Pullback attractors of nonautonomous and stochastic multivalued dynamical systems

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Abstract

In this paper we study the existence of pullback global attractors for multivalued processes generated by differential inclusions. First, we define multivalued dynamical processes, prove abstract results on the existence of ω-limit sets and global attractors and study their topological properties (compactness, conectedness). Further, we apply the abstract results to nonautonomous differential inclusions of the reaction-diffusion type in which the forcing term can grow polynomially in time, and to stochastic differential inclusions as well.

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1 Introduction

In this paper we study the existence of pullback global attractors for multivalued processes generated by differential inclusions. The theory of pullback attractors has been developed for stochastic and nonautonomous systems in which the trajectories can be unbounded when times rises to infinite. In such systems the classical theory of global attractors is not applicable. Hence, a different approach has been considered (see [8, 9] for the stochastic case and [9, 14, 18] for the nonautonomous case). The global attractor is defined as a parameterized family of sets $\mathcal{A}(\sigma)$, which attracts the solutions of the system "from $-\infty$". This means that the initial moment of time goes to $-\infty$ and the final time remains fixed.

A new difficulty appears if the solution corresponding to each initial state can be non-unique. The classical results on attractors in the autonomous and nonautonomous cases are generalized to the multivalued case in [16] and [17], respectively, with applications to evolution inclusions.

In [4, 5, 6] the study of multivalued dynamical systems is extended to the stochastic case, generalizing in this way the results of [8, 9].

In this paper we are mainly concerned with nonautonomous multivalued dynamical systems in which the trajectories can be unbounded in time and also with nonautonomous stochastic multivalued dynamical systems.

In the second section we define multivalued dynamical processes, prove abstract results on the existence of ω-limit sets and global attractors and study their topological properties (compactness, conectedness). In the third section we apply the abstract results to nonautonomous differential inclusions of the reaction-diffusion type in which the forcing term can grow polynomially in time. It is worth pointing out that the multivalued dynamical process is defined as a two-parameter family of multivalued maps. The attraction of any bounded set of the phase space to the global attractor is uniform with respect to the first one, whereas the rate of attraction and the attractor itself can depend on the second one. In the applications the nonlinear and forcing terms are split in the sum of two functions. The first one satisfies some good properties which allow to obtain a compact global attractor (in the classical sense) if the second function vanishes (see [17]). However, the second one can growth unboundedly when time goes to infinity. The uniform attraction with respect to the first parameter means that, if we take translations in time of the first function, the rate of attraction and the global attractor itself do not change. However, they can depend on the translations in time with respect to the second function. Finally, in Section 4, we extend the previous theory to cover the cases in which some stochastic terms may appear in the model.

2 Attractors for multivalued processes

In this section we shall define multivalued dynamical processes in metric spaces. Maps of this kind appear in differential equations for which, although we are able to prove the existence of at least one global solution for each inital condition in some phase space, we do not know if it is unique or not. Hence, multivalued processes generalize the concept of processes, for which the uniqueness property holds [7].

In this way we prove the existence of the so called "pullback" attractors [9, 14, 18], generalizing similar results for processes.

2.1 Multivalued dynamical processes in infinite-dimensional spaces

Let X be a complete metric space with the metric ρ and let $P(X)$ be the set of all non-empty subsets of X. Denote

$$\begin{aligned}
\mathcal{B}(X) &= \{A \in P(X) : A \text{ is bounded}\},\\
C(X) &= \{A \in P(X) : A \text{ is closed}\},\\
C_v(X) &= \{A \in P(X) : A \text{ is bounded, closed and convex}\}\\
K(X) &= \{A \in P(X) : A \text{ is compact}\},\\
\mathbb{R}_d &= \left\{(t,s) \in \mathbb{R}^2 : t \geq s\right\},\\
\mathbb{R}(\tau) &= \{t \in \mathbb{R} : t \geq \tau\},\\
dist(A,B) &= \sup_{x\in A}\inf_{y\in B} \rho(x,y), \text{ for } A, B \subset X\\
\text{dist}_H(A,B) &= \max\{dist(A,B), dist(B,A)\}, \text{ for } A, B \subset X.
\end{aligned}$$

Definition 1 *The map $U : \mathbb{R}_d \times X \to P(X)$ is called a multivalued dynamical process (MDP) on X if:*

1. *$U(t,t,\cdot) = Id$ is the identity map;*
2. *$U(t,s,x) \subset U(t,\tau,U(\tau,s,x))$, for all $x \in X, s \leq \tau \leq t$.*

The MDP U is called strict if:

$$U(t,s,x) = U(t,\tau,U(\tau,s,x)), \textit{for all } x \in X, s \leq \tau \leq t.$$

Consider a parameter set Σ. The following proposition is straightforward to prove.

Proposition 2 *Let $\{U_\sigma : \sigma \in \Sigma\}$ be an arbitrary family of MDP. Then the map $U_\Sigma : \mathbb{R}_d \times X \to P(X)$ defined by*

$$U_\Sigma(t,s,x) = \bigcup_{\sigma\in\Sigma} U_\sigma(t,s,x)$$

is a MDP.

Let $\Sigma = \Sigma_1 \times \Sigma_2$. For any $\sigma_2 \in \Sigma_2$ consider the MDP $U_{\Sigma_1,\sigma_2} : \mathbb{R}_d \times X \to P(X)$, where

$$U_{\Sigma_1,\sigma_2}(t,s,x) = \bigcup_{\sigma_1\in\Sigma_1} U_{\sigma_1,\sigma_2}(t,s,x).$$

Definition 3 *Let $t \in \mathbb{R}, \sigma_2 \in \Sigma_2$. The set $D(t, \sigma_2) \subset X$ attracts the set $B \in \mathcal{B}(X)$ uniformly with respect to Σ_1 at time t if:*

$$\lim_{s \to -\infty} dist\left(U_{\Sigma_1, \sigma_2}(t, s, B), D(t, \sigma_2)\right) = 0. \tag{1}$$

Definition 4 *Let $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$. The set $D(t, \sigma_2)$ is said to be Σ_1-uniformly attracting at time t if (1) is satisfied for any $B \in \mathcal{B}(X)$.*

For $B \in \mathcal{B}(X), \sigma_2 \in \Sigma_2$ and $t \in \mathbb{R}$ put

$$\begin{aligned} \gamma_{\Sigma_1}^{s}(t, \sigma_2, B) &= \bigcup_{\tau \le s} U_{\Sigma_1, \sigma_2}(t, \tau, B), \\ \omega_{\Sigma_1}(t, \sigma_2, B) &= \bigcap_{s \le t} \overline{\gamma_{\Sigma_1}^{s}(t, \sigma_2, B)}. \end{aligned}$$

The set $\omega_{\Sigma_1}(t, \sigma_2, B)$ is called the $\omega-$limit set of B for σ_2 at time t (with respect to Σ_1).

Lemma 5 *The following properties are equivalent:*

1. *$y \in \omega_{\Sigma_1}(t, \sigma_2, B)$;*
2. *There exists a sequence (τ_n, ξ_n) such that $\xi_n \in U_{\Sigma_1, \sigma_2}(t, \tau_n, B)$, $\xi_n \to y$ in X and $\tau_n \to -\infty$.*

Proof. In the space X consider the sequence of sets $\left\{\gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)\right\}$ for $s_n \to -\infty$. Then y belongs to the lower topological limit of Kuratowski

$$\underline{Lim}_{s_n \to -\infty} \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)$$

if for any $\varepsilon > 0$ there exists s_{n_0} such that $O_\varepsilon(y) \cap \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B) \neq \emptyset$, for all $s_n < s_{n_0}$, where $O_\varepsilon(y)$ is an ε-neighborhood of y. On the other hand, y belongs to the upper topological limit of Kuratowski

$$\overline{Lim}_{s_n \to -\infty} \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)$$

if for any $\varepsilon > 0$ there exists a subsequence $\{s_{n_k}\}$ and s_{n_0} such that $O_\varepsilon(y) \cap \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B) \neq \emptyset$, for all $s_{n_k} < s_{n_0}$. Since $\gamma_{\Sigma_1}^{s_1}(t, \sigma_2, B) \subset \gamma_{\Sigma_1}^{s_2}(t, \sigma_2, B)$ if $s_1 \le s_2$, we have

$$\underline{Lim}_{s_n \to -\infty} \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B) = \overline{Lim}_{s_n \to -\infty} \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)$$

$$= Lim_{s_n \to -\infty} \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B) = \bigcap_{s \le t} \overline{\gamma_{\Sigma_1}^{s}(t, \sigma_2, B)} = \omega_{\Sigma_1}(t, \sigma_2, B). \tag{2}$$

Let now $y \in \omega_{\Sigma_1}(t, \sigma_2, B)$. Then (2) implies that for $\varepsilon_n \to 0$ we can find a sequence (s_n, ξ_n) such that $\xi_n \in O_{\varepsilon_n}(y) \cap \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)$, $s_n \to -\infty$. It follows that $\xi_n \in U_{\Sigma_1, \sigma_2}(t, \tau_n, B)$, for some $\tau_n \le s_n$, and $\xi_n \to y$, as $n \to \infty$. We have proved in this way the implication $1 \Longrightarrow 2$.

Conversely, let the sequence $\{\xi_n\}$ satisfy the second condition. Then $\xi_n \in \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B)$ for some $s_n \to -\infty$. The convergence $\xi_n \to y$ implies that for any $\varepsilon-$neighborhood $O_\varepsilon(y)$ of y there exists s_{n_0} for which $O_\varepsilon(y) \cap \gamma_{\Sigma_1}^{s_n}(t, \sigma_2, B) \neq \emptyset$, for all $s_n < s_{n_0}$, so that in view of (2), $y \in \omega_{\Sigma_1}(t, \sigma_2, B)$. ■

Theorem 6 *Suppose that for $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$ and $B \in \mathcal{B}(X)$ there exists $D(t, \sigma_2, B) \in K(X)$ such that*

$$\lim_{s \to -\infty} dist\left(U_{\Sigma_1, \sigma_2}(t, s, B), D(t, \sigma_2, B)\right) = 0. \tag{3}$$

Then $\omega_{\Sigma_1}(t, \sigma_2, B)$ is non-empty, compact and the minimal closed set Σ_1-uniformly attracting B at time t.

Proof. First we shall show that $\omega_{\Sigma_1}(t,\sigma_2,B)$ is non-empty. If it is empty, then in view of Lemma 5, the sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, where $s_n \to -\infty$, has not converging subsequences. But from (3) we get

$$dist\,(\xi_n, D(t,\sigma_2,B)) \to 0, \text{ as } s_n \to \infty. \tag{4}$$

Hence, there exist $\alpha_n \to 0$ and $\zeta_n \in D(t,\sigma_2,B)$ such that $\rho\left(\xi_{n,}\zeta_n\right) < \alpha_n$, for all n. It follows from the compactness of the set $D(t,\sigma_2,B)$ that $\{\xi_n\}$ has a converging subsequence ξ_{n_k}, which is a contradiction.

¿From (4) it follows that if $y = \lim_{s_n\to-\infty}\xi_n$, where $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, then $y \in D(t,\sigma_2,B)$. Hence, using again Lemma 5 we obtain $\omega_{\Sigma_1}(t,\sigma_2,B) \subset D(t,\sigma_2,B)$, so that $\omega_{\Sigma_1}(t,\sigma_2,B)$ is compact.

Suppose now that $\omega_{\Sigma_1}(t,\sigma_2,B)$ does not attract B at time t uniformly with respect to Σ_1. Then we can find $\varepsilon > 0$ and a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, where $s_n \to -\infty$, such that $dist\,(\xi_n, \omega_{\Sigma_1}(t,\sigma_2,B)) > \varepsilon$, for all n. But condition (3) implies, arguing as before, that $\{\xi_n\}$ has a converging subsequence $\left\{\xi_{n_k}\right\}$. Finally, thanks to Lemma 5 we have $\xi_{n_k} \to y \in \omega_{\Sigma_1}(t,\sigma_2,B)$, which gives us a contradiction.

Further, let us consider a closed set Y satisfying

$$\lim_{s\to-\infty} dist\,(U_{\Sigma_1,\sigma_2}(t,s,B), Y) = 0. \tag{5}$$

We have to prove that $\omega_{\Sigma_1}(t,\sigma_2,B) \subset Y$. By Lemma 5 for any $y \in \omega_{\Sigma_1}(t,\sigma_2,B)$ we can obtain a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$ converging to y as $s_n \to -\infty$. Take an arbitrary $\varepsilon > 0$. In view of (5) there exists n_0 such that $dist\,(\xi_n, Y) < \frac{\varepsilon}{2}$ and $\rho(y,\xi_n) < \frac{\varepsilon}{2}$, for all $n > n_0$. Therefore,

$$dist\,(y,Y) \le \rho(y,\xi_n) + dist\,(\xi_n,Y) < \varepsilon.$$

Since Y is closed, we finally obtain that $y \in Y$. ■

Definition 7 *The family of MDP $\{U_\sigma\}$ is called Σ_1-uniformly asymptotically upper semicompact if for any $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$ and $B \in \mathcal{B}(X)$ there exists $t_0 = t_0(t,\sigma_2,B)$ such that $\gamma_{\Sigma_1}^{t_0}(t,\sigma_2,B) \in \mathcal{B}(X)$ and any sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, where $s_n \to -\infty$, is precompact.*

Lemma 8 *The family of MDP $\{U_\sigma\}$ is Σ_1-uniformly asymptotically upper semicompact if and only if for any $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$ and $B \in \mathcal{B}(X)$ there exists $D(t,\sigma_2,B) \in K(X)$ satisfying (3).*

Proof. Let the family $\{U_\sigma\}$ be Σ_1-uniformly asymptotically upper semicompact. Then, in view of Lemma 5, the ω-limit set $\omega_{\Sigma_1}(t,\sigma_2,B)$ is non-empty. We shall first prove that it is compact. Indeed, for any sequence $\{\xi_n\} \subset \omega_{\Sigma_1}(t,\sigma_2,B)$ we have $\{\xi_n\} \subset \overline{\gamma_{\Sigma_1}^{s}(t,\sigma_2,B)}$, for all $s \le t$, and then there exist $\zeta_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, $s_n \to -\infty$, such that $\rho(\xi_n,\zeta_n) < \frac{1}{n}$. Since U_σ is Σ_1-uniformly asymptotically upper semicompact, it is possible to extract a subsequence $\left\{\zeta_{n_k}\right\}$ converging to some $y \in X$. By Lemma 5, $y \in \omega_{\Sigma_1}(t,\sigma_2,B)$, so that the compactness follows.

Further, we have to check that $\omega_{\Sigma_1}(t,\sigma_2,B)$ is Σ_1-uniformly attracting. Suppose the opposite. Then we can find $\varepsilon > 0$ and a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, where $s_n \to -\infty$, such that $dist\,(\xi_n, \omega_{\Sigma_1}(t,\sigma_2,B)) > \varepsilon$, for all n. Since U_σ is Σ_1-uniformly asymptotically upper semicompact, $\{\xi_n\}$ has a converging subsequence $\left\{\xi_{n_k}\right\}$. Thanks to Lemma 5 we have $\xi_{n_k} \to y \in \omega_{\Sigma_1}(t,\sigma_2,B)$, which gives us a contradiction.

Now we can see that the set $D(t,\sigma_2,B) = \omega_{\Sigma_1}(t,\sigma_2,B) \in K(X)$ satisfies (3).

Conversely, let for any $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$ and $B \in \mathcal{B}(X)$ there exist $D(t,\sigma_2,B) \in K(X)$ satisfying (3). We note that for $\varepsilon > 0$ there exists s_0 for which $U_{\Sigma_1,\sigma_2}(t,s,B) \subset O_\varepsilon(D(t,\sigma_2,B))$, for all $s \le s_0$, where $O_\varepsilon(A) = \{z \in X : dist\,(z,A) < \varepsilon\}$ is an ε-neighborhood. Since $O_\varepsilon(D(t,\sigma_2,B))$ is a bounded set, we have $\gamma_{\Sigma_1}^{s_0}(t,\sigma_2,B) \in \mathcal{B}(X)$.

Finally, let us take an arbitrary sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$, where $s_n \to -\infty$. From (3) we get

$$dist\,(\xi_n, D(t,\sigma_2,B)) \to 0, \text{ as } s_n \to -\infty.$$

Hence, there exist $\alpha_n \to 0$ and $\zeta_n \in D(t,\sigma_2,B)$ such that $\rho\left(\xi_{n,}\zeta_n\right) < \alpha_n$, for all n. It follows from the compacity of the set $D(t,\sigma_2,B)$ that $\{\xi_n\}$ has a converging subsequence ξ_{n_k}, so that U_σ is Σ_1-uniformly asymptotically upper semicompact. ■

2.2 Global attractors of multivalued dynamical processes

In this section we define the concept of global attractor of a family of MDP, prove its existence and study its topological properties.

Definition 9 *The family of sets* $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R}}$ *is called a* Σ_1*-uniform global attractor of the MDP* $\{U_\sigma\}$ *for* $\sigma_2\in\Sigma_2$ *if:*

1. $\Theta_{\Sigma_1}(t,\sigma_2)$ *is* Σ_1*-uniformly attracting at time* t *for all* $t\in\mathbb{R}$*;*
2. *It is semi-invariant, that is,*
$$\Theta_{\Sigma_1}(t,\sigma_2)\subset U_{\Sigma_1,\sigma_2}(t,s,\Theta_{\Sigma_1}(s,\sigma_2)),\ \text{for any } (t,s)\in\mathbb{R}_d;$$
3. *It is minimal, that is, for any closed* Σ_1*-uniformly attracting set* Y *at time* t*, we have*
$$\Theta_{\Sigma_1}(t,\sigma_2)\subset Y.$$

Definition 10 *Let* X,Y *be metric spaces. The multivalued map* $F:X\to P(Y)$ *is said to be upper semicontinuous if for all* $x\in X$ *and any neighbourhood of* $F(x)$*,* $\mathcal{O}(F(x))$*, there exists* $\delta>0$ *such that if* $\rho(x,z)<\delta$*, then*
$$F(z)\subset\mathcal{O}(F(x)).$$
On the other hand, F *is called lower semicontinuous if for all* $x\in X$*,* $x_n\to x$ *and* $y\in F(x)$*, there exists a sequence* $\{y_n\}$ *such that* $y_n\in F(x_n)$ *and* $y_n\to y$*.*

It is said to be continuous if it is upper and lower semicontinuous.

Theorem 11 *Let* X *be a complete metric space in which every compact set is nowhere dense and let the family of MDP* $\{U_\sigma\}$ *be* Σ_1*-uniformly asymptotically upper semicompact. Then the following statements hold:*

1. *If for all* $(t,s)\in\mathbb{R}_d$ *and* $\sigma_2\in\Sigma_2$ *the graph of the map* $x\mapsto U_{\Sigma_1,\sigma_2}(t,\tau,x)\in P(X)$ *is closed, then there exists the* Σ_1*-uniform global attractor* $\{\Theta_{\Sigma_1}(t,\sigma_2)\}$*. Moreover,*
$$\Theta_{\Sigma_1}(t,\sigma_2)=\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(t,\sigma_2,B)\neq X,$$
and for each $t\in\mathbb{R}$*,* $\sigma_2\in\Sigma_2$*,* $\Theta_{\Sigma_1}(t,\sigma_2)$ *is a Lindelöf, normal space. It is locally compact in some topology* $\tau_\oplus$*, which is stronger than the topology induced by* X *in* $\Theta_{\Sigma_1}(t,\sigma_2)$*.*
2. *If, in addition,* Σ_1 *is a compact metric space, the map*
$$\Sigma_1\times X\ni(\sigma_1,x)\longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x)\in P(X)$$
is upper semicontinuous for any $(t,\tau)\in\mathbb{R}_d$*,* $\sigma_2\in\Sigma_2$*,* U_σ *has connected values for any* $\sigma\in\Sigma$*,* $(t,\tau)\in\mathbb{R}_d$*,* $x\in X$*,* Σ_1 *is a connected space and*
$$\Theta_{\Sigma_1}(t,\sigma_2)\subset B_1(t,\sigma_2),$$
where $B_1(t,\sigma_2)$ *is a connected set for any* $t\in\mathbb{R},\sigma_2\in\Sigma_2$ *and* $\cup_{s\le t}B_1(s,\sigma_2)\in\mathcal{B}(X)$*, then the set* $\Theta_{\Sigma_1}(t,\sigma_2)$ *is connected for any* $t\in\mathbb{R},\sigma_2\in\Sigma_2$*.*

Remark 12 *We note that the condition of being* X *a space in which every compact set is nowhere dense is only used to prove that* $\Theta_{\Sigma_1}(t,\sigma_2)$ *does not coincide with the whole space.*

To prove this theorem we shall need the following result.

Proposition 13 *Let Σ_1 be a compact metric space with metric ρ_{Σ_1} and let the map*

$$\Sigma_1 \times X \ni (\sigma_1, x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x) \in P(X)$$

be upper semicontinuous for any $(t,\tau) \in \mathbb{R}^2, \sigma_2 \in \Sigma_2$. Then the map

$$X \ni x \longmapsto U_{\Sigma_1,\sigma_2}(t,\tau,x) \in P(X)$$

is also upper semicontinuous.

Proof. Let $x \in X$ be fixed. Take an arbitrary neighborhood U of $U_{\Sigma_1,\sigma_2}(t,\tau,x)$. It is obviously a neighborhood of each $U_{\sigma_1,\sigma_2}(t,\tau,x)$. For any $\sigma_1 \in \Sigma_1, \varepsilon > 0$ we can find $\delta(\varepsilon,\sigma_1) > 0$ such that if $\rho(y,x) < \delta$, $\rho_{\Sigma_1}(\sigma_1,\sigma_1^*) < \delta$, then $U_{\sigma_1^*,\sigma_2}(t,\tau,y) \subset U$. From the open cover of Σ_1 defined by $\{O_{\delta(\varepsilon,\sigma_1)}(\sigma_1)\}_{\sigma_1\in\Sigma_1}$ we can extract a finite subcover $\left\{O_{\delta_i(\varepsilon,\sigma_1^i)}(\sigma_1^i)\right\}_{i=1}^n$. Hence, for $\delta(\varepsilon) = \min \delta_i$ we have that if $\rho(y,x) < \delta$, then $U_{\sigma_1,\sigma_2}(t,\tau,x) \subset U$, for all $\sigma_1 \in \Sigma_1$. Therefore, $U_{\Sigma_1,\sigma_2}(t,\tau,x) \subset U$. ■

Now we shall prove Theorem 11.

