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NUMERICAL AND FINITE DELAY APPROXIMATIONS OF ATTRACTORS FOR LOGISTIC DIFFERENTIAL-INTEGRAL EQUATIONS WITH INFINITE DELAY

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Abstract. The upper semi-continuous convergence of approximate attractors for an infinite delay differential equation of logistic type is proved, first for the associated truncated delay equation with finite delay and then for a numerical scheme applied to the truncated equation.

1. Introduction.

The aim of this paper is to establish some approximation results for the attractors for infinite delay differential equations and is motivated by the difficulty in approximating such equations numerically.

There exists a wide literature on numerical approximations for delay differential equations, see the monograph [1]. However, to our knowledge, this mostly concerns finite delay problems rather than the infinite delay case. A rare exception is the paper [10] which uses spectral methods and Galerkin approximations for an infinite delay problem.

Our goal is to investigate the asymptotic behaviour of systems governed by infinite delay differential equations in terms of the attractors of associated truncated finite delay equations and their numerical approximations. See [6] for a survey of the numerical dynamics of finite delay functional differential equations.

The existence of several types of nonautonomous attractors, both forward and pullback, was established in [5] for (generally multivalued) semi-flows and processes generated by general equations of the type

$$x'(t) = F_0(t, x(t)) + F_1(t, x(t - \rho(t))) + \int_{-\infty}^{0} b(t, s, x(t+s))\mathrm{d}s.$$

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The asymptotic dynamics of these systems is characterized by their attractors and the question arises as to how one can approximate them.

The upper semi-continuity of attractors with respect to parameter dependent delays has been extensively investigated by Hines [9]. However, she used infinite delay differential equations to approximate a finite delay equation and our goal is the opposite: to approximate infinite delay DE by finite delay DE.

The paper is organized as follows. In Section 2 we introduce notation and recall basic concepts on dynamical systems and global attractors for DDE with finite and infinite delays. Logistic models with finite and infinite delay and their corresponding semi-flows and attractors are then considered. Section 3 is devoted to several auxiliary results to provide estimates for comparing the solutions of both problems. In Section 4 the first main result is proved, namely the upper semi-continuous convergence of the attractors of the truncated problems to that of the infinite delay problem as the truncated delay increases to infinity. The second main result on the upper semi-continuous convergence of the numerical attractors to that of a truncated delay system for a fixed delay as the stepsize decreases to zero is then presented in Section 6 after the numerical scheme has been introduced and its properties have been discussed in Section 5. For this numerical work we need stronger assumptions on the coefficients of the logistic delay differential equations. For clarity of the exposition, a technical lemma on the global discretization error bound for the numerical scheme is proved in the appendix at the end of the paper.

2. **Statement of the problem.** Throughout the paper we will use the following notation.

The Euclidean norm on $\mathbb{R}^d$ will be denoted $|\cdot|$ and $C([-T,0];\mathbb{R}^d)$ will denote the Banach space of continuous functions $x:[-T,0]\to\mathbb{R}^d$ with the supremum norm

$$\|x\|_{C([-T,0];\mathbb{R}^d)} = \sup_{t\in[-T,0]} |x(t)|.$$

However, for DDE with infinite delay the state space must satisfy certain additional conditions (cf. [7]). A typical example (which will suffice for us here) is

$$C_\gamma = \{x \in C((-\infty,0];\mathbb{R}^d) \ : \ \sup_{\theta\in(-\infty,0]} \mathrm{e}^{\gamma\theta}|x(\theta)| < +\infty,\ \exists \lim_{\theta\to-\infty} e^{\gamma\theta}x(\theta)\},$$

which is a Banach space with the weighted norm

$$\|x\|_\gamma = \sup_{\theta\in(-\infty,0]} \mathrm{e}^{\gamma\theta}|x(\theta)|.$$

We will also consider the positive cones $C([-T,0];\mathbb{R}^d_+)$ and C_γ^+ of the above Banach spaces.

The Hausdorff semi-distance between two non-empty sets A and B in a metric space (X,d) will be denoted by

$$H_X^*(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b)$$

and the closed ball in X of center 0 and radius r by $B_X(0,r)$, while $P(X)$ and $B(X)$ will denote the families of non-empty subsets and the non-empty bounded subsets of the space X, respectively. A multivalued function $F : X \to P(X)$ is said upper semi-continuous if for every $x \in X$ and every neighbourhood M of $F(x)$, there exists a neighbourhood N of x such that $F(y) \subset M$ for any $y \in N$. A map is said bounded if it maps bounded sets into bounded sets.

2.1. **Dynamical systems and DDEs.** For the sake of brevity, we will not repeat well known results on existence and uniqueness of solutions for delay differential equations (DDEs) here, nor concepts of absorbing, attracting sets and global attractor, but refer the reader, for instance, to [8, 4] for details. We recall briefly only the main points here. We will write $x_t(s) = x(t+s)$ for a function $x(\cdot)$ defined on a subset of $\mathbb{R}$ and taking values in $\mathbb{R}^d$. Let f be an operator from an infinite dimensional space X (such as $C([-T,0];\mathbb{R}^d)$ or C_γ) to $\mathbb{R}^d$ and consider the initial value problem for the delay differential equation

$$x'(t) = f(x_t), \qquad x_0 = \phi \in X. \tag{1}$$

The continuity of f is sufficient (cf. [8]) to ensure the existence of at least one local solution in an interval $(0,\delta)$ [solutions here are understood through an integral equation formulation] and, if f maps bounded sets into bounded sets, a priori bounds then suffice to avoid blow-ups and to ensure the existence of solutions defined global in time.

