# On the asymptotic behaviour of a stochastic 3D LANS-$\alpha$ model *

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**Abstract**

The long-time behaviour of a stochastic 3D LANS-$\alpha$ model on a bounded domain is analysed. First, we reformulate the model as an abstract problem. Next, we establish sufficient conditions ensuring the existence of stationary (steady state) solutions of this abstract nonlinear stochastic evolution equation, and study the stability properties of the model. Finally, we analyse the effects produced by stochastic perturbations in the deterministic version of the system (persistence of exponential stability as well as possible stabilisation effects produced by the noise). The general results are applied to our stochastic LANS-$\alpha$ system throughout the paper.

## 1 Introduction

In this paper we are mainly interested in the study of the asymptotic behaviour of solutions of the 3D-Lagrangian averaged Navier-Stokes (LANS-$\alpha$) equations, with homogeneous Dirichlet boundary condition in a bounded domain, in the case in which random perturbations appear. To be more precise, let $D$ be a connected and bounded open subset of $\mathbb{R}^3$, with a $C^2$ boundary $\partial D$. We denote by $A$ the Stokes operator, and consider the system

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p = F(t,u)+G(t,u)\dot{W}(t), \quad \text{in} \quad D\times(0,+\infty), \\ \nabla\cdot u=0, \quad \text{in} \quad D\times(0,+\infty), \\ u=0, \quad Au=0, \quad \text{on } \partial D\times(0,+\infty), \\ u(0)=u^0, \quad \text{in} \quad D, \end{cases} \tag{1}$$

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where $u = (u_1, u_2, u_3)$ and $p$ are unknown random fields on $D \times (0, +\infty)$, representing, respectively, the large-scale (or averaged) velocity and the pressure, in each point of $D \times (0, +\infty)$, of an incompressible viscous fluid with constant density filling the domain $D$. The constants $\nu > 0$ and $\alpha > 0$ represent respectively the kinematic viscosity of the fluid, and the square of the spatial scale at which fluid motion is filtered. The terms $F(t, u)$ and $G(t, u)\dot{W}(t)$ are external forces depending eventually on $u$, where $\dot{W}(t)$ denotes the time derivative of a cylindrical Wiener process. Finally, $u^0$ is a given initial velocity field.

The deterministic version of (1), i.e. when $G = 0$, has received much attention over the last years. The main reason is that this model has become very useful in order to approximate the 3D Navier-Stokes equations (notice that when $\alpha$ goes to zero, this problem converges to the usual 3D Navier-Stokes model). More precisely, the global well-posedness of weak solutions for the deterministic Lagrangian averaged Navier-Stokes equations on bounded domains has been established in [8] and [14] amongst others, and the asymptotic behaviour can be found in [8]. Similar results have been proved by Foias *et al.* in [10] in the case of periodic boundary conditions.

However, in order to consider a more realistic model for our problem, it is sensible to consider some kind of 'noise' in the equations. This may reflect, for instance, some environmental effects on the phenomenon, some external random forces, etc. To the best of our knowledge, the existence and uniqueness of solution of the stochastic version (1) we will consider in this paper has been analysed in [7] (see also [6]).

We start in this paper the analysis of the asymptotic behaviour of the stochastic version, and point out that our analysis, in particular, also provides information on the long-time dynamics of the deterministic model (by simply setting $G = 0$ in the appropriate formulas).

After reading this paper, one could immediately wonder about the possibility of doing a similar analysis in the case of general unbounded domains (e.g. channels, pipes, etc.). This is beyond the scope of this paper since one would first need some results on the existence of solutions of such a model, and, as far as we know, this still has not been proved.

Being possible to carry out our analysis working directly with the 3D LANS-$\alpha$ model, we have preferred to establish a theory for an abstract stochastic model and then apply it to our system. In this way, with only a little extra work one may be able to apply these abstract results to other models of interest.

Although the techniques we use in the present paper are similar to those used in [5] for the stochastic 2D Navier-Stokes model (in fact, these can be considered, in certain sense, as standard techniques for the investigation of stability properties for stochastic PDEs, see e.g. [2]), it is worth pointing out that neither the stochastic 2D Navier-Stokes equations from [5] falls within the abstract framework in the present paper, nor this abstract setting is a particular case of the model in [5]. Nevertheless, the results we will prove may be applied to other interesting models as, for example, some stochastic reaction-diffusion equations (by simply setting operator $\widetilde{B} = 0$ in problem (23) below), or the same model but for periodic boundary conditions, etc.

The content of the paper is as follows. In Section 2 we show how our problem can be reformulated as an abstract stochastic model. In Section 3 we first establish the existence and eventual uniqueness of stationary steady state solutions when the viscosity is large enough, and prove a result ensuring the exponential stability (in mean square and pathwise) of the stationary solution. Finally, the stabilising effects produced by the noise in the deterministic model is stated in Section 4.

# 2 Variational and Abstract formulation of the problem

In this section we will rewrite our problem as an abstract model. The main reasons are the following. On the one hand, the results are presented, in our opinion, with more clarity and the computations are done in a more simplified way. On the other hand, the abstract formulation may be applied, in addition, to other models as we commented in Section 1.

## 2.1 The cylindrical Wiener process

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of sub $\sigma$-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the $P$ null sets of $\mathcal{F}$. Let $\{\beta^j(t),\ t \geq 0,\ j = 1, 2, ...\}$ be a sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on this space, and suppose that $K$ is a given separable Hilbert space, and $\{e_j;\ j = 1, 2, ...\}$, an orthonormal basis of $K$. We denote by $\{W(t);\ t \geq 0\}$, the cylindrical Wiener process with values in $K$ defined formally as

$$W(t) = \sum_{j=1}^{\infty} \beta^j(t) e_j$$

It is well known that this series does not converge in $K$, but rather in any Hilbert space $\widetilde{K}$ such that $K \subset \widetilde{K}$, and the injection of $K$ in $\widetilde{K}$ is Hilbert-Schmidt (see e.g. [9]).

Let $T > 0$ be given. For any separable Banach space $X$, we will denote by $M^2_{\mathcal{F}_t}(0, T; X)$ the space of all processes $\varphi \in L^2(\Omega \times (0, T), dP \times dt; X)$ that are $\mathcal{F}_t$-progressively measurable. The space $M^2_{\mathcal{F}_t}(0, T; X)$ is a Hilbert subspace of $L^2(\Omega \times (0, T), dP \times dt; X)$.

We will write $L^2(\Omega; C([0, T]; X))$ to denote the space of all continuous and $\mathcal{F}_t$-progressively measurable $X$-valued processes $\{\varphi(t);\ 0 \leq t \leq T\}$ satisfying

$$E\left(\sup_{0\leq t\leq T} \|\varphi(t)\|^2_X\right) < \infty.$$

For another separable Hilbert space $\widetilde{H}$, with scalar product $(\cdot,\cdot)_{\widetilde{H}}$, let us denote by $\mathcal{L}^2(K;\widetilde{H})$ the separable Hilbert space of Hilbert-Schmidt operators from $K$ into $\widetilde{H}$, and by $((\cdot,\cdot))_{\mathcal{L}^2(K;\widetilde{H})}$and $\|\cdot\|_{\mathcal{L}^2(K;\widetilde{H})}$the scalar product and norm in $\mathcal{L}^2(K;\widetilde{H})$,where for all $R$ and $S$ in $\mathcal{L}^2(K;\widetilde{H})$,

$$((R,S))_{\mathcal{L}^2(K;\widetilde{H})} = \sum_{j=1}^{\infty}(Re_j, Se_j)_{\widetilde{H}}.$$

