballo *et al.* [7] (among others) when $F(0) = 0$, since our result holds for any initial datum, while in the mentioned papers it was proved only for those initial values u_0 so that its corresponding solution satisfies $|u(t)| > 0$ for all $t > 0$ and almost surely, which may be a severe restriction in general.

Theorem 3. *Assume that B_i, $i = 1, \cdots, N$ are linear bounded operators on H and that there exists β_i such that*

$$\beta_i|u|^2 \leq (B_i u, u) \;\; (\leq b_i|u|^2), \qquad \text{for } i = 1, \cdots, N, \; u \in H.$$

Then, there exists a random variable X^* generating a stationary solution to (10) which is almost surely exponentially stable provided the constants β_i are such that

$$2a + 2L + \sum_{i=1}^{N}(b_i^2 - 2\beta_i^2) = \mu - 2\sum_{i=1}^{N}\beta_i^2 =: \bar{\mu} < 0. \tag{11}$$

Proof. Let

$$X_1(\cdot) = \Phi(\cdot, 0, x),\ X_2(\cdot) = \Phi(\cdot, 0, \Phi(1, -1, x)) = \Phi(\cdot + 1, -1, x)$$

be two continuous solutions of (5) with initial conditions x, $\Phi(1,-1,x)$ where x is any element in H. To avoid later on that some logarithm might be $-\infty$, we modify the initial condition $X_1(0)$ as

$$X_1^0(\omega) = \begin{cases} x & : \quad \Phi(1,-1,x)(\omega) \neq x \\ x_1 & : \quad \Phi(1,-1,x)(\omega) = x \end{cases}$$

for some $x_1 \neq x$. This random variable belongs to $L_{2,0}$. The solution for this initial condition is denoted by X_1^0.
We introduce the stopping times T_0, T_0^0, T_n^0, $n \in \mathbb{N}$

$$\begin{aligned} T_0(\omega) &= \inf\{t \geq 0 : |X_1(t,\omega) - X_2(t,\omega)|^2 = 0\} \\ T_n^0(\omega) &= \inf\{t \geq 0 : |X_1^0(t,\omega) - X_2(t,\omega)|^2 \leq \frac{1}{n}\} \\ T_0^0(\omega) &= \inf\{t \geq 0 : |X_1^0(t,\omega) - X_2(t,\omega)|^2 = 0\}. \end{aligned} \tag{12}$$

We set $T_n^0(\omega) = \infty$ and $T_0^0(\omega) = \infty$ if there does not exist any $t \geq 0$ such that $|X_1^0(t,\omega) - X_2(t,\omega)|^2 \leq \frac{1}{n}$ and $|X_1^0(t,\omega) - X_2(t,\omega)|^2 = 0$.
We deduce from Ito's formula

$$\begin{aligned} \log&|X_1^0(t\wedge T_n^0) - X_2(t\wedge T_n^0)|^2 = \log|X_1^0 - \Phi(1,-1,x)|^2 \\ &+ \int_0^{t\wedge T_n^0} \frac{2\langle A(X_1^0 - X_2), X_1^0 - X_2\rangle + 2(X_1^0 - X_2, F(X_1^0) - F(X_2))}{|X_1^0 - X_2|^2} ds \\ &+ \int_0^{t\wedge T_n^0} \sum_{i=1}^{N} \frac{\|B_i(X_1^0 - X_2)\|^2_{\mathcal{L}(H)}}{|X_1^0 - X_2|^2} ds + M_n(t) - \frac{1}{2}q_n(t), \end{aligned}$$

where

$$\begin{aligned} M_n(t) &= \sum_{i=1}^{N} \int_0^{t\wedge T_n^0} \frac{2(X_1^0 - X_2, B_i(X_1^0 - X_2))}{|X_1^0 - X_2|^2} dW_i, \\ q_n(t) &= \sum_{i=1}^{N} \int_0^{t\wedge T_n^0} \frac{4(X_1^0 - X_2, B_i(X_1^0 - X_2))^2}{|X_1^0 - X_2|^4} ds = \langle M_n\rangle_t. \end{aligned}$$