Definition 1. *Let $D(\phi)$ be the set of all global solutions of the initial value problem* (1)*. Then, the associated (multivalued) semi-flow $G(t,\phi)$ is given by*

$$G(t,\phi) = \{x_t \ : \ x(\cdot) \in D(\phi)\}.$$

Remark 1.
(i) *Under mild assumptions on f (continuity, boundedness, and a priori estimates) it is easy to see that G defines a (multivalued) semi-dynamical system or semi-flow $G : \mathbb{R}_+ \times X \to P(X)$, i.e. with $G(0,\cdot) = Id.$ and $G(t_1+t_2,\phi) = G(t_1, G(t_2,\phi))$ for all $t_1, t_2 \geq 0$ and $\phi \in X$.*

(ii) *It will sometimes be convenient to restrict to solutions (and semi-flows) to satisfy certain additional conditions such as positivity when dealing with biological models, in which case the positive cones $C([-T,0];\mathbb{R}^d_+)$ and C_γ^+ are the appropriate state spaces.*

Different kinds of differential equations generate semi-flows with different compactness properties and this is reflected in the conditions which ensure the existence of attractors in each case. For delay differential equations with a finite delay the semi-flow is compact and a bounded absorbing set is enough to give the existence of an attractor. On the other hand, the semi-flow for infinite delay DDE is usually not compact, but it is often asymptotically compact.

Definition 2. *A semi-flow $G : \mathbb{R}_+ \times X \to P(X)$ is said to be asymptotically sequentially compact if given any bounded sequence $\{\phi^n\}$ and $t_n \to +\infty$, every sequence $\{\psi^n\}$ with $\psi^n \in G(t_n, \phi^n)$ is relatively compact.*

The following proposition can be found in [5]:

Proposition 1. *Suppose that $f : C_\gamma \to \mathbb{R}^d$ is continuous and bounded and that the initial value problem* (1) *has globally defined solutions, which generate a uniformly bounded semi-flow G, i.e.,such that for every $R > 0$, there exists a constant $M_R > 0$ such that $\{u(t,\phi) \ : \ u(\cdot,\phi) \in D(\phi),\ \phi \in B_{C_\gamma}(0,R)\} \subset B_{\mathbb{R}^d}(0,M_R)$ for all $t \geq 0$.*

Then, $G(t,\cdot)$ has closed values for each $t \geq 0$, is upper semi-continuous and G is asymptotically sequentially compact.

2.2. **Autonomous logistic models with finite and infinite delay.** Consider the one-dimensional delayed logistic problems with infinite delay

$$(P_\infty)\ \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x_t) = rx\,(t)\left(1 - K^{-1}\displaystyle\int_{-\infty}^{0} w\,(s)\,P\,(x\,(s+t))\,\mathrm{d}s\right) \\ x_0 = \psi \in C_\gamma^+, \end{cases}$$

and finite delay

$$(P_T)\ \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = f_T(x_t) = rx\,(t)\left(1 - K^{-1}\displaystyle\int_{-T}^{0} w\,(s)\,P\,(x\,(s+t))\,\mathrm{d}s\right), \\ x_0 = \xi \in C([-T,0];\mathbb{R}_+), \end{cases}$$

with constants $r, K > 0$, where $x\,(t) \geq 0$. The coefficient functions $P \in C\,(\mathbb{R};\mathbb{R})$ with $P\,(x) \geq 0$ if $x \geq 0$ and $w \in L^1\,((-\infty,0);\mathbb{R}_+)$ are such that

$$L\,|x| \leq |P\,(x)| \leq C_1\,|x|^m + C_2, \qquad \text{for all} \quad x \in \mathbb{R}, \tag{2}$$

for certain constants $C_i, L > 0$, $m \geq 1$, and

$$\int_{-\infty}^{0} w\,(s)\,\mathrm{e}^{-\eta s}\mathrm{d}s < \infty, \tag{3}$$

for some $\eta > 0$. In particular, the latter implies that $\int_{-\infty}^{0} w\,(s)\,\mathrm{d}s < \infty$.

The following propositions summarize results from [5] on the existence of solutions and an attractor for the dynamical system associated to (P_∞):

Proposition 2 (Existence of global positive solution)**.** *Under assumptions* (2)-(3), *the functional* $M : C_\gamma \to \mathbb{R}$ *defined by* $\psi \mapsto M\,(\psi) = \int_{-\infty}^{0} w\,(s)\,P\,(\psi\,(s))\,\mathrm{d}s$ *is continuous on* C_γ *provided* $\gamma = \frac{\eta}{m}$*.*

Then (P_∞) *has at least one local solution and, moreover, there exists at least one global positive solution.*

The following definition is useful for biological applications.

Definition 3. *Let* $D^+(\psi)$ *be the set of all global positive solutions of* (P_∞) *with initial data* ψ*. Then,*

$$G_\infty^+ : C_\gamma^+ \to P(C_\gamma^+)$$

is the (eventually multivalued) semi-flow given by $G_\infty^+(t,\psi) = \{u_t\,:\,u(\cdot) \in D^+(\psi)\}$.

Next, we detail the existence of a bounded absorbing set for G_∞^+.