For any process $\Psi \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\widetilde{H}))$ one can define the stochastic integral of $\Psi$with respect to the cylindrical Wiener process $W(t)$, denoted

$$\int_0^t \Psi(s)\, dW(s), \quad 0 \le t \le T,$$

as the unique continuous $\widetilde{H}$-valued $\mathcal{F}_t$-martingale such that for all $h \in \widetilde{H}$,

$$(\int_0^t \Psi(s)\, dW(s), h)_{\widetilde{H}} = \sum_{j=1}^{\infty}\int_0^t (\Psi(s)e_j, h)_{\widetilde{H}}\, d\beta^j(s), \quad 0 \le t \le T,$$

where the integral with respect to $\beta^j(s)$ is the Itô integral, and the series converges in $L^2(\Omega; C([0,T]))$. See e.g. [9] for the properties of the stochastic integral defined in this way. In particular, we note that if $\Psi \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\widetilde{H}))$ and $\phi \in L^2(\Omega; L^\infty(0,T;\widetilde{H}))$ is $\mathcal{F}_t$-progressively measurable, then the series

$$\sum_{j=1}^{\infty}\int_0^t (\Psi(s)e_j, \phi(s))_{\widetilde{H}}\, d\beta^j(s), \quad 0 \le t \le T,$$

converges in $L^1(\Omega; C([0,T]))$,and defines a real valued continuous $\mathcal{F}_t$-martingale. We will use the notation

$$\int_0^t (\Psi(s)dW(s), \phi(s)) := \sum_{j=1}^{\infty}\int_0^t (\Psi(s)e_j, \phi(s))_{\widetilde{H}}\, d\beta^j(s), \quad 0 \le t \le T.$$

## 2.2 Notations and properties of the nonlinear term

We first establish some notations and recall some properties regarding the nonlinear term $(u\cdot\nabla)(u-\alpha\Delta u) - \alpha\nabla u^*\cdot\Delta u$ appearing in (1).

We will denote $(\cdot,\cdot)$and $|\cdot|$, respectively, the scalar product and associated norm in $(L^2(D))^3$, and by $(\nabla u, \nabla v)$the scalar product in $((L^2(D))^3)^3$of the gradients of $u$and $v$. We consider the scalar product in $(H^1_0(D))^3$defined by

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v), \quad u,v \in (H^1_0(D))^3, \qquad (2)$$

where its associated norm $\|\cdot\|$ is, in fact, equivalent to the usual gradient norm.
Let us denote by $H$ the closure in $(L^2(D))^3$ of the set

$$\mathcal{V} = \{v \in (\mathcal{D}(D))^3 : \nabla \cdot v = 0 \text{ in } D\},$$

and by $V$ the closure of $\mathcal{V}$ in $(H_0^1(D))^3$. Then, $H$ is a Hilbert space equipped with the inner product of $(L^2(D))^3$, and $V$ is a Hilbert subspace of $(H_0^1(D))^3$.

Denote by $A$ the Stokes operator, with domain $D(A) = (H^2(D))^3 \cap V$, defined by

$$Aw = -\mathcal{P}(\Delta w), \quad w \in D(A),$$

where $\mathcal{P}$ is the projection operator from $(L^2(D))^3$ onto $H$. Recall that as $\partial D$ is $C^2$, $|Aw|$ defines in $D(A)$ a norm which is equivalent to the $(H^2(D))^3$-norm, i.e., there exists a constant $c_1(D) > 0$, depending only on $D$, such that

$$\|w\|_{(H^2(D))^3} \leq c_1(D)|Aw|, \quad \forall\, w \in D(A), \tag{3}$$

and so $D(A)$ is a Hilbert space with respect to the scalar product

$$(v, w)_{D(A)} = (Av, Aw).$$

For $u \in D(A)$ and $v \in (L^2(D))^3$, we define $(u \cdot \nabla)v$ as the element of $(H^{-1}(D))^3$ given by

$$\langle (u \cdot \nabla)v, w \rangle = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j \rangle, \quad \text{for all } w \in (H_0^1(D))^3. \tag{4}$$

Observe that (4) is meaningful, since $H^2(D) \subset L^\infty(D)$, and $H_0^1(D) \subset L^6(D)$, with continuous injections. This implies that $u_i w_j \in H_0^1(D)$, and there exists a constant $c_2(D) > 0$, depending only on $D$, such that

$$|\langle (u \cdot \nabla)v, w \rangle| \leq c_2(D)|Au||v|\|w\|, \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{5}$$

Observe also that if $v \in (H^1(D))^3$, then the definition above coincides with the definition of $(u \cdot \nabla)v$ as the vector function whose components are $\sum_{i=1}^3 u_i \partial_i v_j$, for $j = 1, 2, 3$. However, as it is not known whether the solutions of the stochastic problem (1) have the same regularity as in the deterministic case (we only can ensure $H^2$ instead of $H^3$), it is necessary the present extension.

Now, if $u \in D(A)$, then $\nabla u^* \in (H^1(D))^{3\times 3} \subset (L^6(D))^{3\times 3}$, and consequently, for $v \in (L^2(D))^3$, we have that $\nabla u^* \cdot v \in (L^{3/2}(D))^3 \subset (H^{-1}(D))^3$, with

$$\langle \nabla u^* \cdot v, w \rangle = \sum_{i,j=1}^{3} \int_D (\partial_j u_i) v_i w_j \, dx, \quad \text{for all } w \in (H_0^1(D))^3. \tag{6}$$

It follows that there exists a constant $c_3(D) > 0$, depending only on $D$, such that

$$|\langle \nabla u^* \cdot v, w\rangle| \le c_3(D)|Au||v|\|w\|,\ \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{7}$$

We have the following results (see [7] for the proofs).

**Proposition 2.1** *For all $(u, w) \in D(A) \times D(A)$ and all $v \in (L^2(D))^3$, it follows*

$$\langle (u \cdot \nabla)v, w\rangle = -\langle \nabla w^* \cdot v, u\rangle. \tag{8}$$

Consider now the trilinear form defined by

$$b^{\#}(u,v,w) = \langle (u\cdot\nabla)v, w\rangle + \langle \nabla u^* \cdot v, w\rangle,\ (u,v,w) \in D(A)\times(L^2(D))^3\times(H_0^1(D))^3.$$

**Proposition 2.2** *The trilinear form $b^{\#}$ satisfies*

$$b^{\#}(u,v,w) = -b^{\#}(w,v,u), \quad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times D(A), \tag{9}$$

*and consequently,*

$$b^{\#}(u,v,u) = 0,\ \forall (u,v) \in D(A) \times (L^2(D))^3. \tag{10}$$

*Moreover, there exists a constant $c(D) > 0$, depending only on $D$, such that*

$$|b^{\#}(u,v,w)| \le c(D)|Au||v|\|w\|,\ \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3, \tag{11}$$

$$|b^{\#}(u,v,w)| \le c(D)\|u\||v||Aw|,\ \forall (u,v,w) \in D(A)\times(L^2(D))^3\times D(A). \tag{12}$$

*Thus, in particular, $b^{\#}$ is continuous on $D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.*

### 2.3 Existence and uniqueness of a variational solution

Assume that $F$ and $G$ are measurable, Lipschitz mappings from $\Omega\times(0,+\infty)\times V$ into $(H^{-1}(D))^3$ and from $\Omega \times (0,+\infty) \times V$ into $\mathcal{L}^2(K;(L^2(D))^3)$, respectively. More precisely, suppose that, for all $u, v \in V$, $F(\cdot,u)$ and $G(\cdot,v)$ are $\mathcal{F}_t-$progressively measurable, and

$$\|F(t,u) - F(t,v)\|_{(H^{-1}(D))^3} \le L_F \|u - v\|, \quad dP \times dt - \text{a.e.}, \tag{13}$$

$$\|G(t,u) - G(t,v)\|_{\mathcal{L}^2(K;(L^2(D))^3)} \le L_G \|u - v\|, \quad dP \times dt - \text{a.e.}. \tag{14}$$

We also suppose

$$F(t,0) \in L^4(\Omega; L^2(0,T;(H^{-1}(D))^3), \quad \text{for all } T > 0, \tag{15}$$

$$G(t,0) \in L^4(\Omega; L^2(0,T;\mathcal{L}^2(K;(L^2(D))^3)), \quad \text{for all } T > 0, \tag{16}$$

$$u_0 \in L^4(\Omega, \mathcal{F}_0, P; V). \tag{17}$$

**Definition 2.3** *A variational solution to problem (1) is a stochastic process* $u \in M^2_{\mathcal{F}_t}(0,T;D(A)) \cap L^2(\Omega; L^\infty(0,T;V))$*, for all* $T>0$*, weakly continuous with values in* $V$*, such that for all* $w \in D(A)$*,*

$$((u(t),w)) + \nu \int_0^t (u(s) + \alpha Au(s), Aw)\,ds + \int_0^t b^{\#}(u(s), u(s) - \alpha\Delta u(s), w)\,ds$$
$$= ((u^0, w)) + \int_0^t \langle F(s,u(s)), w\rangle\,ds + (\int_0^t G(s,u(s))\,dW(s), w),\ t \geq 0. \quad (18)$$

Observe that (18) follows from (1) by multiplying the first equation in (1) by $w \in D(A)$, taking into account the definition of the scalar product $((\cdot,\cdot))$, the definition of $b^{\#}$, and the equality (8).