Note that M_n is a continuous local martingale and q_n is its quadratic variation. Now, we apply the exponential martingale inequality Lemma 6 taking $\gamma = \varepsilon$, $\delta = \frac{2}{\varepsilon}\log k$, $k \geq 1$ for an $\varepsilon > 0$ to be fixed later on. We obtain, since $t \mapsto q_n(t)$ is non-decreasing,

$$\mathbb{P}\left(\omega \in \Omega : \sup_{\tau \in [k,k+1]} \Big(M_n(\tau,\omega) - \frac{\varepsilon}{2}q_n(k+1,\omega)\Big) > \delta\right) \leq \frac{1}{k^2} \tag{13}$$

and, similarly, by the θ_{-k}-invariance of $\mathbb{P}$,

$$\mathbb{P}\Big(\omega \in \Omega :\ M_n(k, \theta_{-k}\omega) - \frac{\varepsilon}{2}q_n(k, \theta_{-k}\omega) > \delta\Big) \leq \frac{1}{k^2}.$$

On account of the Borel Cantelli Lemma and (13), we can assure that for almost all $\omega\in\Omega$ there exists a $k_0(\omega)\in\mathbb{N}$ such that

$$M_n(k,\theta_{-k}\omega)\le\frac{\varepsilon}{2}q_n(k,\theta_{-k}\omega)+\frac{2}{\varepsilon}\log(k+1),$$

for $k\ge k_0(\omega)$.
Hence, we have for k large enough

$$\begin{aligned}&|X_1^0(k\wedge T_n^0(\theta_{-k}\omega),\theta_{-k}\omega)-X_2(k\wedge T_n^0(\theta_{-k}\omega),\theta_{-k}\omega)|^2\\&\le|X_1^0(\theta_{-k}\omega)-\Phi(1,-k-1,x)(\omega)|^2\times\\&\times\exp\left(\bar\mu(k\wedge T_n^0(\theta_{-k}\omega))+\frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)((k+1)\wedge T_n^0(\theta_{-k}\omega))+\frac{2}{\varepsilon}\log(k+1)\right)\end{aligned}$$

and, since X_1^0, X_2 have continuous paths, it follows for $n\to\infty$

$$\begin{aligned}&|X_1^0(k\wedge T_0^0(\theta_{-k}\omega),\theta_{-k}\omega)-X_2(k\wedge T_0^0(\theta_{-k}\omega),\theta_{-k}\omega)|^2\\&\le|X_0^1-\Phi(1,-k-1,x)(\omega)|^2\times\\&\times\exp\left(\bar\mu(k\wedge T_0^0(\theta_{-k}\omega))+\frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)((k+1)\wedge T_0^0(\theta_{-k}\omega))+\frac{2}{\varepsilon}\log(k+1)\right).\end{aligned}$$

Let $R_k(\omega)=T_0(\omega)\wedge k$. If $R_k(\theta_{-k}\omega)=k$ then $T_0(\theta_{-k}\omega)\ge k$, hence we have $X_1^0(\cdot,\theta_{-k}\omega)=X_1(\cdot,\theta_{-k}\omega)$ on $[0,k]$ by Lemma 1. Thus

$$\begin{aligned}&|X_1(k,\theta_{-k}\omega)-X_2(k,\theta_{-k}\omega)|^2\\&\le\ |x-\Phi(1,-k-1,x)(\omega)|^2\exp\left(\bar\mu\,k+\frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)(k+1)+\frac{2}{\varepsilon}\log(k+1)\right).\end{aligned}$$

