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ON DIFFERENTIAL EQUATIONS WITH DELAY IN BANACH SPACES AND ATTRACTORS FOR RETARDED LATTICE DYNAMICAL SYSTEMS

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Abstract. In this paper we first prove a rather general theorem about existence of solutions for an abstract differential equation in a Banach space by assuming that the nonlinear term is in some sense weakly continuous.

We then apply this result to a lattice dynamical system with delay, proving also the existence of a global compact attractor for such system.

1. **Introduction.** Lattice differential equations arise naturally in a wide variety of applications where the spatial structure possesses a discrete character. These systems are used to model, for instance, cellular neural networks with applications to image processing, pattern recognition, and brain science [18, 19, 20, 21]. They are also used to model the propagation of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites (see for example, [8], [9], [41], [40], [30, 31]). Lattice differential equations can be found in chemical reaction theory [23, 28, 33] as well. Also, it can appear after a spatial discretization of a differential equation, as it is the case we are interested in the present paper.

Recently, there have been published many works on deterministic lattice dynamical systems. For traveling waves, we refer the readers to [14, 36, 15, 54, 1, 5] and the references therein. The chaotic properties of solutions for such systems have been investigated by [14] and [17, 42, 16, 22]. The existence and properties of the global attractor for lattice differential equations have been established, for example, in [2], [7], [10], [38], [39], [44], [45], [51], [52], [53].

Also, one can find several papers considering stochastic versions of lattice dynamical systems (see, e.g., [6], [11], [12] [13], [26], [27], [35], [49], [46]).

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On the other hand, the consideration of some kind of delay, memory or retarded terms in the models are a sensible fact, as they are present in many aspects of real models (e.g. in control problems). Therefore, our main aim in this paper is to analyze the existence (and eventually the uniqueness) of solutions and their asymptotic behavior of the following retarded lattice differential equation

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + f_i\left(u_{it}\right) = 0,\ t > 0,\ i \in \mathbb{Z}, \\ \\ u_i\left(s\right) = \psi_i\left(s\right),\ \forall s \in [-h, 0], \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$, which is the discretization of the following scalar retarded reaction-diffusion equation:

$$\begin{cases} \dfrac{du}{dt} - \dfrac{\partial^2 u}{\partial x^2} + \lambda u + f(u_t) = 0,\ t > 0, x \in \mathbb{R}, \\ \\ u\left(s\right) = \psi\left(s\right), \forall s \in [-h, 0]. \end{cases}$$

Here $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $\mathbb{Z}$ denotes the integers set and for a continuous function $u : [-h, T] \to Y$ (where Y is some space), u_t denotes the segment of the solution, i.e., the element in $C\left([-h, 0], Y\right)$ defined by $u_t\left(s\right) = u\left(t + s\right)$, $s \in [-h, 0]$.

Problem (1) has been considered in [48] under some kind of Lipschitz assumption (even under integral formulation). However, those are not stated in a clear way (in our opinion) and we do not even see how the existence of solution of the lattice system can be proved using those assumptions, by following the scheme carried out by the authors. Subsequently, the same kind of assumptions have been used in other papers (see, e.g. [46], [50], [47]).

In the present paper, we impose some general assumptions (only some continuity assumptions and growth conditions on the term containing the delay) and prove the existence of solutions of our problem, and additionally the uniqueness when we also assume a local one-sided Lipschitz hypothesis. The asymptotic behavior is also analyzed by proving the existence of a global attractor, even when our problem generates a set-valued or multi-valued dynamical system due to the lack of uniqueness of solutions of our model.

To do this, we first prove some general abstract results on the existence of solutions for a differential equation with delays in a Banach space (see Section 2). We prove that if the nonlinear function containing the delay is weakly sequentially continuous in bounded sets, then at least one local solution exists for every initial data in a suitable space. This result generalizes a previous one [29] for the case of differential equation in Banach spaces without delay. As far as we know, in other papers (see e.g. [25], [34], [43]) some extra assumptions are considered (as for example compactness conditions).

Next, in Section 3 we apply this general theory to our model (1) under rather general assumptions of the nonlinear term f.

Finally, we analyze in Section 4 the particular case of a lattice dynamical system with a nonlinear term of the form

$$f_i(u_{it}) = F_{0,i}\left(u_i\left(t\right)\right) + F_{1,i}\left(u_i\left(t - h_1\right)\right) + \int_{-h}^{0} b_i\left(s, u_i\left(t + s\right)\right) ds,$$

with $0 < h_1 \leq h$. Under some dissipative and sublinear growth conditions, we define for this problem a multivalued semiflow and prove the existence of a global compact attractor. Additionally, with extra Lipschitz conditions we obtain uniqueness of the Cauchy problem, so that the semiflow is in fact a semigroup of operators.

2. **Existence and uniqueness of solutions for differential equations with delays in Banach spaces.** Let E be a real Banach space with its dual E^* and let $E_0 = C([-h,0],E)$, with norms $\|\cdot\|$, $\|\cdot\|_*$ and $\|\cdot\|_{E_0}$, respectively, where $\|\varphi\|_{E_0} = \max_{t\in[-h,0]}\|\varphi(t)\|$. Also,

$$B_X(y_0,r) = \{y \in X : \|y-y_0\|_X \le r\},$$

where $X = E$ or E_0, and $(\cdot,\cdot)$ will denote pairing between E and E^*.
Let us consider the following Cauchy problem for a functional differential equation in a Banach space:

$$\begin{cases} \dfrac{du}{dt} = f(t,u_t), \\ \\ u_0 = \psi \in E_0, \end{cases} \tag{2}$$

where $f : [0,\infty)\times E_0 \to E$. Also, for any $u \in C([-h,+\infty),E)$, the function $u_t \in E_0$, $t \ge 0$, is defined by $u_t(s) = u(t+s)$, $s \in [-h,0]$.
Let E_w be the space E endowed with the weak topology. We consider the space $E_{0,w} = C([-h,0],E_w)$. We say that $u_n \to u \in E_{0,w}$ in $E_{0,w}$ if

$$u_n(s_n) \to u(s) \text{ in } E_w \text{ for all } s_n \to s \in [-h,0].$$

We will say that the function f is sequentially weakly continuous in bounded sets if $t_n \to t$, $u_n \to u$ in $E_{0,w}$ and $\|u_n\|_{E_0} \le M$, for all n, imply $f(t_n,u_n) \to f(t,u)$ in E_w.
On the other hand, we will say that the function f is bounded if it maps bounded subsets of $[0,\infty)\times E_0$ onto bounded subsets of E.

Definition 1. *The map $u : [-h,T] \to E$ is called a solution of problem (2) if $u_0 = \psi$, $u(\cdot)$ is continuous, once weakly continuously differentiable in $[0,T]$ and satisfies*

$$u(t) = u(0) + \int_0^t f(s,u_s)\,ds, \quad \textit{for all } t \in [0,T].$$

Remark 2. *It follows from this definition that for any solution u of (2), the map $t \mapsto u_t \in E_0$ is continuous.*

Remark 3. *We note that if $f : [0,\infty)\times E_0 \to E$ is sequentially weakly continuous in bounded sets and the map $t \mapsto u_t \in E_0$ is continuous, then $t \mapsto f(t,u_t)$ is weakly continuous, hence weakly measurable. If E is separable, we obtain that $t \mapsto f(t,u_t)$ is strongly measurable. If we assume, moreover, that the map f is bounded, then we have that $f(\cdot,u_\cdot) \in L^1(0,T;E)$.*
If $f : [0,\infty)\times E_0 \to E$ and $t \mapsto u_t \in E_0$ are continuous, then the map $t \mapsto f(t,u_t)$ is continuous, hence strongly measurable. If we assume, moreover, that the map f is bounded, then we have that $f(\cdot,u_\cdot) \in L^1(0,T;E)$.

We shall obtain now the existence of solutions for problem (2).

Theorem 4. *Assume that E is reflexive and separable. Let $f : [0,\infty)\times E_0 \to E$ be sequentially weakly continuous in bounded sets, and let f be a bounded map. Then, for each $r > 0$, there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, then*

problem (2) possesses at least one solution defined on $[0, a(r)]$. *Moreover,* $u(\cdot)$ *is a.e. differentiable and* $\dfrac{du}{dt} = f(t, u_t)$ *for a.a.* $t \in (0, a(r))$.
If we assume additionally that $f : [0, \infty) \times E_0 \to E$ *is continuous, then* $u \in C^1([0, a]; E)$ *and the separability of* E *is not needed.*

Proof. Since f is bounded, for any $r > 0$ there is $M(r)$ such that

$$\|f(t, v)\| \leq M \text{ for all } t \in [0, 1], v \in B_{E_0}(0, r).$$

Define $a(r) = \min\{1, r/M(2r)\}$. For any n we take a partition of the interval $[0, a]$:

$$\Delta_n : 0 = t_0^n < t_1^n < ... < t_{N(n)}^n = a,$$

where

$$|\Delta_n| = \max_{1 \leq k \leq N} \{|t_k^n - t_{k-1}^n|\} \to 0 \text{ as } n \to \infty.$$

For every n we define inductively an approximation sequence by Euler's method:

$$u^n(s) = \psi(s) \text{ for } s \in [-h, 0],$$
$$u^n(t) = u^n(t_k^n) + (t - t_k^n) f\left(t_k^n, u_{t_k^n}^n\right) \text{ for } t_k^n \leq t \leq t_{k+1}^n, 0 \leq k \leq N-1.$$

Let $t, s \in [0, t_1^n]$. Then

$$\|u^n(t) - u^n(s)\| = \|(t - s) f(0, \psi)\| \leq |t - s| M(2r) \tag{3}$$

and, in particular,

$$\|u^n(t) - u^n(0)\| = \|t f(0, \psi)\| \leq |t| M(2r) \leq r, \tag{4}$$

so that

$$\|u^n(t)\| \leq 2r. \tag{5}$$

We shall prove (3)-(5) for all $t, s \in [0, a(r)]$. Assume that these properties are satisfied for $t, s \in [0, t_{k-1}^n]$. Then, if $t \in [t_{k-1}^n, t_k^n]$, we have

$$u^n(t) - u^n(0) = \left(t - t_{k-1}^n\right) f\left(t_{k-1}^n, u_{t_{k-1}^n}^n\right) + \sum_{i=0}^{k-2} \left(t_{i+1}^n - t_i^n\right) f\left(t_i^n, u_{t_i^n}^n\right),$$

$$\|u^n(t) - u^n(0)\| \leq t M(2r) \leq r,$$

so that

$$\|u^n(t)\| \leq 2r. \tag{6}$$

Now, for $t, s \in [t_{k-1}^n, t_k^n]$, we obtain

$$\|u^n(t) - u^n(s)\| = \left\|(t - s) f\left(t_{k-1}^n, u_{t_{k-1}^n}^n\right)\right\| \leq |t - s| M(2r).$$