Now, as a consequence of Theorem 3.3 in [7] we have the following result.

**Theorem 2.4** *Under the hypotheses (13)-(17), there exists a unique variational solution* $u$ *of (1), and moreover,*

$$u \in L^4(\Omega; C([0,T];V)) \cap L^4(\Omega; L^2(0,T;D(A))), \quad \text{for all } T > 0.$$

## 2.4 Formulation of problem (1) as an abstract problem

As we commented at the beginning of Section 2, we are going to rewrite our model (1) as an abstract problem.

Let us set $\mathcal{H} = V$, with scalar product $(u,v)_{\mathcal{H}} = ((u,v))$, and associated norm $|u|_{\mathcal{H}} = \|u\|$, and $\mathcal{U} = D(A)$, with scalar product $((u,v))_{\mathcal{U}} = (Au, Av)$, and associated norm $\|u\|_{\mathcal{U}} = |Au|$. Then, $\mathcal{H}$ and $\mathcal{U}$ are two separable real Hilbert spaces, such that $\mathcal{U} \subset \mathcal{H}$ with compact injection, and $\mathcal{U}$ is dense in $\mathcal{H}$.

We identify $\mathcal{H}$ with its topological dual $\mathcal{H}^*$, but we consider $\mathcal{U}$ as a subspace of $\mathcal{H}^*$, identifying $v \in \mathcal{U}$ with the element $f_v \in \mathcal{H}^*$, defined by

$$f_v(h) = (v,h)_{\mathcal{H}}, \quad \in \mathcal{H}.$$

We denote by $\langle\cdot,\cdot\rangle$ the duality product between $\mathcal{U}^*$ and $\mathcal{U}$. Let us define

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av), \quad u, v \in D(A).$$

It is clear that for all $v \in D(A)$,

$$2\langle \widetilde{A}v, v\rangle = 2\nu(Av, v) + 2\nu\alpha(Av, Av) \geq 2\nu\alpha|Av|^2,$$

and, if we denote by $\lambda_k$ and $w_k$, $k \geq 1$, the eigenvalues and their corresponding eigenvectors associated to $A$,

$$\langle \widetilde{A}w_k, v\rangle = \nu\lambda_k((w_k, v)).$$

Thus, taking

$$\widetilde{\alpha} = 2\nu\alpha, \quad (19)$$

we have that

a) $\widetilde{A}$ is a linear continuous operator $\widetilde{A} \in \mathcal{L}(\mathcal{U},\mathcal{U}^*)$, such that

a1) $\widetilde{A}$ is self adjoint,

a2) there exists $\widetilde{\alpha} > 0$, such that

$$2\langle \widetilde{A}v, v\rangle \geq \widetilde{\alpha}\|v\|_{\mathcal{U}}^2, \quad \text{for all } v \in \mathcal{U}, \tag{20}$$

On the other hand, denote

$$\langle \widetilde{B}(u,v), w\rangle = b^{\#}(u, v - \alpha\Delta v, w), \quad (u,v,w) \in D(A) \times D(A) \times D(A),$$

$$((\widetilde{F}(t,u), w)) = \langle F(t,u), w\rangle, \quad (u,w) \in V \times V.$$

Then, it is straightforward to check that if we take

$$c_1 = (1+\alpha)c_1(D)c(D), \qquad L_{\widetilde{F}} = L_F, \tag{21}$$

we obtain that

b) $\widetilde{B} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}^*$ is a bilinear mapping such that

b1) $\langle \widetilde{B}(u,v), u\rangle = 0$, for all $u, v \in \mathcal{U}$,

b2) $\|\widetilde{B}(u,v)\|_{\mathcal{U}^*} \leq c_1 |u|_{\mathcal{H}} \|v\|_{\mathcal{U}}$, for all $(u,v) \in \mathcal{U} \times \mathcal{U}$,

b3) $|\langle \widetilde{B}(u,v), w\rangle| \leq c_1 \|u\|_{\mathcal{U}} \|v\|_{\mathcal{U}} |w|_{\mathcal{H}}$, for all $u, v, w \in \mathcal{U}$,

c) $\widetilde{F} : \Omega \times (0,+\infty) \times \mathcal{H} \to \mathcal{H}$ is a random mapping such that

c1) for all $v \in \mathcal{H}$, $\widetilde{F}(\cdot, v)$ is $\mathcal{F}_t-$progressively measurable,

c2) $\widetilde{F}(\cdot, 0) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{H})$, for all $T \geq 0$,

c3) $|\widetilde{F}(t,u) - \widetilde{F}(t,v)|_{\mathcal{H}} \leq L_{\widetilde{F}} |u-v|_{\mathcal{H}}, \quad dP \times dt$ a.e. for all $u, v \in \mathcal{H}$.

Now, let $I$ denote the identity operator in $H$, and define $\widetilde{G}(t,u)$ as

$$\widetilde{G}(t,u) = (I + \alpha A)^{-1} \circ \mathcal{P} \circ G(t,u), \quad u \in V.$$

First, observe that $I + \alpha A$ is bijective from $D(A)$ onto $H$, and

$$(((I+\alpha A)^{-1} f, w)) = (f, w), \quad \text{for all } f \in H,\, w \in V.$$

Thus, for each $f \in H$,

$$\|(I+\alpha A)^{-1} f\|^2 = (f, u) \leq |f|\,|u|\,,$$

where $u = (I+\alpha A)^{-1} f$, i.e., $(u, w_k) + \alpha(Au, w_k) = (f, w_k)$, for all $k \geq 1$ so, $(1+\alpha\lambda_k)(u, w_k) = (f, w_k)$, which implies

$$(u, w_k) = \frac{1}{(1+\alpha\lambda_k)}(f, w_k) \leq \frac{1}{(1+\alpha\lambda_1)}(f, w_k),$$

$$|u|^2 = \sum_{k=1}^{\infty}(u,w_k)^2 \le \frac{1}{(1+\alpha\lambda_1)^2}\sum_{k=1}^{\infty}(f,w_k) = \frac{1}{(1+\alpha\lambda_1)^2}|f|^2.$$

Therefore,

$$\|(I+\alpha A)^{-1}f\|^2 \le \frac{1}{1+\alpha\lambda_1}|f|^2,$$

and consequently, taking

$$L_{\widetilde{G}} = \frac{L_G}{\sqrt{1+\alpha\lambda_1}}, \tag{22}$$

we obtain that

d) $\widetilde{G} : \Omega \times (0,+\infty) \times \mathcal{H} \to \mathcal{L}^2(K;\mathcal{H})$ is a measurable random mapping such that

   d1) for any $v \in \mathcal{H}$, $\widetilde{G}(\cdot, v)$ is $\mathcal{F}_t$-progressively measurable,
   d2) $\widetilde{G}(\cdot,0) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\mathcal{H}))$, for all $T \ge 0$,
   d3) $\|\widetilde{G}(t,u)-\widetilde{G}(t,v)\|_{\mathcal{L}^2(K;\mathcal{H})} \le L_{\widetilde{G}}\,|u-v|_{\mathcal{H}}$, $dP\times dt$ a.e. for all $u,v \in \mathcal{H}$.