If $R_k(\theta_{-k}\omega)<k$ then by Lemma 5 for $\omega=\theta_{-k}\omega$ and $k=K$ this inequality is trivially fulfilled for almost all ω with $R_k(\theta_{-k}\omega)<k$.
As $\mu-2\sum_{i=1}^N\beta_i^2<0$, we can choose $\varepsilon>0$ small enough such that

$$\frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)+\mu-2\sum_{i=1}^N\beta_i^2<0.$$

We also have that $\mathbb{E}|\Phi(1,0,x)|^2<\infty$ so that

$$\lim_{k\to\infty}|x-\Phi(1,-k-1,x)|^2e^{\frac{1}{2}k\nu}=0\qquad\text{almost surely,}$$

for every $\nu>0$, by Chebyshev's inequality and the Borel–Cantelli lemma. It is easily seen that

$$(\Phi(k,-k,x))_{k\in\mathbb{N}}\tag{14}$$

is a Cauchy sequence which has the limit $X^*(\omega)$ almost surely if $\varepsilon>0$ is sufficiently small.

We also have

$$X^*(\theta_t\omega)=\Phi(t,0,X^*)(\omega)\qquad\text{almost surely}$$

for every $t\ge0$. First we note the convergence for the sequence of (14) is also true if we replace k by $k+t$ and $t>0$ fixed. In particular, we have the limit (almost surely) which exists on a set $\Omega_{0,x}$ of full measure. In addition, we obtain

$$\lim_{k\to\infty}\Phi(t+k,-t-k,x)(\theta_t\cdot)=X^*(\theta_t\omega)$$

on some set of measure one. In particular, the limit exists for $\omega \in \theta_{-t}\Omega_{0,x}$ with measure one.
On the other hand, from Lemma 4 and Lemma 3, it follows

$$\begin{aligned}(\mathbb{P}) \lim_{k\to\infty} \Phi(t+k,-t-k,x)(\theta_t\cdot) &= (\mathbb{P}) \lim_{k\to\infty} \Phi(t+k,-k,x)(\cdot)\\ &= (\mathbb{P}) \lim_{k\to\infty} \Phi(t,0,\Phi(k,-k,x))(\cdot) = \Phi(t,0 \lim_{k\to\infty} \Phi(k,-k,x))(\cdot)\\ &= \Phi(t,0,X^*)(\cdot)\end{aligned}$$

where this chain of equations is satisfied almost surely.

It remains to prove that for any $(\bar{\mathcal{F}}_0,\mathcal{B}(H))$-measurable random variable X

$$\lim_{t\to\infty} |\Phi(t,0,X)-\Phi(t,0,X^*)| = 0, \qquad \text{almost surely.}$$

Let X^0 be an $(\bar{\mathcal{F}}_0,\mathcal{B}(H))$-measurable modification so that $X^0(\omega) \neq X^*(\omega)$.

$$X^0(\omega) = \begin{cases} X(\omega) & : \quad X(\omega) \neq X^*(\omega)\\ x & : \quad X(\omega) = X^*(\omega) \neq x\\ x_1 & : \quad X(\omega) = X^*(\omega) = x\end{cases}$$

for some $x \neq x_1 \in H$. Let us abbreviate

$$X_1 = \Phi(\cdot,0,X),\ X_2 = \Phi(\cdot,0,X^*),\ X_1^0 = \Phi(\cdot,0,X^0)$$

For these processes we will define the stopping times T_n^0, T_0^0, T_0 similar (12). Following the first part of the proof we apply Ito's formula to $\log|X_1^0(t,\omega)-X_2(t,\omega)|^2$. We obtain from (13)

$$\begin{aligned}\sup_{t\in[k,k+1]} |X_1^0(t,\omega)-X_2(t,\omega)|^2 &\leq |X^0(\omega)-X^*(\omega)|^2\times\\ &\times \exp\left(\bar{\mu}(k\wedge T_0^0(\omega)) + \frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)((k+1)\wedge T_0^0(\omega)) + \frac{2}{\varepsilon}\log(k+1)\right)\end{aligned}$$