Proof of Theorem 11. By Theorem 6 and Lemma 8, for any $B \in \mathcal{B}(X)$, $\sigma_2 \in \Sigma_2, t \in \mathbb{R}$ we obtain that the omega-limit set $\omega_{\Sigma_1}(t,\sigma_2,B)$ is non-empty, compact and attracts B at time t uniformly with respect to Σ_1. Hence, $\Theta_{\Sigma_1}(t,\sigma_2)$ is non-empty and a Σ_1-uniformly attracting set at time t. The minimality property is an easy consequence of Theorem 6.

Let us show further that the omega-limit set $\omega_{\Sigma_1}(t,\sigma_2,B)$ and $\Theta_{\Sigma_1}(t,\sigma_2)$ are semi-invariant. Lemma 5 implies that for an arbitrary $y \in \omega_{\Sigma_1}(t,\sigma_2,B)$ we can find a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,s_n,B)$ converging to y as $s_n \to -\infty$. For any $s_n \le \tau \le t$ we have

$$U_{\Sigma_1,\sigma_2}(t,s_n,B) \subset U_{\Sigma_1,\sigma_2}(t,\tau,U_{\Sigma_1,\sigma_2}(\tau,s_n,B)),$$

so that $\xi_n \in U_{\Sigma_1,\sigma_2}(t,\tau,\zeta_n)$, where $\zeta_n \in U_{\Sigma_1,\sigma_2}(\tau,s_n,B)$. Since U_σ is Σ_1-uniformly asymptotically upper semicompact, we can assume (taking a subsequence if necessary) that $\zeta_n \to \zeta \in \omega_{\Sigma_1}(\tau,\sigma_2,B)$. Since the graph of $x \mapsto U_{\Sigma_1,\sigma_2}(t,\tau,x)$ is closed, we get $y \in U_{\Sigma_1,\sigma_2}(t,\tau,\zeta) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\omega_{\Sigma_1}(\tau,\sigma_2,B))$. It follows

$$\omega_{\Sigma_1}(t,\sigma_2,B) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\omega_{\Sigma_1}(\tau,\sigma_2,B)) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2)),$$

for any $B \in \mathcal{B}(X)$, and then $\omega_{\Sigma_1}(t,\sigma_2,B)$ and $\Theta_{\Sigma_1}(t,\sigma_2)$ are semi-invariant.

We have proved that $\Theta_{\Sigma_1}(t,\sigma_2)$ is a Σ_1-uniform global attractor. Let us prove now that the global does not coincide with the whole space and its topological properties, as well. Consider in the space X the sequence of balls

$$B_i = \{y \in X : \rho(y,a) < i\},$$

with the fixed center a. It is clear that for any $B \in \mathcal{B}(X)$ there exists B_i such that $B \subset B_i$. Since in such case $\omega_{\Sigma_1}(t,\sigma_2,B) \subset \omega_{\Sigma_1}(t,\sigma_2,B_i)$, we have

$$\Theta_{\Sigma_1}(t,\sigma_2) \subset \bigcup_{i=1}^{\infty} \omega_{\Sigma_1}(t,\sigma_2,B_i).$$

On the other hand, the sets B_i are bounded, so that the converse inclusion follows. Hence,

$$\Theta_{\Sigma_1}(t,\sigma_2) = \bigcup_{i=1}^{\infty} \omega_{\Sigma_1}(t,\sigma_2,B_i).$$

Since $\Theta_{\Sigma_1}(t,\sigma_2)$ is a countable union of compact sets, Baire's theorem implies that $\Theta_{\Sigma_1}(t,\sigma_2) \neq \emptyset$. It follows also immediately that $\Theta_{\Sigma_1}(t,\sigma_2)$ is a Lindelöf space. Hence, since a metric space is regular, $\Theta_{\Sigma_1}(t,\sigma_2)$ is normal.

Put $S_i = S^i_{\Sigma_1}(t,\sigma_2) = \omega_{\Sigma_1}(t,\sigma_2,B_i) \times \{i\}$. It is clear that $S_i \cap S_j = \emptyset$, for all $i \neq j$. It is evident that for any i there exists a homeomorphism h_i between S_i and $\omega_{\Sigma_1}(t,\sigma_2,B_i)$. The set S_i is a topological space with the topology induced by the metric of X, which will be denoted by τ_i.

Accurate to the homeomorphism I_i we can write that $S_i = \omega_{\Sigma_1}(t,\sigma_2,B_i)$. In the set $\Theta_{\Sigma_1}(t,\sigma_2)$ (Θ for short) we consider the family of subsets

$$\beta_{\oplus} = \{V \subset \Theta : V \cap S_i \in \tau_i \text{ for some } i\}. \tag{6}$$

This family define on Θ a subbase of the topology $\tau_{\oplus}$. In this topology each $\omega_{\Sigma_1}(t,\sigma_2,B_i)$ is open and closed. On the set Θ we have also the topology τ induced by the metric ρ if X, obtaining in this way two topological spaces $(\Theta,\tau), (\Theta,\tau_{\oplus})$. The topology $\tau_{\oplus}$ is stronger than τ ($\tau_{\oplus} \leq \tau$).

We hall check now that the each space (S_i,τ_i) is compact. Using the homeomorphism h_i we identify the sets S_i and $\omega_{\Sigma_1}(t,\sigma_2,B_i)$. Let $\{W_\alpha\}$ be an arbitrary open cover of S_i, where $W_\alpha = S_i \cap V_\alpha$ and V_α is open in X. Since S_i is a compact set in X, we can extract a finite subset $\{V_{s_j}\}_{j=1}^n$ defining the finite subcover $W_{s_j} = S_i \cap V_{s_j}$. It follows that S_i is compact in the topology $\tau_{\oplus}$, since $\tau_{\oplus}$ is the strongest topology in Θ for which the canonical embedding $I_i : S_i \to \Theta$ is continuous.

Let now take an arbitrary $x \in \Theta$. Then $x \in S_i$ for some i and S_i is a neighborhood of x. Since a compact set is regular, there exists a neighborhood $O(x) \in \tau_i$ such that $\overline{O(x)} \subset S_i$, i.e. $\overline{O(x)}$ is compact in (S_i,τ_i) and also in the topology $\tau_{\oplus}$. Therefore, the space $(\Theta,\tau_{\oplus})$ is locally compact.

It is obvious that the space $(\Theta,\tau_{\oplus})$ is Lindelöf. The proof of the first statement is now complete.

For the second statement, suppose that $\Theta_{\Sigma_1}(t,\sigma_2)$ is not connected. Then there exist two open sets U_1, U_2 satisfying $\Theta_{\Sigma_1}(t,\sigma_2) \cap U_i \neq \emptyset$, for $i = 1,2$, $\Theta_{\Sigma_1}(t,\sigma_2) \subset U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. It is well known (see [3], [10]) that an upper semicontinuous map with connected values maps any connected set into a connected one. Since the set $B_1(\tau,\sigma_2)$ is connected, Proposition 13 implies then that the set $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\tau,\sigma_2))$ is connected. The semi-invariance property gives

$$\Theta_{\Sigma_1}(t,\sigma_2) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2)) \subset U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\tau,\sigma_2)).$$

Hence, $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\tau,\sigma_2)) \cap U_i \neq \emptyset$, for $i = 1,2$, and by the connectedness of $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\tau,\sigma_2))$ we obtain that $U_1 \cup U_2$ does not contain $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\tau,\sigma_2))$. Hence, $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\sigma_2)) \cap U_i \neq \emptyset$ and $U_1 \cup U_2$ does not contain $U_{\Sigma_1,\sigma_2}(t,\tau,B_1(\sigma_2))$, where $B_1(\sigma_2) = \cup_{\tau \leq t} B_1(\tau,\sigma_2)$. There exists a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,\tau_n,B_1(\sigma_2))$, where $\tau_n \to -\infty$, and $\xi_n \notin U_1 \cup U_2$. Then since $B_1(\sigma_2) \in \mathcal{B}(X)$ and the family $\{U_\sigma\}$ is Σ_1-uniform upper asymptotically semicompact, we can extract a converging subsequence $\xi_m \to y \in \omega_{\Sigma_1}(t,\sigma_2,B_1(\sigma_2)) \subset \Theta_{\Sigma_1}(t,\sigma_2) \subset U_1 \cup U_2$. But in such a case there exists m_0 for which $\xi_m \in U_1 \cup U_2$, for all $m > m_0$, which is a contradiction. ■

Remark 14 *We note that under the conditions of point 1 the set $\Theta_{\Sigma_1}(t,\sigma_2)$ can not be locally compact in the topology of the space X as shown in [21] with an example for an autonomous system.*

Remark 15 *We note that in [16, p.89] and [17, p.382] there is a mistake in the definition of the subsets $\beta_{\oplus}$ (see (6)), where it is written "for any i" instead of "for some i".*

The following proposition is useful in applications.

Proposition 16 *Let Σ_1 be a compact metric space and let the map*

$$\Sigma_1 \times X \ni (\sigma_1,x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x) \in P(X)$$

be closed for any $(t,\tau) \in \mathbb{R}^2, \sigma_2 \in \Sigma_2$. Then the map

$$X \ni x \longmapsto U_{\Sigma_1,\sigma_2}(t,\tau,x) \in P(X)$$

is also closed.

Proof. For fixed $(t,\tau) \in \mathbb{R}^2$ and $\sigma_2 \in \Sigma_2$ consider the sequences $x_n \to x, y_n \to y$, where $y_n \in U_{\Sigma_1,\sigma_2}(t,\tau,x_n)$. Then there exist $\sigma_{1n} \in \Sigma_1$ for which $y_n \in U_{\sigma_{1n},\sigma_2}(t,\tau,x_n)$, for each n, and, in view of the compactness of Σ_1, we can extract a converging subsequence $\sigma_{1m} \to \sigma_0$. Therefore, since the map $(\sigma_1,x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x)$ is closed, we have

$$y \in U_{\sigma_0,\sigma_2}(t,\tau,x) \subset U_{\Sigma_1,\sigma_2}(t,\tau,x).$$

■

In the previous theorem we have proved the existence of a global attractor for U_σ. However, although it satisfies some good topological properties, it can be an unbounded set of the space X. In applications it is desirable to obtain a more regular attractor. Namely, by adding a stronger dissipative condition we are able to prove the existence of a compact global attractor.

Theorem 17 *Let us suppose that for all* $(t,s) \in \mathbb{R}_d$ *and* $\sigma_2 \in \Sigma_2$ *the graph of the map* $x \mapsto U_{\Sigma_1,\sigma_2}(t,\tau,x) \in P(X)$ *is closed. If, moreover, for any* $t \in \mathbb{R}$, $\sigma_2 \in \Sigma_2$ *there exists a set* $D(t,\sigma_2) \in K(X)$, *which is* Σ_1-*uniformly attracting, then the set*

$$\Theta_{\Sigma_1}(t,\sigma_2) = \overline{\bigcup_{B \in \mathcal{B}(X)} \omega_{\Sigma_1}(t,\sigma_2,B)}$$

is the Σ_1-*uniform global attractor of* U_σ. *Moreover, the sets* $\Theta_{\Sigma_1}(t,\sigma_2)$ *are compact and, if the conditions of the second statement in Theorem 11 hold, then they are connected.*

Proof. It is easy to see that $\omega_{\Sigma_1}(t,\sigma_2,B) \subset D(t,\sigma_2)$, for all $B \in \mathcal{B}(X)$. Indeed, since $D(t,\sigma_2)$ attracts any $B \in \mathcal{B}(X)$, the limit of any sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(t,\tau_n,B)$, $\tau_n \to -\infty$, belongs to $D(t,\sigma_2)$. Lemma 5 gives then the required inclusion. Hence, the set $\Theta_{\Sigma_1}(t,\sigma_2)$, as a closed subset of a compact one, is compact.

The Σ_1-uniformly attracting and minimality properties follow from the first statement of Theorem 11. It remains to show that $\Theta_{\Sigma_1}(t,\sigma_2)$ is semi-invariant. Let $y \in \Theta_{\Sigma_1}(t,\sigma_2)$ be arbitrary. Then there exists a sequence $y_n \in \omega_{\Sigma_1}(t,\sigma_2,B_n)$, $B_n \in \mathcal{B}(X)$, converging to y. Since omega-limit sets are semi-invariant (see the proof of Theorem 11), for any $\tau < t$ we can obtain a sequence $\zeta_n \in \omega_{\Sigma_1}(\tau,\sigma_2,B_n)$ such that $y_n \in U_{\Sigma_1,\sigma_2}(t,\tau,\zeta_n)$. By the compactness of $D(\tau,\sigma_2)$ we can assume that $\zeta_n \to \zeta \in \Theta_{\Sigma_1}(t,\sigma_2)$. Finally, using the fact that the map $X \ni x \longmapsto U_{\Sigma_1,\sigma_2}(t,\tau,x)$ is closed (this follows from Proposition 16), we have $y \in U_{\Sigma_1,\sigma_2}(t,\tau,\zeta) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(t,\sigma_2))$. Hence, $\Theta_{\Sigma_1}(t,\sigma_2) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(t,\sigma_2))$.

Finally, suppose that the conditions of the second statement in Theorem 6 hold. Since in view of Theorem 6 the set $\cup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(t,\sigma_2,B)$ is connected, we obtain that $\Theta_{\Sigma_1}(t,\sigma_2)$ is connected. ■

It is also interesting to consider the situation where the sets $\Theta_{\Sigma_1}(t,\sigma_2)$ defined above are strictly invariant.

Proposition 18 *Let the MDP* U_σ *be strict,* Σ_1 *be a compact metric space and let the map*

$$\Sigma_1 \times X \ni (\sigma_1,x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x) \in P(X)$$

be lower semicontinuous. Then, the global attractors obtained in Theorems 11 and 17 are invariant, that is, $\Theta_{\Sigma_1}(t,\sigma_2) = U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2))$, *for all* $\tau \le t, \sigma_2 \in \Sigma_2$.

Proof. We have to prove the inclusion $U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2)) \subset \Theta_{\Sigma_1}(t,\sigma_2)$. Consider first the global attractor defined in Theorem 11. Let $y \in U_{\Sigma_1,\sigma_2}(t,\tau,\omega_{\Sigma_1}(\tau,\sigma_2,B))$, $B \in \mathcal{B}(X)$ be arbitrary. Then Lemma 5 implies the existence of a sequence $\xi_n \in U_{\Sigma_1,\sigma_2}(\tau,s_n,B)$ converging to $\xi \in \omega_{\Sigma_1}(\tau,\sigma_2,B)$, as $s_n \to -\infty$, and $y \in U_{\Sigma_1,\sigma_2}(t,\tau,\xi)$. We claim that there exists a sequence $\{y_n\}$ such that $y_n \in U_{\Sigma_1,\sigma_2}(t,\tau,\xi_n)$ and $y_n \to y$. Indeed, take a sequence $\sigma_{1n} \to \sigma_1 \in \Sigma_1$, where $y \in U_{\sigma_1,\sigma_2}(t,\tau,\xi)$. Now the lower semicontinuity of the map $(\sigma_1,x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x)$ provides us a sequence $y_n \in U_{\sigma_{1n},\sigma_2}(t,\tau,\xi_n)$ such that $y_n \to y$. Since U_σ is strict, we get

$$y_n \in U_{\sigma_{1n},\sigma_2}(t,\tau,\xi_n) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\xi_n) \subset U_{\Sigma_1,\sigma_2}(t,\tau,U_{\Sigma_1,\sigma_2}(\tau,s_n,B))$$

$$\subset U_{\Sigma_1,\sigma_2}(t,s_n,B),$$

so that we have $y \in \omega_{\Sigma_1}(t,\sigma_2,B) \subset \Theta_{\Sigma_1}(t,\sigma_2)$. Hence, $U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2)) \subset \Theta_{\Sigma_1}(t,\sigma_2)$.

Further, let now $\Theta_{\Sigma_1}(t,\sigma_2)$ be the global attractor defined in Theorem 17. Denote by $F(t,\sigma_2)$ the union $\cup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(t,\sigma_2,B)$. We have already proved that $F(t,\sigma_2) = U_{\Sigma_1,\sigma_2}(t,\tau,F(\tau,\sigma_2))$. For an arbitrary

$$y \in U_{\sigma_1,\sigma_2}(t,\tau,\xi) \subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2))$$

we take sequences $\sigma_{1n} \in \Sigma_1$, $\xi_n \in F(\tau,\sigma_2)$ converging to σ_1 and ξ, respectively. Using again the lower semicontinuity of $(\sigma_1,x) \longmapsto U_{\sigma_1,\sigma_2}(t,\tau,x)$ we obtain the existence of a sequence

$$y_n \in U_{\sigma_{1n},\sigma_2}(t,\tau,\xi_n) \subset U_{\Sigma_1,\sigma_2}(t,\tau,F(\tau,\sigma_2)) \subset F(t,\sigma_2)$$

converging to y. Hence, $y \in \Theta_{\Sigma_1}(t,\sigma_2)$. ■

2.3 Shift on Σ

Suppose that we are given a one-parameter group $T(h) : \Sigma \to \Sigma$, where $\Sigma = \Sigma_1 \times \Sigma_2$, $h \in \mathbb{R}$ and $T(h) = (T_1(h), T_2(h))$, $T_i(h) : \Sigma_i \to \Sigma_i$, $i = 1, 2$. This is called the shift operator.

The problem we are now going to study was considered in [17] in the case where the maps U_σ, T are defined only for positive moments of time (in such case the family U_σ is called a semiprocess instead of process). There are some subtle differences between the two approaches which will be pointed out. So, we generalize in this way the results of [17] on Σ-uniform attractors. Further we study again Σ_1-uniform attractors (but now using the shift operator T_2 on Σ_2) and prove that, when a Σ-uniform attractor exists, it coincides with the Σ_1-uniform attractor.

2.3.1 Σ-uniform attractors of the family $\{U_\sigma : \sigma \in \Sigma\}$

In the sequel we shall assume:

$(T1)$ For any $(t, s) \in \mathbb{R}_d$, $x \in X$, $\sigma \in \Sigma$, $h \in \mathbb{R}$ the following inclusion holds:

$$U_{\sigma_1,\sigma_2}(t, s, x) \subset U_{T_1(h)\sigma_1,T_2(h)\sigma_2}(t - h, s - h, x).$$

Lemma 19 *Condition $(T1)$ implies*

$$U_{\sigma_1,\sigma_2}(t, s, x) = U_{T_1(h)\sigma_1,T_2(h)\sigma_2}(t - h, s - h, x).$$

Proof. Denoting $\widetilde{\sigma}_i = T_1(h)\sigma_i$, $i = 1, 2$, and using $(T1)$ we have

$$U_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}(t - h, s - h, x) \subset U_{T_1(-h)T_1(h)\sigma_1 T_2(-h),T_2(h)\sigma_2}(t - h + h, s - h + h, x)$$

$$= U_{\sigma_1,\sigma_2}(t, s, x).$$

■

Lemma 20 $T(h)\Sigma = \Sigma$, *for all* $h \in \mathbb{R}$.

Proof. It is obvious that $T(h)\Sigma \subset \Sigma$. Conversely, if $\sigma \in \Sigma$ then for $\widetilde{\sigma} = T(-h)\sigma \in \Sigma$ we have $T(h)\widetilde{\sigma} = T(h)T(-h)\sigma = \sigma$, so that $\Sigma \subset T(h)\Sigma$. ■

Remark 21 *These results are not true in the case considered in [17]. In particular, we have to write $T(h)\Sigma \subset \Sigma$, where the inclusion can be strict.*

For a fixed $\tau \in \mathbb{R}$ we define the map $G_\tau : \mathbb{R}_+ \times \Sigma \times X \to P(X)$ by

$$G_\tau(t, \sigma, x) = U_\sigma(t + \tau, \tau, x).$$

Proposition 22 *If $(T1)$ holds, then the map $G_\tau : \mathbb{R}_+ \times \Sigma \times X \to P(X)$ satifies:*

1. $G_\tau(0, \sigma, x) = x$, *for all* $x \in X, \sigma \in \Sigma$;
2. $G_\tau(t_1 + t_2, \sigma, x) \subset G_\tau(t_1, T(t_2)\sigma, G_\tau(t_2, \sigma, x))$, *for all* $t_1, t_2 \in \mathbb{R}_+, \sigma \in \Sigma, x \in X$;
3. $G_\tau(t, \sigma, x) = G_0(t, T(\tau)\sigma, x)$.