Proposition 3 (Uniform estimates on the solutions)**.** *Under the assumptions of Proposition* 2*, there exists a uniform bound in the Euclidean norm for the solutions of* (P_∞) *in the sense that*

$$\forall B \in B(C_\gamma),\ \exists\, T(B) \geq 0 \text{ such that } |\varphi(t)| \leq R(T_0)\ \forall t \geq T(B), \forall \varphi \in D(B),$$

where

$$R(T_0) = \frac{K}{L}\left(\int_{-T_0}^{0} w(s)\mathrm{d}s\right)^{-1} \exp(rT_0), \quad T_0 > 0, \tag{4}$$

provided $\int_{-T_0}^{0} w(s)\mathrm{d}s > 0$.

In particular, there exists a bounded absorbing set for G_∞^+ *and, in addition,* $\bigcup_{t\geq 0} G_\infty^+\,(t,B)$ *is bounded for any bounded set* B *in* C_γ^+*.*

Remark 2.

(i) *The existence of a global attractor* $\mathcal{A}_\infty$ *in* C_γ *for* G^+_∞ *follows from Propositions* 1, 2 *and* 3 *and standard results on dynamical systems.*

(ii) *It is possible to determine an optimal absorbing radius in* (4). *Observe that the mapping* $T_0 \mapsto R(T_0)$ *is continuous with* $\lim_{T_0\to 0^+} R(T_0) = \lim_{T_0\to+\infty} R(T_0) = +\infty$ *and thus achieves its minimum at some* T_0^*, *i.e.*

$$\exists\ T_0^* \in (0,+\infty)\ :\ R(T_0^*) = \min_{T>0} R(T). \tag{5}$$

This value cannot be computed in general, since it would correspond to solve the equation $r\int_{-T_0}^0 w(s)\mathrm{d}s = w(-T_0)$ *(which has sense only almost everywhere).*

(iii) *The following estimate holds:* $\sup_{s\le 0}|\varphi(s)| \le R(T_0)\ \forall\varphi\in\mathcal{A}_\infty$.

3. **Flows, attractors and error bound for DDEs.** Results on the existence of solutions and global attractor for (P_∞) proved in [5] carry over easily to the finite delay problem (P_T). After presenting them, we will compare the solutions of the two problems (P_∞) and (P_T) and finish with an upper semi-continuity result relating their attractors.

Corollary 1. *Assume that* (2) *holds. Then, Problem* (P_T) *has at least one global positive solution. Thus, it is possible to consider, analogously to Definition* 3, *a semi-flow* G_T^+ *on* $C([-T,0];\mathbb{R}_+)$, *which has a compact global attractor* $\mathcal{A}_T \subset C([-T,0];\mathbb{R}_+)$.

Moreover, there exists an extension of (P_T) *to a problem of the form* (P_∞), *that we will call* $(P_{T,\infty})$, *by the way of embedding* $C([-T,0];\mathbb{R})$ *into* C_γ. *Let* $\tilde{x}$ *denote the backward extension through a constant of* $x\in C([-T,0];\mathbb{R})$ *to* C_γ. *Then it is possible to define a semi-flow* $G^+_{T,\infty} : C_\gamma^+ \to P(C_\gamma^+)$ *such that*

$$G_T^+(t,x) = G^+_{T,\infty}(t,\tilde{x}))|_{[-T,0]}, \qquad \forall x\in C([-T,0];\mathbb{R}), \tag{6}$$

and $G^+_{T,\infty}$ *has a global attractor,* $\mathcal{A}_{T,\infty}$, *which satisfies*

$$\mathcal{A}_T = \mathcal{A}_{T,\infty}|_{[-T,0]}. \tag{7}$$

Proof. Firstly, any element $\xi\in C([-T,0];\mathbb{R})$ can be extended backwards as $\tilde{\xi}(\theta) = \xi(-T)$ for every $\theta < -T$ to obtain an element $\tilde{\xi}\in C_\gamma$. Analogously, any function $\tilde{w}\in L^1(-T,0)$ can be extended by zero to the interval $(-\infty,-T)$ to obtain an element of $L^1(\mathbb{R}_-)$ satisfying (3).

Problem (P_T) can thus be embedded into "a problem of the form (P_∞)", so all of the statements up to (6) follow from this argument and the results from the previous section (cf. Propositions 2 and 3).

To prove (7), observe that $A_{T,\infty} = \omega(\mathcal{K})$, w.r.t. the semi-flow $G^+_{T,\infty}$, where

$$\mathcal{K} = \{\phi\in C((-\infty,0];[0,R(T_0^*)])\} \subset B_{C_\gamma^+}(0,R(T_0^*)).$$

The characterization of any omega-limit set ensures that there exist sequences x_n with values in $[0,R(T_0^*)]$ and $t_n\to+\infty$, such that

$$x = \lim_{n\to+\infty} x^n \text{ in } C_\gamma \quad \text{with } x^n \in G^+_{T,\infty}(t_n,\psi^n).$$

But the convergence in C_γ implies that in $C([-T,0];\mathbb{R})$ for any fixed $T>0$, so we conclude that

$$x|_{[-T,0]} = \lim_{n\to+\infty} x^n|_{[-T,0]} \text{ with } x^n|_{[-T,0]} \in G^+_{T,\infty}(t_n,\psi^n)|_{[-T,0]} = G_T^+(t_n,\psi^n|_{[-T,0]}).$$