Finally, if $t \in [t_{k-1}^n, t_k^n]$, $s \in [0, t_{k-1}^n]$, then

$$\begin{aligned} \|u^n(t) - u^n(s)\| &= \left\|u\left(t_{k-1}^n\right) + \left(t - t_{k-1}^n\right) f\left(t_{k-1}^n, u_{t_{k-1}^n}^n\right) - u^n(s)\right\| \\ &\leq \left(\left|t - t_{k-1}^n\right| + \left|t_{k-1}^n - s\right|\right) M(2r) = |t - s| M(2r). \end{aligned} \tag{7}$$

Hence, the result follows.
Since E is reflexive, from (6) we deduce that every sequence $\{u^n(t)\}$ is relatively compact in E_w. It follows the existence of a continuous function $u(\cdot)$ and a subsequence of $\{u^n(\cdot)\}$ (denoted again u^n) such that $u^n(t) \to u(t)$ in E_w for all $t \in [0, a]$. Indeed, using the diagonal method one can choose a subsequence of $\{u^n(\cdot)\}$ and a

function $u:\mathbb{Q}\cap[0,a]\rightarrow E$ such that $u^{n}(t)\rightarrow u(t)$ in E_{w} for any rational number $t\in\mathbb{Q}$. Since

$$\|u(t)-u(s)\|\leq\liminf\ \|u^{n}(t)-u^{n}(s)\|\leq|t-s|\,M(2r)\,,\text{ for all }t,s\in\mathbb{Q}\cap[0,a],$$

the function u can be extended to a continuous function (denoted again $u:[0,a]\rightarrow E$) such that

$$\|u(t)-u(s)\|\leq|t-s|\,M(2r)\,,\text{ for all }t,s\in[0,a]. \tag{8}$$

We shall prove that $u^{n}(t_{0})\rightarrow u(t_{0})$ in E_{w}. Indeed, for any $t_{0}\in[0,a]\backslash\mathbb{Q}$, $v\in E^{*}$ we have

$$\begin{aligned}(u^{n}(t_{0})-u(t_{0}),v)&=(u^{n}(t_{0})-u^{n}(t_{m}),v)+(u^{n}(t_{m})-u(t_{m}),v)\\&\quad+(u(t_{m})-u(t_{0}),v)\,,\end{aligned}$$

where $t_{m}\in\mathbb{Q}$ are such that $t_{m}\rightarrow t_{0}$. For any $\varepsilon>0$ there exist $m(\varepsilon)$ and $N(m(\varepsilon),\varepsilon)$ such that

$$\begin{aligned}|(u^{n}(t_{0})-u^{n}(t_{m}),v)|&\leq\|u^{n}(t_{0})-u^{n}(t_{m})\|\,\|v\|<\frac{\varepsilon}{3},\\|(u(t_{m})-u(t_{0}),v)|&\leq\|u(t_{m})-u(t_{0})\|\,\|v\|<\frac{\varepsilon}{3},\\|(u^{n}(t_{m})-u(t_{m}),v)|&<\frac{\varepsilon}{3}.\end{aligned}$$

Thus, $|(u^{n}(t_{0})-u(t_{0}),v)|<\varepsilon$, so that $u^{n}(t_{0})\rightarrow u(t_{0})$ in E_{w}. In fact, we have that

$$u^{n}(t_{n})\rightarrow u(t_{0})\ \text{ in }E_{w}\text{ if }t_{n}\rightarrow t_{0},$$

which follows by a similar argument from the equality

$$(u^{n}(t_{n})-u(t_{0}),v)=(u^{n}(t_{n})-u^{n}(t_{0}),v)+(u^{n}(t_{0})-u(t_{0}),v)\,.$$

Thus

$$u_{t_{n}}^{n}(\tau_{n})=u^{n}(t_{n}+\tau_{n})\rightarrow u(t+\tau)=u_{t}(\tau)\,,$$

for $t_{n}\rightarrow t\in[0,a],\tau_{n}\rightarrow\tau\in[-h,0]$, implies that

$$u_{t_{n}}^{n}\rightarrow u_{t}\text{ in }E_{0,w}\text{ if }t_{n}\rightarrow t\in[0,a].$$

It remains to show that $u(\cdot)$ is a solution of (2). For this aim we will pass to the limit in the integral

$$u^{n}(t)=u^{n}(0)+\int_{0}^{t}f_{\Delta_{n}}(\tau)\,d\tau,\ t\in[0,a],$$

where

$$f_{\Delta_{n}}(\tau)=f\left(t_{i}^{n},u_{t_{i}^{n}}^{n}\right)\text{ for }\tau\in[t_{i}^{n},t_{i+1}^{n}),\ 0\leq i\leq N-1.$$

Since f is sequentially weakly continuous in bounded sets, for any $\tau\in[0,t]$ we have

$$f_{\Delta_{n}}(\tau)=f\left(t_{i}^{n},u_{t_{i}^{n}}^{n}\right)\rightarrow f(\tau,u_{\tau}^{n})\text{ in }E_{w}.$$

Then by $\|f_{\Delta_{n}}(\tau)\|\leq M(2r)$ and Lebesgue's theorem we obtain for any $v\in E^{*}$ that

$$\left(\int_{0}^{t}f_{\Delta_{n}}(\tau)\,d\tau,v\right)=\int_{0}^{t}(f_{\Delta_{n}}(\tau),v)\,d\tau\rightarrow\int_{0}^{t}(f(\tau,u_{\tau}),v)\,d\tau=\left(\int_{0}^{t}f(\tau,u_{\tau})\,d\tau,v\right),$$

and then

$$(u(t),v)=(u(0),v)+\left(\int_{0}^{t}f(\tau,u_{\tau})\,d\tau,v\right),$$

where we have used that $f(\cdot, u_\cdot) \in L^1(0,T;E)$ (see Remark 3). As $v \in E^*$ is arbitrary, we get the equality

$$u(t) = u(0) + \int_0^t f(\tau, u_\tau)\, d\tau \text{ for all } t \in [0,a].$$

This implies that

$$\frac{du}{dt} = f(t, u_t)$$

in the distribution sense, that is, for any $\phi \in C_0^\infty(0,a)$,

$$\int_0^a u(\tau)\phi'(\tau)\, d\tau = -\int_0^a f(\tau, u_\tau)\phi(\tau)\, d\tau,$$

and also that $g(t) = f(t, u_t)$ is the weak derivative of u, that is,

$$\frac{d}{dt}(u, v) = (g, v), \text{ for all } v \in E^*,$$

in the scalar distribution sense on $(0,a)$. Since $t \mapsto f(t, u_t)$ is weakly continuous, u is weakly continuously differentiable.

Also, since u is absolutely continuous on $[0,a]$ and $\frac{du}{dt} \in L^1(0,a;E)$, we obtain that u is a.e. differentiable and $\frac{du}{dt} = f(t, u_t)$ for a.a. $t \in (0,a)$.

Finally, if $f : [0,\infty) \times E_0 \to E$ is continuous, then $t \mapsto f(t, u_t)$ is continuous, so that $u \in C^1([-h,\infty);E)$. □

If the space E is not assumed to be reflexive, we need to assume an extra compactness condition on f.

Theorem 5. *Assume that E is separable. Let $f : [0,\infty) \times E_0 \to E$ be sequentially weakly continuous in bounded sets. Assume that $f([0,T] \times B_{E_0}(0,r))$ is relatively compact in E_w for any $T, r > 0$. Then, for each $r > 0$, there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, problem (2) has at least one solution defined on $[0, a(r)]$. Moreover, $u(\cdot)$ is a.e. differentiable and $\frac{du}{dt} = f(t, u_t)$ for a.a. $t \in (0, a(r))$.*

If we additionally assume that $f : [0,\infty) \times E_0 \to E$ is continuous, then $u \in C^1([0,a];E)$ and the separability of E is not needed.

Proof. Since $f([0,T] \times B_{E_0}(0,r))$ is relatively compact in E_w, the map f is bounded. Then for any $R > 0$ there is $M(R)$ such that

$$\|f(t,v)\| \le M \text{ for all } t \in [0,1], v \in B_{E_0}(0,R).$$

We define $a(r) = \min\{1, r/M(2r)\}$ and exactly the same approximation sequence $\{u^n\}$ as in the proof of Theorem 4, which satisfies (6), (7). Also, we have that

$$\frac{u^n(t) - u(0)}{t} = f(0,\psi) \text{ if } t \in (0, t_1^n],$$

$$\frac{u^n(t) - u(0)}{t} = \frac{t - t_k^n}{t} f\left(t_k^n, u_{t_k^n}^n\right) + \sum_{i=0}^{k-1} \frac{t_{i+1}^n - t_i^n}{t} f\left(t_i^n, u_{t_i^n}^n\right) \text{ if } t \in [t_k^n, t_{k+1}^n],\ k \ge 1.$$

Hence, using (6) we have $\frac{u^n(t) - u(0)}{t} \in co(f([0,a] \times B_{E_0}(0,2r))$, where $co(B)$ is the convex hull of the set B. By the Krein-Smulian theorem $co(f([0,a] \times B_{E_0}(0,2r))$

is relatively compact in E_w. Hence, the sequence $\left\{\dfrac{u^n(t)-u(0)}{t}\right\}$ contains a subsequence converging in E_w, and then the same convergence property is satisfied by $\{u^n(t)\}$. Arguing as in the proof of Theorem 4 we obtain the existence of a continuous function $u(\cdot)$ satisfying (8) and a subsequence of $\{u^n(\cdot)\}$ such that $u^n(t_n) \to u(t_0)$ in E_w if $t_n \to t_0$. Also, exactly in the same way it is proved that $u(\cdot)$ is a solution of (2), and the additional regularity properties, as well. □

Theorem 6. *Assume either the conditions of Theorem 4 or 5. If a solution $u(\cdot)$ of (2) has a maximal interval of existence $[0,b)$ and there exists $K>0$ such that $\|u(t)\| \leq K$, for all $t \in [0,b)$, then $b=+\infty$, that is, $u(\cdot)$ is a globally defined solution.*

Proof. Since the map f is bounded, from the definition of solution it follows that the function $u(\cdot)$ is uniformly continuous on $[0,b)$. Hence, the limit $\lim_{t\to b^-} u(t) = u^*$ exists. Then, using the initial condition

$$\psi^*(s) = \begin{cases} u^* \text{ if } s=0, \\ u(s+b) \text{ if } s \in [-h,0), \end{cases}$$

and either Theorem 4 or 5 we obtain that the solution $u(\cdot)$ can be extended to the interval $[0,b+\alpha)$, $\alpha>0$, which is a contradiction. □

Let $J : E \to 2^{E^*}$ be the duality map, i.e. $J(y) = \{\xi \in E^* \mid (y,\xi) = \|y\|^2 = \|\xi\|_*^2\}$, $\forall y \in E$. We will prove two results concerning uniqueness of solutions.