Proof. The first property is evident. Further, for any $t_1, t_2 \in \mathbb{R}_+$ it follows from $(T1)$ and the properties of the map U_σ that

$$G_\tau(t_1 + t_2, \sigma, x) = U_\sigma(t_1 + t_2 + \tau, \tau, x) \subset U_{T(t_2)\sigma}(t_1 + \tau, \tau - t_2, x)$$

$$\subset U_{T(t_2)\sigma}\big(t_1 + \tau, \tau, U_{T(t_2)\sigma}(\tau, \tau - t_2, x)\big) = U_{T(t_2)\sigma}(t_1 + \tau, \tau, U_\sigma(\tau + t_2, \tau, x))$$

$$= G_\tau (t_1, T(t_2)\sigma, G_\tau (t_2, \sigma, x)).$$

Finally, using Lemma 19 we have

$$G_\tau (t, \sigma, x) = U_\sigma (t + \tau, \tau, x) = U_{T(\tau)\sigma} (t, 0, x) = G_0 (t, T(\tau)\sigma, x).$$

■

Let us define the map $U_\Sigma : \mathbb{R}_d \times X \to P(X)$ by

$$U_\Sigma (t, \tau, x) = \bigcup_{\sigma \in \Sigma} U_\sigma (t, \tau, x), \text{ for all } (t, \tau) \in \mathbb{R}_d, x \in X.$$

Proposition 23 *If $(T1)$ holds, then the map $U_\Sigma : \mathbb{R}_d \times X \to P(X)$ is a MDP for which the following formula holds:*

$$U_\Sigma (t + h, \tau + h, x) = U_\Sigma (t, \tau, x), \text{ for all } (t, \tau) \in \mathbb{R}_d, x \in X, h \in \mathbb{R}. \tag{7}$$

Proof. It is an easy consequence of Lemma 19. Indeed,

$$U_\sigma (t + h, \tau + h, x) = U_{T(h)\sigma} (t, \tau, x) \subset U_\Sigma (t, \tau, x), \text{ for all } \sigma \in \Sigma, x \in X, (t, \tau) \in \mathbb{R}_d.$$

The converse inequality follows changing h by $-h$. ■

In a similar way as before for $B \in \mathcal{B}(X)$ and $t \in \mathbb{R}$ we set

$$\begin{aligned} \gamma^s_\Sigma (t, B) &= \bigcup_{\tau \leq s} U_\Sigma (t, \tau, B), \ t \geq s, \\ \omega_\Sigma (t, B) &= \bigcap_{s \leq t} \overline{\gamma^s_\Sigma (t, B)}. \end{aligned}$$

Definition 24 *The family of MDP U_σ is called point dissipative if for any $t \in \mathbb{R}$ there exists $B_0(t) \in \mathcal{B}(X)$ such that*

$$dist (U_\Sigma (t, \tau, x), B_0(t)) \to 0, \text{ as } \tau \to -\infty, \text{ for all } x \in X.$$

Proposition 25 *If $(T1)$ holds, then the following statements are equivalent:*

1. *U_σ is point dissipative;*
2. *There exists $B_0 \in \mathcal{B}(X)$ such that*

$$dist (U_\Sigma (0, \tau, x), B_0) \to 0, \text{ as } \tau \to -\infty, \text{ for all } x \in X.$$

3. *There exists $B_0 \in \mathcal{B}(X)$ such that*

$$dist (U_\Sigma (t, \tau, x), B_0) \to 0, \text{ as } t \to +\infty, \text{ for all } x \in X, \tau \in \mathbb{R}.$$

Proof. The implication $\{1 \Longrightarrow 2\}$ is obvious. For $\{2 \Longrightarrow 3\}$ it is sufficient to apply Proposition 23 to have

$$dist (U_\Sigma (t, \tau, x), B_0) = dist (U_\Sigma (0, \tau - t, x), B_0) \to 0, \text{ as } t \to +\infty.$$

Finally, $\{3 \Longrightarrow 1\}$ is proved as follows

$$dist (U_\Sigma (t, \tau, x), B_0) = dist (U_\Sigma (t - \tau, 0, x), B_0) \to 0, \text{as } \tau \to -\infty.$$

Note that as a consequence the set $B_0(t)$ in Definition 24 does not depend on t. ■

Definition 26 *The family of MDP* $\{U_\sigma\}$ *is called* Σ*-uniformly asymptotically upper semicompact if for any* $t \in \mathbb{R}$ *and* $B \in \mathcal{B}(X)$ *there exists* $t_0 = t_0(t, B)$ *such that* $\gamma_\Sigma^{t_0}(t, B) \in \mathcal{B}(X)$ *and any sequence* $\xi_n \in U_\Sigma(t, s_n, B)$, *where* $s_n \to -\infty$, *is precompact.*

Proposition 27 *Let* $(T1)$ *hold. Then the following statements are equivalent:*

1. *The MDP* U_σ *is* Σ*-uniformly asymptotically upper semicompact;*
2. *For any* $t \in \mathbb{R}, B \in \mathcal{B}(X)$ *there exists* $D(t, B) \in K(X)$ *such that*
$$dist\left(U_\Sigma(t, \tau, B), D(t, B)\right) \to 0, \text{ as } \tau \to -\infty.$$
3. *For any* $B \in \mathcal{B}(X)$ *there exists* $D(B) \in K(X)$ *such that*
$$dist\left(U_\Sigma(0, \tau, B), D(B)\right) \to 0, \text{ as } \tau \to -\infty.$$
4. *For any* $B \in \mathcal{B}(X)$ *there exists* $D(B) \in K(X)$ *such that*
$$dist\left(U_\Sigma(t, \tau, B), D(B)\right) \to 0, \text{ as } t \to +\infty, \text{ for all } \tau \in \mathbb{R}.$$

Proof. For $\{1 \Longrightarrow 2\}$ in the same way as in Lemma 8 for any $t \in \mathbb{R}$, $B \in \mathcal{B}(X)$ we can prove the existence of $D(t, B) \in K(X)$ such that

$$dist\left(U_\Sigma(t, \tau, B), D(t, B)\right) \to 0, \text{ as } \tau \to -\infty.$$

The implication $\{2 \Longrightarrow 3\}$ is evident.
For $\{3 \Longrightarrow 4\}$ using (7) we have that

$$dist\left(U_\Sigma(t, \tau, B), D(B)\right) = dist\left(U_\Sigma(0, \tau - t, B), D(B)\right) \to 0, \text{ as } t \to +\infty.$$

Finally, let us prove $\{4 \Longrightarrow 1\}$. We note that for any $\varepsilon > 0, l \in \mathbb{R}$ there exists s_0 for which $U_\Sigma(s, l, B) \subset O_\varepsilon(D(B))$, for all $s \geq s_0$. Since $O_\varepsilon(D(B))$ is a bounded set and in view of (7),

$$U_\Sigma(t, \tau, B) = U_\Sigma(t - \tau, 0, B) \subset O_\varepsilon(D(B)), \text{ for all } \tau \leq t - s_0 = t_0,$$

we have $\gamma_\Sigma^{t_0}(t, B) \in \mathcal{B}(X)$.
Finally, let us take an arbitrary sequence $\xi_n \in U_\Sigma(t, s_n, B)$, where $s_n \to -\infty$. Using (7) again we get

$$dist\left(\xi_n, D(B)\right) \leq dist\left(U_\Sigma(t, s_n, B), D(B)\right) = dist\left(U_\Sigma(t - s_n, 0, B), D(B)\right) \to 0, \text{ as } s_n \to -\infty.$$

Hence, there exist $\alpha_n \to 0$ and $\zeta_n \in D(B)$ such that $\rho\left(\xi_n, \zeta_n\right) < \alpha_n$, for all n. It follows from the compacity of the set $D(B)$ that $\{\xi_n\}$ has a converging subsequence ξ_{n_k}, so that U_σ is Σ-uniformly asymptotically upper semicompact. ■

Definition 28 *The set* Θ_Σ *is called a* Σ*-uniform global attractor of the MDP* $\{U_\sigma\}$ *if*

1. Θ_Σ *is* Σ*-uniformly attracting at time* 0*, that is,*
$$dist\left(U_\Sigma(0, s, B), \Theta_\Sigma\right) \to 0, \text{ as } s \to -\infty, \text{ for all } B \in \mathcal{B}(X).$$
2. *It is semi-invariant, that is,*
$$\Theta_\Sigma \subset U_\Sigma(t, s, \Theta_\Sigma), \text{ for any } (t, s) \in \mathbb{R}_d.$$
3. *It is minimal, that is, for any closed* Σ*-uniformly attracting set* Y *at time* 0*, we have*
$$\Theta_\Sigma \subset Y.$$

As a consequence of Proposition 23 we can prove, in the same way as in the previous propositions, the following proposition.

Proposition 29 *If $(T1)$ holds, the following statements are equivalent:*

1. Θ_Σ *is* Σ*-uniformly attracting at time* 0*, that is,*

$$dist\left(U_\Sigma\left(0,s,B\right),\Theta_\Sigma\right)\rightarrow 0,\ \text{as } s\rightarrow -\infty,\ \text{for all } B\in\mathcal{B}\left(X\right).$$

2. Θ_Σ *is* Σ*-uniformly attracting at time* t*, for any* $t\in\mathbb{R}$*, that is,*

$$dist\left(U_\Sigma\left(t,s,B\right),\Theta_\Sigma\right)\rightarrow 0,\ \text{as } s\rightarrow -\infty,\ \text{for all } B\in\mathcal{B}\left(X\right),\forall t\in\mathbb{R}.$$

3. Θ_Σ *is* Σ*-uniformly attracting, that is,*

$$dist\left(U_\Sigma\left(t,s,B\right),\Theta_\Sigma\right)\rightarrow 0,\ \text{as } t\rightarrow +\infty,\ \text{for all } B\in\mathcal{B}\left(X\right),\forall s\in\mathbb{R}.$$

Definition 30 *Let X,Y be metric spaces. The multivalued map $F:X\rightarrow P\left(Y\right)$ is called w-upper semicontinuous if for any $x_0\in X$ and $\varepsilon>0$ there exists $\delta>0$ such that $F\left(x\right)\subset O_\varepsilon\left(F\left(x_0\right)\right)$, for all $x\in O_\delta\left(x_0\right)$.*

Theorem 31 *Let X be a complete metric space in which every compact set is nowhere dense and let the family of MDP $\{U_\sigma\}$ be Σ-uniformly asymptotically upper semicompact. Then the following statements hold:*

1. *If for all $s\leq 0$ the graph of the map $x\mapsto U_\Sigma\left(0,s,x\right)\in P\left(X\right)$ is closed, then there exists the Σ-uniform global attractor Θ_Σ. Moreover,*

$$\Theta_\Sigma=\bigcup_{B\in\mathcal{B}(X)}\omega_\Sigma\left(0,D\left(B\right)\right)=\bigcup_{B\in\mathcal{B}(X)}\omega_\Sigma\left(0,B\right)=\bigcup_{B\in\mathcal{B}(X)}\omega_\Sigma\left(t,B\right)\neq X,\ \text{for all } t\in\mathbb{R},$$

   where $D\left(B\right)$ is the set defined in Proposition 27, and Θ_Σ is a Lindelöf, normal space. It is locally compact in some topology $\tau_\oplus$, which is stronger than the topology induced by X in Θ_Σ.

2. *If for all $s\leq 0$ the map $x\mapsto U_\Sigma\left(0,s,x\right)\in P\left(X\right)$ is closed and there exists a set $D\in K\left(X\right)$, which is Σ-uniformly attracting, then the set Θ_Σ is compact.*

3. *If the family U_σ is point dissipative and for all $s\leq 0$ the map $x\mapsto U_\Sigma\left(0,s,x\right)\in P\left(X\right)$ has closed graph and is w-upper semicontinuous, then Θ_Σ is compact.*

4. *If, in addition, Σ is a compact metric space, the map*

$$\Sigma\times X\ni\left(\sigma,x\right)\longmapsto U_\sigma\left(0,s,x\right)\in P\left(X\right)$$

   is upper semicontinuous for any $s\leq 0$, U_σ has connected values for any $\sigma\in\Sigma$, $\left(t,\tau\right)\in\mathbb{R}_d$, $x\in X$, Σ is a connected space and $\Theta_\Sigma\subset B_1\in\mathcal{B}\left(X\right)$, where B_1 is connected, then the set Θ_Σ is connected.

Proof. It follows from Theorem 11 (replacing Σ_1 by Σ) that the set $\Theta_\Sigma=\cup_{B\in\mathcal{B}(X)}\omega_\Sigma\left(0,B\right)$ is the Σ-uniform global attractor of U_σ and the topological properties, as well. Note that

$$\omega_\Sigma\left(t,B\right)=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}U_\Sigma\left(t,\tau,B\right)}=\bigcap_{s-t\leq 0}\overline{\bigcup_{\tau-t\leq s-t}U_\Sigma\left(0,\tau-t,B\right)}=\omega_\Sigma\left(0,B\right).$$

Hence, $\Theta_\Sigma=\cup_{B\in\mathcal{B}(X)}\omega_\Sigma\left(t,B\right)$, for all $t\in\mathbb{R}$. Further, it follows from Proposition 27 that $D\left(B\right)$ is Σ-attracting at time 0. The omega-limit set $\omega_\Sigma\left(0,B\right)$ belongs to $D\left(B\right)$ and is semi-invariant (these facts are proved exactly in the same way as in Theorems 11, 17 replacing Σ_1 by Σ and taking into account the equality $\omega_\Sigma\left(0,B\right)=\omega_\Sigma\left(t,B\right)$). Therefore,

$$\omega_\Sigma\left(0,B\right)\subset U_\Sigma\left(t,\tau,\omega_\Sigma\left(0,B\right)\right)\subset U_\Sigma\left(t,\tau,D\left(B\right)\right),\ \text{for all } \left(t,\tau\right)\in\mathbb{R}_d,$$

so that $\omega_\Sigma(0,B) \subset \omega_\Sigma(0,D(B))$ and then $\Theta_\Sigma \subset \cup_{B\in\mathcal{B}(X)}\omega_\Sigma(0,D(B))$. Since $D(B) \in \mathcal{B}(X)$ the converse inclusion is obvious. The proof of the first statement is complete.

For the second statement note that we have proved $\omega_\Sigma(0,B) \subset \omega_\Sigma(0,D(B))$. Since $D(B) = D$, we have $\Theta_\Sigma = \omega_\Sigma(0,D)$. Hence, Θ_Σ is compact.

Let us prove now the third statement. In view of Proposition 25 there exists $B_0 \in \mathcal{B}(X)$ such that

$$dist\left(U_\Sigma(t,\tau,x),B_0\right) \to 0, \text{ as } t \to +\infty, \text{ for all } x \in X, \tau \in \mathbb{R}.$$

Set $B_1 = O_{\varepsilon_1}(B_0)$ for some fixed $\varepsilon_1 > 0$. We shall show that $\Theta_\Sigma = \omega_\Sigma(0,B_1)$. Take an arbitrary $B \in \mathcal{B}(X)$. Fix $\tau \in \mathbb{R}$.

For any $x \in \omega(0,B)$ there exists $T_1(x)$ such that $U_\Sigma(T_1(x),\tau,x) \subset B_1$. Note that since the map $x \mapsto U_\Sigma(0,\tau - T_1(x),x)$ is w-upper semicontinuous and $U_\Sigma(0,\tau - T_1(x),x) = U_\Sigma(T_1(x),\tau,x)$ (see again Proposition 23), the map $x \mapsto U_\Sigma(T_1(x),\tau,x)$ is w-upper semicontinuous. Then we can find a neighborhood $O_{\delta(x)}(x)$ for which $U_\Sigma\left(T_1(x),\tau,O_{\delta(x)}(x)\right) \subset B_1$.

The set $\left\{O_{\delta(x)}(x) : x \in \omega_\Sigma(0,B)\right\}$ is an open cover of the compact set $\omega_\Sigma(0,B)$. Let $\left\{O_{\delta(x_i)}(x_i)\right\}_{i=1}^n$ be a finite subcover. Then $O(\omega_\Sigma(0,B)) = \cup_{i=1}^n O_{\delta(x_i)}(x_i)$ is a neighborhood of $\omega_\Sigma(0,B)$. Fix $\varepsilon_2 > 0$. Since $\omega_\Sigma(0,B_1)$ is Σ-uniformly attracting B_1 at time 0 (this fact can be proved exactly in the same way as in Theorem 6 replacing Σ_1 by Σ), for any x_i there exists $T_2(x_i)$ such that

$$U_\Sigma(t,T_1(x_i),B_1) = U_\Sigma(0,T_1(x_i)-t,B_1) \subset O_{\varepsilon_2}(\omega_\Sigma(0,B_1)), \text{ for all } t \geq T_2(x_i).$$

Then for $T_2 = T_2(\varepsilon_2,\varepsilon_1,B) = \max\{T_2(x_i)\}$ we have

$$U_\Sigma\left(t,\tau,O_{\delta(x_i)}(x_i)\right) \subset U_\Sigma\left(t,T_1(x_i),U_\Sigma\left(T_1(x_i),\tau,O_{\delta(x_i)}(x_i)\right)\right)$$

$$\subset U_\Sigma(t,T_1(x_i),B_1) \subset O_{\varepsilon_2}(\omega_\Sigma(0,B_1)), \text{ for all } t \geq T_2, i = 1,...,n.$$

Hence,

$$U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset U_\Sigma(t,\tau,O(\omega_\Sigma(0,B)))$$

$$\subset O_{\varepsilon_2}(\omega_\Sigma(0,B_1)), \text{ for all } t \geq T_2.$$

Therefore, the semi-invariance of $\omega_\Sigma(0,B)$ implies

$$\omega_\Sigma(0,B) \subset U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset O_{\varepsilon_2}(\omega_\Sigma(0,B_1)), \text{ for all } t \geq T_2,$$

so that $\omega_\Sigma(0,B) \subset \omega_\Sigma(0,B_1)$. It follows the desired equality $\omega_\Sigma(0,B_1) = \Theta_\Sigma$.

The last statement is a consequence of Theorems 11, 17 replacing again Σ_1 by Σ. Note that in our case $B_1(t) = B_1 \in \mathcal{B}(X)$, so that condition $\cup_{s\leq t}B_1(s) \in \mathcal{B}(X)$ holds. ■

Remark 32 *We note that the condition of being X a space in which every compact is nowhere dense is only use to prove that Θ_Σ does not coincide with the whole space.*

Proposition 33 *If the family of MDP U_σ is strict, then the global attractor obtained in Theorem 31 is invariant, that is,*

$$\Theta_\Sigma = U_\Sigma(t,\tau,\Theta_\Sigma), \textit{ for all } (t,\tau) \in \mathbb{R}_d.$$

Proof. We have to prove the inclusion $U_\Sigma(t,\tau,\Theta_\Sigma) \subset \Theta_\Sigma$, for all $(t,\tau) \in \mathbb{R}_d$. Since $\omega_\Sigma(0,B) \subset U_\Sigma(\tau,s,\omega_\Sigma(0,B))$ for any $B \in \mathcal{B}(X), (\tau,s) \in \mathbb{R}_d$ (see the proof of Theorem 31), we have, using Proposition 23, that

$$U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset U_\Sigma(t,\tau,U_\Sigma(\tau,s,\omega_\Sigma(0,B))) = U_\Sigma(t,s,\omega_\Sigma(0,B))$$

$$= U_\Sigma(0,s-t,\omega_\Sigma(0,B)), \text{ for all } s \leq \tau.$$

Hence,

$$U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset \bigcup_{s\leq l} U_\Sigma(0,s-t,\omega_\Sigma(0,B)), \text{ for all } l \leq \tau,$$

so that $U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset \omega_\Sigma(0,\omega_\Sigma(0,B))$. Since $\omega_\Sigma(0,B)$ is bounded, $U_\Sigma(t,\tau,\omega_\Sigma(0,B)) \subset \Theta_\Sigma$. It follows then that $U_\Sigma(t,\tau,\Theta_\Sigma) \subset \Theta_\Sigma$. ■

2.3.2 Σ_1-uniform attractors of the family $\{U_\sigma : \sigma \in \Sigma\}$

We shall consider again the existence of a global Σ_1-uniform attractor, but having now the shift operator T satisfying condition $(T1)$. The main difference with the previous results consists in the existence of an equivalence between the parameter σ_2 and the final moment of time t, so that it is only necessary to keep one of them.

Proposition 34 *Let $(T1)$ hold and let for any $B \in \mathcal{B}(X)$, $\sigma_2 \in \Sigma_2$ there exist a compact set $D(\sigma_2, B) \subset X$ such that*

$$\lim_{s \to -\infty} dist\left(U_{\Sigma_1,\sigma_2}(0,s,B), D(\sigma_2,B)\right) = 0. \tag{8}$$

Then the family of MDP U_σ is Σ_1-uniformly asymptotically upper semicompact.

Proof. For $B \in \mathcal{B}(X), \sigma_2 \in \Sigma_2$ and $t \in \mathbb{R}$ consider a sequence $\xi_n \in U_{\sigma_{1n},\sigma_2}(t, s_n, B)$, where $\sigma_{1n} \in \Sigma_1$, $s_n \to -\infty$. Then condition $(T1)$ implies

$$\xi_n \in U_{T_1(t)\sigma_{1n},T_2(t)\sigma_2}(0, s_n - t, 0, B) \subset U_{\Sigma_1,T_2(t)\sigma_2}(0, s_n - t, 0, B)$$

and then it follows from (8) (in a similar way as in the proof of Lemma 8) that the sequence $\{\xi_n\}$ is precompact.

Further, note that

$$\gamma_{\Sigma_1}^{t_0}(t,\sigma_2,B) = \cup_{\tau \le t_0} U_{\Sigma_1,\sigma_2}(t,\tau,B) \subset \cup_{\tau - t \le t_0 - t} U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau - t, B).$$

Then in view of (8) there exists t_0 for which $\gamma_{\Sigma_1}^{t_0}(t,\sigma_2,B)$ is bounded. ■

Let us now study in detail the relationship between the parameter σ_2 and the final moment of time t.

Proposition 35 *Let $(T1)$ hold. Then*

$$\omega_{\Sigma_1}(t,\sigma_2,B) = \omega_{\Sigma_1}(0, T_2(t)\sigma_2, B), \text{ for all } t \in \mathbb{R}.$$

Proof. Using Lemma 19 we have

$$\omega_{\Sigma_1}(t,\sigma_2,B) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U_{\Sigma_1,\sigma_2}(t,\tau,B)} = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau - t,B)}$$

$$= \bigcap_{s-t \le 0} \overline{\bigcup_{\tau - t \le s - t} U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau - t,B)} = \omega_{\Sigma_1}(0,T_2(t)\sigma_2,B).$$

■

Definition 36 *Let $(T1)$ hold. Then the family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2 \in \Sigma_2}$ is called a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ if:*

1. *$\Theta_{\Sigma_1}(\sigma_2)$ is Σ_1-uniformly attracting at time 0 for any $\sigma_2 \in \Sigma_2$;*
2. *It is semi-invariant, that is,*

$$\Theta_{\Sigma_1}(T_2(t)\sigma_2) \subset U_{\Sigma_1,\sigma_2}(t,s,\Theta_{\Sigma_1}(T_2(s)\sigma_2)), \text{ for any } (t,s) \in \mathbb{R}_d, \sigma_2 \in \Sigma_2;$$

3. *It is minimal, that is, for any $\sigma_2 \in \Sigma_2$ and any closed Σ_1-uniformly attracting set $Y(\sigma_2)$ at time 0, we have*

$$\Theta_{\Sigma_1}(\sigma_2) \subset Y.$$

This definition is justified by the following propositions.