Next, for each $j \ge 1$, and all $(t,u,w) \in (0,+\infty) \times V \times D(A)$, we have

$$(G(t,u)e_j, w) = ((I+\alpha A)(\widetilde{G}(t,u)e_j), w) = ((\widetilde{G}(t,u)e_j, w)),$$

and, for all $u \in M^2_{\mathcal{F}_t}(0,T;V)$, $(t,w) \in (0,T)\times D(A)$, it follows

$$(\int_0^t G(s,u(s))\,dW(s), w) = \sum_{j=1}^{\infty}\int_0^t (G(s,u(s))e_j, w)\,d\beta^j(s)$$

$$= \sum_{j=1}^{\infty}\int_0^t ((\widetilde{G}(s,u(s))e_j, w))\,d\beta^j(s) = ((\int_0^t \widetilde{G}(s,u(s))\,dW(s), w)).$$

Consequently, in a abstract framework, a variational solution of problem (1) is, equivalently, a stochastic process

$$u \in M^2_{\mathcal{F}_t}(0,T;\mathcal{U}) \cap L^2(\Omega; L^\infty(0,T;\mathcal{H})), \quad \text{for all } T > 0,$$

such that the equation

$$\begin{aligned} u(t) &+ \int_0^t \widetilde{A}u(s)\,ds + \int_0^t \widetilde{B}(u(s),u(s))\,ds \\ &= u^0 + \int_0^t \widetilde{F}(s,u(s))\,ds + \int_0^t \widetilde{G}(s,u(s))\,dW(s), \quad t \ge 0, \end{aligned} \tag{23}$$

is satisfied in $\mathcal{U}^*$, a.s. for all $t \ge 0$.

Finally, observe that if $u \in D(A)$, then

$$\lambda_1^2 |u|^2 \leq |Au|^2, \qquad |\nabla u|^2 = (Au, u) \leq |Au||u|,$$

and consequently,

$$\|u\|^2 = |u|^2 + \alpha |\nabla u|^2 \leq \frac{1+\alpha\lambda_1}{\lambda_1^2} |Au|^2, \quad \text{for all } u \in D(A).$$

Thus, if we take

$$c_2 = \frac{\sqrt{1+\alpha\lambda_1}}{\lambda_1}, \tag{24}$$

we can write

$$|\, u \,|_{\mathcal{H}} \leq c_2 \, \|u\|_{\mathcal{U}}\,, \quad \text{for all } u \in \mathcal{U}. \tag{25}$$

# 3 A pathwise stability result for stationary solutions

In this section we analyse the stability properties of the stationary solutions to (1). For this reason, we suppose that $\widetilde{F}(t, v) = \widetilde{F}(v)$ (i. e. $F(t, v) = F(v)$) is independent of $\omega$ and $t$. Associated to (23), we consider its *deterministic* version

$$u(t) + \int_0^t \widetilde{A} u(s)\, ds + \int_0^t \widetilde{B}(u(s), u(s))\, ds = u^0 + \int_0^t \widetilde{F}(u(s))\, ds, \quad t \geq 0. \tag{26}$$

**Definition 3.1** *It is said that $u_\infty \in \mathcal{U}$ is a stationary solution of (26) if*

$$\widetilde{A} u_\infty + \widetilde{B}(u_\infty, u_\infty) = \widetilde{F}(u_\infty). \tag{27}$$

Observe that a stationary solution of (26) is a stationary solution of the corresponding variational formulation of (1) when $G = 0$.

We have the following result.

**Lemma 3.2** *Suppose that*

$$\widetilde{\alpha} > 2c_2^2 L_{\widetilde{F}}. \tag{28}$$

*Then there exists a stationary solution $u_\infty \in \mathcal{U}$ of (26).*

*If moreover*

$$(\widetilde{\alpha} - 2c_2^2 L_{\widetilde{F}})^2 > 4c_1 c_2^2 |\widetilde{F}(0)|_{\mathcal{H}}, \tag{29}$$

*then the stationary solution is unique.*

**Proof.** As the proof follows the same lines as those of Theorem 10.1 in [17] (see also [5]), we omit it. ■

Observe that, for example, condition (28) can be written for the problem (1) as

$$\nu\alpha\lambda_1^2 > (1+\alpha\lambda_1)L_F.$$

Thus, the existence of stationary solutions is guaranteed provided the viscosity dominates the external forcing term, and the stationary solution is unique when the viscosity is large enough (see condition (29)).

Now we can study the pathwise stability properties of the stationary solutions.

**Theorem 3.3** *Let $u_\infty \in \mathcal{U}$ be a stationary solution of problem* (26), *and suppose there exist constants $\widetilde{M} \geq 0$, $\theta > 0$, such that a.e. in $\Omega \times (0,+\infty)$,*

$$\|\widetilde{G}(t,u_\infty)\|^2_{\mathcal{L}^2(K;\mathcal{H})} \leq \widetilde{M}e^{-\theta t}, \tag{30}$$

*and*

$$\widetilde{\alpha} > c_1\,|u_\infty|_{\mathcal{H}} + c_2^2(2L_{\widetilde{F}} + L^2_{\widetilde{G}})\,. \tag{31}$$

*Then, there exist constants $\gamma \in (0,\theta)$ and $M \geq 0$ such that*

$$E\,|u(t)-u_\infty|^2_{\mathcal{H}} \leq \left(E|u^0-u_\infty|^2_{\mathcal{H}} + \frac{M}{\theta-\gamma}\right)e^{-\gamma t}, \quad \text{for all } t \geq 0, \tag{32}$$

*for any solution $u(t)$ of* (23). *Furthermore, $u_\infty$ is the unique stationary solution of problem* (26), *and there exists $\widetilde{\gamma} > 0$ such that for any solution $u(t)$ of* (23) *there is a random time $T(u^0,\omega) \geq 0$, such that for almost all $\omega \in \Omega$*

$$|u(t,\omega)-u_\infty|^2_{\mathcal{H}} \leq e^{-\widetilde{\gamma}t}, \qquad \text{for all } t \geq T(u^0,\omega). \tag{33}$$

**Proof.** Taking into account (23) and (27), it is easy to see, by applying Itô's formula to the process $e^{\gamma t}\,|u(t)-u_\infty|^2_{\mathcal{H}}$, and using a2), c3), d3) and (30), that

$$\begin{aligned}
&e^{\gamma t}E\,|u(t)-u_\infty|^2_{\mathcal{H}}\\
&\quad\leq E|u^0-u_\infty|^2_{\mathcal{H}} + \gamma\int_0^t e^{\gamma s}E\,|u(s)-u_\infty|^2_{\mathcal{H}}\,ds\\
&\quad-\widetilde{\alpha}\int_0^t e^{\gamma s}E\,\|u(s)-u_\infty\|^2_{\mathcal{U}}\,ds\\
&\quad+2E\left|\int_0^t e^{\gamma s}\langle\widetilde{B}(u(s),u(s))-\widetilde{B}(u_\infty,u_\infty),u(s)-u_\infty\rangle\,ds\right|\\
&\quad+2c_2^2L_{\widetilde{F}}\int_0^t e^{\gamma s}E\,\|u(s)-u_\infty\|^2_{\mathcal{U}}\,ds\\
&\quad+\int_0^t e^{\gamma s}\left(M_\varepsilon e^{-\theta s} + (1+\varepsilon)c_2^2L^2_{\widetilde{G}}E\,\|u(s)-u_\infty\|^2_{\mathcal{U}}\right)ds,
\end{aligned} \tag{34}$$

with $M_\varepsilon = (1+1/\varepsilon)\widetilde{M}$ and $\varepsilon > 0$.