Theorem 7. *Assume either the conditions of Theorem 4 or 5. Also, suppose that*

$$(f(t,v) - f(t,w), j) \leq \beta(t) \|v-w\|_{E_0}^2,$$

for all $j \in J(v(0)-w(0))$, $v,w \in E_0$ and a.a. $t \in (0,\infty)$, where $\beta \in L^1_{loc}(0,\infty)$, $\beta(t) \geq 0$. Then, for every $\psi \in E_0$, problem (2) possesses a unique solution $u(\cdot)$ defined on $[0,\infty)$.

Proof. By either Theorem 4 or 5 there exists at least one solution defined in some maximal interval $[0,\alpha)$. We will show that for this solution $\alpha=\infty$.
Since $\dfrac{du}{dt}$ exists for a.a. $t \in (0,\alpha)$, Lemma 1.2 in [4, p.100] implies that

$$\|u(t)\| \frac{d}{dt}\|u(t)\| = \left(\frac{d}{dt}u(t), j\right) = (f(t,u_t), j) \text{ for all } j \in J(u(t)) \text{ and a.a. } t.$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= \|u(t)\| \frac{d}{dt}\|u(t)\| = (f(t,u_t) - f(t,0), j) + (f(t,0), j) \\ &\leq \beta(t)\|u_t\|_{E_0}^2 + \frac{1}{4}\|f(t,0)\|^2 + \|u(t)\|^2 \\ &\leq (\beta(t)+1)\|u_t\|_{E_0}^2 + \frac{1}{4}\|f(t,0)\|^2.\end{aligned}$$

Thus,

$$\|u(t)\|^2 \leq \|u(0)\|^2 + \int_0^t \frac{1}{2}\|f(s,0)\|^2 ds + 2\int_0^t (\beta(s)+1)\|u_s\|_{E_0}^2 ds,$$

and

$$\|u_t\|_{E_0}^2 \leq \|\psi\|_{E_0}^2 + \int_0^t \frac{1}{2} \|f(s,0)\|^2 \, ds + 2\int_0^t (\beta(s)+1) \|u_s\|_{E_0}^2 \, ds.$$

Denote $C(t) = \|\psi\|_{E_0}^2 + \int_0^t \frac{1}{2} \|f(s,0)\|^2 \, ds$, $\gamma(t) = 2(\beta(t)+1)$. By Gronwall's lemma,

$$\|u_t\|_{E_0}^2 \leq C(t) + \int_0^t \gamma(s) C(s) e^{\int_s^t \gamma(s)ds} ds \leq K \text{ for all } t \in [0,\alpha).$$

Therefore, Theorem 6 implies that $\alpha = +\infty$.
We shall prove that this solution is unique. If $u(\cdot), v(\cdot)$ are two solutions with the initial data ψ, then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|u(t) - v(t)\|^2 &= \|u(t) - v(t)\| \frac{d}{dt} \|u(t) - v(t)\| \\ &= (f(t,u_t) - f(t,v_t), j) \leq \beta(t) \|u_t - v_t\|_{E_0}^2, \end{aligned}$$

where $j \in J(u(t) - v(t))$. Thus,

$$\|u(t) - v(t)\|^2 \leq \int_0^t 2\beta(s) \|u_s - v_s\|_{E_0}^2 \, ds,$$

and then

$$\|u_t - v_t\|_{E_0}^2 \leq \int_0^t 2\beta(s) \|u_s - v_s\|_{E_0}^2 \, ds \text{ for all } t \geq 0.$$

Again by Gronwall's lemma we have that $u \equiv v$. □

Theorem 8. *Assume either the hypotheses of Theorem 4 or 5. Also, suppose that, for any $M > 0$, there exists $\beta(\cdot, M) \in L^1_{loc}(0,\infty)$ such that $\beta(t,M) \geq 0$ for a.a. $t \in (0,\infty)$, and the following inequality holds:*

$$(f(t,v) - f(t,w), j) \leq \beta(t,M) \|v - w\|_{E_0}^2, \tag{9}$$

for all $j \in J(v(0) - w(0))$, and all $v, w \in E_0$ with $\|v\|_{E_0}, \|u\|_{E_0} \leq M$, and a.a. $t \in (0,\infty)$. Then, for each $r > 0$, there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \leq r$, problem (2) has a unique solution defined on $[0, a(r)]$.

Proof. We know by Theorems 4 or 5 that there exists at least one solution defined in $[0, a(r)]$. Suppose that we have two different solutions u, v defined in $[0, a(r)]$. Then, arguing as in the proof of the previous theorem, we have

$$\frac{1}{2}\frac{d}{dt} \|u(t) - v(t)\|^2 \leq \beta(t,M) \|u_t - v_t\|_{E_0}^2, \tag{10}$$

where $M > 0$ is such that $\|u_t\|_{E_0}, \|v_t\|_{E_0} \leq M$ for all $t \in [0, a(r)]$. Then, using Gronwall's lemma we obtain that $u \equiv v$. □

3. **Lattice dynamical systems with delay: setting of the problem.** Consider the following first order lattice dynamical system with finite delay

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + f_i(u_{it}) = 0, \ t > 0, \ i \in \mathbb{Z}, \\ \\ u_i(s) = \psi_i(s), \ \forall s \in [-h, 0], \end{cases} \tag{11}$$

where $\lambda \in \mathbb{R}$, which is the discretization of the following scalar retarded reaction-diffusion equation:

$$\begin{cases} \dfrac{du}{dt} - \dfrac{\partial^2 u}{\partial x^2} + \lambda u + f(u_t) = 0, \ t > 0, x \in \mathbb{R}, \\ \\ u(s) = \psi(s), \forall s \in [-h, 0]. \end{cases}$$

We consider the separable Hilbert space $\ell^2 = \{v = (v_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} v_i^2 < \infty\}$ with norm $\|v\| = \sqrt{\sum_{i\in\mathbb{Z}} v_i^2}$ and scalar product $(w, v) = \sum_{i\in\mathbb{Z}} w_i v_i$, and also the Banach space $\ell^\infty = \{v = (v_i)_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |v_i| < \infty\}$ with norm $\|v\|_\infty = \sup_{i\in\mathbb{Z}} |v_i|$.
Further, we shall use the notation $E = \ell^2$, $E_0 = C\left([-h, 0], \ell^2\right)$, $E_1 = C([-h, 0], \mathbb{R})$, with the norms $\|u\|_{E_0} = \max_{s\in[-h,0]} \|u(s)\|$, $\|u\|_{E_1} = \max_{s\in[-h,0]} |u(s)|$. Also, put $E_\infty = C([-h, 0], \ell^\infty)$ with norm $\|u\|_{E_\infty} = \max_{s\in[-h,0]} \|u(s)\|_\infty$. We note that $E_0 \subset E_\infty$, as

$$\begin{aligned} \|u(t) - u(s)\|_\infty &= \sup_{i\in\mathbb{Z}} |u_i(t) - u_i(s)| \leq \sqrt{\sum_{i\in\mathbb{Z}} |u_i(t) - u_i(s)|^2} \\ &= \|u(t) - u(s)\|, \ \forall t, s \in [-h, 0], \end{aligned}$$

and

$$\|u\|_{E_\infty} = \max_{s\in[-h,0]} \sup_{i\in\mathbb{Z}} |u_i| \leq \max_{s\in[-h,0]} \sqrt{\sum_{i\in\mathbb{Z}} |u_i|^2} = \|u\|_{E_0}.$$

We consider the following conditions:

- $(H1)$ The operator $f : E_0 \to E$ given by the rule $(f(v))_i = f_i(v_i)$, $i \in \mathbb{Z}$, is well defined and bounded.
- $(H2)$ The maps $f_i : C([-h, 0], \mathbb{R}) \to \mathbb{R}$ are continuous.

We shall first prove the existence of solutions for problem (11). For this aim we shall rewrite it in abstract form. We define the operator $A : E \to E$ by

$$(Av)_i := -v_{i-1} + 2v_i - v_{i+1}, \ i \in \mathbb{Z}.$$

Also, we define the operators B, $B^* : E \to E$ by

$$(Bv)_i := v_{i+1} - v_i, \ \ (B^* v)_i := v_{i-1} - v_i.$$

It is easy to check that

$$\begin{aligned} A = B^* B = BB^*, \\ (B^* w, v) = (w, Bv). \end{aligned}$$

Then the operator $F : E_0 \to E$ is defined by

$$F(v) = -Av(0) - f(v) - \lambda v(0)$$

and (11) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} = F\left(u_t\right), \ t > 0, \\ \\ u\left(s\right) = \psi\left(s\right), \ \forall s \in [-h, 0]. \end{cases} \tag{12}$$

Lemma 9. *Let (H1)-(H2) hold. Then the map $f : E_0 \to E$ is sequentially weakly continuous in bounded sets. Also, the map $A : E \to E$ is weakly continuous.*

Proof. Let $v^n \to v \in E_{0,w}$, $\|v^n\|_{E_0} \le M_1$ for all n, and let $w \in \ell^2$ be arbitrary. For any $\varepsilon > 0$ we take $K_0\left(\varepsilon\right) > 0$ such that $\sum_{|i|\ge K_0} |w_i|^2 < \varepsilon$. Since f is bounded, there exists $M_2 > 0$ such that $\|f\left(v^n\right)\| \le M_2$, $\|f\left(v\right)\| \le M_2$, for all n. Also, as $v_i^n \to v_i$ in $C([-h,0],\mathbb{R})$, for all i, (H2) imply the existence of $N(K_0,\varepsilon)$ such that $\sum_{|i|<K_0} |f_i\left(v_i^n\right) - f_i\left(v_i\right)|^2 < \varepsilon^2$ if $n \ge N$. Hence,

$$\begin{aligned} |(f\left(v^n\right) - f\left(v\right), w)| &\le \sqrt{\sum_{|i|<K_0} |f_i\left(v_i^n\right) - f_i\left(v_i\right)|^2}\, \|w\| \\ &\quad + (\|f\left(v\right)\| + \|f\left(v^n\right)\|) \sqrt{\sum_{|i|\ge K_0} |w_i|^2} \\ &\le \varepsilon \|w\| + 2M_2\varepsilon. \end{aligned}$$

The result for the operator A can be proved similarly. This completes the proof. □

Theorem 10. *Let (H1)-(H2) hold. Then for each $r > 0$ there exists $a\left(r\right) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, then problem (11) has at least one solution defined on $[0, a\left(r\right)]$. Moreover, $u\left(\cdot\right)$ is a.e. differentiable and $\dfrac{du}{dt} = F\left(u_t\right)$ for a.a. $t \in (0, a(r))$.*

Proof. Lemma 9 implies that the operator F is sequentially weakly continuous in bounded sets. Since f is bounded, F is also bounded. The result follows from Theorem 4. □

In order to obtain that the map f is continuous, we need an assumption which is stronger than (H1).