Proposition 37 *Let $(T1)$ hold and let the family of sets $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R},\sigma_2\in\Sigma_2}$ be a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ for each fixed $\sigma_2\in\Sigma_2$ in the sense of Definition 9. Then $\overline{\Theta_{\Sigma_1}(t,\sigma_2)} = \overline{\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)}$, for any $t\in\mathbb{R},\sigma_2\in\Sigma_2$.*

Proof. Using Lemma 19 for any $B\in\mathcal{B}(X)$ we have

$$dist\left(U_{\Sigma_1,\sigma_2}(t,\tau,B),\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)\right)$$

$$= dist\left(U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau-t,B),\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)\right)\to 0, \text{ as } \tau\to-\infty,$$

so that $\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)$ is Σ_1-uniformly attracting at time t for σ_2. The minimality property of $\Theta_{\Sigma_1}(t,\sigma_2)$ implies then that $\Theta_{\Sigma_1}(t,\sigma_2)\subset\overline{\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)}$.

In a similar way we get

$$dist\left(U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau,B),\Theta_{\Sigma_1}(t,\sigma_2)\right)$$

$$= dist\left(U_{\Sigma_1,\sigma_2}(t,\tau+t,B),\Theta_{\Sigma_1}(t,\sigma_2)\right)\to 0, \text{ as } \tau\to-\infty,$$

so that $\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)\subset\overline{\Theta_{\Sigma_1}(t,\sigma_2)}$. ■

Proposition 38 *Let $(T1)$ hold.*

1. *If the family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}$ is a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ in the sense of Definition 36, then the family of sets $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R}}$ defined by $\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(T_2(t)\sigma_2)$ is a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ for each fixed $\sigma_2\in\Sigma_2$ in the sense of Definition 9.*

2. *Conversely, if the family of sets $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R}}$ is a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ for each fixed $\sigma_2\in\Sigma_2$ (in the sense of Definition 9) such that $\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)$, for all $t\in\mathbb{R}$, then the family of sets $\{\Theta_{\Sigma_1}(0,\sigma_2)\}_{\sigma_2\in\Sigma_2}$ is a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ in the sense of Definition 36.*

3. *If the families $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R},\sigma_2\in\Sigma_2}$, $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}$ are Σ_1-uniform global attractors in the sense of Definitions 9 and 36, respectively, then*

$$\overline{\Theta_{\Sigma_1}(t,\sigma_2)}=\overline{\Theta_{\Sigma_1}(T_2(t)\sigma_2)}, \text{ for any } t\in\mathbb{R},\sigma_2\in\Sigma_2.$$

Proof. Let the family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}$ be a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ in the sense of Definition 36. Define the family $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R},\sigma_2\in\Sigma_2}$ by $\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(T_2(t)\sigma_2)$. Using Lemma 19 and the fact that the family $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}$ is Σ_1-uniformly attracting at time 0, for any $B\in\mathcal{B}(X)$ we have

$$dist\left(U_{\Sigma_1,\sigma_2}(t,\tau,B),\Theta_{\Sigma_1}(t,\sigma_2)\right)=dist\left(U_{\Sigma_1,\sigma_2}(t,\tau,B),\Theta_{\Sigma_1}(T_2(t)\sigma_2)\right)$$

$$= dist\left(U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau-t,B),\Theta_{\Sigma_1}(T_2(t)\sigma_2)\right)\to 0, \text{ as } \tau\to-\infty, \tag{9}$$

so that $\Theta_{\Sigma_1}(T_2(t)\sigma_2)$ is Σ_1-uniformly attracting at time t for σ_2.

For the semi-invariance property note that

$$\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(T_2(t)\sigma_2)\subset U_{\Sigma_1,\sigma_2}(t,s,\Theta_{\Sigma_1}(T_2(s)\sigma_2))=U_{\Sigma_1,\sigma_2}(t,s,\Theta_{\Sigma_1}(s,\sigma_2)).$$

Finally, let Y be a closed Σ_1-uniformly attracting set at time t for σ_2. Since

$$dist\left(U_{\Sigma_1,\sigma_2}(t,\tau,B),Y\right)=dist\left(U_{\Sigma_1,T_2(t)\sigma_2}(0,\tau-t,B),Y\right)\to 0, \text{ as } \tau\to-\infty, \tag{10}$$

the minimality property of $\Theta_{\Sigma_1}(T_2(t)\sigma_2)$ implies $\Theta_{\Sigma_1}(t,\sigma_2)\subset Y$.

Conversely, let the family of sets $\{\Theta_{\Sigma_1}(t,\sigma_2)\}_{t\in\mathbb{R}}$ be a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ for each fixed $\sigma_2\in\Sigma_2$ in the sense of Definition 9. It is obvious that the family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}=$

$\{\Theta_{\Sigma_1}(0,\sigma_2)\}_{\sigma_2\in\Sigma_2}$ is Σ_1-uniformly attracting and satisfies the minimality property. For the semi-invariance property we have

$$\Theta_{\Sigma_1}(T_2(t)\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)=\Theta_{\Sigma_1}(t,\sigma_2)\subset U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2))$$

$$=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(0,T_2(\tau)\sigma_2))=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(T_2(\tau)\sigma_2)).$$

It remains to prove the equality $\overline{\Theta_{\Sigma_1}(t,\sigma_2)}=\overline{\Theta_{\Sigma_1}(T_2(t)\sigma_2)}$, for any $t\in\mathbb{R},\sigma_2\in\Sigma_2$. Since $\Theta_{\Sigma_1}(t,\sigma_2)$ is minimal and, in view of (9), $\Theta_{\Sigma_1}(T_2(t)\sigma_2)$ is Σ_1-uniformly attracting at time t for σ_2, we get $\Theta_{\Sigma_1}(t,\sigma_2)\subset\overline{\Theta_{\Sigma_1}(T_2(t)\sigma_2)}$. The converse inclusion $\Theta_{\Sigma_1}(T_2(t)\sigma_2)\subset\overline{\Theta_{\Sigma_1}(t,\sigma_2)}$ follows from (10) and the minimality property of $\Theta_{\Sigma_1}(T_2(t)\sigma_2)$. ∎

Remark 39 *We note that if the sets $\Theta_{\Sigma_1}(t,\sigma_2)$ are closed in the second statement, then Proposition 37 implies that the condition $\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)$ is satisfied. This is the case where the sets $\Theta_{\Sigma_1}(t,\sigma_2)$ are compact.*

Theorem 40 *Let X be a complete metric space in which every compact set is nowhere dense, $(T1)$ hold and let (8) be satisfied. Then the following statements hold:*

1. *If for all $\tau\leq 0$ and $\sigma_2\in\Sigma_2$ the graph of the map $x\mapsto U_{\Sigma_1,\sigma_2}(0,\tau,x)\in P(X)$ is closed, then there exists the Σ_1-uniform global attractor $\{\Theta_{\Sigma_1}(\sigma_2)\}$. Moreover,*

$$\Theta_{\Sigma_1}(\sigma_2)=\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(0,\sigma_2,B)\neq X,$$

   and for each $t\in\mathbb{R}$, $\sigma_2\in\Sigma_2$, $\Theta_{\Sigma_1}(\sigma_2)$ is a Lindelöf, normal space. It is locally compact in some topology $\tau_{\oplus}$, which is stronger than the topology induced by X in $\Theta_{\Sigma_1}(\sigma_2)$.

2. *If, in addition, Σ_1 is a compact metric space, the map*

$$\Sigma_1\times X\ni(\sigma_1,x)\longmapsto U_{\sigma_1,\sigma_2}(0,\tau,x)\in P(X)$$

   is upper semicontinuous for any $\tau\leq 0$, $\sigma_2\in\Sigma_2$, U_σ has connected values for any $\sigma\in\Sigma$, $(0,\tau)\in\mathbb{R}_d$, $x\in X$, Σ_1 is a connected space and

$$\Theta_{\Sigma_1}(T_2(\tau)\sigma_2)\subset B_1(\sigma_2),\text{ for all }\tau\leq 0,$$

   where $B_1(\sigma_2)$ is a bounded connected set for any $\sigma_2\in\Sigma_2$, then the set $\Theta_{\Sigma_1}(\sigma_2)$ is connected for any $\sigma_2\in\Sigma_2$.

Proof. In view of Proposition 34 Lemma 19 the conditions of the first statement are equivalent to those of Theorem 11. Then the family

$$\Theta_{\Sigma_1}(t,\sigma_2)=\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(t,\sigma_2,B)$$

is a Σ_1-global attractor in the sense of Definition 9. It follows from Proposition 35 the equality

$$\Theta_{\Sigma_1}(t,\sigma_2)=\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(t,\sigma_2,B)=\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(0,T_2(t)\sigma_2,B)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2).\tag{11}$$

The first statement is then a consequence of the first statement in Proposition 38.

For the second statement we note that using

$$\Theta_{\Sigma_1}(t+\tau,\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t+\tau)\sigma_2)=\Theta_{\Sigma_1}(T_2(t+\tau)\sigma_2)$$

$$=\Theta_{\Sigma_1}(T_2(\tau)T_2(t)\sigma_2)\subset B_1(T_2(t)\sigma_2),\text{ for all }\tau\leq 0,$$

and Lemma 19 we obtain that the conditions of the second statement of Theorem 11 are also satisfied. It follows that the sets $\Theta_{\Sigma_1}(\sigma_2)$ are connected. ∎

Similarly, we can obtain the following consequence of Theorem 17.

Theorem 41 *Let us suppose that for all* $(0,s)\in\mathbb{R}_d$ *and* $\sigma_2\in\Sigma_2$ *the graph of the map* $x\mapsto U_{\Sigma_1,\sigma_2}(0,s,x)\in P(X)$ *is closed. If, moreover, for any* $\sigma_2\in\Sigma_2$ *there exists a set* $D(\sigma_2)\in K(X)$*, which is* Σ_1*-uniformly attracting, then the set*

$$\Theta_{\Sigma_1}(\sigma_2)=\overline{\bigcup_{B\in\mathcal{B}(X)}\omega_{\Sigma_1}(0,\sigma_2,B)}$$

is the Σ_1*-uniform global attractor of* U_σ*. Moreover, the sets* $\Theta_{\Sigma_1}(\sigma_2)$ *are compact and, if the conditions of the second statement in Theorem 11 hold, then they are connected.*

Finally, we have:

Proposition 42 *Let the MDP* U_σ *be strict,* Σ_1 *be a compact metric space and let the map*

$$\Sigma_1\times X\ni(\sigma_1,x)\longmapsto U_{\sigma_1,\sigma_2}(0,\tau,x)\in P(X)$$

be lower semicontinuous. Then the global attractors obtained in Theorems 40 and 41 are invariant, that is, $\Theta_{\Sigma_1}(T_2(t)\sigma_2)=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(T_2(\tau)\sigma_2))$*, for all* $\tau\leq t,\sigma_2\in\Sigma_2$*.*

Proof. By using the equality $\Theta_{\Sigma_1}(t,\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)$ (proved in (11)) and Proposition 18 we obtain

$$\Theta_{\Sigma_1}(T_2(t)\sigma_2)=\Theta_{\Sigma_1}(0,T_2(t)\sigma_2)=\Theta_{\Sigma_1}(t,\sigma_2)=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(\tau,\sigma_2))$$

$$=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(0,T_2(\tau)\sigma_2))=U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(T_2(\tau)\sigma_2)).$$

■

3 Applications to nonautonomous evolution inclusions

Let $\Omega\subset\mathbb{R}^n$ be a bounded open subset with smooth boundary $\partial\Omega$. Consider the parabolic inclusion

$$\begin{cases}\dfrac{\partial u}{\partial t}-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)\in f_1(t,u)+f_2(t,u)+g_1(t)+g_2(t)\,,\ \text{in }\Omega\times(\tau,T)\,,\\ u\mid_{\partial\Omega}=0,\\ u\mid_{t=\tau}=u_\tau,\end{cases}\tag{12}$$

where $\tau\in\mathbb{R}$, $p\geq2$, $f_i:\mathbb{R}\times\mathbb{R}\to C_v(\mathbb{R})$, $i=1,2$, $g_1\in L_\infty(\mathbb{R},L_2(\Omega))$, $g_2\in L_2^{loc}(\mathbb{R},L_2(\Omega))$ and the following conditions hold:

(F1) There exists $C\geq0$ such that

$$\text{dist}_H(f_1(t,u),f_1(t,v))\leq C|u-v|\,,\ \text{for all }t\in\mathbb{R},u,v\in\mathbb{R}.$$

(F2) For any $t,s\in\mathbb{R}$ and $u\in\mathbb{R}$, it holds

$$\text{dist}_H(f_1(t,u),f_1(s,u))\leq l(|u|)\,\alpha(|t-s|)\,,$$

where α is a continuous function such that $\alpha(t)\to0$, as $t\to0^+$, and l is a continuous nondecreasing function. Moreover, there exist $K_1,K_2\geq0$ such that

$$|l(u)|\leq K_1+K_2|u|\,,\ \text{for all }u\in\mathbb{R}.$$

(F3) There exist $D\in\mathbb{R}_+$, $v_0\in\mathbb{R}$ for which

$$|f_1(t,v_0)|_+\leq D,\ \text{for all }t\in\mathbb{R},$$

where $|f_1(t,v_0)|_+=\sup_{\zeta\in f_1(t,v_0)}|\xi|$.

(F4) There exist $\alpha_1(t), \alpha_2(t) \geq 0,\ \alpha_1(\cdot), \alpha_2(\cdot) \in L_2^{loc}(-\infty, \infty)$, such that

$$\sup_{y \in f_2(t,u)} |y| \leq \alpha_1(t) + \alpha_2(t)|u|, \text{ for all } u, t \in \mathbb{R}.$$

(F5) For each $t \in \mathbb{R}$, the map $f_2(t, \cdot)$ is upper semicontinuous.

(F6) For each $s \in \mathbb{R}$, the map $f_2(\cdot, s)$ is measurable.

(F7) If $p = 2$, there exist $\epsilon > 0$ and $M \geq 0$ such that

$$yu \leq (\lambda_1 - \epsilon) u^2 + M, \text{ for all } u \in \mathbb{R}, t \in \mathbb{R}, y \in f_1(t,u) + f_2(t,u),$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

(F8) There exist $R_1, R_2, R_3 > 0$ such that

$$\|g_2(t)\|_{L_2(\Omega)} \leq R_1 + R_2 |t|^{R_3}, \text{ for a.a. } t \in \mathbb{R}.$$

(F9) If $p > 2$, there exist $R_4, R_4, R_6 > 0$ such that

$$|\alpha_i(t)| \leq R_4 + R_5 |t|^{R_6}, \text{ for a.a. } t \in \mathbb{R},\ i = 1, 2.$$

Our aim is to apply the abstract results of the previous section to inclusion (12).

3.1 Abstract setting: construction of the family of multivalued processes

First let us construct the sets Σ_1, Σ_2. The set Σ_1 will be defined exactly in the same way as in [17]. We shall briefly recall how Σ_1 is defined.

Let W be the space $C_v(\mathbb{R})$ endowed with the Hausdorff metric $\rho(x,y) = \text{dist}_H(x,y)$. The space $W \subset K(\mathbb{R})$ is complete.

For any $\psi \in W$ let $|\psi|_+ = \max_{y \in \psi} |y|$. Define also the space

$$\mathcal{M} = \left\{ \psi \in C(\mathbb{R}, W) : |\psi(v)|_+ \leq D_1 + D_2 |v| \right\},$$

where the constants D_1, D_2 are such that

$$|y| \leq D_1 + D_2 |u|, \text{ for all } u \in \mathbb{R}, t \in \mathbb{R}, y \in f_1(t,u),$$

(see [17, Lemma 12].)

If we take $K_i = [-R_i, R_i]$, where $0 < N_1 < N_2 < ... < N_n \to \infty$, we have $\psi^m \to \psi$ if and only if

$$\max_{|v| \leq N_i} dist_H(\psi^m(v), \psi(v)) \to 0, \text{ as } m \to \infty, \text{ for all } N_i.$$

The space $\mathcal{M} \subset C(\mathbb{R}, K(\mathbb{R}))$ is complete. Let $\Phi \subset \mathcal{M}$ be the set

$$\Phi = \{\psi \in \mathcal{M} : dist_H(\psi(u), \psi(v)) \leq C|u - v|, \text{ for all } u, v \in \mathbb{R}\},$$

where C is defined in $(F1)$. This set is compact.

Recall that the hull of $f \in C(\mathbb{R}, \mathcal{M})$ is defined by

$$\mathcal{H}(f) = cl_{C(\mathbb{R},\mathcal{M})} \{f(\cdot + h) : h \in \mathbb{R}\}.$$

Definition 43 *The function $f \in C(\mathbb{R}, \mathcal{M})$ is said to be translation-compact if its hull $\mathcal{H}(f)$ is compact in $C(\mathbb{R}, \mathcal{M})$.*

Lemma 44 *The function f_1 is translation-compact.*

Proof. It follows the same lines as in [17, Lemma 15] but changing $\mathbb{R}_+$ by $\mathbb{R}$. ■

Since $g_1 \in L_\infty(\mathbb{R}, L_2(\Omega))$, in the same way as in [17] we obtain that the symbol $\sigma_{10}(t) = (f_1(t,\cdot), g_1(t))$ is translation-compact in the space $C(\mathbb{R}, \mathcal{M}) \times L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega))$, where $L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega))$ is the space $L_2^{loc}(\mathbb{R}, L_2(\Omega))$ endowed with the weak topology. The hull of this symbol will be denoted by $\Sigma_1 = \mathcal{H}(f_1) \times \mathcal{H}(g_1)$, where $\mathcal{H}(g_1) = cl_{L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega))}\{g_1(\cdot + h) : h \in \mathbb{R}\}$. On the other hand, for any $g_{\sigma_1} \in \mathcal{H}(g_1)$ we have that

$$\|g_{\sigma_1}\|_{L_\infty(\mathbb{R}, L_2(\Omega))} \leq C_0 = \|g_1\|_{L_\infty(\mathbb{R}, L_2(\Omega))}. \tag{13}$$

(See [17, Lemma 12].) It is straightforward to check that for any $f_{\sigma_1} \in \mathcal{H}(f_1)$ conditions $(F1) - (F3)$ hold. We note that all the constants and functions in $(F1) - (F3)$ do not depend on $\sigma_1 \in \Sigma_1$.

The set Σ_1 is then a compact metric space and in the same way as in [17, Lemma 11] we can prove that $T_1(h)\Sigma_1 \subset \Sigma_1$, for all $h \in \mathbb{R}$, where $T_1(h)$ is the shift operator, that is, $T_1(h)\sigma_1(t) = \sigma_1(t+h)$.

For the set Σ_2 we put

$$\Sigma_2 = \bigcup_{h \in \mathbb{R}} (f_2(\cdot + h), g_2(\cdot + h)).$$

It is clear that $T_2(h)\Sigma_2 \subset \Sigma_2$, for all $h \in \mathbb{R}$, and also that if $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2}) \in \Sigma_2$, then f_{σ_2} satisfies $(F4) - (F6)$ and $(F9)$, whereas $g_{\sigma_2} \in L_2^{loc}(\mathbb{R}, L_2(\Omega))$ satisfies $(F8)$. We note that in this case the functions α_1, α_2 and the constants R_i can depend on σ_2.

Finally, we note that for any $\sigma \in \Sigma = \Sigma_1 \times \Sigma_2$ condition $(F7)$ holds (with the same constants, which do not depend on σ).

Now let $X = L_2(\Omega)$ with the norm $\|\cdot\|_X$ and the scalar product $(\cdot,\cdot)$. Consider the abstract evolution inclusion

$$\begin{cases} \dfrac{du(t)}{dt} \in A(u(t)) + F_\sigma(t, u(t)), \ t \in [\tau, \infty), \\ u(\tau) = u_\tau, \end{cases} \tag{14}$$

where $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ and $A : D(A) \subset X \to 2^X$, $F_\sigma : \mathbb{R} \times X \to 2^X$, are multivalued maps defined as follows:

$$A(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right),$$

$$D(A) = \left\{u \in W_0^{1,p}(\Omega) : A(u) \in L_2(\Omega)\right\},$$

$$F_{\sigma_1}(t, u) = \{y \in X : y(x) \in f_{\sigma_1}(t, u(x)) + g_{\sigma_1}(t), \text{ a.e. on } \Omega\},$$

$$F_{\sigma_2}(t, u) = \{y \in X : y(x) \in f_{\sigma_2}(t, u(x)) + g_{\sigma_2}(t), \text{ a.e. on } \Omega\}$$

$$F_\sigma(t, u) = F_{\sigma_1}(t, u) + F_{\sigma_2}(t, u).$$

It is understood that the map F_σ is defined for a.a. $t \in \mathbb{R}$.

The operator A satisfies the following properties (see [16, Section 3.2]):

$(A1)$ The operator A is m-dissipative, i.e. for any $y_1, y_2 \in D(A)$, $\xi_i \in A(y_i)$, $i = 1, 2$, there exists $j(y_i, \xi_i) \in J(y_1 - y_2)$ such that

$$< \xi_1 - \xi_2, j > \leq 0,$$

and $Im(A - \lambda I) = X$, for all $\lambda > 0$, where $J : X \to 2^{X^*}$ is the duality map defined by

$$J(y) = \{\xi \in X^* \mid < y, \xi >= \|y\|_X^2 = \|\xi\|_{X^*}^2\}, \text{ for any } y \in X.$$

(A2) $\overline{D(A)} = L_2(\Omega)$.

(A3) A generates a compact semigroup S.

For F_{σ_1} it holds (see [17, p.395 and Lemma 18]):

(G1) $F_{\sigma_1} : \mathbb{R} \times X \to C_v(X)$, for all $\sigma_1 \in \Sigma_1$ (recall that $C_v(X)$ is the set of all nonempty, closed, bounded, convex subsets of X).