By b1) and b2), we have

$$
\begin{aligned}
&\langle \widetilde{B}(u(s),u(s)) - \widetilde{B}(u_\infty,u_\infty), u(s)-u_\infty\rangle \\
&\quad = \langle \widetilde{B}(u_\infty, u(s)-u_\infty), u(s)-u_\infty\rangle \\
&\quad \le c_1 \, |u_\infty|_{\mathcal{H}} \, \|u(s)-u_\infty\|_{\mathcal{U}}^2 ,
\end{aligned}
$$

and thus, by (25), we obtain from (34)

$$
\begin{aligned}
e^{\gamma t} E|u(t)-u_\infty|_{\mathcal{H}}^2 &\le E|u^0 - u_\infty|_{\mathcal{H}}^2 + \frac{M_\varepsilon}{\gamma-\theta}\left(e^{(\gamma-\theta)t}-1\right) \qquad (35)\\
&+ \left(\gamma c_2^2 - \widetilde{\alpha} + c_1\, |u_\infty|_{\mathcal{H}} + 2c_2^2 L_{\widetilde{F}} + (1+\varepsilon)c_2^2 L_{\widetilde{G}}^2\right) \int_0^t e^{\gamma s} E\, \|u(s)-u_\infty\|_{\mathcal{U}}^2 \, ds.
\end{aligned}
$$

By assumption (31), we can choose $\varepsilon > 0$ and $0 < \gamma < \theta$ such that

$$
\gamma c_2^2 - \widetilde{\alpha} + c_1\, |u_\infty|_{\mathcal{H}} + 2c_2^2 L_{\widetilde{F}} + (1+\varepsilon)c_2^2 L_{\widetilde{G}}^2 \le 0,
$$

and (32) holds.

As for the uniqueness of $u_\infty$, notice that setting $\widetilde{G} = 0$ as well as its corresponding associated constants $\widetilde{M} = 0$ and $L_{\widetilde{G}} = 0$, condition (31) ensures that any solution of the deterministic model approaches the stationary solution $u_\infty$ with an exponential rate.

The proof of the pathwise exponential convergence follows from the standard and well known technique based on the application of the Itô formula, the Burkholder-Davis-Gundy and Tchebyshev inequalities, and the Borel-Cantelli lemma (see, e.g. [5] for a similar proof in the case of a stochastic 2D Navier-Stokes model). ■

**Remark 3.4** *Observe that, for problem* (1)*, conditions* (30) *and* (31) *become*

$$
\|G(t,u_\infty)\|_{\mathcal{L}^2(K;\mathcal{H})}^2 \le \widetilde{M}(1+\alpha\lambda_1)e^{-\theta t},
$$

$$
2\alpha\nu\lambda_1^2 > c_1\lambda_1^2 \|u_\infty\|^2 + 2(1+\alpha\lambda_1)L_F + L_G^2.
$$

*And the conclusion of Theorem 3.3 ensures the pathwise exponential decay of the $L^2$ gradient norm of the solution.*

**Remark 3.5** *Although we content ourselves in this paper with the analysis of the exponential asymptotic behaviour of the stationary solutions of our problem, it is also possible to prove more general results concerning different convergence rate, e.g. polynomial or even super-exponential (see, for instance, [2]).*

# 4 Stabilisation results.

It is well known that stochastic perturbations can produce stabilising as well as destabilising effects on the long-term behaviour of a deterministic evolution system (see for example [1], [16] for the finite dimensional framework, and [3], [4] for the infinite dimensional context). We will not discuss in this paper the suitability of considering one kind of noise or other (say Itô versus Stratonovich) and refer the reader to the previous papers (especially [3] and [4]) for more details on this topic. Instead, we will be interested in improving the stability properties of our model when a stochastic perturbation appears in the system.

As an auxiliary result, we are going to prove a lemma which states that, when two initial data coincide in a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) > 0$, then the corresponding solutions to (23) coincide in a subset $\Omega_1 \subset \Omega_0$ with $P(\Omega_1) = P(\Omega_0)$. This result is a nontrivial consequence of the uniqueness of solutions of problem (23), and will be crucial in the rigorous proof of our main stabilisation result, namely Theorem 4.3.

**Lemma 4.1** *Let $u_i^0 : \Omega \to \mathcal{H}$, $i = 1, 2$, be two $\mathcal{F}_0-$measurable random variables such that $E|u_i^0|_{\mathcal{H}}^2 < +\infty$. Suppose there exist their corresponding solutions $u^i$ to problem (23) with initial datum $u_i^0$.*

*Let us denote*

$$\Gamma = \left\{ \omega \in \Omega;\ u_1^0(\omega) = u_2^0(\omega) \right\}.$$

*Then,*

$$1_\Gamma(\omega) u^1(\omega, t) = 1_\Gamma(\omega) u^2(\omega, t), \quad \text{for all } t \geq 0, \text{ a.s.} \tag{36}$$

**Proof.** Let us denote

$$\widehat{F}(s, u^i(s)) = -\widetilde{A} u^i(s) - \widetilde{B}(u^i(s), u^i(s)) + \widetilde{F}(s, u^i(s)).$$

Then, for all $t \geq 0$,

$$\begin{aligned} 1_\Gamma u^i(t) &= 1_\Gamma u_i^0 + 1_\Gamma \int_0^t (\widehat{F}(s, u^i(s)) - \widetilde{F}(s, 0)\,)\, ds \\ &\quad + 1_\Gamma \int_0^t \widetilde{F}(s, 0)\, ds + 1_\Gamma \int_0^t (\widetilde{G}(s, u^i(s)) - \widetilde{G}(s, 0))\, dW(s) \\ &\quad + 1_\Gamma \int_0^t \widetilde{G}(s, 0)\, dW(s), \end{aligned} \tag{37}$$

with

$$1_\Gamma \int_0^t (\widehat{F}(s, u^i(s)) - \widetilde{F}(s, 0)\,)\, ds = \int_0^t 1_\Gamma (\widehat{F}(s, u^i(s)) - \widetilde{F}(s, 0)\,)\, ds,$$

$$1_\Gamma \int_0^t \widetilde{F}(s, 0)\, ds = \int_0^t 1_\Gamma \widetilde{F}(s, 0)\, ds,$$

and, by the $\mathcal{F}_0-$measurability of the set $\Gamma$,

$$1_\Gamma \int_0^t (\widetilde{G}(s, u^i(s)) - \widetilde{G}(s, 0))\, dW(s) = \int_0^t 1_\Gamma (\widetilde{G}(s, u^i(s)) - \widetilde{G}(s, 0))\, dW(s),$$

$$1_\Gamma \int_0^t \widetilde{G}(s,0)\,dW(s) = \int_0^t 1_\Gamma \widetilde{G}(s,0)\,dW(s).$$

But, as can be easily checked,

$$\begin{aligned} 1_\Gamma(\widehat{F}(s,u^i(s)) - \widetilde{F}(s,0)) &= \widehat{F}(s,1_\Gamma u^i(s)) - \widetilde{F}(s,0), \\ 1_\Gamma(\widetilde{G}(s,u^i(s)) - \widetilde{G}(s,0)) &= \widetilde{G}(s,1_\Gamma u^i(s)) - \widetilde{G}(s,0), \end{aligned}$$

for all $t \geq 0$.

Consequently, by (37),

$$\begin{aligned} 1_\Gamma u^i(t) &= 1_\Gamma u_i^0 - \int_0^t \widetilde{A} 1_\Gamma u^i(s)\,ds - \int_0^t \widetilde{B}(1_\Gamma u^i(s), 1_\Gamma u^i(s))\,ds \\ &\quad + \int_0^t \widetilde{F}(s,1_\Gamma u^i(s))\,ds - \int_0^t 1_{\Gamma^c}\widetilde{F}(s,0)\,ds \\ &\quad + \int_0^t \widetilde{G}(s,1_\Gamma u^i(s))\,dW(s) - \int_0^t 1_{\Gamma^c}\widetilde{G}(s,0)\,dW(s), \end{aligned} \tag{38}$$

for all $t \geq 0$, where $\Gamma^c = \Omega \setminus \Gamma$.