$(H3)$ The operator $f : E_0 \to E$ given by the rule $\left(f\left(v\right)\right)_i = f_i\left(v_i\right)$, $i \in \mathbb{Z}$, is well defined, and for any $v \in E_0$, we have

$$\sum_{|i|\ge K} |f_i\left(v_i\right)|^2 \le C\left(\|v\|_{E_0}\right) \left(\max_{s\in[-h,0]} \sum_{|i|\ge K} v_i^2\left(s\right) + b_K \right), \text{ for all } K \in \mathbb{Z}^+,$$

where $b_K \to 0^+$ as $K \to \infty$, and $C\left(\cdot\right) \ge 0$ is a continuous non-decreasing function.

Remark 11. *Condition (H3) implies that the map f is bounded.*

Lemma 12. *Let (H2)-(H3) hold. Then, the map $f : E_0 \to E$ is continuous.*

Proof. Let $v^n \to v$ in E_0. Then for any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$\max_{s\in[-h,0]} \sum_{|i|\geq K} |v_i^n(s)|^2 < \varepsilon, \quad \max_{s\in[-h,0]} \sum_{|i|\geq K} |v_i(s)|^2 < \varepsilon.$$

Then by (H3) one can choose $K_1(\varepsilon) \geq K(\varepsilon)$ such that

$$\sum_{|i|\geq K} |f_i(v_i^n)|^2 \leq R\varepsilon, \quad \sum_{|i|\geq K} |f_i(v_i)|^2 \leq R\varepsilon,$$

for some $R > 0$. On the other hand, by (H2) we obtain the existence of $N(\varepsilon, K)$ such that

$$\sum_{|i|<K_1} |f_i(v_i^n) - f_i(v_i)|^2 < \varepsilon \text{ if } n \geq N.$$

Thus,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |f_i(v_i^n) - f_i(v_i)|^2 &\leq \sum_{|i|<K_1} |f_i(v_i^n) - f_i(v_i)|^2 \\ &\quad + 2\sum_{|i|\geq K_1} |f_i(v_i^n)|^2 + 2\sum_{|i|\geq K_1} |f_i(v_i)|^2 \\ &\leq \varepsilon + 2R\varepsilon, \text{ if } n \geq N.\end{aligned}$$

□

Corollary 13. *Under conditions (H2)-(H3) the solution given in Theorem 10 belongs to the space* $C^1([0,a];E)$.

In order to obtain the uniqueness of solutions we need an additional Lipschitz assumption.

$(H4)$ For any $M > 0$ there exists $\beta(M) \geq 0$ such that

$$(f(z) - f(v), z(0) - v(0)) \geq -\beta(M) \|z - v\|_{E_0}^2,$$

if $\|z\|_{E_0}, \|v\|_{E_0} \leq M$.

Theorem 14. *Assume (H1)-(H2) and (H4). Then the solution given in Theorem 10 is unique.*

Proof. Let $z, v \in E_0$, $\|z\|_{E_0}, \|v\|_{E_0} \leq M$, and $w = z - v$. It follows from (H4) and $(Aw(0), w(0)) = (Bw(0), Bw(0)) \geq 0$ that

$$\begin{aligned}(F(z) - F(v), z(0) - v(0)) &= -(Aw(0), w(0)) - \lambda \|w\|_{E_0} - (f(z) - f(v), w(0)) \\ &\leq \beta(M) \|w\|_{E_0}.\end{aligned}$$

Then the result follows from Theorem 8. □

We now aim to study the asymptotic behaviour of solutions for problem (11). In particular, we will show the existence of a global attractor.

When conditions (H1)-(H2), (H4) hold, if we assume that every solution is global (this is true if we obtain an estimate of the solutions by Theorem 6), then we can define a semigroup of operators $S : \mathbb{R}^+ \times E_0 \to E_0$ by

$$S(t, \psi) = u_t,$$

where $u(\cdot)$ is the unique solution to (11) with $u_0 = \psi$. Moreover, it is easy to prove using (10) and Gronwall's lemma that the map S is continuous with respect to the initial data u_0.

On the other hand, if we assume only (H1)-(H2) and that every solution is global, then we can define a multivalued semiflow by $G : \mathbb{R}^+ \times E_0 \to P(E_0)$ ($P(E_0)$ is the set of all non-empty subsets of E_0) by

$$G(t, \psi) = \{u_t : u(\cdot) \text{ is a solution of (11) with } u_0 = \psi\}. \tag{13}$$

Since we do not have uniqueness of the Cauchy problem, this map is in general multivalued. In a standard way (see [38, Lemma 13]) one can prove that it is a multivalued semiflow, that is:

1. $G(0, \cdot) = Id$ (the identity map);
2. $G(t + s, u_0) \subset G(t, G(s, u_0))$ for all $u_0 \in E_0$, $t, s \in \mathbb{R}^+$.

Moreover, it is strict, that is, $G(t + s, u_0) = G(t, G(s, u_0))$ for all $u_0 \in E_0$, $t, s \in \mathbb{R}^+$.

In the following sections we will show, for more particular cases of the map f, the existence of global attractors for (11). For this aim, we recall now some well known results of the general theory of attractors for semigroups and multivalued semiflows.

Let $S : \mathbb{R}^+ \times X \to X$ ($G : \mathbb{R}^+ \times X \to P(X)$) be a semigroup (a multivalued semiflow) in the complete metric space X. The set B_0 is called absorbing for the semigroup S (the semiflow G) if for any bounded set B there is a time $T(B)$ such that $S(t, B) \subset B_0$ ($G(t, B) \subset B_0$) for any $t \geq T$.

The semigroup S (the semiflow G) is asymptotically compact if for any bounded set B such that $\cup_{t\geq T(B)} S(t, B)$ ($\cup_{t\geq T(B)} G(t, B)$) is bounded for some $T(B)$, any arbitrary sequence $y_n \in S(t_n, B)$ ($y_n \in G(t_n, B)$), where $t_n \to \infty$, is relatively compact.

Recall that $dist(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|$ is the Hausdorff semi-distance from the set A to the set B.

The set $\mathcal{A}$ is called a global attractor of S if it is invariant ($S(t, \mathcal{A}) = \mathcal{A}$ for any $t \geq 0$) and attracts any bounded set B, that is, $dist(S(t, B), \mathcal{A}) \to 0$ as $t \to \infty$.

The set $\mathcal{A}$ is called a global attractor of G if it is negatively semi-invariant ($\mathcal{A} \subset G(t, \mathcal{A})$ for any $t \geq 0$) and attracts any bounded set B, that is, $dist(G(t, B), \mathcal{A}) \to 0$ as $t \to \infty$. It is invariant if $\mathcal{A} = G(t, \mathcal{A})$ for any $t \geq 0$.

We state two well–known results about the existence of global attractors.

Theorem 15. *([32] and [24]) Let $x \mapsto S(t, x)$ be continuous for any $t \geq 0$. Assume that S is asymptotically compact and possesses a bounded absorbing set B_0. Then there exists a global compact attractor $\mathcal{A}$, which is the minimal closed set attracting any bounded set. If, moreover, the space X is connected and the map $t \mapsto S(t, x)$ is continuous for any $x \in X$, then the set $\mathcal{A}$ is connected.*

We recall that the map $x \mapsto G(t, x)$ is called upper semicontinous if for any neighborhood O of $G(t, x)$ there exists $\delta > 0$ such that if $\|y - x\| < \delta$, then $G(t, y) \subset O$.

Theorem 16. *([37]) Assume that G is asymptotically compact and has a bounded absorbing set B_0. Also, let the map $x \mapsto G(t, x)$ be upper semicontinuous and have closed values. Then there exists a global compact attractor $\mathcal{A}$, which is the minimal closed set attracting any bounded set. If, moreover, the semiflow G is strict, then $\mathcal{A}$ is invariant.*

4. **A lattice system with sublinear retarded terms.**

We shall consider a function $f : E_0 \to E$ given by the rule $(f(v))_i = f_i(v_i)$ and

$$f_i(v_i) = F_{0,i}(v_i(0)) + F_{1,i}(v_i(-h_1)) + \int_{-h}^{0} b_i(s, v_i(s))\, ds,$$

where $h \geq h_1 > 0$, that is, putting $v = u_t = u(t + \cdot)$, problem (11) can be rewritten as

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + F_{0,i}(u_i(t)) + F_{1,i}(u_i(t - h_1)) \\ \qquad + \int_{-h}^{0} b_i(s, u_i(t+s))\, ds = 0, \ t > 0, \ i \in \mathbb{Z}, \\ \qquad u_i(s) = \psi_i(s), \ \forall s \in [-h, 0]. \end{cases} \tag{14}$$

We consider the following conditions:

(C1) $\lambda > 0$.

(C2) $F_{0,i}$ are continuous and satisfy that $F_{0,i}(x)\, x \geq -C_{0,i}$, $C_0 \in \ell^1$.

(C3) $|F_{0,i}(x)| \leq H(|x|)\,|x| + C_{1,i}$, for all $x \in \mathbb{R}$, where $C_1 \in \ell^2$, and $H(\cdot) \geq 0$ is a continuous and non-decreasing function.

(C4) $F_{1,i}$ are continuous and verify that $|F_{1,i}(x)| \leq K_1 |x| + C_{2,i}$, for all $x \in \mathbb{R}$, where $C_2 \in \ell^2$, $K_1 > 0$.

(C5) $|b_i(s, x)| \leq m_{0,i}(s) + m_{1,i}(s)\,|x|$, for all $x \in \mathbb{R}$ and a.a. $s \in (-h, 0)$, where b_i are Caratheodory, that is, measurable in s and continuous in x.

Also, $m_{0,i}(\cdot)$, $m_{1,i}(\cdot) \in L^1(-h, 0)$, $m_{0,i}(s), m_{1,i}(s) \geq 0$ and defining $M_{0i} = \int_{-h}^{0} m_{0,i}(s)\, ds$ and $M_{1i} = \int_{-h}^{0} m_{1,i}(s)\, ds$ we assume that $M_r^2 := \sum_{i\in\mathbb{Z}} M_{ri}^2 < \infty$, $r = 0, 1$.