(G2) For any $(T,\tau) \in \mathbb{R}_d$, $x \in X$, $\sigma_1 \in \Sigma$ the map $t \mapsto F_{\sigma_1}(t,x)$ is measurable and

$$dist_H(F_{\sigma_1}(t,x_1), F_{\sigma_1}(t,x_2)) \le C \|x_1 - x_2\|_X, \text{ for all } x_1, x_2 \in X, t \in \mathbb{R},\ \sigma_1 \in \Sigma_1.$$

(G3) For any $x \in X$ there exist $\gamma_1, \gamma_2 \ge 0$ such that

$$\|F_{\sigma_1}(t,x)\|_+ \le \gamma_1 + \gamma_2 \|x\|_X + C_0, \text{ a.e. } t \in \mathbb{R}, \text{ for all } \sigma \in \Sigma_1,$$

where $\|K\|_+ = \sup_{y \in K} \|y\|_X$ and C_0 is taken from (13).

Define the map $\widetilde{F}_{\sigma_2}$ by

$$\widetilde{F}_{\sigma_2}(t,u) = \{y \in X : y(x) \in f_{\sigma_2}(t,u(x)), \text{ a.e. on } \Omega\}.$$

We know from [12, Proposition 2.5] that, for any fixed $t \in \mathbb{R}$, the map $u \longmapsto \widetilde{F}_{\sigma_2}(t,u) \in C_v(X)$ is w-upper semicontinuous. On the other hand, in view of $(F4)$ we get

$$\|y\|_X^2 \le \int_\Omega (\alpha_1(t) + \alpha_2(t)|u|)^2 dx \le 2\left(\alpha_1^2(t)\mu(\Omega) + \alpha_2^2(t)\|u\|_X^2\right)$$

$$\le 2\left(\alpha_1(t)(\mu(\Omega))^{\frac{1}{2}} + \alpha_2(t)\|u\|_X\right)^2, \text{ for all } y \in \widetilde{F}_{\sigma_2}(t,u) \text{ and a.a. } t \in \mathbb{R},$$

so that there exist $\widetilde{\alpha}_1(t), \widetilde{\alpha}_2(t) \ge 0$, $\widetilde{\alpha}_1(\cdot), \widetilde{\alpha}_2(\cdot) \in L_2^{loc}(-\infty,\infty)$, such that

$$\left\|\widetilde{F}_{\sigma_2}(t,u)\right\|_+ \le \widetilde{\alpha}_1(t) + \widetilde{\alpha}_2(t)\|u\|_X, \text{ for all } u \in X \text{ and a.a. } t \in \mathbb{R}. \tag{15}$$

Hence, for F_{σ_2} we have:

(G4) $F_{\sigma_2} : \mathbb{R} \times X \to C_v(X)$, for all $\sigma_2 \in \Sigma_2$.

(G5) For any fixed $t \in \mathbb{R}$ and σ_2 the map $u \longmapsto F_{\sigma_2}(t,u)$ is w-upper semicontinuous.

(G6) For any $\sigma_2 \in \Sigma_2$ we have

$$\|F_{\sigma_2}(t,u)\|_+ \le \widetilde{\alpha}_1(t) + \widetilde{\alpha}_2(t)\|u\|_X + \|g_{\sigma_2}(t)\|_X, \text{ for all } u \in X \text{ and a.a. } t \in \mathbb{R}.$$

Let us now study the properties of the map F_σ.

Lemma 45 *The map F_σ satisfies:*

(S1) *$F_\sigma : \mathbb{R} \times X \to C_v(X)$, for all $\sigma \in \Sigma$.*

(S2) *For any fixed $t \in \mathbb{R}$ and $\sigma \in \Sigma$ the map $u \longmapsto F_\sigma(t,u)$ is w-upper semicontinuous.*

(S3) *For any $\sigma \in \Sigma$ there exist $\beta_1, \beta_2 \ge 0$, $\beta_1, \beta_2 \in L_2^{loc}(-\infty,\infty)$ (depending on σ_2 but not on σ_1), such that*

$$\|F_\sigma(t,u)\|_+ \le \beta_1(t) + \beta_2(t)\|u\|_X, \textit{for all } u \in X \textit{and a.a. } t \in \mathbb{R}.$$

(S4) *For any $(T,\tau) \in \mathbb{R}_d$, $x \in X$, $\sigma \in \Sigma$, the map $t \mapsto F_\sigma(t,x)$ has a measurable selection.*

Proof. For $(S1)$ note that it follows immediately from $(G1)$ and $(G4)$ that F_σ has non-empty, bounded, convex values. Finally, for the closedness take a sequence $y_n \in F_\sigma(t,u)$ such that $y_n \to y$. Note that $y_n = x_n + z_n$, where $x_n \in F_{\sigma_1}(t,u), z_n \in F_{\sigma_2}(t,u)$. Passing to a subsequence, if necessary, we obtain that $x_n \to x$, $z_n \to z$ weakly in $L_2(\Omega)$. Since $F_{\sigma_1}(t,u), F_{\sigma_2}(t,u)$ are closed and convex, they are weakly closed, so that $x \in F_{\sigma_1}(t,u), z \in F_{\sigma_2}(t,u)$. Hence, $y = x + z \in F_\sigma(t,u)$.

For $(S2)$ note that in view of $(G2)$ and $(G5)$ for any $\varepsilon > 0$ and $v \in X$ we have

$$dist\left(F_\sigma(t,u), F_\sigma(t,v)\right) = dist\left(F_{\sigma_1}(t,u) + F_{\sigma_2}(t,u), F_{\sigma_1}(t,v) + F_{\sigma_2}(t,v)\right)$$

$$\leq dist\left(F_{\sigma_1}(t,u), F_{\sigma_1}(t,v)\right) + dist\left(F_{\sigma_2}(t,u), F_{\sigma_2}(t,v)\right) \leq \varepsilon,$$

provided that $\|u - v\| \leq \delta(\varepsilon, v)$.

Further, by $(G3)$ and $(G6)$ we get

$$\|F_\sigma(t,u)\|_+ \leq \gamma_1 + \gamma_2 \|u\|_X + C_0 + \widetilde{\alpha}_1(t) + \widetilde{\alpha}_2(t)\|u\|_X + \|g_{\sigma_2}(t)\|_X,$$

so that $(S3)$ holds.

Consider now $(S4)$. First let us prove that the map $\widetilde{F}_{\sigma_2}(\cdot,u)$ has a measurable selection for all $\sigma_2 \in \Sigma_2,\ u \in X$. Take first a constant function $u(x) \equiv u \in \mathbb{R}$. The map $t \mapsto f_2(t,u)$ is measurable by assumption $(F6)$, so that it has a measurable selection $g(t)$ (see [1, Theorem 8.3.1]). Define the map $G : \mathbb{R} \to L_2(\Omega)$ by $G(t,x) = g(t)$, for all $x \in \Omega$. We claim that $G(t)$ is a measurable selection of $\widetilde{F}_{\sigma_2}(\cdot,u)$. Indeed, for any $v \in L_2(\Omega)$ we have

$$(G(t), v) = \int_\Omega g(t) v(x)\, dx = g(t) \int_\Omega v(x)\, dx = g(t) v_0.$$

Since the last map is measurable and the space $L_2(\Omega)$ is separable, $G(t)$ is a measurable map (see [22]). The inclusion $G(t) \in \widetilde{F}_{\sigma_2}(t,u)$ is obvious. Further, let u be a step function, that is,

$$u(x) = \begin{cases} u_1, & \text{if } x \in \Omega_1, \\ \vdots & \\ u_m, & \text{if } x \in \Omega_m. \end{cases}$$

For each u_i we can take a measurable selection $g_i(t)$ of the map $f_2(t,u_i)$. Define the map $G : \mathbb{R} \to L_2(\Omega)$ by

$$G(t,x) = \begin{cases} g_1(t), & \text{if } x \in \Omega_1, \\ \vdots & \\ g_m(t), & \text{if } x \in \Omega_m. \end{cases}.$$

We claim that $G(t)$ is a measurable selection of $\widetilde{F}_{\sigma_2}(\cdot,u)$. Indeed, for any $v \in L_2(\Omega)$ we have

$$(G(t), v) = \sum_{i=1}^m \int_{\Omega_i} g_i(t) v(x)\, dx = \sum_{i=1}^m g_i(t) v_i,$$

so that $G(t)$ is measurable and again the inclusion $G(t) \in \widetilde{F}_{\sigma_2}(t,u)$ is obvious.

Further, take a sequence of step functions u_n converging to u in $L_2(\Omega)$. In view of (15) the sequence of selections $G_n(t) \in \widetilde{F}_{\sigma_2}(t,u_n)$ satisfies the inequality

$$\|G_n(t)\|_X \leq \left\|\widetilde{F}_{\sigma_2}(t,u_n)\right\|_X \leq \widetilde{\alpha}_1(t) + \widetilde{\alpha}_2(t)\|u_n\|_X \leq \widetilde{\alpha}_1(t) + \widetilde{\alpha}_2(t) C.$$

Hence, choosing a subsequence if necessary we can assume that $G_n \to G$ weakly in $L_2(\tau, T; L_2(\Omega))$.

We have to prove further that $G(t) \in \widetilde{F}_{\sigma_2}(t,u)$, a.e. on (τ, T). In view of [19, Proposition 1.1] we have

$$G(t) \in \cap_{m=1}^\infty \overline{co} \cup_{n \geq m} G_n(t), \text{ for a.a. } t \in (\tau, T).$$

Fix $t\in(\tau,T)$. Since the map $u\mapsto\widetilde{F}_{\sigma_2}(t,u)$ is w-upper semicontinuous (see $(G5)$), we have

$$dist\left(G_n(t),\widetilde{F}_{\sigma_2}(t,u)\right)\to 0,\text{ as } n\to\infty.$$

Hence, the convexity of the set $\widetilde{F}_{\sigma_2}(t,u)$ implies that for any $\delta>0$ there exists n_0 such that

$$dist\left(\overline{co}\cup_{n\geq n_0}G_n(t),\widetilde{F}_{\sigma_2}(t,u)\right)<\delta,\ \text{if } n\geq n_0.$$

Therefore, the closedness of $\widetilde{F}_{\sigma_2}(t,u)$ gives $G(t)\in\widetilde{F}_{\sigma_2}(t,u)$, a.e. on (τ,T), as required.

By $(G2)$ the map $t\mapsto F_{\sigma_1}(t,u)$ is measurable, so that it has a measurable selection $K(t)$ (see [1, Theorem 8.3.1]). Hence, the map

$$\eta(t)=K(t)+G(t)+g_{\sigma_2}(t)$$

is a measurable selection of $F_\sigma(t,u)$. ■

Now we can construct the multivalued process corresponding to (12).

Definition 46 *The continuous function $u_\sigma(\cdot)\in C([\tau,T],X)$ is called an integral solution of (14) if $u_\sigma(\tau)=u_\tau$ and there exists $l(\cdot)\in L_1([\tau,T],X)$ such that $l(t)\in F_\sigma(t,u_\sigma(t))$, a.e. on (τ,T), and for any $\xi\in D(A)$, $v\in A(\xi)$ one has*

$$\|u_\sigma(t)-\xi\|_X^2\leq\|u_\sigma(s)-\xi\|_X^2+2\int_s^t(l(r)+v,u_\sigma(r)-\xi)\,dr,\ t\geq s. \tag{16}$$

It follows from $(A1)-(A2)$, $(S1)-(S4)$ that for any $u_\tau\in\overline{L_2(\Omega)}$ there exists at least one integral solution u_σ to (14) for any $T>\tau$ (such that $u_\sigma(t)\in\overline{D(A)}$ for a.a. $t\in(\tau,T)$) [20, Theorem 2.1]. We shall denote any integral solution by $u_\sigma(\cdot)=I(u_\tau)\,l(\cdot)$. For a fixed $\sigma\in\Sigma$ let $\mathcal{D}_{\sigma,\tau}(x)$ be the set of all integral solutions corresponding to the initial condition $u(\tau)=x$.

For any integral solutions $u_\sigma(\cdot)=I(u_\tau)\,l_1(\cdot)$, $v_\sigma(\cdot)=I(v_\tau)\,l_2(\cdot)$, the following inequality holds

$$\|u_\sigma(t)-v_\sigma(t)\|_X\leq\|u_\sigma(s)-v_\sigma(s)\|_X+\int_s^t\|l_1(r)-l_2(r)\|_X\,dr,\ t\geq s. \tag{17}$$

In the sequel we shall write u instead of u_σ for simplicity of notation if no confussion is possible. We shall define the map $U_\sigma:\mathbb{R}_d\times X\to P(X)$ by

$$U_\sigma(t,\tau,x)=\{z:\text{there exists } u(\cdot)\in\mathcal{D}_{\sigma,\tau}(x)\text{ such that } u(t)=z\}.$$

Proposition 47 *For each $\sigma\in\Sigma$, $h\in\mathbb{R}$, $\tau\leq s\leq t$, $x\in X$ we have*

$$U_\sigma(t,s,U_\sigma(s,\tau,x))=U_\sigma(t,\tau,x),$$

$$U_{T(h)\sigma}(t,\tau,x)=U_\sigma(t+h,\tau+h,x).$$

Hence, U_σ is a multivalued process for each $\sigma\in\Sigma$ and condition $(T1)$ holds.

Proof. It follows the same lines of [17, Proposition 4]. ■

3.2 Existence of the global Σ_1-uniform attractor

We shall check further that the conditions of Theorem 41 are satisfied.

First we shall prove that the graph of the map U is closed.

Proposition 48 *For all $(0,\tau)\in\mathbb{R}_d$ and $\sigma_2\in\Sigma_2$ the graph of the map $x\mapsto U_{\Sigma_1,\sigma_2}(0,\tau,x)\in P(X)$ is closed.*

Proof. In view of Proposition 16 it is sufficient to prove that the map

$$\Sigma_1 \times X \ni (\sigma_1, x) \longmapsto U_{\sigma_1,\sigma_2}(0,\tau,x) \in P(X)$$

is closed for any $(0,\tau) \in \mathbb{R}_d, \sigma_2 \in \Sigma_2$.

Let $y_n \in U_{\sigma_{1n},\sigma_2}(0,\tau,u_\tau^n)$ be such that

$$y_n \to y \text{ in } L_2(\Omega),$$

$$u_\tau^n \to u_\tau \text{ in } L_2(\Omega),$$

$$\sigma_{1n} = (f_{\sigma_{1n}}, g_{\sigma_{1n}}) \to \sigma_1 = (f_{\sigma_1}, g_{\sigma_1}) \text{ in } C(\mathbb{R}, \mathcal{M}) \times L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega)).$$

We have to prove that $y \in U_{\sigma_1,\sigma_2}(0,\tau,u_0)$.

There exist sequences $u_n(\cdot) = I(u_\tau^n) l_n(\cdot)$, $l_n(s) \in F_{\sigma_{1n},\sigma_2}(s, u_n(s))$, a.e. in $(\tau, 0)$, such that $y_n = u_n(0)$.

In view of $(S3)$ we get

$$\|l_n(s)\|_X \leq \|F_{\sigma_{1n},\sigma_2}(s,u_n(s))\|_+ \leq \beta_1(s) + \beta_2(s)\|u_n(s)\|_X, \text{ a.e. on } (\tau,0), \tag{18}$$

where the functions β_1, β_2 may depend on σ_2, but not on σ_{1n}.

We shall show first the existence of a function $m(\cdot) \in L_2(\tau,0)$, $m(s) \geq 0$, such that $\|l_n(s)\|_X \leq m(s)$, a.e. in $(\tau,0)$. Let us introduce the sequence $v_n(\cdot) = I(u_\tau) l_n(\cdot)$ and let $z(\cdot)$ be the unique solution to

$$\begin{cases} \dfrac{dz(t)}{dt} = A(z(t)), \text{ on } (0,T), \\ z(0) = u_0. \end{cases}$$

Let $r_0 = \max\{\|z(s)\|_X : s \in [0,T]\}$ and $r_2 = r_1 + r_0$, where $\|u_\tau - u_\tau^n\|_{L_2} \leq r_1$, for all n. From (17) we have

$$\|u_n(s) - z(s)\|_X \leq \|u_0 - u_0^n\|_X + \int_\tau^s \|l_n(r)\|_{L_2} dr$$

and then by (18),

$$\|u_n(s)\|_{L_2} \leq \|z(s)\|_{L_2} + r_1 + \int_\tau^s (\beta_1(r) + \beta_2(r)\|u_n(r)\|_X) dr$$

$$\leq r_2 + K_1(\tau,\beta_1) + \int_\tau^s \beta_2(r)\|u_n(s)\|_{L_2} dr.$$

Hence, by Gronwall lemma we have

$$\|u_n(s)\|_X \leq (r_2 + K_1(\tau,\beta_1)) \exp\left(\int_\tau^s \beta_2(r) dr\right) = r(s), \text{ for all } s \in [\tau, 0]. \tag{19}$$

Therefore using (18) again we obtain

$$\|l_n(s)\|_X \leq \beta_1(s) + \beta_2(s) r(s) = m(s), \text{ a.e. in } (\tau,0). \tag{20}$$

The sequence $\{l_n\}$ is then precompact in the space $L_2(\tau,0;L_2(\Omega))$ endowed with the weak topology. Hence, it is precompact in the space $L_1(\tau,0;L_2(\Omega))$ endowed with the weak topology and, since the semigroup generated by A is compact, this implies that the sequence $\{v_n\}$ is precompact in $C([\tau,0], L_2(\Omega))$ (see [11, Theorem 2.3]). We obtain that there exist subsequences such that

$$v_n \to v \text{ in } C([\tau,0], L_2(\Omega)),$$

$$l_n \to l \text{ weakly in } L_2(\tau, 0; L_2(\Omega)).$$

Since $l_n \to l$ weakly in $L_1(\tau, 0; L_2(\Omega))$, Lemma 1.3 from [19] implies $v(\cdot) = I(u_\tau) l(\cdot)$. Using again (17) we have $\|u_n(s) - v_n(s)\|_X \leq \|u_0^n(s) - u_0(s)\|_X$, for all $s \in [\tau, 0]$, so that $u_n \to v$ in $C([\tau, 0], L_2(\Omega))$ and $y = v(t)$. To conclude the proof we have to check that $l(s) \in F_\sigma(s, v(s))$, a.e. on $(\tau, 0)$.

Since $l_n \to l$, $g_{\sigma_{1n}} \to g_{\sigma_1}$, weakly in $L_2(\tau, 0; L_2(\Omega))$, we have $l_n - g_{\sigma_{1n}} - g_{\sigma_2} = d_n(\cdot) \to l - g_{\sigma_1} - g_{\sigma_2} = d_\sigma(\cdot)$, weakly in $L_2(\tau, 0; L_2(\Omega))$. Then we need to obtain $d_\sigma(s) \in \widetilde{F}_\sigma(s, v(s)) = F_\sigma(s, v(s)) - g_{\sigma_1}(s) - g_{\sigma_2}(s)$, a.e. on $(\tau, 0)$.

Fix $s \in (\tau, 0)$ and denote $\widetilde{F}_{\sigma_i}(s, v(s)) = F_{\sigma_i}(s, v(s)) - g_{\sigma_i}(s)$, $i = 1, 2$, $d_n(s) = d_{1n}(s) + d_{2n}(s)$, where $d_{in}(s) \in L_2(\Omega)$, $i = 1, 2$, are such that

$$d_{in}(s) \in \widetilde{F}_{\sigma_i}(s, u_n(s)).$$

Note that since $u_n(s) \to v(s)$ in $L_2(\Omega)$, passing to a subsequence if necessary $u_n(s, x) \to v(s, x)$ for a.a. $x \in \Omega$. Hence by $(F1)$ and $(F5)$ we have

$$dist(f_{\sigma_1}(s, u_n(s, x)), f_{\sigma_1}(s, v(s, x))) \leq C|u_n(s, x) - v(s, x)| \to 0,$$

$$dist(f_{\sigma_2}(s, u_n(s, x)), f_{\sigma_2}(s, v(s, x))) \to 0, \quad \text{as } n \to +\infty,$$

for a.a. $x \in \Omega$. On the other hand, since $\{u_n(s, x)\}$ is bounded, that is $|u_n(s, x)| \leq C(x)$, for all n, and $f_{\sigma_{1n}}$ converges to f_{σ_1} in $C(\mathbb{R}, \mathcal{M})$, we get

$$dist(f_{\sigma_{1n}}(s, u_n(s, x)), f_{\sigma_1}(s, u_n(s, x))) \to 0, \text{ as } n \to +\infty,$$

for a.a. $x \in \Omega$. Then

$$dist(d_{1n}(s, x), f_{\sigma_1}(s, v(s, x)))$$

$$\leq dist(f_{\sigma_1}(s, u_n(s, x)), f_{\sigma_1}(s, v(s, x))) + dist(f_{\sigma_{1n}}(s, u_n(s, x)), f_{\sigma_1}(s, u_n(s, x))) \to 0,$$

$$dist(d_{2n}(s, x), f_{\sigma_2}(s, v(s, x))) \leq dist(f_{\sigma_2}(s, u_n(s, x)), f_{\sigma_2}(s, v(s, x))) \to 0, \tag{21}$$

for a.a. $x \in \Omega$.

In view of [19, Proposition 1.1] for a.a. $s \in (\tau, 0)$ we have

$$d(s) \in \bigcap_{n=1}^{\infty} \overline{co} \bigcup_{k \geq n}^{\infty} d_k(s) = \mathcal{A}(s).$$

Fix s. Denote $\mathcal{A}_n(s) = co \bigcup\limits_{k \geq n}^{\infty} d_k(s)$. It is easy to see that $z \in \mathcal{A}(s)$ if and only if there exist $z_n \in \mathcal{A}_n(s)$ such that $z_n \to z$, as $n \to \infty$, in $L_2(\Omega)$. Taking a subsequence we have $z_n(x) \to z(x)$, a.e. in Ω. Since $z_n \in \mathcal{A}_n(s)$, we get

$$z_n(s) = \sum_{i=1}^{N(n)} \lambda_i d_{k_i}(s),$$

where $\lambda_i \in [0, 1]$, $\sum_{i=1}^{N} \lambda_i = 1$ and $k_i \geq n$, for all i.