Taking into account that $1_\Gamma u_1^0 = 1_\Gamma u_2^0$, it follows from (38) that $1_\Gamma u^1$ and $1_\Gamma u^2$ are solutions of the same equation with the same initial datum, and then, by uniqueness of solution, we obtain (36). ∎

**Corollary 4.2** *Under the assumptions of the preceding lemma, suppose in addition that* $\widetilde{F}(t,0) = 0$ *and* $\widetilde{G}(t,0) = 0$ $dP \times dt$*-a.e. Let* $u^0 \in L^2(\Omega, \mathcal{F}_0, P; \mathcal{H})$ *such that there exists* $u(t)$*, solution of* (23) *with initial datum* $u^0$*, and denote*

$$\Omega_0 = \left\{\omega \in \Omega;\ u^0(\omega) = 0\right\}.$$

*Then,*

$$1_{\Omega_0} u(t) = 0 \quad \text{for all } t \geq 0, \text{ a.s.}$$

**Proof.** As $\widetilde{u} = 0$ is the solution of problem (23) with initial datum $\widetilde{u}^0 = 0$, the result follows from Lemma 4.1. ∎

**Theorem 4.3** *Suppose that conditions a1), a2), b1), b2), b3), c1), c2), c3), d1), d2), and d3) hold. Suppose also that*

$$\widetilde{F}(t,0) = 0, \quad \widetilde{G}(t,0) = 0, \quad \text{for all } t \geq 0,$$

*and there exists* $\delta > 0$ *such that*

$$\sum_{k=1}^{\infty} (\widetilde{G}(t,v)e_k, v)_{\mathcal{H}}^2 \geq \delta\, |v|_{\mathcal{H}}^4, \quad \text{for all } v \in \mathcal{H},\ dP \times dt\text{-a.e.}, \tag{39}$$

*and*

$$\widetilde{\alpha} c_2^{-2} + 2\delta - 2L_{\widetilde{F}} - L_{\widetilde{G}}^2 > 0. \tag{40}$$

*Then, there exists a constant $\gamma > 0$ such that for all $u^0 \in L^4(\Omega, \mathcal{F}_0, P; \mathcal{H})$ there is a $T(u^0, \omega) \geq 0$ for which the solution $u(t)$ of (23) with corresponding initial datum $u^0$ satisfies*

$$|u(t,\omega)|_{\mathcal{H}}^2 \leq e^{-\gamma t} \left|u^0(\omega)\right|_{\mathcal{H}}^2 \quad \text{for all } t \geq T(u^0,\omega), \text{ a.s.} \tag{41}$$

**Proof.** Although we could use a heuristic argument which would avoid some technicalities in the proof, we prefer to include all the necessary tools to prove the result in a rigorous way. This requires the use of the exponential martingale inequality, and the consideration of several stopping times.

Let $u^0 \in L^4(\Omega, \mathcal{F}_0, P; \mathcal{H})$ be fixed.

Thanks to Lemma 4.1 we can assume, without lose of generality, that $u^0$ satisfies

$$u^0(\omega) \neq 0, \quad \text{for any } \omega \in \Omega. \tag{42}$$

Otherwise, denoting $\Omega_0 = \left\{\omega \in \Omega;\ u^0(\omega) = 0\right\}$, and taking $\xi \in \mathcal{H} \setminus \{0\}$, we can consider the solution $\widetilde{u}(t)$ of problem of (23) with initial datum $\widetilde{u}^0$ defined as

$$\widetilde{u}^0(\omega) = \begin{cases} u^0(\omega), & \text{if } \omega \in \Omega \setminus \Omega_0, \\ \xi, & \text{if } \omega \in \Omega_0. \end{cases}$$

Then, by Lemma 4.1 and Corollary 4.2, we have

$$u(t,\omega) = \widetilde{u}(t,\omega), \quad \text{for all } t \geq 0, \text{ a.s. in } \Omega \setminus \Omega_0,$$

$$u(t,\omega) = 0, \quad \text{for all } t \geq 0, \text{ a.s. in } \Omega_0,$$

and, consequently, (41) follows from the corresponding inequality for $\widetilde{u}(t)$.

Thus, from now on we suppose that $u^0$ satisfies (42).

Let us define

$$\tau_0(\omega) = \inf\left\{t \geq 0;\ u(t,\omega) = 0\right\},$$

$$v(t) = u(t \wedge \tau_0).$$

From the fact that if $\tau_0(\omega)$ is finite then $u(\tau_0(\omega), \omega) = 0$, it is easy to see that $v(t)$ satisfies the same equation than $u(t)$, and consequently, by uniqueness,

$$v(t) = u(t \wedge \tau_0) = u(t), \quad \text{for all } t \geq 0, \text{ a.s.},$$

i.e., if we denote

$$\widetilde{\Omega} = \left\{\omega \in \Omega;\ \tau_0(\omega) < +\infty\right\}, \tag{43}$$

$$u(t,\omega) = u(\tau_0(\omega), \omega) = 0, \quad \text{for all } t \geq \tau_0(\omega), \text{ a.s. in } \widetilde{\Omega}. \tag{44}$$

Consider now the sequence of $\mathcal{F}_t$-stopping times $\{\tau_n,\ n \geq 1\}$ defined by

$$\tau_n(\omega) = \inf\{t \geq 0;\ |u(t,\omega)|_{\mathcal{H}} \leq \frac{1}{n}\}.$$

This is an increasing sequence, almost surely convergent to $\tau_0$.

Denote

$$v_n(t) = u(t \wedge \tau_n).$$

Then,

$$\begin{aligned} &v_n(t) + \int_0^t 1_{[0,\tau_n]}(s)\widetilde{A}v_n(s)\,ds + \int_0^t 1_{[0,\tau_n]}(s)\widetilde{B}(v_n(s),v_n(s))\,ds \\ &= u^0 + \int_0^t 1_{[0,\tau_n]}(s)\widetilde{F}(s,v_n(s))\,ds + \int_0^t 1_{[0,\tau_n]}(s)\widetilde{G}(s,v_n(s))\,dW(s), \end{aligned}$$

for all $t \geq 0$, and consequently,

$$\begin{aligned} &|v_n(t)|_{\mathcal{H}}^2 + 2\int_0^t 1_{[0,\tau_n]}(s)\langle \widetilde{A}v_n(s), v_n(s)\rangle\,ds \\ &\quad +2\int_0^t 1_{[0,\tau_n]}(s)\langle \widetilde{B}(v_n(s),v_n(s)), v_n(s)\rangle\,ds \\ &\qquad = \left|u^0\right|_{\mathcal{H}}^2 + 2\int_0^t 1_{[0,\tau_n]}(s)\langle \widetilde{F}(s,v_n(s)), v_n(s)\rangle\,ds \\ &\qquad +2\int_0^t \Big(1_{[0,\tau_n]}(s)\widetilde{G}(s,v_n(s))dW(s), v_n(s)\Big) \\ &\qquad + \int_0^t 1_{[0,\tau_n]}(s)\|\widetilde{G}(s,v_n(s))\|_{\mathcal{L}^2(K;\mathcal{H})}^2 ds, \quad \text{for all } t \geq 0. \end{aligned} \tag{45}$$

Let us denote

$$\Omega_n = \left\{\omega \in \Omega;\ |u^0(\omega)|_{\mathcal{H}} \leq \frac{1}{n}\right\}.$$

It is clear that

$$\tau_n(\omega) = 0 \quad \text{and} \quad v_n(t,\omega) = u^0(\omega) \quad \text{if } \omega \in \Omega_n, \tag{46}$$

$$|v_n(t,\omega)|_{\mathcal{H}} \geq \frac{1}{n} \quad \text{if } \omega \in \Omega \setminus \Omega_n. \tag{47}$$