Therefore $x|_{[-T,0]} \in \mathcal{A}_T$. For the opposite inclusion, take $y \in \mathcal{A}_T$ and we have to prove that $y = \psi|_{[-T,0]}$ for some $\psi \in \mathcal{A}_{T,\infty}$. As before, we know that there exist sequences $\{t_n\}$, increasing to $+\infty$, and $\{\varphi^n\} \subset B_{C([-T,0];\mathbb{R}_+)}(0, R(T_0^*))$ such that

$$y = \lim_{n\to+\infty} y^n \quad \text{with } y^n \in G_T^+(t_n, \varphi^n).$$

We extend the elements φ^n backwards as constants to $\tilde{\varphi}^n$ and consider $G_{T,\infty}^+(t_n, \tilde{\varphi}^n)$. Since $y^n = \tilde{y}^n|_{[-T,0]}$ with $\tilde{y}^n \in G_{T,\infty}^+(t_n, \tilde{\varphi}^n)$, by the asymptotic sequential compactness of $G_{T,\infty}^+$ (Proposition 1) there exists a subsequence $\tilde{\varphi}^{n'}$ such that $\tilde{y}_n'$ converges to some $\psi \in \mathcal{A}_{T,\infty}$.

Then, the $\tilde{y}_n'|_{[-T,0]}$ converge to $\psi|_{[-T,0]}$ and, since $y^{n'} = \tilde{y}^{n'}|_{[-T,0]}$, we conclude that $y = \psi|_{[-T,0]}$. Hence $\mathcal{A}_T = \mathcal{A}_{T,\infty}|_{[-T,0]}$. □

Remark 3. *The uniform estimate for the elements in the attractors given in Remark* 2 (iii) *remains valid for those in $\mathcal{A}_T$ and $\mathcal{A}_{T,\infty}$.*

We aim to compare the attractor A_∞ with the attractors $\mathcal{A}_{T,\infty}$ and $\mathcal{A}_T$ associated to (P_T). For this, we first need to obtain a bound error relating the solutions of these problems, since other approaches such as in [11] require special uniform conditions on the phase space for the whole set of problems under consideration, which are not valid here.

Lemma 1. *Suppose that* (2)-(3) *hold and let f and f_T be the functionals on the right hand sides of (P_∞) and (P_T), respectively.*

Let $T(\varepsilon)$ be the value of T such that $\int_{-\infty}^{-T} w(s)\mathrm{d}s = \varepsilon$. Then, for every uniformly bounded element $\psi \in C_b(\mathbb{R}_-;\mathbb{R})$ there exists a constant $C(\|\psi\|_\infty)$ such that

$$|f(\psi) - f_{T(\varepsilon)}(\psi|_{[-T,0]})| \leq C(\|\psi\|_\infty)\varepsilon. \tag{8}$$

Proof. Subtracting the two functions we have

$$\begin{aligned} |f(\psi) - f_{T(\varepsilon)}(\psi|_{[-T,0]})| &= \left| -K^{-1} r \psi(0) \int_{-\infty}^{-T(\varepsilon)} w(s) P(\psi(s))\mathrm{d}s \right| \\ &\leq K^{-1} r \|\psi\|_\infty \int_{-\infty}^{-T(\varepsilon)} w(s)(C_1\|\psi\|_\infty^m + C_2)\mathrm{d}s. \end{aligned}$$

The lemma is proved and (8) is satisfied with

$$C(\|\psi\|_\infty) = K^{-1} r \|\psi\|_\infty (C_1 \|\psi\|_\infty^m + C_2). \tag{9}$$

□

In anticipation of the numerical approximations, we now introduce a Lipschitz condition on P. This will give uniqueness of solutions inside any "tube" in the next lemma, i.e. a bounded set with uniform bound in the Euclidean sense as in Remark 2 (iii).

Lemma 2. *Assume that the function P is Lipschitz with Lipschitz constant L_P. Then, the functionals f and f_T defined in (P_∞) and (P_T) satisfy the following properties:*

(i) *For any pair* $\psi_1, \psi_2 \in C_\gamma$

$$\begin{aligned} |f(\psi_1) - f(\psi_2)| \quad \leq \quad & r|\psi_1(0) - \psi_2(0)| \\ & + rK^{-1} L_P |\psi_1(0)| \|\psi_1 - \psi_2\|_\gamma \int_{-\infty}^0 w(s) \mathrm{e}^{-\gamma s} \mathrm{d}s \\ & + |\psi_1(0) - \psi_2(0)| \int_{-\infty}^0 w(s) \left| P(\psi_2(s)) \right| \mathrm{d}s \end{aligned} \tag{10}$$

(ii) *For any pair* $\phi_1, \phi_2 \in C([-T, 0]; \mathbb{R})$

$$\begin{aligned} |f_T(\phi_1) - f_T(\phi_2)| \quad \leq \quad & r|\phi_1(0) - \phi_2(0)| + rK^{-1} \int_{-T}^0 w(s) \mathrm{d}s \times \\ & \times \Bigg(|\phi_1(0)| L_p \|\phi_1 - \phi_2\|_{C([-T,0];\mathbb{R})} \\ & \quad + |\phi_1(0) - \phi_2(0)| \max_{B(0, \|\phi_2\|_{C([-T,0];\mathbb{R})})} |P| \Bigg). \end{aligned} \tag{11}$$