Let us check conditions (H1)-(H3). First, in order to obtain (H1) we prove that f is well defined and bounded. We note that

$$|f_i(v_i)| \leq |F_{0,i}(v_i(0))| + |F_{1,i}(v_i(-h_1))| + \int_{-h}^{0} |b_i(s, v_i(s))|\, ds. \tag{15}$$

For the first term by (C3) we have

$$|F_{0,i}(v_i(0))|^2 \leq 2\left(H^2(|v_i(0)|)\,|v_i(0)|^2 + C_{1,i}^2\right) \leq 2\chi(\|v\|_{E_0})\,|v_i(0)|^2 + 2C_{1,i}^2, \tag{16}$$

where $\chi(\|v\|_{E_0}) = \max_{i\in\mathbb{Z}}\left(H^2(|v_i(0)|)\right)$, which exists because $H(\cdot)$ is non-decreasing and $v \in E_0$. Then,

$$\sum_{i\in\mathbb{Z}} |F_{0,i}(v_i(0))|^2 \leq 2\chi(\|v\|_{E_0})\,\|v\|_{E_0}^2 + 2\,\|C_1\|^2. \tag{17}$$

For the second term, by (C4), we obtain

$$\sum_{i\in\mathbb{Z}} |F_{1,i}(v_i(-h_1))|^2 \leq 2K_1^2 \sum_{i\in\mathbb{Z}} |v_i(-h_1)|^2 + 2\,\|C_2\|^2 \leq 2K_1^2\,\|v\|_{E_0}^2 + 2\,\|C_2\|^2. \tag{18}$$

Now, for the term with the integral delay, by (C5), we proceed as follows:

$$\int_{-h}^{0} |b_i(s, v_i(s))|\, ds \leq \int_{-h}^{0} (m_{0,i}(s) + m_{1,i}(s)\,|v_i(s)|)\, ds \leq M_{0,i} + \|v\|_{E_\infty} M_{1,i}.$$

Then

$$\sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} |b_i(s, v_i(s))|\, ds\right)^2 \leq 2\sum_{i\in\mathbb{Z}} M_{0,i}^2 + 2\,\|v\|_{E_\infty}^2 \sum_{i\in\mathbb{Z}} M_{1,i}^2 \leq 2M_0^2 + 2\,\|v\|_{E_0}^2\, M_1^2. \tag{19}$$

Then, using (17)-(19) in (15) we obtain that f is well defined and bounded.

Now, we check (H2), i.e., that the maps $f_i : C([-h, 0], \mathbb{R}) \to \mathbb{R}$ are continuous. We consider $\{v^n\}_{n\in\mathbf{N}} \subset C([-h, 0], \mathbb{R})$ and $v^0 \in C([-h, 0], \mathbb{R})$ such that $v^n \to v^0$ in

$C([-h,0],\mathbb{R})$. Now, we take

$$\left|f_i\left(v^n\right)-f_i\left(v^0\right)\right| \leq \left|F_{0,i}\left(v^n\left(0\right)\right)-F_{0,i}\left(v^0\left(0\right)\right)\right| + \left|F_{1,i}\left(v^n\left(-h_1\right)\right)-F_{1,i}\left(v^0\left(-h_1\right)\right)\right| + \left|\int_{-h}^{0} b_i\left(s,v^n\left(s\right)\right)ds - \int_{-h}^{0} b_i\left(s,v^0\left(s\right)\right)ds\right|.$$

From (C2) and (C4), $F_{0,i}$ and $F_{1,i}$ are continuous functions. Also, from (C5) and Lebesgue's theorem the last term converges to 0. Thus, the continuity of f_i follows. To check (H3) we observe that

$$\begin{aligned}\sum_{|i|\geq K}\left(\int_{-h}^{0}\left|b_i\left(s,v_i\left(s\right)\right)\right|ds\right)^2 &\leq 2\sum_{|i|\geq K}\left(\int_{-h}^{0}m_{0,i}\left(s\right)ds\right)^2 \\ &\quad + 2\sum_{|i|\geq K}\left(\int_{-h}^{0}m_{1,i}\left(s\right)\left|v_i\left(s\right)\right|ds\right)^2 \\ &\leq 2\sum_{|i|\geq K}M_{0,i}^2 + 2\left\|v\right\|_{E_0}^2\sum_{|i|\geq K}M_{1,i}^2\end{aligned}$$

Also, by (15), (16) and (C4) we have

$$\begin{aligned}\sum_{|i|\geq K}\left|f_i\left(v_i\right)\right|^2 &\leq R\Bigg(\chi(\left\|v\right\|_{E_0})\sum_{|i|\geq K}\left|v_i\left(0\right)\right|^2 + \sum_{|i|\geq K}C_{1,i}^2 + K_1^2\sum_{|i|\geq K}\left|v_i\left(-h_1\right)\right|^2 \\ &\quad + \sum_{|i|\geq K}C_{2,i}^2 + \sum_{|i|\geq K}M_{0,i}^2 + \left\|v\right\|_{E_0}^2\sum_{|i|\geq K}M_{1,i}^2\Bigg) \\ &\leq C\left(\left\|v\right\|_{E_0}\right)\left(\max_{s\in[-h,0]}\sum_{|i|\geq K}v_i^2\left(s\right)+b_K\right),\end{aligned}$$

where $b_K \to 0^+$ as $K\to\infty$, and $C\left(\cdot\right)\geq 0$ is a continuous non-decreasing function. Thus, (H3) holds.

Then Theorem 10 and Corollary 13 imply that for any $\psi\in E_0$ there exists, at least, one solution $u\left(\cdot\right)\in C^1\left([0,\alpha),E\right)$ in a maximal interval $[0,\alpha)$. In order to obtain that every solution is globally defined we need to get some estimates. This will be done in the next section.

4.1. **Estimate of solutions.** Now, we shall obtain some estimates of solutions. Such estimates will imply that the solutions are bounded uniformly with respect to bounded sets of initial conditions and positive values of time. This result allows us to define also a bounded absorbing set.

Proposition 17. *Assume (C1)-(C5). Also, let*

$$2M_1eh < 1, \tag{20}$$

$$K_1^2 < e^{-\eta h}\lambda\left(\lambda-\eta\right), \tag{21}$$

where $\eta\in(\eta_0,\eta_1)$ *and* η_j *are the two solutions of the equation* $\eta e^{-\eta h}=2M_1$. *Then, every solution* $u\left(\cdot\right)$ *with* $u_0=\psi\in E_0$ *verifies*

$$\left\|u_t\right\|_{E_0}^2 \leq R_1e^{-(\eta-L)t}\left\|\psi\right\|_{E_0}^2 + R_2,\ \forall t\in[0,T^*), \tag{22}$$

where T^* *is the maximal time of existence,* $L=2M_1e^{\eta h}$ *and* $R_j>0$ *are some constants depending on the parameters of the problem.*

Remark 18. *We note that (20) implies that* $\eta e^{-\eta h} > 2M_1$ *if* $\eta \in (\eta_0, \eta_1)$*, so that* $\eta > L$*. Also, (21) implies that* $\lambda > \eta$*.*

Proof. We multiply (14) by $u = (u_i)_{i\in\mathbb{Z}}$ in ℓ^2. Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2 + (Au, u) + \lambda \|u(t)\|^2 \\
&= -\sum_{i\in\mathbb{Z}} F_{0,i}(u_i(t))\, u_i(t) - \sum_{i\in\mathbb{Z}} F_{1,i}(u_i(t-h_1))\, u_i(t) \\
&- \sum_{i\in\mathbb{Z}} \int_{-h}^{0} b_i(s, u_i(t+s))\, ds\ u_i(t). \qquad (23)
\end{aligned}$$

Multiplying (23) by $e^{\eta t}$, and using $(Au, u) = \|Bu\|^2$ and (C1)-(C4), we have, for any $\epsilon > 0$ to be determined later on,

$$\begin{aligned}
\frac{d}{dt}\left(e^{\eta t}\|u(t)\|^2\right) &\le (\eta - 2\lambda + \epsilon)\, e^{\eta t}\|u(t)\|^2 + 2e^{\eta t}\|C_0\|_{\ell^1} \\
&+ 2\frac{e^{\eta t}}{\epsilon}\left(K_1^2\|u(t-h_1)\|^2 + \|C_2\|^2\right) \\
&- 2e^{\eta t}\sum_{i\in\mathbb{Z}} \int_{-h}^{0} b_i(s, u_i(t+s))\, ds\ u_i(t). \qquad (24)
\end{aligned}$$

Now, integrating the last inequality over $[0, t]$ we obtain

$$\begin{aligned}
e^{\eta t}\|u(t)\|^2 &\le \|u(0)\|^2 + (\eta - 2\lambda + \epsilon)\int_0^t e^{\eta s}\|u(s)\|^2 ds + \frac{2}{\eta}\|C_0\|_{\ell^1}\left(e^{\eta t} - 1\right) \\
&+ \frac{2}{\epsilon\eta}\|C_2\|^2\left(e^{\eta t} - 1\right) + \frac{2K_1^2}{\epsilon}\int_0^t e^{\eta s}\|u(s-h_1)\|^2 ds \\
&- 2\int_0^t e^{\eta s}\left(\sum_{i\in\mathbb{Z}} \int_{-h}^{0} b_i(r, u_i(s+r))\, dr\ u_i(s)\right) ds. \qquad (25)
\end{aligned}$$

We proceed to estimate the two last terms in (25). First,

$$\begin{aligned}
\int_0^t e^{\eta s}\|u(s-h_1)\|^2 ds &= \int_{-h_1}^{t-h_1} e^{\eta(l+h_1)}\|u(l)\|^2 dl \\
&\le e^{\eta h_1}\int_{-h}^{0} e^{\eta l}\|u(l)\|^2 dl + e^{\eta h_1}\int_0^t e^{\eta l}\|u(l)\|^2 dl \\
&\le \frac{e^{\eta h}}{\eta}\|\psi\|_{E_0}^2\left(1 - e^{-\eta h}\right) + e^{\eta h}\int_0^t e^{\eta l}\|u(l)\|^2 dl. \qquad (26)
\end{aligned}$$

Next, we analyze the last term in (25). By (C5),

$$\begin{aligned}
\left|\sum_{i\in\mathbb{Z}} \int_{-h}^{0} b_i(s, u_i(t+s))\, ds\ u_i(t)\right| &\le \sum_{i\in\mathbb{Z}} \int_{-h}^{0} \left(m_{0,i}(s)\, |u_i(t)|\right) ds \qquad (27) \\
&+ \sum_{i\in\mathbb{Z}} \int_{-h}^{0} \left(m_{1,i}(s)\, |u_i(t+s)|\, |u_i(t)|\right) ds.
\end{aligned}$$

Now, we estimate the two terms in (27) separately. On the one hand,

$$\sum_{i\in\mathbb{Z}} \int_{-h}^{0} \left(m_{0,i}(s)\, |u_i(t)|\right) ds = \sum_{i\in\mathbb{Z}} M_{0,i}\, |u_i(t)| \le \|u(t)\|\, M_0. \qquad (28)$$

On the other,

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\int_{-h}^{0}\left(m_{1,i}\left(s\right)\left|u_i\left(t+s\right)\right|\left|u_i\left(t\right)\right|\right)ds &\leq \left\|u_t\right\|_{E_\infty}\sum_{i\in\mathbb{Z}}\left(\int_{-h}^{0}\left(m_{1,i}\left(s\right)\right)ds\right)\left|u_i\left(t\right)\right| \\ &\leq \left\|u_t\right\|_{E_\infty}M_1\left\|u\left(t\right)\right\| \\ &\leq \left\|u_t\right\|_{E_0}^2 M_1. \end{aligned} \tag{29}$$