Let us take any function $\phi_n \in C^2(\mathbb{R})$ such that

$$\phi_n(r) = \log r, \quad \text{for all } r \geq \frac{1}{n^2}.$$

Applying the Itô formula to $\phi_n(|v_n(t,\omega)|_{\mathcal{H}}^2)$, and taking into account (45), (46) and (47), we obtain

$$\begin{aligned} \log|v_n(t)|_{\mathcal{H}}^2 &= \log\left|u^0\right|_{\mathcal{H}}^2 + 2\int_0^t 1_{[0,\tau_n]}(s)\frac{\langle -\widetilde{A}v_n(s) + \widetilde{F}(s,v_n(s)), v_n(s)\rangle}{|v_n(s)|_{\mathcal{H}}^2}\,ds \\ &+ \int_0^t 1_{[0,\tau_n]}(s)\frac{\|\widetilde{G}(s,v_n(s))\|_{\mathcal{L}^2(K;\mathcal{H})}^2}{|v_n(s)|_{\mathcal{H}}^2}\,ds + M_n(t) - \frac{1}{2}q_n(t), \quad t \geq 0, \end{aligned} \tag{48}$$

where

$$M_n(t) = 2\int_0^t \left( 1_{[0,\tau_n]}(s)\widetilde{G}(s, v_n(s))\, dW(s), \frac{v_n(s)}{|v_n(s)|_{\mathcal{H}}^2} \right), \tag{49}$$

$$q_n(t) = 4\int_0^t \frac{1_{[0,\tau_n]}(s)}{|v_n(s)|_{\mathcal{H}}^4} \left( \sum_{k=1}^{\infty} (\widetilde{G}(s, v_n(s))e_k, v_n(s))_{\mathcal{H}}^2 \right) ds. \tag{50}$$

Thus $M_n(t)$ is a real continuous square integrable $\mathcal{F}_t$-martingale such that $M_n(0) = 0$, with associated increasing process $\langle M_n \rangle_t = q_n(t)$.

Let us now define

$$M(t) = 2\int_0^{t\wedge\tau_0} \left( \widetilde{G}(s, u(s))\, dW(s), \frac{u(s)}{|u(s)|_{\mathcal{H}}^2} \right),$$

$$q(t) = 4\int_0^{t\wedge\tau_0} \frac{1}{|u(s)|_{\mathcal{H}}^4} \left( \sum_{k=1}^{\infty} (\widetilde{G}(s, u(s))e_k, u(s))_{\mathcal{H}}^2 \right) ds.$$

Observe that, as $u = v_n$ on $[0, \tau_n(\omega)]$, then

$$M(t \wedge \tau_n) = M_n(t), \quad q(t \wedge \tau_n) = q_n(t),$$

$$\lim_{n\to+\infty} M_n(t) = M(t), \quad \lim_{n\to+\infty} q_n(t) = q(t),$$

and consequently, $M(t)$ is a real continuous square integrable local $\mathcal{F}_t$-martingale such that $M(0) = 0$, with associated increasing process $\langle M \rangle_t = q(t)$.

Thus, from the exponential martingale inequality (see [12] and [15]) we have that for any positive numbers $T$, $\varepsilon$ and $k > 0$,

$$P(\max_{t\in[0,T]} (M(t) - \frac{\varepsilon}{2} q(t)) \geq k) \leq e^{-\varepsilon k}. \tag{51}$$

Let us fix $\varepsilon > 0$ such that

$$\lambda := \widetilde{\alpha} c_2^{-2} + 2\delta - 2L_{\widetilde{F}} - L_{\widetilde{G}}^2 - 2\varepsilon L_{\widetilde{G}}^2 > 0, \tag{52}$$

whose existence is guaranteed by (40).

Taking $k = (2/\varepsilon) \log m$, with $m \geq 1$ any integer number, we obtain from (51),

$$P\left( \max_{t\in[0,m+1]} (M(t) - \frac{\varepsilon}{2} q(t)) \geq \frac{2}{\varepsilon} \log m \right) \leq \frac{1}{m^2}. \tag{53}$$

Consequently, we deduce from the Borel-Cantelli lemma that there exists $F_0 \in \mathcal{F}$, with $P(F_0) = 0$, such that for any $\omega \in \Omega \setminus F_0$ there is an integer $m_0(\omega) \geq 1$ for which

$$\max_{t\in[0,m+1]} (M(t,\omega) - \frac{\varepsilon}{2} q(t,\omega)) < \frac{2}{\varepsilon} \log m, \quad \text{for all } m \geq m_0(\omega). \tag{54}$$

In particular, if we take any $\omega \in \Omega \setminus F_0$, $m \geq m_0(\omega)$, and take into account that $q_n(t)$ is increasing, we obtain from (54),

$$\begin{aligned} M_n(t,\omega) - \frac{\varepsilon}{2} q_n(m+1,\omega) &\leq M_n(t,\omega) - \frac{\varepsilon}{2} q_n(t,\omega) \\ &= M(t \wedge \tau_n(\omega),\omega) - \frac{\varepsilon}{2} q(t \wedge \tau_n(\omega),\omega) \\ &\leq \max_{s \in [0,m+1]} (M(s,\omega) - \frac{\varepsilon}{2} q(s,\omega)) \\ &< \frac{2}{\varepsilon} \log m, \end{aligned}$$

for all $n \geq 1$, $t \in [0, m+1]$.

Consequently, for all $\omega \in \Omega \setminus F_0$, there exists $m_0(\omega) \geq 1$ such that

$$M_n(t,\omega) - \frac{\varepsilon}{2} q_n(m+1,\omega) < \frac{2}{\varepsilon} \log m, \tag{55}$$

for all $n \geq 1$, $m \geq m_0(\omega)$, $t \in [0, m+1]$.

From (39), (48), (55) and the hypotheses on $\widetilde{A}$, $\widetilde{F}$ and $\widetilde{G}$, we have that almost surely

$$\begin{aligned} &\log |v_n(t,\omega)|^2_{\mathcal{H}} \\ &\quad \leq \log |u^0(\omega)|^2_{\mathcal{H}} + \int_0^{t \wedge \tau_n(\omega)} (-\widetilde{\alpha} c_2^{-2} + 2L_{\widetilde{F}} + L^2_{\widetilde{G}} - 2\delta)\, ds \\ &\qquad + \frac{2}{\varepsilon} \log m + \frac{\varepsilon}{2} q_n(m+1,\omega), \end{aligned} \tag{56}$$

for all $n \geq 1$, $m \geq m_0(\omega)$, $t \in [0, m+1]$.

But from the expression for $q_n$, we deduce

$$\frac{\varepsilon}{2} q_n(m+1,\omega) \leq 2\varepsilon L^2_{\widetilde{G}} \left((m+1) \wedge \tau_n(\omega)\right).$$

Also observe that if $t \in [m, m+1]$, then $(m+1) \wedge \tau_n(\omega) \leq (t \wedge \tau_n(\omega)) + 1$, and, hence, $\log m \leq \log t$.