Proof. From the definition, (10) follows straightforwardly:

$$\begin{aligned} |f(\psi_1) - f(\psi_2)| \quad = \quad & \Bigg| r\psi_1(0) \left(1 - K^{-1} \int_{-\infty}^0 w(s) P(\psi_1(s)) \mathrm{d}s \right) \\ & - r\psi_2(0) \left(1 - K^{-1} \int_{-\infty}^0 w(s) P(\psi_2(s)) \mathrm{d}s \right) \Bigg| \\ \leq \quad & r|\psi_1(0) - \psi_2(0)| \\ & + rK^{-1} \left| \psi_1(0) \int_{-\infty}^0 w(s) P(\psi_1(s)) \mathrm{d}s - \psi_2(0) \int_{-\infty}^0 w(s) P(\psi_2(s)) \mathrm{d}s \right| \\ = \quad & r|\psi_1(0) - \psi_2(0)| \\ & + rK^{-1} \Bigg| \psi_1(0) \int_{-\infty}^0 w(s) \left(P(\psi_1(s)) - P(\psi_2(s)) \right) \mathrm{d}s \\ & + (\psi_1(0) - \psi_2(0)) \int_{-\infty}^0 w(s) P(\psi_2(s)) \mathrm{d}s \Bigg| \\ \leq \quad & r|\psi_1(0) - \psi_2(0)| + rK^{-1} L_P |\psi_1(0)| \|\psi_1 - \psi_2\|_\gamma \int_{-\infty}^0 w(s) \mathrm{e}^{-\gamma s} \mathrm{d}s \\ & + |\psi_1(0) - \psi_2(0)| \int_{-\infty}^0 w(s) \left| P(\psi_2(s)) \right| \mathrm{d}s. \end{aligned}$$

The second part follows similarly,

$$\begin{aligned} |f_T(\phi_1) - f_T(\phi_2)| \quad = \quad & \Bigg| r\phi_1(0) \left(1 - K^{-1} \int_{-T}^0 w(s) P(\phi_1(s)) \mathrm{d}s \right) \\ & - r\phi_2(0) \left(1 - K^{-1} \int_{-T}^0 w(s) P(\phi_2(s)) \mathrm{d}s \right) \Bigg| \\ \leq \quad & r|\phi_1(0) - \phi_2(0)| \\ & + rK^{-1} \left| \phi_1(0) \int_{-T}^0 w(s) P(\phi_1(s)) \mathrm{d}s - \phi_2(0) \int_{-T}^0 w(s) P(\phi_2(s)) \mathrm{d}s \right|. \end{aligned}$$

Rewriting the term within the absolute value on the last line

$$\phi_1(0)\int_{-T}^{0} w(s)\left[P(\phi_1(s))-P(\phi_2(s))\right]\mathrm{d}s+(\phi_1(0)-\phi_2(0))\int_{-T}^{0} w(s)P(\phi_2(s))\mathrm{d}s,$$

and using the Lipschitz property of P, it immediately yields the sought after result. □

Remark 4.

(i) *Henceforth we will assume that* $C_1 = L_P$, $m = 1$, *and* $C_2 = |P(0)|$ *in Condition* (2).

(ii) *We could go deeper instead of* (10) *and use the bound*

$$\int_{-\infty}^{0} w(s)|P(\psi_2(s))|\mathrm{d}s \leq \|\psi_2\|_\gamma \int_{-\infty}^{0} w(s)\mathrm{e}^{-\gamma s}\mathrm{d}s.$$

However, we prefer to use a different bound (thanks to Remark 2 (iii)*). This will be enough for the Lipschitz character of functional* f.

Corollary 2. *Suppose* P *is Lipschitz with Lipschitz constant* L_P *and that* (2)-(3) *hold. Let* X_1 *and* X_2 *be bounded subsets of* C_γ *and* $C([-T,0];\mathbb{R})$, *respectively, and that* $\sup_{s\leq 0}|\psi(s)| \leq C(X_1)$ *for all* $\psi \in X_1$. *Then,* $f|_{X_1}$ *and* $f_T|_{X_2}$ *are Lipschitz.*

More precisely, the following inequality holds for any pair $\psi_1, \psi_2 \in X_1$*:*

$$\begin{aligned} |f(\psi_1)-f(\psi_2)| \quad \leq \quad & r|\psi_1(0)-\psi_2(0)| \\ & + rK^{-1}L_P C(X_1)\int_{-\infty}^{0} w(s)\mathrm{e}^{-\gamma s}\mathrm{d}s\|\psi_1-\psi_2\|_\gamma \\ & rK^{-1}(L_P C(X_1)+|P(0)|)\|w\|_{L^1}|\psi_1(0)-\psi_2(0)| \end{aligned} \tag{12}$$

Proof. This is a consequence of Lemma 2 and Remark 4 (i). □

We proceed now to estimate solutions of our problems corresponding to the "same initial data".

Proposition 4. *Under the assumptions of Corollary* 2*, consider Problem* (P_∞) *with initial data* $\psi \in C_\gamma$ *satisfying* $|\psi(\theta)| \leq D$ *for all* $\theta < 0$*, and consider the corresponding Problem* (P_T) *for a fixed value* T *with the restricted initial data* $\psi|_{[-T,0]} \in C([-T,0];\mathbb{R})$*. Denote by* x *the solution to* (P_∞)*, and* $\hat{x}$ *the solution to* (P_T)*. Then, there exist positive constants* $d_1 = d_1(r,K,P,D)$, $d_2 = d_2(r,K,P,D,w,T)$ *and* $d_3 = d_3(r,K,P,D,w,T)$ *such that*

$$\|x_t-\hat{x}_t\|_{C([-T,0];\mathbb{R})} \leq d_1\varepsilon\left((r+d_2+d_3)\mathrm{e}^{(r+d_2+d_3)t}-\frac{1}{r+d_2+d_3}\right) \qquad \forall t \geq 0, \tag{13}$$

where $\varepsilon = \int_{-\infty}^{-T} w(s)\mathrm{d}s$*. Moreover, it is possible to find a* $\widehat{C} = \widehat{C}(r,K,P,D,w)$ *independent of* T *such that*

$$\|x_t-\hat{x}_t\|_{C([-T,0];\mathbb{R})} \leq d_1\varepsilon\left(\widehat{C}\mathrm{e}^{\widehat{C}t}-\frac{1}{\widehat{C}}\right) \qquad \forall t \geq 0. \tag{14}$$