Now (21) implies that for any $\varepsilon > 0$ and a.a. $x \in \Omega$ there exists $n(x, \varepsilon)$ such that

$$d_k(s, x) \subset [a(s, x) - \varepsilon, b(s, x) + \varepsilon], \text{ for all } k \geq n,$$

where $[a(s, x), b(s, x)] = f_\sigma(s, v(s, x)) = f_{\sigma_1}(s, v(s, x)) + f_{\sigma_2}(s, v(s, x))$ (note that the map f_σ has convex closed values). Hence,

$$z_n(s, x) \subset [a(s, x) - \varepsilon, b(s, x) + \varepsilon],$$

as well. Passing to the limit we obtain

$$z(s,x) \in [a(s,x), b(s,x)] = f_\sigma(s, v(s,x)), \text{ a.e. on } \Omega.$$

Further, note that

$$z_n(s) = \sum_{i=1}^{N(n)} \lambda_i d_{1k_i}(s) + \sum_{i=1}^{N(n)} \lambda_i d_{2k_i}(s) = z_{1n}(s) + z_{2n}(s),$$

and in view of $(G3)$, the inclusion $d_{1k_i}(s) \in \widetilde{F}_{\sigma_1}(s, u_{k_i}(s))$, (13) and (19) we get

$$\|z_{1n}(s)\|_X \leq \sum_{i=1}^{N(n)} \lambda_i \|d_{1k_i}(s)\|_X \leq \sum_{i=1}^{N(n)} \lambda_i \left\|\widetilde{F}_{\sigma_1}(s, u_{k_i}(s))\right\|_+$$

$$\leq \gamma_1 + \gamma_2 \|u_{k_i}(s)\|_X + C_0 + \|g_{\sigma_1}(s)\| \leq \gamma_1 + \gamma_2 r(s) + 2C_0.$$

Hence, passing to a subsequence if necessary we have $z_{1n}(s) \to z_1(s)$, weakly in $L_2(\Omega)$. Mazur's theorem implies that

$$z_1(s) \in \bigcap_{m=1}^{\infty} \overline{co} \bigcup_{n\geq m}^{\infty} z_{1n}(s).$$

Denote now $\mathcal{Z}_m(s) = co \bigcup_{n\geq m}^{\infty} z_{1n}(s)$. As before, there exist $x_m \in \mathcal{Z}_m(s)$ such that $x_m \to z_1(s)$, as $m \to \infty$, in $L_2(\Omega)$. Taking a subsequence we have $x_m(x) \to z_1(x,s)$, a.e. in Ω. Since $x_m \in \mathcal{Z}_m(s)$, we get

$$x_m(s) = \sum_{i=1}^{N_1(m)} \lambda_i z_{1n_i}(s),$$

where $\lambda_i \in [0,1]$, $\sum_{i=1}^{N_1} \lambda_i = 1$ and $n_i \geq m$, for all i.

Now (21) implies that for any $\varepsilon > 0$ and a.a. $x \in \Omega$ there exists $m(x,\varepsilon)$ such that

$$d_{1k}(s,x) \subset [a_1(s,x) - \varepsilon, b_1(s,x) + \varepsilon], \text{ for all } k \geq m,$$

where $[a_1(s,x), b_1(s,x)] = f_{\sigma_1}(s, v(s,x))$. Hence,

$$z_{1n_i}(x) = \sum_{j=1}^{N(n_i)} \gamma_j d_{k_j}(s) \subset [a_1(x) - \varepsilon, b_1(x) + \varepsilon],$$

where $\sum_{j=1}^{N(n_i)} \gamma_j = 1$, $k_j \geq n_i \geq m$, and then

$$x_m(s,x) \subset [a_1(s,x) - \varepsilon, b_1(s,x) + \varepsilon],$$

as well. Passing to the limit we obtain that

$$z_1(s,x) \in [a_1(s,x), b_1(s,x)] = f_{\sigma_1}(s, v(s,x)), \text{ a.e. on } \Omega.$$

Therefore, we get $z(s) = z_1(s) + z_2(s)$, where $z_i(s) \in \widetilde{F}_{\sigma_i}(s, v(s))$, $i = 1, 2$. Hence, $d(s) \in \mathcal{A}(s) \subset \widetilde{F}_\sigma(s, v(s))$, a.e. on $(\tau, 0)$. It follows that $l(s) \in F_\sigma(s, v(s))$, a.e. on $(\tau, 0)$, as required.

Therefore, $y = v(0) \in U_\sigma(0, \tau, x)$. ■

Corollary 49 *For all $(0,\tau) \in \mathbb{R}_d$ and $\sigma_2 \in \Sigma_2$ the map $x \mapsto U_{\Sigma_1,\sigma_2}(0,\tau,x)$ has closed values.*

Further, we shall check the existence of a compact Σ_1-uniformly attracting set at time 0. For this aim we shall use that integral solutions are in fact strong ones.

Indeed, consider the equation

$$\left\{ \begin{array}{c} \dfrac{du(t)}{dt} = A(u(t)) + l(t), \\ u\mid_{t=\tau} = u_\tau, \end{array} \right. \tag{22}$$

where $l \in L_2([\tau, T], X)$. The operator $-A$ is the subdifferential of the proper convex lower semicontinuous function

$$\varphi(u) = \left\{ \begin{array}{c} \frac{1}{p}\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx, \text{ if } u \in W_0^{1,p}(\Omega), \\ +\infty, \text{ otherwise.} \end{array} \right.$$

It is well known that if $l(\cdot) \in L_2([\tau, T], X)$, then the integral solution $u(\cdot)$ to the problem (22), which is unique, is in fact a strong one, that is, $u(\cdot)$ is absolutely continuous on compact sets of (τ, T), a.e. differentiable on (τ, T) and satisfies (22) a.e. in (τ, T) (see [16, Section 3.2]).

Therefore, if we take an arbitrary integral solution to (14), say $u(\cdot) = I(u_\tau) l(\cdot)$, then $(S3)$ implies that $l \in L_2([\tau, T], X)$, so that u is a strong solution to (22).

Lemma 50 *For any $\sigma_2 \in \Sigma_2$ there exists a set $B_1(\sigma_2)$, bounded in X, such that for any $B \in \mathcal{B}(X)$ there exists $T = T(B) < -1$ for which*

$$U_{\Sigma_1,\sigma_2}(-1, \tau, B) \subset B_1, \text{ for all } \tau \leq T. \tag{23}$$

Proof. Fix $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2}) \in \Sigma_2$. First let $p = 2$. Take an arbitrary solution $u(\cdot) = I(u_\tau) l(\cdot)$ defined on $[\tau, -1]$, where $l(t) \in F_{\sigma_1}(t, u(t)) + F_{\sigma_2}(t, u(t))$, $\sigma_1 \in \Sigma_1$ and $u_\tau \in B \in \mathcal{B}(X)$. Multiplying (22) by $u(s)$ and using $(F7)$, $(F8)$ and (13) we have

$$\frac{1}{2}\frac{d}{ds}\|u\|_X^2 + \lambda_1\|u\|_X^2 \leq \frac{1}{2}\frac{d}{ds}\|u\|_X^2 + \|\nabla u\|_X^2$$

$$\leq (\lambda_1 - \epsilon)\|u\|_X^2 + M\mu(\Omega) + \|u\|_X\|g_{\sigma_1}(s)\|_X + \|u\|_X\|g_{\sigma_2}(s)\|_X$$

$$\leq (\lambda_1 - \epsilon)\|u\|_X^2 + M\mu(\Omega) + \left(C_0 + R_1 + R_2|s|^{R_3}\right)\|u\|_X$$

$$\leq \left(\lambda_1 - \frac{\epsilon}{2}\right)\|u\|_X^2 + M\mu(\Omega) + \frac{\left(C_0 + R_1 + R_2|s|^{R_3}\right)^2}{2\varepsilon},$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω in $\mathbb{R}^n$. Therefore,

$$\frac{d}{ds}\|u\|_X^2 + \varepsilon\|u\|_X^2 \leq 2M\mu(\Omega) + \frac{\left(C_0 + R_1 + R_2|s|^{R_3}\right)^2}{\varepsilon}. \tag{24}$$

By the Gronwall lemma

$$\begin{aligned} \|u(-1)\|_X^2 \leq\ & \exp(\varepsilon(1+\tau))\,||u_\tau||_X^2 \\ & + \int_\tau^{-1}\exp(\varepsilon(1+s))\left(2M\mu(\Omega) + \frac{1}{\varepsilon}\left(C_0 + R_1 + R_2|s|^{R_3}\right)^2\right)ds, \end{aligned}$$

so that the ball

$$B_1(\sigma_2) = \left\{y \in X : \|y\| \leq \sqrt{K(\sigma_2) + \alpha}\right\},$$

with $\alpha > 0$, $K(\sigma_2) = \int_{-\infty}^{-1} \exp(\varepsilon(1+s)) \left(2M\mu(\Omega) + \frac{1}{\varepsilon}\left(C_0 + R_1 + R_2 |s|^{R_3}\right)^2\right) ds$, satisfies (23). Indeed, we can find $T(B) < -1$ such that $\exp(\varepsilon(1+\tau)) ||u_\tau||_X^2 \leq \alpha$, for all $\tau \leq T(B)$, $u_\tau \in B$.

Now let $p > 2$. Note that the operator A satisfies Poincaré's inequality $(-A(u), u) \geq \gamma \|u\|_{L_p}^p \geq D \|u\|_{L_2}^p$, where $\gamma, D > 0$. Multiplying (22) by $u(s)$ and using $(G3)$, (15), $(F8)$, $(F9)$ and (13) we have

$$\frac{1}{2}\frac{d}{ds}\|u\|_X^2 + D\|u\|_X^p \leq \frac{1}{2}\frac{d}{ds}\|u\|_X^2 + \gamma\|u\|_{L_p}^p \leq \|F_\sigma(s, u(s))\|_+ \|u\|_X$$

$$\leq \sqrt{2}\left(\alpha_1(s)(\mu(\Omega))^{\frac{1}{2}} + \alpha_2(s)\|u\|_X\right)\|u\|_X + (\gamma_1 + \gamma_2\|u\|_X)\|u\|_X$$

$$+ \|u\|_X \|g_{\sigma_1}(s)\|_X + \|u\|_X \|g_{\sigma_2}(s)\|_X$$

$$\leq \left(\sqrt{2}\left(\alpha_1(s)(\mu(\Omega))^{\frac{1}{2}} + \alpha_2(s)\|u\|_X\right) + \gamma_1 + \gamma_2\|u\|_X + C_0 + \|g_{\sigma_2}(s)\|_X\right)\|u\|_X$$

$$\leq \eta_1(s) + \eta_2(s)\|u\|_X^2,$$

where $\eta_i(t) \geq 0$ are locally integrable functions with polynomial growth at most.

Using Young inequality and $(F8) - (F9)$ we obtain

$$\frac{1}{2}\frac{d}{ds}\|u\|_X^2 + \frac{\widetilde{D}}{2}\|u\|_X^2 - K \leq \frac{1}{2}\frac{d}{ds}\|u\|_X^2 + \frac{D}{2}\|u\|_X^p \leq \eta_3(s),$$

where $\widetilde{D}, K > 0$ and $\eta_3(t) \geq 0$ is a locally integrable function such that

$$|\eta_3(t)| \leq \delta_1 + \delta_2 |t|^{\delta_3}, \text{ for a.a. } t \in \mathbb{R},\ i = 1, 2,$$

for some $\delta_j > 0$, $j = 1, 2, 3$.

The final part of the proof repeats the same steps of the case $p = 2$. ■

Remark 51 *We note that $T(B)$ does not depend upon σ_2, so that the rate of attraction is in this case uniform with respect to this paramenter.*

For any bounded set B, $\sigma_2 \in \Sigma_2$ and $\tau, t \in \mathbb{R}$, $\tau \leq t$, let us introduce the set

$$M(B, \sigma_2, \tau, t)$$
$$= \{l \in L_1(\tau, t; L_2(\Omega)) : u_\sigma(\cdot) = I(u_\tau) l(\cdot), u_\sigma \in \mathcal{D}_{\sigma,\tau}(u_\tau),\ u_\tau \in B,\ \sigma = (\sigma_1, \sigma_2), \sigma_1 \in \Sigma\}.$$

Lemma 52 *For any $R_0 > 0$, $\sigma_2 \in \Sigma$ and $(t, \tau) \in \mathbb{R}_d$, there exists $R \geq R_0$ such that*

$$\|U_{\Sigma_1,\sigma_2}(s, \tau, u)\|_+ \leq R,$$

for all $\tau \leq s \leq t$ and $u \in X$ such that $\|u\|_X \leq R_0$.

Proof. Fix $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2}) \in \Sigma_2$. First let $p = 2$. Take an arbitrary solution $u(\cdot) = I(u_\tau) l(\cdot)$ defined on $[\tau, t]$, where $l(r) \in F_{\sigma_1}(r, u(r)) + F_{\sigma_2}(r, u(r))$, $\sigma_1 \in \Sigma_1$ and u_τ satisfies $\|u_\tau\|_X \leq R_0$. Arguing as in Lemma 50 we have

$$\begin{aligned}
\|u(s)\|_X^2 &\leq \exp(\varepsilon(-s+\tau)) ||u_\tau||_X^2 \\
&\quad + \int_\tau^s \exp(\varepsilon(-s+r)) \left(2M\mu(\Omega) + \frac{1}{\varepsilon}\left(C_0 + R_1 + R_2|r|^{R_3}\right)^2\right) dr \\
&\leq R_0^2 + \int_\tau^t \exp(\varepsilon(-\tau+r)) \left(2M\mu(\Omega) + \frac{1}{\varepsilon}\left(C_0 + R_1 + R_2|s|^{R_3}\right)^2\right) ds = R^2,
\end{aligned}$$

for all $\tau \leq s \leq t$.

For $p > 2$ the proof is similar. ■

Lemma 53 *For any bounded set B, $\sigma_2 \in \Sigma_2$ and $\tau, t \in \mathbb{R}$, $\tau \leq t$, the set $M(B, \sigma_2, \tau, t)$ is bounded in the space $L_2(\tau, t; L_2(\Omega))$.*

Proof. Let $R_0 > 0$ be such that $\|u\|_X \leq R_0$, for any $u \in B$. As shown before in Lemma 45 for any $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ there exist $\beta_1, \beta_2 \geq 0$, $\beta_1, \beta_2 \in L_2^{loc}(-\infty, \infty)$ (depending on σ_2 but not on σ_1) such that

$$\|F_\sigma(s, u_\sigma)\|_+ \leq \beta_1(s) + \beta_2(s) \|u_\sigma\|_X, \text{ for all } u \in X \text{ and a.a. s} \in \mathbb{R}.$$

Hence, for any $l \in M(B, \sigma_2, \tau, t)$ one has

$$\|l(s)\|_X \leq \beta_1(s) + \beta_2(s) \|u_\sigma(s)\|_X, \text{ for a.a. } s \in (\tau, t),$$

where $u_\sigma(\cdot) = I(u_\tau) l(\cdot)$. But Lemma 52 implies the existence of $R \geq R_0$ such that $\|u_\sigma(s)\|_X \leq R$, for any $\sigma_1 \in \Sigma_1$, $\tau \leq s \leq t$, $u_\sigma(s) \in U_\sigma(s, \tau, B)$, so that the statement follows. ■

Proposition 54 *For any $\sigma_2 \in \Sigma_2$ there exists a compact set $D(\sigma_2)$ such that for any $B \in \mathcal{B}(X)$ there exists $T(B) \leq 0$ for which*

$$U_{\Sigma_1, \sigma_2}(0, \tau, B) \subset D(\sigma_2), \text{ if } \tau \leq T.$$

Proof. We take $D(\sigma_2) = \overline{U_{\Sigma_1, \sigma_2}(0, -1, B_1(\sigma_2))}$, where $B_1(\sigma_2)$ is the set defined in Lemma 50, and claim that it is the desired set. First let us prove that it is compact. Let $y \in U_{\Sigma_1, \sigma_2}(0, -1, B_1(\sigma_2))$ be arbitrary. Then there exists $u_\sigma(\cdot) = I(u_0) l(\cdot)$, with $\sigma_2 \in \Sigma_2$, $u_0 \in B_1(\sigma_2)$, such that $y = u_\sigma(0)$, $u_\sigma(-1) = u_0$. Multiplying the equation

$$\frac{du_\sigma}{dt} - A(u_\sigma) = l \tag{25}$$

by u_σ and using the inequality $(-A(u), u) \geq \gamma \|u\|_{W^{1,p}}^p$, for all $u \in D(A)$, where $\gamma > 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|u_\sigma(s)\|_{L_2}^2 + \gamma \|u_\sigma(s)\|_{W^{1,p}}^p \leq \|l(s)\|_{L_2} \|u_\sigma(s)\|_{L_2} \leq \frac{1}{2D} \|l(s)\|_{L_2}^2 + \frac{1}{2} D \|u_\sigma(s)\|_{L_2}^2,$$

for any $D > 0$.

The continuous injections $W_0^{1,p}(\Omega) \subset L_p(\Omega) \subset L_2(\Omega)$ allows us to choose $D > 0$ such that $D \|u_\sigma(s)\|_{L_2}^p \leq \gamma \|u_\sigma(s)\|_{W^{1,p}}^p$. Hence, integrating over $(-1, 0)$ and using Lemma 53 and Young inequality we obtain

$$\|u_\sigma(0)\|_{L_2}^2 + 2\gamma \int_{-1}^0 \|u_\sigma(s)\|_{W^{1,p}}^p ds \leq C + \gamma \int_{-1}^0 \|u_\sigma(s)\|_{W^{1,p}}^p ds + \|u_0\|_{L_2}^2, \tag{26}$$

where C is some positive constant.

Recall that $\varphi(u) = \frac{1}{p} \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} u \right\|_{L_p}^p$, if $u \in W_0^{1,p}(\Omega)$. Consider first the case where $u_0 \in D(\varphi) = W_0^{1,p}(\Omega)$. In this case since $l(\cdot) \in L_2(-1, 0; L_2(\Omega))$, it is known (see [2, p.189]) that $\varphi(u(t))$ is absolutely continuous in $[-1, 0]$ and $\frac{d}{ds} \varphi(u(s)) = \left(\partial \varphi(u(s)), \frac{du(s)}{ds} \right)$, a.e. on $(-1, 0)$. Further, multiplying (25) by $(1+s) \frac{du_\sigma}{ds}$ we have

$$(1+s) \left\| \frac{d}{dt} u_\sigma(s) \right\|_{L_2}^2 + (1+s) \frac{d}{ds} \varphi(u(s)) \leq (1+s) \|l(s)\|_{L_2} \left\| \frac{d}{ds} u_\sigma(s) \right\|_{L_2}.$$

$$\leq \frac{1}{2}(1+s) \|l(s)\|_{L_2}^2 + \frac{1}{2}(1+s) \left\| \frac{d}{dt} u_\sigma(s) \right\|_{L_2}^2$$

Integrating by parts over $(-1, 0)$ and using Lemma 53 we get

$$\int_{-1}^0 \frac{1}{2}(1+s) \left\| \frac{d}{dt} u_\sigma(s) \right\|_{L_2}^2 ds + \varphi(u_\sigma(0)) \leq \int_{-1}^0 \varphi(u(s)) ds + K,$$

where $K>0$. Using the fact that the norms $\|u\|_{W^{1,p}}$ and $\left(\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}u\right\|_{L_p}^{p}\right)^{\frac{1}{p}}$ are equivalent in $W_0^{1,p}(\Omega)$ and (26) we have

$$\varphi(u_\sigma(0)) \leq \alpha\left(C+\|u_0\|_{L_2}^2\right)+K, \tag{27}$$

for some $\alpha>0$.

Let now consider the general case $u_0\in L_2(\Omega)$. We take $u_0^n\to u_0$ with $u_0^n\in D(\varphi), u_0^n\in B_1(\sigma_2)$. From [19, Theorem 3.1] we obtain the existence of a sequence $u_n(\cdot)=I(u_0^n)\,l_n(\cdot)$, $l_n(s)\in F_{\sigma(s)}(s,u_n(s))$, such that $u_n\to u_\sigma$ in $C([-1,0],L_2(\Omega))$. Hence by (27) and using the lower semicontinuity of φ we obtain

$$\varphi(u_\sigma(0)) \leq \liminf \varphi(u_n(0)) \leq \alpha\left(C+\|u_0\|_{L_2}^2\right)+K.$$

This implies that the set $U_{\Sigma_1,\sigma_2}(0,-1,B_1(\sigma_2))$ is bounded in the space $W^{1,p}(\Omega)$. Since the injection $W^{1,p}(\Omega)\subset L_2(\Omega)$ is compact, the set $D(\sigma_2)$ is compact.

Further, let $B\in\mathcal{B}(X)$ be arbitrary. Lemma 50 implies the existence of some $T(\tau,B)<-1$ for which $U_{\Sigma_1,\sigma_2}(-1,\tau,B)\subset B_1(\sigma_2)$, if $\tau\leq T$. Then by Proposition 47 we have

$$U_{\Sigma_1,\sigma_2}(0,\tau,B)=U_{\Sigma_1,\sigma_2}(0,-1,U_{\Sigma_1,\sigma_2}(-1,\tau,B))\subset D(\sigma_2).$$

■

We have proved that the family of semiprocesses generated by (12) satisfies all conditions of Theorem 41. We can then state the main result of this paper.

Theorem 55 *If $(F1)-(F9)$ hold and $g_1\in L_\infty(\mathbb{R},L_2(\Omega))$, then the family of semiprocesses U_σ has the Σ_1-uniform global compact attractor $\Theta_{\Sigma_1}(\sigma_2)$.*

Let us consider now the connectivity of the global attractor.

Theorem 56 *In the conditions of Theorem 55, let $f_2\equiv 0$ and let there exist a non-decreasing map $C(t)$ such that $\|g_2(t)\|_X\leq C(t)$, for a.a. $t\in\mathbb{R}$. Then the set $\Theta_{\Sigma_1}(\sigma_2)$ is connected in X for each $\sigma_2\in\Sigma_2$.*

Proof. We have to check the conditions of point 2 in Theorem 40.