Consequently, (52) and (56) imply

$$\log |v_n(t,\omega)|^2_{\mathcal{H}} \leq \log \left|u^0(\omega)\right|^2_{\mathcal{H}} - \lambda \left(t \wedge \tau_n(\omega)\right) + 2\varepsilon L^2_{\widetilde{G}} + \frac{2}{\varepsilon} \log t,$$

or, in other words,

$$|u(t \wedge \tau_n(\omega),\omega)|^2_{\mathcal{H}} \leq |u^0(\omega)|^2_{\mathcal{H}} t^{2/\varepsilon} e^{-\lambda(t \wedge \tau_n(\omega))} e^{2\varepsilon L^2_{\widetilde{G}}}, \tag{57}$$

for all $t \geq m_0(\omega)$, $n \geq 1$,

Letting $n \to +\infty$ in (57), we deduce

$$|u(t \wedge \tau_0(\omega),\omega)|^2_{\mathcal{H}} \leq |u^0(\omega)|^2_{\mathcal{H}} t^{2/\varepsilon} e^{-\lambda(t \wedge \tau_0(\omega))} e^{2\varepsilon L^2_{\widetilde{G}}}, \quad \text{for all } t \geq m_0(\omega),$$

and, in particular, as

$$\lim_{t\to+\infty}(e^{-\frac{\lambda}{2}t}e^{2\varepsilon L^2_{\widetilde{G}}}t^{2/\varepsilon})=0,$$

then there exists $T_0(\omega)\geq m_0(\omega)$ such that

$$|u(t,\omega)|^2_{\mathcal{H}}\leq e^{-\frac{\lambda}{2}t}\left|u^0(\omega)\right|^2_{\mathcal{H}}\quad \text{for all } t\geq T_0(\omega), \text{ a.s. in } \Omega\setminus\widetilde{\Omega}, \tag{58}$$

where $\widetilde{\Omega}$ is defined by (43). Now, (41) follows from (44) and (58). ■

**Remark 4.4** *Observe that this result shows how the noise can produce a stabilising effect on the deterministic problem when the intensity is large enough. Indeed, thanks to Theorem 3.3, we observe that if*

$$\widetilde{\alpha}c_2^{-2}-2L_{\widetilde{F}}>0,$$

*then the trivial stationary solution of the deterministic problem is exponentially stable, but if this condition does not hold, and the noise appearing in the model (or the noise added to the model) is such that*

$$\widetilde{\alpha}c_2^{-2}+2\delta-2L_{\widetilde{F}}-L^2_{\widetilde{G}}>0,$$

*then the trivial solution of the stochastic system becomes pathwise exponentially stable. Furthermore, if we are interested in stabilising the deterministic system and we are allowed to choose appropriate stochastic terms to do this, then we can always use a very simple one, namely, a one dimensional Brownian motion multiplied by a linear operator. From this point on we consider this special noise.*

*We will describe below in more details this stabilisation procedure in the general case of considering a non-trivial stationary solution.*

**Remark 4.5** *Observe that Theorem* 4.3 *affirms for* (1) *that, if* $F(t,0)=0$, $G(t,0)=0$, *and there exists* $\delta>0$ *such that*

$$\sum_{k=1}^{\infty}(G(t,v)e_k,v)^2\geq\delta\|v\|^4,\quad \textit{for all } v\in V,\ dP\times dt\textit{-a.e.},$$

*with*

$$2\nu\alpha\lambda_1^2+2\delta(1+\alpha\lambda_1)>2L_F(1+\alpha\lambda_1)+L_G^2,$$

*then, there exists a constant* $\gamma>0$ *such that for all* $u^0\in L^4(\Omega,\mathcal{F}_0,P;V)$ *there is a* $T(u^0,\omega)\geq 0$ *for which the variational solution* $u(t)$ *of (1) with corresponding initial datum* $u^0$*, satisfies*

$$\|u(t,\omega)\|^2\leq e^{-\gamma t}\|u^0(\omega)\|^2\quad \textit{for all } t\geq T(u^0,\omega),\ \textit{a.s.}$$

Suppose now that $\widetilde{F} : \mathcal{H} \to \mathcal{H}$ is independent of $\omega$ and $t$, and

$$\widetilde{\alpha} > 2c_2^2 L_{\widetilde{F}}. \tag{59}$$

Under this condition, there exists a stationary solution of problem (26), say $u_\infty \in \mathcal{U}$.

If, in addition,

$$\widetilde{\alpha} > c_1 \, |u_\infty|_{\mathcal{H}} + 2c_2^2 L_{\widetilde{F}}, \tag{60}$$

then, as a consequence of Theorem 3.3, taking $\widetilde{G} \equiv 0$, we deduce that $u_\infty$ is exponentially stable, i.e., there exists $\gamma > 0$ such that for all $u^0 \in \mathcal{H}$,

$$|\widetilde{u}(t) - u_\infty|_{\mathcal{H}}^2 \le \left|u^0 - u_\infty\right|_{\mathcal{H}}^2 e^{-\gamma t}, \quad \text{for all } t \ge 0,$$

where $\widetilde{u}(t)$ is the solution of problem (26) with initial datum $u^0$.

However, if the Lipschitz constant $L_{\widetilde{F}}$ is large enough, then (60) might not hold, and we would not know if $u_\infty$ is exponentially stable. Nevertheless, we can always choose a very simple noise to stabilise our problem (26).

To this end, let us consider a fixed $\mathcal{F}_t$-Wiener process $\beta(t)$. For any $\sigma \in \mathbb{R}$, define

$$\widetilde{G}_\sigma(t, v) = \sigma(v - u_\infty), \quad v \in \mathcal{H}, \tag{61}$$

and consider the problem

$$\begin{aligned} u(t) + \int_0^t \widetilde{A}u(s)\, ds + \int_0^t \widetilde{B}(u(s), u(s))\, ds \\ = u^0 + \int_0^t \widetilde{F}(u(s))\, ds + \int_0^t \widetilde{G}_\sigma(s, u(s))\, d\beta(s), \text{ a.s., } \forall\, t \ge 0. \end{aligned} \tag{62}$$

We can prove the following result.

**Theorem 4.6** *Suppose that $\widetilde{F} : \mathcal{H} \to \mathcal{H}$ and satisfies* (59). *Let $u_\infty \in \mathcal{U}$ be a stationary solution of problem* (26), *and suppose that*

$$\widetilde{\alpha} - 2c_1 c_2 \, \|u_\infty\|_{\mathcal{U}} > 0. \tag{63}$$

*Let $\sigma$ be any real number such that*

$$\widetilde{\alpha} - 2c_1 c_2 \, \|u_\infty\|_{\mathcal{U}} + c_2^2 \sigma^2 > 2c_2^2 L_{\widetilde{F}}. \tag{64}$$

*Then, there exists $\gamma > 0$ satisfying that for any $u^0 \in L^4(\Omega, \mathcal{F}_0, P; \mathcal{H})$, there exists $T(u^0, \omega) \ge 0$ such that*

$$|u(t,\omega) - u_\infty|_{\mathcal{H}}^2 \le e^{-\gamma t} |u^0(\omega) - u_\infty|_{\mathcal{H}}^2, \quad \textit{for all } t \ge T(u^0, \omega), \textit{ a.s.}, \tag{65}$$

*where $u(t,\omega)$ is the corresponding solution of* (62).

**Proof.** The proof is similar to that of Theorem 4.3 but arguing with $u^0 - u_\infty$ and $u(t) - u_\infty$ instead of $u^0$ and $u(t)$ respectively, and taking into account that the special form of the noise allows us to finish the proof either using the subexponential decay of the Wiener process instead of the exponential martingale inequality, or repeating the analysis already done in that proof (see [13] for more details). ∎

**Remark 4.7** *As for problem* (1), *Theorem 4.6 states that, under conditions*

$$\nu\alpha > c_2^2 L_F, \qquad \nu\alpha\lambda_1 > c_1\sqrt{1+\alpha\lambda_1}|Au_\infty|,$$

*for a suitable choice of* $\sigma$, *there exists* $\gamma > 0$ *satisfying that for any* $u^0 \in L^4(\Omega, \mathcal{F}_0, P; V)$, *there exists* $T(u^0, \omega) \geq 0$ *such that*

$$\|u(t,\omega) - u_\infty\|^2 \leq e^{-\gamma t}\|u^0(\omega) - u_\infty\|^2, \quad \textit{for all } t \geq T(u^0,\omega), \textit{ a.s.},$$

*where* $u(t,\omega)$ *is the corresponding variational solution of* (1), *with* $G(t,u)\dot{W}(t) = \sigma(u - u_\infty)\dot{\beta}(t)$.

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