Proof. By Lemma 1, with $\varepsilon = \int_{-\infty}^{-T} w(s)\mathrm{d}s$, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-\hat{x}(t)|^2 &= (x(t)-\hat{x}(t))\cdot(f(x_t)-f_T(\hat{x}_t)) \\
&\leq |x(t)-\hat{x}(t)|\left(|f(x_t)-f_T(x_t|_{[-T,0]})|+|f_T(x_t|_{[-T,0]})-f_T(\hat{x}_t)|\right) \\
&\leq d_1\varepsilon|x(t)-\hat{x}(t)|+(r+d_3)|x(t)-\hat{x}(t)|^2 \\
&\quad +d_2|x(t)-\hat{x}(t)|\|x_t-\hat{x}_t\|_{C([-T,0];\mathbb{R})}. \qquad (15)
\end{aligned}$$

In the first term here, we apply inequality (8) with (9) adapted after Remark 4 to obtain

$$d_1 = rK^{-1}D(L_P D + |P(0)|).$$

In the second term we use inequality (11) from Lemma 2 and obtain

$$d_2 = rK^{-1}\left(\int_{-T}^{0} w(s)\mathrm{d}s\right)DL_P, \quad d_3 = rK^{-1}\left(\int_{-T}^{0} w(s)\mathrm{d}s\right)\max_{|x|\leq D}|P|.$$

Cancelling $|x(t)-\hat{x}(t)|$ from both sides we obtain for all $t\geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-\hat{x}(t)| \leq d_1\varepsilon+(r+d_3)|x(t)-\hat{x}(t)|+d_2\|x_t-\hat{x}_t\|_{C([-T,0];\mathbb{R})},$$

which can be integrated to give

$$|x(t)-\hat{x}(t)| \leq d_1\varepsilon t+(r+d_3)\int_0^t|x(\tau)-\hat{x}(\tau)|\mathrm{d}\tau+d_2\int_0^t\|x_\tau-\hat{x}_\tau\|_{C([-T,0];\mathbb{R})}\mathrm{d}\tau.$$

Replacing t by $t+\theta$ for $\theta\in[-T,0]$ and taking the supremum then gives

$$\begin{aligned}
\|x_t-\hat{x}_t\|_{C([-T,0];\mathbb{R})} &\leq d_1\varepsilon t+(r+d_3)\int_0^t|x(\tau)-\hat{x}(\tau)|\mathrm{d}\tau+d_2\int_0^t\|x_\tau-\hat{x}_\tau\|_{C([-T,0];\mathbb{R})}\mathrm{d}\tau \\
&\leq d_1\varepsilon t+(r+d_2+d_3)\int_0^t\|x_\tau-\hat{x}_\tau\|_{C([-T,0];\mathbb{R})}\mathrm{d}\tau.
\end{aligned}$$

Finally, an application of the Gronwall inequality gives (13).

Observe that d_2 and d_3 depend on T, but this can be disregarded as the dependence is through the integral $\int_{-T}^{0} w(s)\mathrm{d}s$, which is majorized by $\|w\|_{L^1}$. Denote $\tilde{d}_2$ and $\tilde{d}_3$ with this upper bound. Then, (14) holds with $\widehat{C}=(r+\tilde{d}_2+\tilde{d}_3)$. □

4. **Comparing attractors: First main result.** Problems (P_∞), (P_T), and $(P_{T,\infty})$ after the trivial embedding in Corollary 1, were proved to have attractors $\mathcal{A}_\infty$, $\mathcal{A}_T$ and $\mathcal{A}_{T,\infty}$ under the semi-flows G_∞^+, G_T^+ and $G_{T,\infty}^+$ respectively.

We seek for an u.s.c. result relating the above attractors. There are two options for comparison in the same phase space: $H^*_{C_\gamma}(\mathcal{A}_{T,\infty},\mathcal{A}_\infty)$ or $H^*_{C([-T,0];\mathbb{R})}(\mathcal{A}_T,\mathcal{A}_\infty|_{[-T,0]})$. We will use Proposition 4 for that purpose.

Theorem 1. *Assume that conditions* (2)-(3) *hold and that P is Lipschitz with Lipschitz constant L_P. Then, the following upper semi-continuous convergence results hold:*

$$\lim_{T\to+\infty} H^*_{C_\gamma}(\mathcal{A}_{T,\infty},\mathcal{A}_\infty)=0,$$

$$\lim_{T\to+\infty} H^*_{C([-T,0];\mathbb{R})}(\mathcal{A}_T,\mathcal{A}_\infty|_{[-T,0]})=0.$$