Now, using (28) and (29) in (27), we have

$$\begin{aligned}&\left|2\int_0^t e^{\eta s}\left(\sum_{i\in\mathbb{Z}}\int_{-h}^{0}b_i\left(r,u_i\left(s+r\right)\right)dr\ \ u_i\left(s\right)\right)\right|ds \\ &\quad\leq 2\int_0^t e^{\eta s}\left(\left\|u\left(s\right)\right\|M_0+\left\|u_s\right\|_{E_0}^2 M_1\right)ds \\ &\quad\leq \hat{\epsilon}\int_0^t e^{\eta s}\left\|u\left(s\right)\right\|^2 ds+\frac{M_0^2}{\hat{\epsilon}\eta}\left(e^{\eta t}-1\right)+2M_1\int_0^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds, \end{aligned} \tag{30}$$

with $\hat{\epsilon}>0$ arbitrary.
Using (26) and (30) in (25) we obtain

$$\begin{aligned}e^{\eta t}\left\|u\left(t\right)\right\|^2 \leq \left\|u\left(0\right)\right\|^2 &+\left(\eta-2\lambda+\epsilon+\hat{\epsilon}+\tfrac{2K_1^2e^{\eta h}}{\epsilon}\right)\textstyle\int_0^t e^{\eta s}\left\|u\left(s\right)\right\|^2 ds \\ &+\left(\tfrac{2\left\|C_2\right\|_E^2}{\epsilon\eta}+\tfrac{M_0^2}{\hat{\epsilon}\eta}+\tfrac{2}{\eta}\left\|C_0\right\|_{\ell^1}\right)\left(e^{\eta t}-1\right) \\ &+\tfrac{2K_1^2e^{\eta h}}{\epsilon\eta}\left\|\psi\right\|_{E_0}^2\left(1-e^{-\eta h}\right)+2M_1\textstyle\int_0^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.\end{aligned}$$

Taking $\epsilon=\lambda$, condition (21) implies that $\eta-\lambda+\hat{\epsilon}+\frac{K_1^2e^{\eta h}}{\lambda}<0$ for $\hat{\epsilon}$ small enough. Then

$$\begin{aligned}e^{\eta t}\left\|u\left(t\right)\right\|^2 \leq \left\|u\left(0\right)\right\|^2 &+\left(\tfrac{2\left\|C_2\right\|_E^2}{\lambda\eta}+\tfrac{M_0^2}{\hat{\epsilon}\eta}+\tfrac{2}{\eta}\left\|C_0\right\|_{\ell^1}\right)\left(e^{\eta t}-1\right) \\ &+\tfrac{2K_1^2}{\lambda\eta}\left\|\psi\right\|_{E_0}^2\left(e^{\eta h}-1\right)+2M_1\textstyle\int_0^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.\end{aligned} \tag{31}$$

Let $\theta\in[-h,0]$. Replacing t by $t+\theta$ in (31), using that $\left\|u\left(t+\theta\right)\right\|=\left\|\psi\left(t+\theta\right)\right\|\leq\left\|\psi\right\|_{E_0}$ if $t+\theta<0$, and multiplying by $e^{-\eta(t+\theta)}$ we have

$$\begin{aligned}\left\|u\left(t+\theta\right)\right\|^2 \leq e^{-\eta(t+\theta)}\left\|\psi\right\|_{E_0}^2&+\left(\tfrac{2\left\|C_2\right\|_E^2}{\lambda\eta}+\tfrac{M_0^2}{\hat{\epsilon}\eta}+\tfrac{2}{\eta}\left\|C_0\right\|_{\ell^1}\right)\left(1-e^{-\eta(t+\theta)}\right) \\ &+e^{-\eta(t+\theta)}\tfrac{2K_1^2}{\lambda\eta}\left\|\psi\right\|_{E_0}^2\left(e^{\eta h}-1\right)+2M_1e^{-\eta(t+\theta)}\textstyle\int_0^{t+\theta}e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.\end{aligned}$$

Using that $\theta\in[-h,0]$ and neglecting the negative terms we get

$$\begin{aligned}e^{\eta t}\left\|u_t\right\|_{E_0}^2 \leq e^{\eta h}\left\|\psi\right\|_{E_0}^2&+\left(\tfrac{2\left\|C_2\right\|_E^2}{\lambda\eta}+\tfrac{M_0^2}{\hat{\epsilon}\eta}+\tfrac{2}{\eta}\left\|C_0\right\|_{\ell^1}\right)e^{\eta t} \\ &+\tfrac{2K_1^2}{\lambda\eta}\left\|\psi\right\|_{E_0}^2 e^{2\eta h}+2M_1e^{\eta h}\textstyle\int_0^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.\end{aligned}$$

We can rewrite this expression as

$$e^{\eta t}\left\|u_t\right\|_{E_0}^2\leq\hat{C}\left(t\right)+L\int_0^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds, \tag{32}$$

where we have used the notation

$$\hat{C}_1 := \frac{2\left\|C_2\right\|_E^2}{\lambda\eta} + \frac{M_0^2}{\hat{\epsilon}\eta} + \frac{2}{\eta}\left\|C_0\right\|_{\ell^1},$$
$$\hat{C}_2 := \frac{2K_1^2}{\lambda\eta}e^{2\eta h},$$
$$L := 2M_1 e^{\eta h},$$
$$\hat{C}(t) := \left(e^{\eta h} + \hat{C}_2\right)\left\|\psi\right\|_{E_0}^2 + \hat{C}_1 e^{\eta t}.$$

Applying Gronwall's inequality and using $\eta - L > 0$ (see Remark 18) yields

$$\begin{aligned} e^{\eta t}\left\|u_t\right\|_{E_0}^2 &\leq \hat{C}(t) + L\int_0^t \hat{C}(s)\,e^{L(t-s)}ds \\ &= \hat{C}(t) + L\left\|\psi\right\|_{E_0}^2\left(e^{\eta h} + \hat{C}_2\right)\frac{1}{L}\left(e^{Lt} - 1\right) + \frac{L\hat{C}_1}{\eta - L}\left(e^{\eta t} - e^{Lt}\right) \\ &\leq \hat{C}(t) + \left\|\psi\right\|_{E_0}^2\left(e^{\eta h} + \hat{C}_2\right)e^{Lt} + \frac{L\hat{C}_1}{\eta - L}e^{\eta t}, \end{aligned}$$

and then

$$\left\|u_t\right\|_{E_0}^2 \leq e^{-\eta t}\left(e^{\eta h} + \hat{C}_2\right)\left\|\psi\right\|_{E_0}^2 + \hat{C}_1 + \left\|\psi\right\|_{E_0}^2\left(e^{\eta h} + \hat{C}_2\right)e^{-(\eta - L)t} + \frac{L\hat{C}_1}{\eta - L}. \tag{33}$$

From here (22) follows. □

Corollary 19. *Assuming the conditions of Proposition 17, Theorem 6 implies that every local solution of (11) can be defined globally. Also, as shown in Section 3, the map G defined by (13) is a strict multivalued semiflow.*

Corollary 20. *The bounded set defined by*

$$B_0 := \left\{\psi \in E_0 : \ \left\|\psi\right\|_{E_0} \leq R_0\right\},$$

with $R_0 := \sqrt{1 + R_2}$, is absorbing for the multivalued semiflow G.

4.2. **Estimate of the tails.** In order to obtain the existence of a global attractor we need to use an estimate of the tails of solutions.

Lemma 21. *We assume the conditions of Proposition 17. Let B be a bounded set of E_0. Then, for any $\epsilon > 0$ there exist $T(\epsilon, B)$, $K(\epsilon, B)$ such that*

$$\max_{s\in[-h,0]}\sqrt{\sum_{|i|>2K(\epsilon,B)}\left|u_i(t+s)\right|^2} < \epsilon,\ t \geq T, \tag{34}$$

for any initial condition $\psi \in B$ and any solution $u(\cdot)$ with $u_0 = \psi$.

Proof. Define a smooth function θ satisfying

$$\theta(s) = \begin{cases} 0,\ 0 \leq s \leq 1, \\ 0 \leq \theta(s) \leq 1,\ 1 \leq s \leq 2, \\ 1,\ s \geq 2. \end{cases}$$

Obviously $|\theta'(s)| \leq C$, for all $s \in \mathbb{R}^+$. For any solution $u(\cdot)$, let $v(t) := (v_i(t))_{i\in\mathbb{Z}}$ be given by $v_i(t) = \rho_{K,i}u_i(t)$, where $\rho_{K,i} := \theta\left(\frac{|i|}{K}\right)$. We multiply (14) by v. We note that $u(\cdot) \in C^1([0,\infty), E)$ implies

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i|^2 = \sum_{i\in\mathbb{Z}}\frac{du_i(t)}{dt}v_i(t), \ \forall t>0$$

Following now the arguments in [38, p.571], and thanks to Proposition 17, there exists another constant C (depending on the bounded subset B and the parameters of the problem) such that

$$(Au(t), v(t)) \geq -\frac{C}{K}, \ \forall t \geq 0.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t)|^2 \leq -\lambda\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t)|^2 - \sum_{i\in\mathbb{Z}}\rho_{K,i}f_i(u_{it})u_i(t) + \frac{C}{K}.$$

Then, arguing as in the proof of Proposition 17 we have

$$\begin{aligned}\frac{d}{dt}\left(e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t)|^2\right) &\leq e^{\eta t}(\eta-2\lambda+\epsilon)\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t)|^2 \\ &+ 2e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i} + \frac{2C}{K}e^{\eta t} \\ &+ \frac{2}{\epsilon}e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2 + \frac{2K_1^2}{\epsilon}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t-h_1)|^2 \\ &+ 2e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}\int_{-h}^0|b_i(s,u_i(t+s))|\,ds\ \ |u_i(t)|\,. \end{aligned} \tag{35}$$

Integrating over $(0,t)$ we get

$$\begin{aligned} e^{\eta t}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(t)|^2\right) &\leq \sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(0)|^2 \\ &+ (\eta-2\lambda+\epsilon)\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(s)|^2\,ds \\ &+ \frac{2}{\eta}\left(e^{\eta t}-1\right)\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i} + \frac{C}{K} + \frac{1}{\epsilon}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2\right) \\ &+ \frac{2K_1^2}{\epsilon}\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(s-h_1)|^2\,ds \\ &+ 2\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\int_{-h}^0|b_i(r,u_i(s+r))|\,dr\ \ |u_i(s)|\,ds\,. \end{aligned} \tag{36}$$

Next, we estimate the last two terms in (36). The first one, arguing as in (26), is estimated by

$$\begin{aligned}\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(s-h_1)|^2\,ds \quad &\leq \quad \frac{e^{\eta h}}{\eta}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(1-e^{-\eta h}\right) \\ &+ e^{\eta h}\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}|u_i(s)|^2\,ds. \end{aligned} \tag{37}$$