almost surely. For $\omega \in \{T_0=\infty\} \subset \{T_0^0=\infty\}$ we have $X_1(\cdot,\omega) = X_1^0(\cdot,\omega)$ such that

$$\begin{aligned}\sup_{t\in[k,k+1]} |X_1(t,\omega)-X_2(t,\omega)|^2 &\leq |X(\omega)-X^*(\omega)|^2\times\\ &\times \exp\left(\bar{\mu}k + \frac{\varepsilon}{2}\left(\sum_{i=1}^N 4b_i^2\right)(k+1) + \frac{2}{\varepsilon}\log(k+1)\right)\end{aligned}$$

almost surely. If $\{T_0<\infty\}$ then by Lemma 5 ii) we have $X_1(t,\omega) = X_2(t,\omega)$ for sufficiently large t so that the above inequality is trivially satisfied.
The exponential convergence follows immediately. □

4. **Stabilization in some applications.** We now illustrate our theorems with several applications. First, we will show how the existence of exponentially stable stationary solutions can be ensured by perturbing a deterministic problem by noise. Then, we will show that, for instance, our results improve those ones in [18], where the linear situation was considered and the stability referred to the null solution of the linear equation (i.e. when $F=0$).

Example 1. Let us consider $H = L^2(\mathcal{O})$ and $V = H_0^1(\mathcal{O})$ where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary. Consider the Laplace operator $A = \Delta$ with the Dirichlet boundary condition. Theorem IX.31 in Brezis [3] ensures the existence of a sequence of real numbers $\{\nu_n\}_{n\geq 1}$ such that $0 < \nu_1 < \nu_2 < \cdots < \nu_n < \cdots$, and $\nu_n \to +\infty$ (the eigenvalues of the negative Laplacian), and a sequence $\{e_n\}_{n\geq 1} \subset V \cap C^\infty(\mathcal{O})$ of associated eigenvectors (i.e. $-\Delta e_n = \nu_n e_n$ on $\mathcal{O}$) which is a complete orthonormal basis in H.
In addition, we consider some linear operators $B_i : H \to H$, $i = 1, \cdots, N$, which are bounded below (and consequently, below and above), i.e. there exist $\beta_1, ..., \beta_N > 0$ such that

$$0 < \beta_k |u|^2 \leq (B_k u, u) \quad (\leq \|B_k\| \, |u|^2), \quad k = 1, ..., N, \quad u \in H.$$

Also, we consider a Lipschitz continuous function f from H into H defined as $f(u)(x) = F(u(x))$, $x \in \mathcal{O}$ with $F : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with constant L. Recall (see [3]) that $\langle Au, u\rangle \leq -\nu_1 |u|^2$ for all $u \in V$, and that

$$\nu_1 |u|^2 \leq \|u\|^2, \quad \text{for all } \ u \in V.$$

Then, the constant $\overline{\mu}$ in Theorem 3 is given by

$$\overline{\mu} = -2\nu_1 + 2L + \sum_{k=1}^{N} \left(\|B_k\|^2 - 2\beta_k^2 \right). \tag{15}$$

Now, if our goal is to find operators B_k so that the deterministic problem

$$\frac{dX}{dt} = AX + f(X), \tag{16}$$

has an exponentially stable stationary solution by adding a stochastic perturbation, i.e., so that the equation

$$dX = (AX + f(X))\, dt + \sum_{k=1}^{N} B_k X dw_k \tag{17}$$

possesses an exponentially stable stationary solution, it is easy to see that we can just use a single operator B_i defined in a very simple way. Indeed, take B_1 defined as $B_1 u = \beta_1 u$, for some positive constant β_1, and define $B_k = 0$ for $k = 2, ..., N$. Then, the constant $\overline{\mu}$ in (15) becomes