Proof. In view of Corollary 1 and the relation between the norms used in $C([-T,0];\mathbb{R})$ and C_γ, it is enough to obtain the u.s.c. result for the first distance, i.e., to prove the u.s.c. convergence of $\mathcal{A}_{T,\infty}$ to $\mathcal{A}_\infty$. Now

$$\begin{aligned} H^*_{C_\gamma}(\mathcal{A}_{T,\infty},\mathcal{A}_\infty) &= H^*_{C_\gamma}(G^+_{T,\infty}(t,\mathcal{A}_{T,\infty}),\mathcal{A}_\infty) \\ &\leq H^*_{C_\gamma}(G^+_{T,\infty}(t,\mathcal{K}),\mathcal{A}_\infty) \\ &\leq H^*_{C_\gamma}(G^+_{T,\infty}(t,\mathcal{K}),G^+_\infty(t,\mathcal{K})) + \mathrm{dist}(G^+_\infty(t,\mathcal{K}),\mathcal{A}_\infty). \end{aligned}$$

Fix $\varepsilon > 0$, and consider a time $T_{K,\infty}(\varepsilon)$ such that

$$H^*_{C_\gamma}(G^+_\infty(t,K),\mathcal{A}_\infty) \leq \varepsilon \qquad \forall t \geq T_{K,\infty}(\varepsilon).$$

The problem now is to find a value $T = T(\varepsilon)$ such that

$$H^*_{C_\gamma}(G^+_{T,\infty}(t,K),G^+_\infty(t,K)) \leq \varepsilon \qquad \text{for } t = T(\varepsilon)). \tag{16}$$

Take

$$T = \max\left\{T_0^*, T(\bar\varepsilon), -\frac{1}{\gamma}\ln(\varepsilon R(T_0^*)^{-1}/2)\right\} \tag{17}$$

with $\bar\varepsilon$ such that $d_1\bar\varepsilon(\widehat{C}\mathrm{e}^{\widehat{C}T_{K,\infty}(\varepsilon)} - 1/\widehat{C}) = \varepsilon$, where $\widehat{C}$ is the constant in Proposition 4.

This choice is due to the following facts: Firstly, since it is larger than T_0^*, we are sure that the uniform Euclidean bound (5) can be used in the setup above. Secondly, by (14) in Proposition 4, we can ensure that

$$\|x_t - \hat{x}_t\|_{C([-T(\varepsilon),0];\mathbb{R})} \leq d_1\bar\varepsilon(\widehat{C}\mathrm{e}^{\widehat{C}t} - 1/\widehat{C}) \quad \text{for } T(\bar\varepsilon) \text{ given by } \bar\varepsilon = \int_{-\infty}^{-T(\varepsilon)} w(s)\mathrm{d}s. \tag{18}$$

However, this only provides a bound in $\|\cdot\|_{C([-T(\varepsilon),0];\mathbb{R})}$, but not directly in the $C_\gamma-$norm.

Taking into account that $\|\varphi\|_{C([-T,0];\mathbb{R})} \leq C$ for any $\varphi \in C_\gamma$ implies that

$$\|\varphi\|_\gamma \leq \max\{\sup_{\theta\leq -T} \mathrm{e}^{\gamma\theta}|\varphi(\theta)|, C\}, \tag{19}$$

we should provide a bound for the tail. But the dynamics starting in a bounded set K lives uniformly bounded by $R(T_0^*)$. Therefore, comparison in norm $\|\cdot\|_\gamma$ of two elements from $G^+_{T,\infty}(t,K)$ and $G^+_\infty(t,K)$ is bounded by $2\mathrm{e}^{-\gamma T}R(T_0^*)$. Joining this to (19) and the third element in the maximum in (17) we obtain (16). □

Remark 5. *Establishing a continuous dependence relationship for the attractors seems a more difficult task. The method relying on equi-attraction properties (e.g. cf.* [13]*) and an equi-dissipative property (cf.* [13, Thm.2.3]*) needs a uniformly compact property for the family of parametrized semi-dynamical systems. However, the truncated finite time delay is the parameter in approximating to infinite delay and a uniform compact property is not suitable for this case.*

5. **A numerical scheme.** We have seen above that the attractor for the problem (P_∞) can be approximated in an u.s.c. sense by attractors of the problems (P_T), as $T \to +\infty$. Now we will consider the numerical approximation of these finite delay attractors.

For this we need to assume that the weight function w satisfies some additional properties to those above: specifically w is defined everywhere (not just almost everywhere) with values $w(t) \in [0, \bar{w}]$ for some $\bar{w} > 0$, and for each T there exists Δ_T such that for any $\Delta \le \Delta_T$ and $N_\Delta = T/\Delta$,

$$\max_{j=0,\dots,N_\Delta - 1} \int_{-(j+1)\Delta}^{-j\Delta} |w(\rho) - w(-j\Delta)| \mathrm{d}\rho \le M\Delta^{2p+1}, \tag{20}$$

for some $M > 0$ and $p \in (0, 1)^*$. This holds, for instance, if w is continuous and has uniform modulus of continuity $\omega_w(\Delta) \le M\Delta^{2p}$.

We apply the following adaptation of the Euler scheme with constant step size Δ to the autonomous logistic equation with finite delay (P_T) (T will henceforth be held fixed and the dependence on T will be omitted):

$$x_{n+1} = x_n + rx_{n+1}\Delta \left(1 - \frac{\Delta}{K} \sum_{j=0}^{N_\Delta - 1} w_j \, P\left(x_{n-j}\right)\right), \tag{21}$$

for $n = 0, 1, 2, \dots$. Here $w_j = w(-j\Delta)$ for $j = 0, 1, \dots, N_\Delta - 1$. Thus we have a mixture of the implicit Euler scheme for ODE and the Riemann sum evaluated at the upper end point of each subinterval for the integral term in the DDE.