As for the second term, using assumption (C5) and

$$\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(s\right)\right|\int_{-h}^0 m_{0,i}\left(r\right)dr\ ds \ \ \leq \ \ \frac{\hat{\epsilon}}{2}\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u\left(s\right)\right\|^2\ ds + \frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}\eta}\left(e^{\eta t}-1\right),$$

$$\begin{aligned}&\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(s\right)\right|\int_{-h}^0 m_{1,i}\left(r\right)\left|u_i\left(s+r\right)\right|dr\ ds\\ &\leq \int_0^t e^{\eta s}\left\|\rho_{K,i}^{\frac{1}{2}}u_s\right\|_{E_\infty}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}^{\frac{1}{2}}\left|u_i\left(s\right)\right|M_{1,i}\right)\ ds\\ &\leq M_1\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2\ ds,\end{aligned}$$

we obtain

$$\begin{aligned}&\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\int_{-h}^0\left|b_i\left(r,u_i\left(s+r\right)\right)\right|dr\ \ \left|u_i\left(s\right)\right|ds \qquad (38)\\ &\quad\leq\frac{\hat{\epsilon}}{2}\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u\left(s\right)\right\|^2\ ds+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}\eta}\left(e^{\eta t}-1\right)\\ &\quad\quad+M_1\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2\ ds.\end{aligned}$$

Taking into account all these estimates together we obtain

$$\begin{aligned}&e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(t\right)\right|^2\\ &\quad\leq\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(0\right)\right|^2+\left(\eta-2\lambda+\epsilon+\frac{2K_1^2}{\epsilon}e^{\eta h}+\hat{\epsilon}\right)\int_0^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(s\right)\right|^2ds\\ &\quad\quad+\frac{2}{\eta}\left(e^{\eta t}-1\right)\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i}+\frac{C}{K}+\frac{1}{\epsilon}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}}\right)\\ &\quad\quad+\frac{2K_1^2}{\epsilon}\frac{e^{\eta h}}{\eta}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(1-e^{-\eta h}\right)+2M_1\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2\ ds.\end{aligned}$$

In a similar way as in Proposition 17 we have

$$\begin{aligned}&e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(t\right)\right|^2\\ &\quad\leq\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i\left(0\right)\right|^2\\ &\quad\quad+\frac{2}{\eta}\left(e^{\eta t}-1\right)\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i}+\frac{C}{K}+\frac{1}{\lambda}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}}\right)\\ &\quad\quad+\frac{2K_1^2}{\lambda}\frac{e^{\eta h}}{\eta}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(1-e^{-\eta h}\right)+2M_1\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2\ ds\end{aligned}$$

and

$$\begin{aligned} e^{\eta t}\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 &\leq \left(e^{\eta h}+\frac{2K_1^2}{\lambda\eta}e^{2\eta h}\right)\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2 \\ &\quad+\frac{2}{\eta}e^{\eta t}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i}+\frac{C}{K}+\frac{1}{\lambda}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}}\right) \\ &\quad+2M_1e^{\eta h}\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds. \end{aligned}$$

We can rewrite this expression as

$$e^{\eta t}\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 \leq \widetilde{C}(t)+\widetilde{L}\int_0^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds, \tag{39}$$

where we have used the notation

$$\begin{aligned} \widetilde{C}_1 &:= \frac{2}{\eta}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i}+\frac{C}{K}+\frac{1}{\lambda}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2}{2\hat{\epsilon}}\right), \\ \widetilde{C}_2 &:= \frac{2K_1^2}{\lambda\eta}e^{2\eta h}, \\ \widetilde{L} &:= 2M_1e^{\eta h}, \\ \widetilde{C}(t) &:= \left(e^{\eta h}+\widetilde{C}_2\right)\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2+\widetilde{C}_1e^{\eta t}. \end{aligned}$$

Applying Gronwall's inequality and using $\eta - L > 0$ (see Remark 18) we obtain

$$e^{\eta t}\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 \leq \widetilde{C}(t)+\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(e^{\eta h}+\widetilde{C}_2\right)e^{\widetilde{L}t}+\frac{\widetilde{L}\widetilde{C}_1}{\eta-\widetilde{L}}e^{\eta t},$$

and then

$$\begin{aligned} \left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 &\leq \left\{\left(e^{\eta h}+\widetilde{C}_2\right)e^{-\eta t}+\left(e^{\eta h}+\widetilde{C}_2\right)e^{-(\eta-\widetilde{L})t}\right\}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2 \\ &\quad+\left(\frac{\eta}{\eta-\widetilde{L}}\right)\widetilde{C}_1. \end{aligned} \tag{40}$$

Thus, there exist $K(\epsilon, B)$, $T(\epsilon, B)$ such that

$$\begin{aligned} \max_{s\in[-h,0]}\sqrt{\sum_{|i|\geq 2K}\left(u_i(t+s)\right)^2} &\leq \max_{s\in[-h,0]}\sqrt{\sum_{i\in\mathbb{Z}}\rho_{K,i}\left(u_i(t+s)\right)^2} \\ &= \left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0} \\ &\leq \epsilon, \text{ if } t\geq T. \end{aligned}$$

□

4.3. **Existence of the global attractor: general case.** We know from Corollary 19 that under the assumptions of Proposition 17, the map G given by (13) is a strict multivalued semiflow. For any initial data $\psi \in E_0$ we denote

$$\mathcal{D}(\psi) = \{u(\cdot) \text{ is a global solution of (14) with initial data } \psi\}.$$

In view of Theorem 16 we need to prove that G is asymptotically compact, upper semicontinuous with respect to the initial data and that has closed values.
For this end, we will need the following auxiliary lemma.

Lemma 22. *Let $\psi^n \to \psi$ in E_0. Then:*

1. *For arbitrary $\epsilon, T > 0$ there exists $K(\epsilon, T)$ such that for any solution $u^n(\cdot) \in \mathcal{D}(\psi^n)$,*
$$\max_{s\in[-h,0]} \sqrt{\sum_{|i|\geq 2K} \left|u_i^n(t+s)\right|^2} \leq \epsilon,\ \forall t \in [0,T]. \tag{41}$$
2. *Also, there exists $u(\cdot) \in \mathcal{D}(\psi)$ and a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ so that*
$$u^{n_k} \to u \text{ in } \mathcal{C}([0,T], E). \tag{42}$$

Proof. It is not difficult to see that there exists $K_1(\epsilon) > 0$ such that

$$\sum_{i\in\mathbb{Z}} \rho_{K,i} \left|\psi_i^n(s)\right|^2 < \epsilon,\ \forall n,\ s \in [-h,0]$$
$$\sum_{i\in\mathbb{Z}} \rho_{K,i} \left|\psi_i^0(s)\right|^2 < \epsilon,\ \forall s \in [-h,0],$$

if $K \geq K_1$. Now, from (40) we obtain the existence of $K(\epsilon, T) \geq K_1$ such that

$$\begin{aligned} \left\|\rho_K^{\frac{1}{2}} u_t^n\right\|_{E_0}^2 \quad &\leq \quad \left\{e^{-\eta t}\left(e^{\eta h} + \widetilde{C}_2\right) + \left(e^{\eta h} + \widetilde{C}_2\right) e^{-\left(\eta - \widetilde{L}\right)t}\right\} \left\|\rho_K^{\frac{1}{2}} \psi^n\right\|_{E_0}^2 \\ &\quad + \left(\frac{\eta}{\eta - \widetilde{L}}\right) \widetilde{C}_1 \leq \epsilon. \end{aligned}$$

Therefore,

$$\max_{s\in[-h,0]} \sqrt{\sum_{|i|\geq 2K} \left(u_i^n(t+s)\right)^2} \leq \max_{s\in[-h,0]} \sqrt{\sum_{i\in\mathbb{Z}} \rho_{K,i} \left(u_i^n(t+s)\right)^2} = \left\|\rho_K^{\frac{1}{2}} u_t^n\right\|_{E_0} \leq \epsilon,$$

proving (41).
Next, from Proposition 17 we have that $\|u^n(t)\| \leq \|u_t^n\|_{E_0} \leq C$. Fix $t \in [0,T]$. Then we can find ω and a subsequence verifying

$$u^{n_k}(t) \to \omega \text{ in } E_w.$$

In fact, the convergence is strong, which follows from (41). Indeed, for any $\mu > 0$ there exist $K_2(\mu)$ and $N(\mu)$ such that $\sum_{|i|>K_2} \left|u_i^n(t)\right|^2 < \mu$, $\sum_{|i|>K_2} \left|\omega_i\right|^2 < \mu$ and $\sum_{|i|\leq K_2} \left|u_i^n(t) - \omega_i\right|^2 < \mu$ if $n \geq N$, so that

$$\left\|u^n(t) - \omega\right\|^2 \leq \sum_{|i|\leq K_2} \left|u_i^n(t) - \omega_i\right|^2 + \sum_{|i|>K_2} \left|u_i^n(t) - \omega_i\right|^2 < 5\mu. \tag{43}$$

Thus, $\{u^n(t)\}$ is precompact in E for any $t \in [0,T]$.
Since F is a bounded map, Proposition 17 and the integral representation of solutions imply that

$$\|u^n(t) - u^n(s)\| \leq \int_s^t \|F(u_\tau^n)\| \, d\tau \leq K(t-s),\ \text{if } 0 \leq s < t \leq T, \tag{44}$$

so that the sequence $\{u^n(\cdot)\}$ is equicontinuous in $[0,T]$. Then, we can apply the Ascoli-Arzelà theorem to obtain a subsequence (denoted again as u^n) such that

$$u^n(\cdot) \to u(\cdot) \text{ in } \mathcal{C}([0,T], E).$$

Now, we need to prove that $u(\cdot) \in \mathcal{D}(\psi)$. It is clear that $u_0 = \psi$. Since the map F is sequentially weakly continuous in bounded sets (see Lemma 9), arguing as in the proof of Theorem 4 we obtain that

$$u(t) = u(0) + \int_0^t F(u_s)\, ds,$$

and then $u(\cdot) \in \mathcal{D}(\psi)$. □

Corollary 23. *Assume the conditions of Proposition 17. Then, the multivalued map $\psi \mapsto G(t,\psi)$ has closed graph and is upper semicontinous. Moreover, it has compact values.*