$$\overline{\mu} = -2\nu_1 + 2L - \beta_1^2,$$

and obviously $\overline{\mu} < 0$ provided β_1 is large enough. Therefore, to ensure the existence of exponentially stable stationary solutions, we only need to add a very simple noise to the deterministic equation.
To clarify a bit more how our results work, we can consider the particular case in which $\mathcal{O} = [0, \pi]$, so that $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and it is straightforward to check that the first eigenvalue of $-\Delta$ with homogeneus Dirichlet boundary condition is $\nu_1 = 1$. As for the function f we consider $f(u) = 3u + \alpha$ for some $\alpha > 0$, which is obviously Lipschitz continuous with constant $L = 3$. Considering again B_1 defined as $B_1 u = \beta_1 u$, for some positive constant β_1, and $B_k = 0$ for $k = 2, ..., N$, it follows that the constant $\overline{\mu}$ in (15) becomes

$$\overline{\mu} = 4 - \beta_1^2,$$

and taking, for example, $\beta_1 > 2$, we obtain the existence of a unique pathwise exponentially stable stationary solution of our problem. Moreover, when $\alpha = 0$, then this unique stationary solution is the null solution. In this latter case, this zero solution is unstable for the deterministic problem (16) and pathwise exponentially stable as solution of (17).

Example 2. As we have already mentioned, our main result provides much more information concerning the stability of stationary solutions to stochastic evolution equations and, in particular, improves several well known results. For instance, let us consider the linear situation from [18]. In other words, we have a linear unbounded operator A which is the generator of a C_0-semigroup $S(t)$ in the Hilbert space H, satisfying

$$|S(t)| \leq e^{at}, \quad t \geq 0,$$

where $a \in \mathbb{R}$, and a finite number of linear operators B_k which are diagonalizable and bounded from above and below, i.e., there exist a basis $\{e_i\}$ of the Hilbert space H, and constants $d_i^k, k = 1, \ldots, N$ such that

$$(B_k e_i, e_j) = d_i^k \delta_{ij}, \qquad k = 1, \ldots, N; \quad i, j = 1, 2, \ldots$$

The assumption that operators B_k are bounded from above and below is equivalent to the condition that there exist positive constants β_k, b_k such that

$$0 < \beta_k \leq |d_i^k| \leq b_k, \quad k = 1, \ldots, N; \quad i = 1, 2, \ldots$$

It is worth pointing out that a commutativity assumption between A and operators B_k is imposed in [18] in order to obtain the stabilization result. However, we do not need this assumption in our framework. To this end, let us now study the linear stochastic evolution equation

$$dX = AXdt + \sigma \sum_{k=1}^{N} B_k X dw_k, \quad \sigma \in \mathbb{R}. \tag{18}$$

In this situation, we have that the constant $\overline{\mu}$ in Theorem 3 becomes

$$\overline{\mu} = 2a + \sigma^2 \sum_{k=1}^{N} \left(b_k^2 - 2\beta_k^2 \right),$$

and if $\overline{\mu} < 0$, Theorem 3 implies the existence of a unique stationary solution to our problem which is exponentially stable with probability one. As the system is linear, then this stationary solution is zero. This result allows us to prove stability of the null solution to (18) for a more general class of operators than the one considered in [18], and it is also valid for semilinear equations, a case in which we have nontrivial stationary solutions.

5. **Conclusions and final comments.** We have proved some results ensuring the existence, uniqueness and exponential stability of non-trivial stationary solutions to a semilinear stochastic evolution equation. In fact, we have proved a kind of stabilization effect produced by the noise considered in the Ito sense when it is added to a deterministic evolution equation. However, it would be very interesting to prove the same type of result by interpreting the noise in the sense of Stratonovich, since in this case the stabilization effect comes properly from the noisy term, and not from the systematic dissipative one arising from the application of the Ito formula (for more comments and results on this see Arnold et al. [2] in the finite-dimensional framework, and Caraballo and Robinson [6] for the infinite-dimensional one).

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