We have already seen that the set Σ_1 is compact. Let us prove that for each $T\geq\tau$, $U_\sigma(T,\tau,\cdot)$ has connected values. It follows from the condition $f_2\equiv 0$ that F_σ satisfies $(G1)-(G2)$. These properties and $(S3)$ imply that the set

$$\begin{gathered} M(u_\tau,\sigma,\tau,T) \\ =\{l\in L_1(\tau,T;L_2(\Omega)) : u_\sigma(\cdot)=I(u_\tau)\,l(\cdot),\, u_\sigma\in\mathcal{D}_{\sigma,\tau}(u_\tau)\} \end{gathered}$$

is connected in the space $L_1(\tau,T;L_2(\Omega))$ (see [20, p.169]). Thanks to inequality (17) we have

$$\|u_1(T)-u_2(T)\|_X \leq \int_\tau^T \|l_1(t)-l_2(t)\|_X\,dt, \text{ for all } l_1,l_2\in M(u_\tau,\sigma_2,\tau,T),$$

where $u_i=I(u_\tau)\,l_i, i=1,2$. Hence, the map $L:L_1(\tau,T;L_2(\Omega))\to L_2(\Omega)$ defined by $L(l)=u(T)$ is continuous. Since $L(M(u_\tau,\sigma,\tau,T))=U_\sigma(T,\tau,u_\tau)$, $U_\sigma(T,\tau,\cdot)$ has connected values.

The space Σ_1 is connected. Indeed, first note that in view of $(F2)$ the map $h\mapsto T_1(h)\,f_1\in C(\mathbb{R},\mathcal{M})$ is continuous. Consider further the continuity of the map $h\mapsto T_1(h)\,g_1\in L_{2,w}^{loc}(\mathbb{R},X)$. If, for example, this function is not continuous at $h=0$, then there exists a neighborhood $\mathcal{U}$ of g_1 in $L_{2,w}^{loc}(\mathbb{R},X)$ and $h_n\to 0$ such that $g_n(t)=g_1(t+h_n)\notin\mathcal{U}$, for all n (for the general case $h_n\to h$ the proof is similar). Take an arbitrary interval $I=[\tau,T]\subset\mathbb{R}$ and $\varphi\in L_\infty(\tau,T;L_2(\Omega))$. Since the scalar product $(g(t),\varphi(t))$ (in X) is measurable on any interval of $\mathbb{R}$, Luzin's theorem implies

$$(g_n(t),\varphi(t))\to(g(t),\varphi(t)) \text{ in measure.}$$

Therefore, choosing a subsequence if necessary, we get

$$\left(g_n(t), \varphi(t)\right) \rightarrow \left(g(t), \varphi(t)\right) \text{ a.e. on } (\tau, T).$$

By the inequality $\int_\tau^T |(g_n(t), \varphi(t))|^2 dt \leq C$ and [15, Chapter 1, Lemma 1.3] we have

$$\int_\tau^T (g_n(t), \varphi(t))\, dt \rightarrow \int_\tau^T (g(t), \varphi(t))\, dt.$$

Note that the compactness of $\mathcal{H}(g_1)$ allow us to assume without loss of generality that $g_n \rightarrow l$ weakly in $L_{2,w}^{loc}(\mathbb{R}, X)$. It follows that $l = g_1$, which is a contradiction. Then the map $h \longmapsto T(h)\sigma_1 \in C(\mathbb{R}, \mathcal{M}) \times L_{2,w}^{loc}(\mathbb{R}, X)$ is continuous on $\mathbb{R}$, so that the set $\cup_{h\in\mathbb{R}}\sigma_1(\cdot + h)$ is connected. Hence, Σ_1, as the closure of a connected set, is connected.

Further, let us prove for any $\sigma_2 \in \Sigma_2$ the existence of a ball containing the sets $\Theta_{\Sigma_1}(T_2(h)\sigma_2)$, for $h \leq 0$. It is clear that for any $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2})$ there exists a non-decreasing function $\widetilde{C}(t)$ such that $\|g_{\sigma_2}(t)\|_X \leq \widetilde{C}(t)$, for a.a. $t \in \mathbb{R}$, and also that $\left\|g_{T_2(h)\sigma_2}(t)\right\|_X \leq \widetilde{C}(t)$, for all $h \leq 0$ and a.a. $t \in \mathbb{R}$. For $p = 2$, arguing as in Lemma 50, we obtain

$$\frac{d}{ds}\|u\|_X^2 + \varepsilon\|u\|_X^2 \leq 2M\mu(\Omega) + \frac{\left(C_0 + \widetilde{C}(0)\right)^2}{\varepsilon},$$

for any $u(\cdot) = I(u_\tau)\, l(\cdot)$ defined on $[\tau, 0]$, where $l(t) \in F_{\sigma_1}(t, u(t)) + F_{\sigma_2}(t, u(t))$, and any $\sigma_1 \in \Sigma_1$, $\widetilde{\sigma}_2 = T_2(h)\sigma_2$, $h \leq 0$, and $u_\tau \in B \in \mathcal{B}(X)$. It follows that the closed set

$$B_1 = \left\{y \in X : \|y\|_X \leq \sqrt{K + \alpha}\right\},$$

with $\alpha > 0$, $K = \frac{2M\mu(\Omega) + \frac{1}{\varepsilon}\left(C_0 + \widetilde{C}(0)\right)^2}{\varepsilon}$, is attracting at 0 for any $\widetilde{\sigma}_2 = T_2(h)\sigma_2$, $h \leq 0$. The minimality property of the global attractor implies then that $\Theta_{\Sigma_1}(T_2(h)\sigma_2) \subset B_1$, for $h \leq 0$. For $p > 2$ the proof is similar.

Finally, let us prove that $(\sigma_1, x) \mapsto U_{\sigma_1,\sigma_2}(t, \tau, x)$ is upper semicontinuous. Suppose that for some (σ_1, x) the map is not upper semicontinuous. Then there exists a neighborhood O of $U_{\sigma_1,\sigma_2}(t, \tau, x)$ and sequences $z_n \in U_{\sigma_{1n},\sigma_2}(t, \tau, x_n)$, $\sigma_{1n} \rightarrow \sigma_1$ in $C(\mathbb{R}_+, \mathcal{M}) \times L_{2,w}^{loc}(\mathbb{R}_+, L_2(\Omega))$, $x_n \rightarrow x$ in $L_2(\Omega)$, such that $z_n \notin O$. Repeating the same lines of the proof of Proposition 48 we can prove that for some subsequence $z_{n_k} \rightarrow z \in U_{\sigma_1,\sigma_2}(t, \tau, x)$, which is a contradiction.

Hence, it follows from the second statement in Theorem 40 that the sets $\Theta_{\Sigma_1}(\sigma_2)$ are connected. ■

4 Stochastic non-autonomous evolution inclusions

4.1 Additive white noise case

Consider the following non-autonomous differential inclusion perturbed by an additive white noise

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \in f(t, u) + g_1(t) + g_2(t) + \sum_{i=1}^m \phi_i \frac{dw_i(t)}{dt}, \text{ on } D \times (\tau, T), \\ u\mid_{\partial D} = 0, \\ u\mid_{t=\tau} = u_\tau, \end{cases} \tag{28}$$

where $\tau \in \mathbb{R}$, $D \subset \mathbb{R}^n$ is an open bounded set with smooth boundary ∂D, $\phi_i \in D(A)$ (where $A(u) = \Delta u$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$), $i = 1, ..., m$, $f : \mathbb{R} \times \mathbb{R} \rightarrow C_v(\mathbb{R})$, $i = 1, 2$, $g_1 \in L_\infty(\mathbb{R}, L_2(D))$, $g_2 \in L_2^{loc}(\mathbb{R}, L_2(D))$. We write $\zeta(t) = \sum_{i=1}^m \phi_i w_i(t)$. Consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega = (w_1(\cdot), ..., w_m(\cdot)) \in C(\mathbb{R}, \mathbb{R}^m) \mid \omega(0) = 0\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$ and the Wiener measure $\mathbb{P}$. Each $\omega \in \Omega$ generates a map $\zeta(\cdot) = \sum_{i=1}^m \phi_i w_i(\cdot) \in C(\mathbb{R}, L_2(D))$ such that $\zeta(0) = 0$.

Suppose that f satisfies $(F1) - (F3), (F7)$, whereas g_2 satisfies $(F8)$.

4.1.1 Construction of the family of multivalued processes

Firstly, let us construct the sets Σ_1, Σ_2. The set Σ_1 will be defined in the same way as in the previous section, that is, $\Sigma_1 = \mathcal{H}(f) \times \mathcal{H}(g_1)$, where $\mathcal{H}(g_1) = cl_{L_{2,w}^{loc}(\mathbb{R}, L_2(D))} \{g_1(t+h) : h \in \mathbb{R}\}$ and $\mathcal{H}(f) = cl_{C(\mathbb{R},\mathcal{M})} \{f(\cdot + h) : h \in \mathbb{R}\}$.

Then $T_1(h)\Sigma_1 = \Sigma_1$, for all $h \in \mathbb{R}$, where $T_1(h)$ is the shift operator, that is, $T_1(h)\sigma_1(t) = \sigma_1(t+h)$.

For the set Σ_2 we write

$$\Sigma_2 = \tilde{\Sigma}_2 \times \Omega,$$

with

$$\tilde{\Sigma}_2 = \underset{h\in\mathbb{R}}{\cup} g_2(\cdot + h).$$

We define the map $\theta_s : \Omega \to \Omega$ as follows

$$\theta_s\omega = (w_1(s+\cdot) - w_1(s), ..., w_m(s+\cdot) - w_m(s)) \in \Omega.$$

Then the function $\widetilde{\zeta}$ corresponding to $\theta_s\omega$ is defined by $\widetilde{\zeta}(\tau) = \zeta(s+\tau) - \zeta(s) = \sum_{i=1}^m \phi_i(w_i(s+\tau) - w_i(s))$.

The operator T_1 is defined as before. We define the shift operator $T_2 : \Sigma_2 \to \Sigma_2$ as

$$T_2(h)\sigma_2 = T_2(h)(\tilde{\sigma}_2, \omega) = (\tilde{\sigma}_2(\cdot + h), \theta_h\omega), \text{ for all } \tilde{\sigma}_2 \in \tilde{\Sigma}_2, \ \omega \in \Omega.$$

Thus, $T_2(h)\Sigma_2 = \Sigma_2$, for all $h \in \mathbb{R}$, and if $\sigma_1 = (f_{\sigma_1}, g_{\sigma_1}) \in \Sigma_1$, $g_{\sigma_2} \in \tilde{\Sigma}_2$, then f_{σ_1} satisfies $(F1) - (F3)$, $(F7)$ and $g_{\sigma_2} \in L_2^{loc}(\mathbb{R}, L_2(\Omega))$ satisfies $(F8)$.

To study (28), we make the change of variable $v(t) = u(t) - \zeta(t)$. Then inclusion (28) turns, for each $\omega \in \Omega$ fixed, into

$$\begin{cases} \dfrac{dv}{dt} \in \Delta v(t) + f(t, v(t) + \zeta(t)) + g_1(t) + g_2(t) + \sum_{i=1}^m \Delta\phi_i w_i(t), \\ v\mid_{\partial D} = 0, \ v(\tau) = v_\tau = u_\tau - \zeta(\tau). \end{cases} \tag{29}$$

Now let $X = L_2(\Omega)$. Consider the abstract evolution inclusion

$$\begin{cases} \dfrac{dv(t)}{dt} \in A(v(t)) + F_\sigma(t, v(t)), \ t \in [\tau, \infty), \\ v(\tau) = v_\tau = u_\tau - \zeta(\tau), \end{cases} \tag{30}$$

where $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ and $A : D(A) \subset X \to X$, $F_\sigma : \mathbb{R} \times X \to 2^X$, are maps defined as follows:

$$A(u) = \Delta u, \ D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$

$$F_\sigma(t, \omega, u) = g_{\sigma_2}(t) + \widehat{F}_{\sigma_1}(t, \omega, u),$$

with

$$\widehat{F}_{\sigma_1}(t, \omega, u) = F_{\sigma_1}(t, u + \zeta(t)) + A\zeta(t),$$

where F_{σ_1} is as in the previous section. It is understood that the map F_σ is defined for a.a. $t \in \mathbb{R}$.

It follows from $(G3)$ the existence of $\gamma_i \geq 0$ such that

$$\left\| \widehat{F}_{\sigma_1}(t, \omega, u) \right\|_+ \leq \gamma_1 + \gamma_2 \|u\|_X + \gamma_2 \|\zeta(t)\|_X + \|A\zeta(t)\|_X + C_0, \text{ for all } u \in X, t \in \mathbb{R}, \ \omega \in \Omega. \tag{31}$$

It is easy to see that F_{σ_1} satisfies $(G1) - (G2)$. As before, the operator A satisfies $(A1) - (A3)$. Note that $(S1) - (S4)$ from Lemma 45 hold for F_σ.

We now construct the multivalued process corresponding to (28). It follows from $(A1) - (A3)$, $(S1) - (S4)$ that for any $v_\tau \in L_2(\Omega)$ there exists at least one integral solution to (30) for any $T > \tau$ [20, Theorem

2.1]. We shall denote this solution by $v_\sigma(\cdot) = I(v_\tau) l(\cdot)$. For a fixed $\sigma \in \Sigma$ let $\mathcal{D}_{\sigma,\tau}(x)$ be the set of all integral solutions corresponding to the initial condition $v(\tau) = x$. We shall write v instead of v_σ for simplicity of notation if no confussion is possible.

We define the map $U_\sigma : \mathbb{R}_d \times X \to P(X)$ by

$$U_\sigma(t,\tau,x) = \{z + \zeta(t) : \text{there exists } v(\cdot) \in \mathcal{D}_{\sigma,\tau}(x - \zeta(\tau)) \text{ such that } v(t) = z\}.$$

Also note that, for a fixed $\tau \in \mathbb{R}$ and arbitrary $t \in \mathbb{R}_+$, $x \in X$, $\sigma \in \Sigma$ we can define the *cocycle* (Kloeden and Schmalfuss [13], Caraballo et. al. [5, 6]) $G_\tau : \mathbb{R}_+ \times \Sigma \times X \to P(X)$ by

$$G_\tau(t,\sigma,x) = U_\sigma(t+\tau,\tau,x).$$

Proposition 57 *For each $\sigma \in \Sigma$, $h \in \mathbb{R}$, $\tau \le s \le t$, $x \in X$, we have*

$$U_{T(h)\sigma}(t,\tau,x) = U_\sigma(t+h,\tau+h,x).$$

$$U_\sigma(t,s,U_\sigma(s,\tau,x)) = U_\sigma(t,\tau,x),$$

Hence, U_σ is a multivalued dynamical process for each $\sigma \in \Sigma$ and condition $(T1)$ holds.

Proof. In view of Lemma 19 we have to prove only the inclusion $U_\sigma(t+h,\tau+h,x) \subset U_{T(h)\sigma}(t,\tau,x)$. Given $\eta \in U_\sigma(t+h,\tau+h,x)$, where $h \in \mathbb{R}$, there exists $y(\cdot) \in \mathcal{D}_{\sigma,\tau+h}(x - \zeta(\tau+h))$ such that $\eta = y(t+h) + \zeta(t+h)$. Let $z(s) = y(s+h) + \zeta(h)$, $l_z(s) = l(s+h) - A\zeta(h)$, where $l(s) \in F_{\sigma_1}(s, y(s) + \zeta(s)) + A\zeta(s) + g_{\sigma_2}(s)$, a.e. on $(\tau+h, t+h)$, $y(\cdot) = I(x - \zeta(\tau+h)) l(\cdot)$, so that we have $l_z(s) \in F_{\sigma_1}(s+h, y(s+h) + \zeta(s+h)) + A\zeta(s+h) - A\zeta(h) + g_{\sigma_2}(s+h) = F_{T_1(h)\sigma_1}\left(s, z(s) + \widetilde{\zeta}(s)\right) + A\widetilde{\zeta}(s) + g_{T_2(h)\sigma_2}(s) = F_{T(h)\sigma}(s, z(s))$, a.e. on (τ,t), where $\widetilde{\zeta}$ corresponds to $\theta_h\omega$, and $z(\tau) = x - \zeta(\tau+h) + \zeta(h) = x - \widetilde{\zeta}(\tau)$. We can show that $z(\cdot) \in \mathcal{D}_{T(h)\sigma,\tau}\left(x - \widetilde{\zeta}(\tau)\right)$ as follows

$$\|z(t) - \xi\|_X^2 = \|y(t+h) + \zeta(h) - \xi\|_X^2 \le \|y(s+h) + \zeta(h) - \xi\|_X^2$$

$$+2\int_{s+h}^{t+h} (l(r) - A\zeta(h) + A\xi, y(r) + \zeta(h) - \xi)\, dr = \|z(s) - \xi\|_X^2 + 2\int_s^t (l_z(r) + A\xi, z(r) - \xi)\, dr,$$

for any $\xi \in D(A)$. Therefore, $\eta = y(t+h) + \zeta(t+h) = z(t) + \zeta(t+h) - \zeta(h) \in U_{T(h)\sigma}(t,\tau,x)$.

In a similar way as in [5, Proposition 4] we can prove $G_0(t+s,\sigma,x)) = G_0(t, T(s)\sigma, G_0(s,\sigma,x))$ (the only difference in the proof is that we have to take into account the translation on time of the map F_σ). Using the first property we obtain $U_\sigma(t,\tau,x) = U_{T(\tau)\sigma}(t-\tau,0,x) = G_0(t-\tau, T(\tau)\sigma, x) = G_0(t-s, T(s), G_0(s-\tau, T(\tau), x)) = U_{T(s)\sigma}\left(t-s, 0, U_{T(\tau)\sigma}(s-\tau,0,x)\right) = U_\sigma(t,s,U_\sigma(s,\tau,x))$. ■

4.1.2 Existence of the global Σ_1-uniform attractor

As in the previous section, the conditions of Theorem 41 providing the existence of a global Σ_1-uniform compact attractor hold.

Theorem 58 *In the preceedings conditions, the family of semiprocesses U_σ has the Σ_1-uniform global compact attractor $\Theta_{\Sigma_1}(\sigma_2)$.*

Proof. First we shall argue as in Lemma 50. Fix $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2}) \in \Sigma_2$. Take an arbitrary solution to (30), $v(\cdot) = I(u_\tau - \zeta(\tau)) l(\cdot)$ defined on $[\tau, -1]$, where $l(s) \in F_{\sigma_1}(s, v(s) + \zeta(s)) + A\zeta(s) + g_{\sigma_2}(s)$, $\sigma_1 \in \Sigma_1$ and $u_\tau \in B \in \mathcal{B}(X)$. Denote $\widetilde{l}(s) = l(s) - A\zeta(s) - g_{\sigma_1}(s) - g_{\sigma_2}(s)$. Multiplying (22) by $v(s)$ and using $(F7)$, $(F8)$, $(G3)$ and (13) we have

$$\frac{1}{2}\frac{d}{ds}\|v\|_X^2 + \lambda_1 \|v(s)\|_X^2 \le \frac{1}{2}\frac{d}{ds}\|v\|_X^2 + \|\nabla v(s)\|_X^2$$

$$= \left(\widetilde{l}(s), v(s)\right) + (A\zeta(s) + g_{\sigma_1}(s) + g_{\sigma_2}(s), v(s))$$

$$= \left(\widetilde{l}(s), v(s) + \zeta(s)\right) - \left(\widetilde{l}(s), \zeta(s)\right) + (A\zeta(s) + g_{\sigma_1}(s) + g_{\sigma_2}(s), v(s))$$

$$\leq (\lambda_1 - \epsilon)\|v(s) + \zeta(s)\|_X^2 + M\mu(\Omega) + (\gamma_1 + \gamma_2\|v(s)\|_X + \gamma_2\|\zeta(s)\|_X + 2C_0)\|\zeta(s)\|_X$$

$$+ \left(C_0 + R_1 + R_2|s|^{R_3} + \|A\zeta(s)\|_X\right)\|v(s)\|_X$$

$$\leq \left(\lambda_1 - \frac{\epsilon}{2}\right)\|v\|_X^2 + K_1 + K_2\|\zeta(s)\|_X^2 + K_3\left(C_0 + R_1 + R_2|s|^{R_3} + \|A\zeta(s)\|_X\right)^2,$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω in $\mathbb{R}^n$. Arguing as in Lemma 50 and taking into account that the map $\zeta(s)$ satisfies $\lim_{s\to\infty}\frac{\zeta(s)}{s} = 0$, for a.a. $\omega \in \Omega$, we obtain that for any $\sigma_2 \in \Sigma_2$ there exists a radius $R_0(\sigma_2)$ such that for any $B \in \mathcal{B}(X)$ there is $T = T(B, \sigma_2) < -1$ for which $\|v(-1)\|_X \leq R_0$, as soon as $\tau \leq T(B)$. Hence, $\|v(-1) + \zeta(-1)\|_X \leq R_0 + \|\zeta(-1)\|_X = \widetilde{R}_0(\sigma_2)$, so that $\|U_{\Sigma_1,\sigma_2}(-1,\tau,B)\|_+ \leq \widetilde{R}_0$.

Using the previous inequality and arguing as in Lemma 52 we prove that for any $R_0 > 0$, $\sigma_2 \in \Sigma$ and $(t,\tau) \in \mathbb{R}_d$, there exists $R \geq R_0$ such that $\|v(s)\|_X \leq R$, for all $\tau \leq s \leq t$, $v \in \mathcal{D}_{\sigma,\tau}(u_\tau)$ and $u_\tau \in X$ such that $\|u_\tau\|_X \leq R_0$. Hence, $\|U_{\Sigma_1,\sigma_2}(s,\tau,u)\|_+ \leq R + \|\zeta(s)\|_X . \leq \widetilde{R}(\sigma_2)$.