Proof. The facts that the map $\psi \mapsto G(t,\psi)$ has closed graph and compact values follow easily from Lemma 22 (see similar results in [38] for more details). In order to prove the upper semicontinuity we proceed by contradiction. Let $t \geq 0$. Consider $\psi \in E_0$, a neighborhood $\mathcal{O}$ of $G(t,\psi)$ and a sequence $\xi^n \in G(t,\psi^n)$, $\psi^n \to \psi$ in E_0, such that $\xi^n \notin \mathcal{O}$. We take $u^n(\cdot) \in \mathcal{D}(\psi^n)$ such that $u_t^n = \xi^n$. Using (42), there exists $u(\cdot) \in \mathcal{D}(\psi)$ such that (up to a subsequence) $u^n(\cdot) \to u(\cdot)$ in $\mathcal{C}([0,T],E)$. Also, $u^n(\cdot) \to u(\cdot)$ in $\mathcal{C}([t-h,t],E)$, so that $\xi^n \to \xi = u_t$ in E_0. Then, $\xi^n \to \xi \in G(t,\psi)$, a contradiction. □

Lemma 24. *Assume the conditions of Proposition 17. Then, the multivalued map G is asymptotically compact.*

Proof. We consider $\xi^n = u_{t_n}^n \in G(t_n,\psi^n)$, where $u^n(\cdot) \in \mathcal{D}(\psi^n)$, $\psi^n \in B$ (a bounded set in E_0). From (33) we have

$$\left\| u_{t_n}^n(s) \right\| \leq C, \ \forall s \in [-h,0], \ \forall n,$$

for some $C > 0$. For fixed $s \in [-h,0]$ we can find a subsequence (denoted again as u^n) such that

$$u^n(t_n + s) \to \omega_s \text{ in } E_w.$$

Using a similar argument as in (43) (with the help of Lemma 21) we obtain that $u^n(t_n+s) \to \omega_s$ in E. From here, we obtain that $\left\{ u_{t_n}^n(s) \right\}$ is a precompact sequence for any $s \in [-h,0]$. In order to apply the Ascoli-Arzelà theorem, we need to obtain the equicontinuity property. To do this, in a similar as in Lemma 22, using Proposition 17 we can obtain that

$$\|u^n(t_n+t) - u^n(t_n+s)\| \leq \int_s^t \left\| F\left(u_{t_n+\tau}^n\right)\right\| d\tau \leq K(t-s), \text{ if } -h \leq s < t \leq 0.$$

Then, the Ascoli-Arzelà theorem implies that ξ^n is relatively compact in E_0. □

In view of Proposition 17, Lemma, 24, Corollaries 20, 23 and Theorem 16 we obtain:

Theorem 25. *Assume the conditions of Proposition 17. Then, the multivalued semiflow G possesses a global compact invariant attractor $\mathcal{A}$.*

We can obtain the same result by changing slightly conditions (20)-(21).

Theorem 26. *Assume conditions (C1)-(C5) and let*

$$2eh(M_1 + \frac{K_1^2}{\lambda}) < 1, \tag{45}$$

$$\lambda - \eta > 0 \tag{46}$$

where $\eta \in (\eta_0, \eta_1)$ *and* η_j *are the two solutions of the equation* $\eta e^{-\eta h} = 2M_1 + \frac{2K_1^2}{\lambda}$. *Then, the multivalued semiflow* G *possesses a global compact invariant attractor* $\mathcal{A}$.

Proof. The only difference in the proof is how to obtain (22) and (34). Indeed, in the proof of Proposition 17 we change (26) by

$$\int_0^t e^{\eta s} \left\| u\left(s - h_1\right)\right\|^2 ds \leq \int_0^t e^{\eta s} \left\| u_s \right\|_{E_0}^2 ds.$$

Then, arguing in the same way as in Lemma 17 we obtain the inequality

$$\begin{gathered} e^{\eta t} \left\| u_t \right\|_{E_0}^2 \leq e^{\eta h} \left\| \psi \right\|_{E_0}^2 + \left(\tfrac{2\|C_2\|_E^2}{\lambda\eta} + \tfrac{M_0^2}{\hat{\epsilon}\eta} + \tfrac{2}{\eta} \left\| C_0 \right\|_{\ell^1} \right) e^{\eta t} \\ +(2M_1 + \tfrac{2K_1^2}{\lambda}) e^{\eta h} \int_0^t e^{\eta s} \left\| u_s \right\|_{E_0}^2 ds \end{gathered}$$

and by Gronwall's lemma we have

$$\left\| u_t \right\|_{E_0}^2 \leq e^{-\eta t} e^{\eta h} \left\| \psi \right\|_{E_0}^2 + \hat{C}_1 + \left\| \psi \right\|_{E_0}^2 e^{\eta h} e^{-(\eta - L)t} + \frac{L\hat{C}_1}{\eta - L},$$

where

$$\begin{aligned} \hat{C}_1 &:= \frac{2 \left\| C_2 \right\|_E^2}{\lambda\eta} + \frac{M_0^2}{\hat{\epsilon}\eta} + \frac{2}{\eta} \left\| C_0 \right\|_{\ell^1}, \\ L &:= (2M_1 + \frac{2K_1^2}{\lambda}) e^{\eta h}, \\ \hat{C}\left(t\right) &:= e^{\eta h} \left\| \psi \right\|_{E_0}^2 + \hat{C}_1 e^{\eta t}. \end{aligned}$$

Similar changes have to be done in order to prove (34).
With these estimates the proof of the result is exactly the same as for Theorem 25. ☐

4.4. **Existence of the global attractor: case of uniqueness.** We can prove uniqueness of the Cauchy problem (14) if we assume the following extra assumption:

(C6) For any $x, y \in \mathbb{R}$ and $s \in [-h, 0]$ we have

$$\left| F_{0,i}\left(x\right) - F_{0,i}\left(y\right) \right| \leq C_1\left(\left|x\right|, \left|y\right|\right) \left|x - y\right|,$$

$$\left| F_{1,i}\left(x\right) - F_{1,i}\left(y\right) \right| \leq C_2\left(\left|x\right|, \left|y\right|\right) \left|x - y\right|,$$

$$\left| b_i\left(s, x\right) - b_i\left(s, y\right) \right| \leq k\left(s\right) C_3\left(\left|x\right|, \left|y\right|\right) \left|x - y\right|,$$

where $C_j\left(\cdot, \cdot\right) \geq 0$ are continuous and non-decreasing functions in both variables and $k\left(\cdot\right) \in L^2\left(-h, 0\right)$.

Lemma 27. *If (C6) holds, the map* $f : E_0 \to E$ *is Lipschitz on bounded subsets of* E_0.

Proof. Let $v, w \in E_0$ be such that $\|v\|_{E_0}, \|w\|_{E_0} \leq R$. Then

$$\begin{aligned}\|f(v) - f(w)\|^2 \leq\ & 3\sum_{i\in\mathbb{Z}} |F_{0,i}(v_i(0)) - F_{0,i}(w_i(0))|^2 \\ & + 3\sum_{i\in\mathbb{Z}} |F_{0,i}(v_i(-h_1)) - F_{0,i}(w_i(-h_1))|^2 \\ & + 3\sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} |b_i(s, v_i(s)) - b_i(s, w_i(s))|\, ds\right)^2.\end{aligned}$$

We have that

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |F_{0,i}(v_i(0)) - F_{0,i}(w_i(0))|^2 &\leq \left(\max_{i\in\mathbb{Z}}(C_1(|v_i(0)|, |w_i(0)|))\right)^2 \sum_{i\in\mathbb{Z}} |v_i(0) - w_i(0)|^2 \\ &\leq \chi_1^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \|v - w\|_{E_0}^2,\end{aligned}$$

$$\begin{aligned}&\sum_{i\in\mathbb{Z}} |F_{0,i}(v_i(-h_1)) - F_{0,i}(w_i(-h_1))|^2 \\ &\leq \left(\max_{i\in\mathbb{Z}}(C_2(|v_i(-h_1)|, |w_i(-h_1)|))\right)^2 \sum_{i\in\mathbb{Z}} |v_i(-h_1) - w_i(-h_1)|^2 \\ &\leq \chi_2^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \|v - w\|_{E_0}^2,\end{aligned}$$

where $\chi_j\left(\|v\|_{E_0}, \|w\|_{E_0}\right) = \max_{i\in\mathbb{Z}, s\in[-h,0]}(C_j(|v_i(s)|, |w_i(s)|))$. Also,

$$\begin{aligned}&\sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} |b_i(s, v_i(s)) - b_i(s, w_i(s))|\, ds\right)^2 \\ &\leq \left(\max_{i\in\mathbb{Z}, s\in[-h,0]}(C_3(|v_i(s)|, |w_i(s)|))\right)^2 \sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} k(s)\, |v_i(s) - w_i(s)|\, ds\right)^2 \\ &\leq \chi_3^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \sum_{i\in\mathbb{Z}} \int_{-h}^{0} k^2(s)\, ds \int_{-h}^{0} |v_i(s) - w_i(s)|^2\, ds \\ &= \chi_3^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \int_{-h}^{0} k^2(s)\, ds \int_{-h}^{0} \sum_{i\in\mathbb{Z}} |v_i(s) - w_i(s)|^2\, ds \\ &\leq \chi_3^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) h \int_{-h}^{0} k^2(s)\, ds\, \|v - w\|_{E_0}^2,\end{aligned}$$

where $\chi_3\left(\|v\|_{E_0}, \|w\|_{E_0}\right) = \max_{i\in\mathbb{Z}, s\in[-h,0]}(C_3(|v_i(s)|, |w_i(s)|))$. The fact that the summatory and the integral can be exchanged can be shown easily using Lebesgue's theorem.

Thus, there exists $K(R)$ such that

$$\|f(v) - f(w)\| \leq K(R)\, \|v - w\|_{E_0},$$

proving the result. □

Corollary 28. *If (C6) holds, the map $F : E_0 \to E$ is Lipschitz on bounded subsets of E_0.*

Corollary 28 implies that (H4) is satisfied. Then, if we assume conditions (C1)-(C6) and (20)-(21), Theorems 6, 10, 14, Corollary 13 and Proposition 17 imply that for any $\psi \in E_0$ there exists a unique global solution $u(\cdot) \in C^1([0,\infty), E)$.
Hence, as shown in Section 3, we can define a semigroup of operators $S : \mathbb{R}^+ \times E_0 \to E_0$ by putting

$$S(t, u_0) = u_t,$$

where $u(\cdot)$ is the unique solution to (14) with $\psi = u_0$. Moreover, this map is continuous with respect to the initial data ψ.
We obtain for it the existence of a global compact attractor.

Theorem 29. *Assume conditions (C1)-(C6) and (20)-(21). Then, the semigroup S possesses a global compact connected attractor $\mathcal{A}$.*

Proof. Proposition 17, Lemma 24, Corollary 20 and Theorem 15 imply the existence of a global compact attractor $\mathcal{A}$. Since the space E_0 is connected and the map $t \mapsto S(t, \psi)$ is continuous, Theorem 15 implies that the set $\mathcal{A}$ is connected. □

Also, in the same way as in Theorem 26 we can change (20)-(21) by (45)-(46).

Theorem 30. *Assume conditions (C1)-(C6) and (45)-(46). Then, the semigroup S possesses a global compact connected attractor $\mathcal{A}$.*

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