Note that if we define,

$$V_\sigma(t,\tau,x) = \{z : \text{there exists } v(\cdot) \in \mathcal{D}_{\sigma,\tau}(x) \text{ such that } v(t) = z\},$$

then, for $\tau < 0$, $U_\sigma(0,\tau,x) = V_\sigma(0,\tau,x)$. Since F_σ satisfies $(S3)$, we can prove exactly in the same way as in Proposition 54 that there exists a compact set $D(\sigma_2)$ which is Σ_1-uniformly attracting..

Finally, we have to prove that the graph of $(\sigma_1, x) \mapsto U_{\sigma_1,\sigma_2}(0,\tau,x)$ is closed. As in the proof of Proposition 48 we take sequences $u_n(\cdot) = I(u_\tau^n - \zeta(\tau))\, l_n(\cdot)$, $l_n(s) \in F_{\sigma_{1n}}(s, u_n(s) + \zeta(s)) + A\zeta(s) + g_{\sigma_2}(s)$, a.e. on $(\tau, 0)$, such that $y_n = u_n(0) \to y$, $u_\tau^n \to u_\tau$, $\sigma_{1n} \to \sigma_1$. Repeating the same steps of the proof of Proposition 48 we obtain $v(\cdot) = I(u_\tau - \zeta(\tau))\, l(\cdot)$ such that $u_n \to v$ in $C([\tau,0], X)$ and $l_n \to l$ weakly in $L_2(\tau, 0; L_2(D))$.

If we prove $l(s) \in F_{\sigma_1}(s, v(s) + \zeta(s)) + A\zeta(s) + g_{\sigma_2}(s)$, a.e. on $(\tau, 0)$, then $y \in U_{\sigma_1,\sigma_2}(0,\tau,u_\tau)$. This is equivalent to prove $l(s) - g_{\sigma_1}(s) - g_{\sigma_2}(s) - A\zeta(s) = d(s) \in F_{\sigma_1}(s, v(s) + \zeta(s)) - g_{\sigma_1}(s) = \widetilde{F}_{\sigma_1}(s, v(s) + \zeta(s))$.

Fix s. Passing to a subsequence and using $(F1)$ we have

$$dist(f_{\sigma_1}(s, u_n(s,x) + \zeta(s)), f_{\sigma_1}(s, v(s,x) + \zeta(s))) \leq C|u_n(s,x) - v(s,x)| \to 0, \text{ for a.a. } x \in D.$$

On the other hand, since $\{u_n(s,x)\}$ is bounded, that is $|u_n(s,x)| \leq C(x)$, for all n, and $f_{\sigma_{1n}}$ converges to f_{σ_1} in $C(\mathbb{R}, \mathcal{M})$, we get

$$dist(f_{\sigma_{1n}}(s, u_n(s,x) + \zeta(s)), f_{\sigma_1}(s, u_n(s,x) + \zeta(s))) \to 0, \text{ as } n \to +\infty,$$

for a.a. $x \in D$. Then for $d_n(s) = l_n(s) - g_{\sigma_{1n}}(s) - g_{\sigma_2}(s) - A\zeta(s)$ it holds

$$dist(d_n(s,x), f_{\sigma_1}(s, v(s,x) + \zeta(s))) \leq dist(f_{\sigma_1}(s, u_n(s,x) + \zeta(s)), f_{\sigma_1}(s, v(s,x) + \zeta(s)))$$

$$+ dist(f_{\sigma_{1n}}(s, u_n(s,x) + \zeta(s)), f_{\sigma_1}(s, u_n(s,x) + \zeta(s))) \to 0, \text{ for a.a. } x \in \Omega. \quad (32)$$

As shown in Proposition 48, there exists a sequence z_n such that $z_n(s) = \sum_{i=1}^{N(n)} \lambda_i d_{k_i}(s)$, where $\lambda_i \in [0,1]$, $\sum_{i=1}^{N} \lambda_i = 1$ and $k_i \geq n$, for all i, and $z_n(s,x) \to d(s,x)$, a.e. in D. Now (32) implies that for any $\varepsilon > 0$ and a.a. $x \in D$ there exists $n(x,\varepsilon)$ such that

$$d_k(s,x) \subset [a(s,x) - \varepsilon, b(s,x) + \varepsilon], \text{ for all } k \geq n,$$

where $[a(s,x), b(s,x)] = f_{\sigma_1}(s, v(s,x) + \zeta(s))$ (note that the map f_{σ_1} has convex closed values). Hence,

$$z_n(s,x) \subset [a(s,x) - \varepsilon, b(s,x) + \varepsilon],$$

as well. Passing to the limit we obtain

$$d(s,x) \in [a(s,x), b(s,x)] = f_{\sigma_1}(s, v(s,x) + \zeta(s)), \text{ a.e. on } \Omega.$$

Thus, we can apply Theorem 41. ■

4.2 Multiplicative white noise case

Finally, consider the following non-autonomous differential inclusion perturbed by a linear multiplicative white noise in the Stratonovich sense

$$\left\{\begin{array}{c} \dfrac{\partial u}{\partial t}-\Delta u\in f\left(t,u\right)+g_{1}\left(t\right)+g_{2}\left(t\right)+u\circ\frac{dw(t)}{dt},\text{ on }D\times\left(\tau,T\right),\\ u\mid_{\partial D}=0,\\ u\mid_{t=\tau}=u_{\tau},\end{array}\right. \tag{33}$$

where $\tau\in\mathbb{R}$, $D\subset\mathbb{R}^{n}$ is and open bounded set with smooth boundar ∂D, $f:\mathbb{R}\times\mathbb{R}\rightarrow C_{v}\left(\mathbb{R}\right)$, $i=1,2$, $g_{1}\in L_{\infty}\left(\mathbb{R},L_{2}\left(D\right)\right)$, $g_{2}\in L_{2}^{loc}\left(\mathbb{R},L_{2}\left(D\right)\right)$. Consider the Wiener probability space $(\Omega,\mathcal{F},\mathbb{P})$ defined by

$$\Omega=\left\{\omega=w\left(\cdot\right)\in C\left(\mathbb{R},\mathbb{R}\right)\mid\omega\left(0\right)=0\right\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$ and the Wiener measure $\mathbb{P}$.

Suppose again that f satisfies $(F1)-(F3),(F7)$, whereas g_{2} satisfies $(F8)$.

4.2.1 Construction of the family of multivalued processes

We define $\Sigma=\Sigma_{1}\times\Sigma_{2}=\Sigma_{1}\times\tilde{\Sigma}_{2}\times\Omega$ and T_{1},T_{2} exactly as in the previous section, with $\theta_{s}:\Omega\rightarrow\Omega$

$$\theta_{s}\omega=\left(w\left(s+\cdot\right)-w\left(s\right)\right)\in\Omega.$$

Thus, if $\sigma_{1}=\left(f_{\sigma_{1}},g_{\sigma_{1}}\right)\in\Sigma_{1}$, $g_{\sigma_{2}}\in\tilde{\Sigma}_{2}$, then $f_{\sigma_{1}}$ satisfies $(F1)-(F3)$, $(F7)$ and $g_{\sigma_{2}}\in L_{2}^{loc}\left(\mathbb{R},L_{2}\left(\Omega\right)\right)$ satisfies $\left(F8\right)$.

To study (33), we make the change of variable $v\left(t\right)=\gamma(t)u\left(t\right)$, with $\gamma(t)=\gamma\left(\omega,t\right)=e^{-w(t)}$ (we shall omit ω if no confusion is possible). Then inclusion (33) turns into

$$\left\{\begin{array}{c} \dfrac{dv}{dt}\in\Delta v\left(t\right)+\gamma(t)f\left(t,\gamma^{-1}(t)v(t)\right)+\gamma(t)(g_{1}\left(t\right)+g_{2}\left(t\right)),\\ v\mid_{\partial D}=0,\ v\left(\tau\right)=v_{\tau}=\gamma\left(\tau\right)u_{\tau}.\end{array}\right. \tag{34}$$

Now let $X=L_{2}\left(\Omega\right)$. Consider

$$\left\{\begin{array}{c} \dfrac{dv\left(t\right)}{dt}\in A\left(v\left(t\right)\right)+F_{\sigma}\left(t,v\left(t\right)\right),\ t\in\left[\tau,\infty\right),\\ v\left(\tau\right)=v_{\tau},\end{array}\right. \tag{35}$$

where $\sigma=\left(\sigma_{1},\sigma_{2}\right)\in\Sigma$, $A:D\left(A\right)\subset X\rightarrow X$ is defined as before, and $F_{\sigma}:\mathbb{R}\times X\rightarrow2^{X}$ is defined as

$$F_{\sigma}\left(t,\omega,u\right)=\gamma\left(t\right)g_{\sigma_{2}}(t)+\widehat{F}_{\sigma_{1}}\left(t,\omega,u\right),$$

with

$$\widehat{F}_{\sigma_{1}}\left(t,\omega,u\right)=\gamma\left(t\right)F_{\sigma_{1}}\left(t,\gamma^{-1}\left(t\right)u\right),$$

where $F_{\sigma_{1}}$ is as in the previous section. It follows from $(G3)$ the existence of $\alpha_{i}\geq0$ such that

$$\left\Vert\widehat{F}_{\sigma_{1}}\left(t,\omega,u\right)\right\Vert_{+}\leq\gamma\left(t\right)\left(\alpha_{1}+\alpha_{2}\gamma^{-1}\left(t\right)\left\Vert u\right\Vert_{X}+C_{0}\right),\text{ for all }u\in X,t\in\mathbb{R},\ \omega\in\Omega. \tag{36}$$

It is easy to see that $F_{\sigma_{1}}$ satisfies $(G1)-(G2)$. As before, the operator A satisfies $(A1)-(A3)$. Note that $(S1)-(S4)$ from Lemma 45 hold for F_{σ}.

We now construct the multivalued process corresponding to (33). It follows from $(A1)-(A3)$, $(S1)-(S4)$ that for any $v_{\tau}\in L_{2}\left(\Omega\right)$ there exists at least one integral solution to (35) for any $T>\tau$. We shall denote this solution by $v_{\sigma}\left(\cdot\right)=I\left(v_{\tau}\right)l\left(\cdot\right)$. For a fixed $\sigma\in\Sigma$ let $\mathcal{D}_{\sigma,\tau}\left(x\right)$ be the set of all integral solutions corresponding to the initial condition $v\left(\tau\right)=x$.

We define the map $U_{\sigma}:\mathbb{R}_{d}\times X\rightarrow P\left(X\right)$ by

$$U_{\sigma}\left(t,\tau,x\right)=\left\{\gamma^{-1}\left(t\right)z:\text{there exists }v\left(\cdot\right)\in\mathcal{D}_{\sigma,\tau}\left(\gamma\left(\tau\right)x\right)\text{ such that }v\left(t\right)=z\right\}.$$

Moreover, for a fixed $\tau\in\mathbb{R}$ and arbitrary $t\in\mathbb{R}_{+}$, $x\in X$, $\sigma\in\Sigma$ we can define the cocycle $G_{\tau}:\mathbb{R}_{+}\times\Sigma\times X\rightarrow P\left(X\right)$ by

$$G_{\tau}\left(t,\sigma,x\right)=U_{\sigma}\left(t+\tau,\tau,x\right).$$

Proposition 59 *For each $\sigma \in \Sigma$, $h \in \mathbb{R}$, $\tau \leq s \leq t$, $x \in X$, we have*

$$U_{T(h)\sigma}(t,\tau,x) = U_{\sigma}(t+h,\tau+h,x).$$

$$U_{\sigma}(t,s,U_{\sigma}(s,\tau,x)) = U_{\sigma}(t,\tau,x),$$

Hence, U_{σ} is a multivalued dynamical process for each $\sigma \in \Sigma$ and condition $(T1)$ holds.

Proof. In view of Lemma 19 for the first equality we have to prove only the inclusion $U_{\sigma}(t+h,\tau+h,x) \subset U_{T(h)\sigma}(t,\tau,x)$. Given $\eta \in U_{\sigma}(t+h,\tau+h,x)$, where $h \in \mathbb{R}$, there exists $y(\cdot) \in \mathcal{D}_{\sigma,\tau+h}(\gamma(\tau+h)x)$ such that $\eta = \gamma^{-1}(t+h)\,y(t+h)$. Let $z(s) = \gamma^{-1}(h)\,y(s+h)$, $l_z(s) = \gamma^{-1}(h)\,l(s+h)$, where $l(s) \in \gamma(s)F_{\sigma_1}\left(s,\gamma^{-1}(s)\,y(s)\right) + \gamma(s)\,g_{\sigma_2}(s)$, a.e. on $(\tau+h,t+h)$, $y(\cdot) = I(\gamma(\tau+h)x)\,l(\cdot)$, so that we have $l_z(s) \in \gamma^{-1}(h)\,\gamma(s+h)\,F_{\sigma_1}\left(s+h,\gamma^{-1}(s+h)\,\gamma(h)\,z(s)\right) + \gamma^{-1}(h)\,\gamma(s+h)\,g_{\sigma_2}(s+h) = F_{T(h)\sigma}(s,z(s))$, a.e. on (τ,t), (note that $\gamma^{-1}(h)\,\gamma(s+h) = \gamma(\theta_h\omega,s) = \widetilde{\gamma}(s)$, and $z(\tau) = \gamma^{-1}(h)\,\gamma(\tau+h)\,x = \widetilde{\gamma}(\tau)\,x$. We can show that $z(\cdot) \in \mathcal{D}_{T(h)\sigma,\tau}(\widetilde{\gamma}(\tau)\,x)$ as follows

$$\|z(t)-\xi\|_X^2 = \left\|\gamma^{-1}(h)\,y(t+h)-\xi\right\|_X^2 \leq \gamma^{-2}(h)\,\|y(s+h)-\gamma(h)\,\xi\|_X^2$$

$$+2\gamma^{-2}(h)\int_{s+h}^{t+h}(l(r)+\gamma(h)\,A\xi,y(r)-\gamma(h)\,\xi)\,dr = \|z(s)-\xi\|_X^2 + 2\int_s^t (l_z(r)+A\xi,z(r)-\xi)\,dr,$$

for any $\xi \in D(A)$. Therefore, $\eta = \gamma^{-1}(t+h)\,y(t+h) = \gamma^{-1}(t+h)\,\gamma(h)\,z(t) \in U_{T(h)\sigma}(t,\tau,x)$.

For the second equality we proceed as in Proposition 57, but taking into account [6, Proposition 13]. ∎

4.2.2 Existence of the global Σ_1-uniform attractor

Once more, the conditions of Theorem 41 providing the existence of a global Σ_1-uniform compact attractor hold.

Theorem 60 *In the preceedings conditions, the family of semiprocesses U_{σ} has the Σ_1-uniform global compact attractor $\Theta_{\Sigma_1}(\sigma_2)$.*

Proof. First we argue as in Lemma 50. Fix $\sigma_2 = (f_{\sigma_2}, g_{\sigma_2}) \in \Sigma_2$. Take an arbitrary solution to (35), $v(\cdot) = I(\gamma(\tau)\,u_{\tau})\,l(\cdot)$ defined on $[\tau,-1]$, where $l(s) \in \gamma(s)\,F_{\sigma_1}\left(s,\gamma^{-1}(s)\,v(s)\right) + \gamma(s)\,g_{\sigma_2}(s)$, $\sigma_1 \in \Sigma_1$ and $u_{\tau} \in B \in \mathcal{B}(X)$. Denote $\widetilde{l}(s) = l(s) - \gamma(s)\,g_{\sigma_1}(s) - \gamma(s)\,g_{\sigma_2}(s)$. Multiplying (22) by $v(s)$ and using $(F7)$, $(F8)$ and (13) we have

$$\frac{1}{2}\frac{d}{ds}\|v\|_X^2 + \lambda_1\|v(s)\|_X^2 \leq \left(\widetilde{l}(s),v(s)\right) + \gamma(s)\,(g_{\sigma_1}(s)+g_{\sigma_2}(s),v(s))$$

$$= \gamma(s)\left(\widetilde{l}(s),\gamma^{-1}(s)\,v(s)\right) + \gamma(s)\,(g_{\sigma_1}(s)+g_{\sigma_2}(s),v(s))$$

$$\leq (\lambda_1-\epsilon)\,\|v(s)\|_X^2 + \gamma^2(s)\,M\mu(\Omega) + \gamma(s)\left(C_0+R_1+R_2\,|s|^{R_3}\right)\|v(s)\|_X$$

$$\leq \left(\lambda_1-\frac{\epsilon}{2}\right)\|v(s)\|_X^2 + \gamma^2(s)\left(K_1+K_2\left(C_0+R_1+R_2\,|s|^{R_3}\right)^2\right).$$

Note that, for all $\varepsilon > 0$, $T > 0$, $p,q > 0$ and a.a. $\omega \in \Omega$, we have

$$\int_{-\infty}^{T} e^{\varepsilon s}\gamma^p(s)|s|^q ds < \infty,\quad \lim_{s\to-\infty}\exp(\varepsilon s)\,\gamma^p(s) = 0,$$

so that arguing as in Lemma 50 we obtain that for any $\sigma_2 \in \Sigma_2$ there exists a radius $R_0(\sigma_2)$ such that for any $B \in \mathcal{B}(X)$ there is $T = T(B, \sigma_2) < -1$ for which $\|v(-1)\|_X \leq R_0$, as soon as $\tau \leq T(B)$. Hence, $\left\|\gamma^{-1}(-1) v(-1)\right\|_X \leq \gamma^{-1}(-1) R_0 = \widetilde{R}_0(\sigma_2)$, so that $\|U_{\Sigma_1,\sigma_2}(-1,\tau,B)\|_+ \leq \widetilde{R}_0$.

Using the previous inequality and arguing as in Lemma 52 we prove that for any $R_0 > 0$, $\sigma_2 \in \Sigma$ and $(t,\tau) \in \mathbb{R}_d$, there exists $R \geq R_0$ such that $\|v(s)\|_X \leq R$, for all $\tau \leq s \leq t$, $v \in \mathcal{D}_{\sigma,\tau}(\gamma(\tau) u_\tau)$ and $u_\tau \in X$ such that $\|u_\tau\|_X \leq R_0$. Hence, $\|U_{\Sigma_1,\sigma_2}(s,\tau,u)\|_+ \leq \gamma^{-1}(s) R \leq \widetilde{R}(\sigma_2)$.

As remarked in Theorem 58 we can follow the same steps of the proof in Proposition 54, so that there exists a compact set $D(\sigma_2)$ which is Σ_1-uniformly attracting.

To prove the statement of Proposition 48 we argue as in Theorem 58. We take sequences $u_n(\cdot) = I(\gamma(\tau) u_\tau^n) l_n(\cdot)$, $l_n(s) \in \gamma(s) F_{\sigma_{1n}}\left(s, \gamma^{-1}(s) u_n(s)\right) + \gamma(s) g_{\sigma_2}(s)$, a.e. on $(\tau, 0)$, such that $y_n = u_n(0) \to y$, $u_\tau^n \to u_\tau$, $\sigma_{1n} \to \sigma_1$, and obtain $v(\cdot) = I(\gamma(\tau) u_\tau) l(\cdot)$ such that $u_n \to v$ in $C([\tau,0],X)$ and $l_n \to l$ weakly in $L_2(\tau,0;L_2(D))$. If we prove that $l(s) \in \gamma(s) F_{\sigma_1}\left(s,\gamma^{-1}(s) v(s)\right) + \gamma(s) g_{\sigma_2}(s)$, a.e. on $(\tau,0)$, then $y \in U_{\sigma_1,\sigma_2}(0,\tau,u_\tau)$. This is equivalent to prove $l(s) - g_{\sigma_1}(s) - g_{\sigma_2}(s) = d(s) \in \gamma(s) F_{\sigma_1}\left(s,\gamma^{-1}(s) v(s)\right) - g_{\sigma_1}(s) = \gamma(s) \widetilde{F}_{\sigma_1}\left(s,\gamma^{-1}(s) v(s)\right)$.

Fix s. Passing to a subsequence and using $(F1)$ we have

$$dist\left(\gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) u_n(s,x)\right), \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) v(s,x)\right)\right) \leq C\left|u_n(s,x) - v(s,x)\right| \to 0,$$

for a.a. $x \in D$. On the other hand, since $\{u_n(s,x)\}$ is bounded and $f_{\sigma_{1n}}$ converges to f_{σ_1} in $C(\mathbb{R},\mathcal{M})$, we get

$$dist\left(\gamma(s) f_{\sigma_{1n}}\left(s,\gamma^{-1}(s) u_n(s,x)\right), \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) u_n(s,x)\right)\right) \to 0, \text{ as } n \to +\infty,$$

for a.a. $x \in D$. Then for $d_n(s) = l_n(s) - g_{\sigma_{1n}}(s) - g_{\sigma_2}(s)$ it holds

$$dist\left(d_n(s,x), \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) v(s,x)\right)\right) \leq dist\left(\gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) u_n(s,x)\right), \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) v(s,x)\right)\right)$$

$$+dist\left(\gamma(s) f_{\sigma_{1n}}\left(s,\gamma^{-1}(s) u_n(s,x)\right), \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) u_n(s,x)\right)\right) \to 0, \text{ for a.a. } x \in D. \quad (37)$$

We conclude the proof as in Theorem 58, but putting $[a(s,x), b(s,x)] = \gamma(s) f_{\sigma_1}\left(s,\gamma^{-1}(s) v(s,x)\right)$. Thus we can apply Theorem 41. ■

5 Conclusions

In this work we have given a general framework of nonautonomous attractors for PDE, which includes the possibility of non-uniqueness of solutions and also the existence of unbounded (in time) trajectories. Hence, random dynamical systems are in fact particular cases of the theory. The splitting of the parameter set Σ in the product $\Sigma_1 \times \Sigma_2$ allows us to consider together the classical nonautonomous attractor and the attractor in the sense of the pull-back attraction.

We note that for the stochastic differential inclusions considered in this paper it is possible to study the measurability of the global attractor $\Theta_{\Sigma_1}(\sigma_2) = \Theta_{\Sigma_1}(\widetilde{\sigma}_2,\omega)$ with respect to the parameter ω (using similar arguments as in [5, 6]) . However, this is out of the aim of this work.

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