We can write the numerical scheme (21) in explicit form as

$$x_{n+1} = \rho_\Delta\left(\mathbf{X}_n\right) x_n, \tag{22}$$

where $\mathbf{X}_n$ is given by the column vector $\mathbf{X}_n = (x_n, x_{n-1}, \dots, x_{n-N_\Delta+1})^\top$ and

$$\rho_\Delta\left(\mathbf{X}_n\right) := \left(1 - r\Delta\left(1 - \frac{\Delta}{K} \sum_{j=0}^{N_\Delta - 1} w_j \, P\left(x_{n-j}\right)\right)\right)^{-1}.$$

In order to obtain a discrete time semi-dynamical system, we reformulate the numerical scheme (22) as an autonomous first order vector valued difference equation

$$\mathbf{X}_{n+1} = G_\Delta\left(\mathbf{X}_n\right) := \mathbf{L}\mathbf{X}_n + \rho_\Delta\left(\mathbf{X}_n\right) \mathbf{E}^{(1,1)} \mathbf{X}_n. \tag{23}$$

with the mapping $G_\Delta : \mathbb{R}^{N_\Delta} \to \mathbb{R}^{N_\Delta}$ defined in terms of the N_Δ dimensional vector $\mathbf{X}_n$ and the $N_\Delta \times N_\Delta$ matrices

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{E}^{(1,1)} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

i.e. $\mathbf{L}$ has 1's on the first subdiagonal and zeros elsewhere and $\mathbf{E}^{(1,1)}$ has 1 in the upper left corner and zeros everywhere else.

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*We use $2p$ here instead of p for later convenience.

5.1. **Non-negativity and the existence of an absorbing set.** Since $0 \le Lx \le P(x)$ when $x \ge 0$ and since the $w_j \ge 0$, we see that $\sum_{j=0}^{N_\Delta-1} w_j P(x_{n-j}) \ge 0$ for $\mathbf{X}_n \in \mathbb{R}_+^{N_\Delta}$. Thus $\rho_\Delta(\mathbf{X}_n)$ is well defined and positive for $\mathbf{X}_n \in \mathbb{R}_+^{N_\Delta}$ as long as $r\Delta < 1$, i.e., as long as the step size satisfies

$$\Delta < \frac{1}{r}. \tag{24}$$

In particular, G_Δ maps the nonnegative cone $\mathbb{R}_+^{N_\Delta}$ into itself provided the step size is small enough. Specifically, if $\mathbf{X}_0 \in \mathbb{R}_+^{N_\Delta}$ and if (24) holds, then it follows from (22) that $x_n = 0$ for all $n \ge 0$ if $x_0 = 0$ and that $x_n \ne 0$ for all $n \ge 0$ if $x_0 \ne 0$.

In order to establish the existence of an absorbing set we adapt some ideas in [5]. Define

$$B_\Delta := \frac{\Delta}{2} \sum_{j=0}^{N_\Delta-1} w_j.$$

Lemma 3. *Suppose that $r\Delta < 1$ and that $\mathbf{X}_0 \in \mathbb{R}_+^{N_\Delta}$ satisfies*

$$x_0 \le \frac{K}{LB_\Delta}. \tag{25}$$

Then, the iterates of the numerical scheme (22) *satisfy*

$$x_n \le R_\Delta := (1 - r\Delta)^{-N_\Delta} \frac{K}{LB_\Delta} \qquad \text{for all} \quad n \ge 0.$$

Proof. First we note that

$$\rho_\Delta(\mathbf{X}_n) = \left(1 - r\Delta\left(1 - \frac{\Delta}{K}\sum_{j=0}^{N_\Delta-1} w_j P(x_{n-j})\right)\right)^{-1} \le (1 - r\Delta)^{-1}$$

for any $\mathbf{X}_n \in \mathbb{R}_+^{N_\Delta}$. Hence

$$x_n \le (1 - r\Delta)^{-N_\Delta} x_0 \qquad \text{for} \quad n = 0, 1, \ldots, N_\Delta$$

when $\mathbf{X}_0 \in \mathbb{R}_+^{N_\Delta}$. In particular,

$$x_n \le R_\Delta \qquad \text{for} \quad n = 0, 1, \ldots, N_\Delta$$

if in addition x_0 satisfies (25).

Suppose now that this last inequality does not hold for all $n \ge 0$. Then there is a solution x_n and integers $N_{\Delta,3} > N_{\Delta,2} > N_{\Delta,1} + N_\Delta$ for which

$$x_{N_{\Delta,1}} \le \frac{K}{LB_\Delta}, \qquad x_n > \frac{K}{LB_\Delta} \quad \text{for} \quad n = N_{\Delta,1} + 1, \ldots, N_{\Delta,2}$$

and

$$x_{N_{\Delta,2}} \le R_\Delta, \qquad x_n > R_\Delta \quad \text{for} \quad n = N_{\Delta,2} + 1, \ldots, N_{\Delta,3}.$$

By the properties of P we have

$$\sum_{j=0}^{N_\Delta-1} w_j P(x_{n-j}) \ge L \sum_{j=0}^{N_\Delta-1} w_j x_{n-j}$$

for $\mathbf{X}_n \in \mathbb{R}_+^{N_\Delta}$.

Moreover $n - j > N_{\Delta,1}$ for $n \geq N_{\Delta,2}$ and $j = 0, 1, \ldots, N_\Delta$, so

$$x_{n-j} > \frac{K}{LB_\Delta},$$

which means

$$\sum_{j=0}^{N_\Delta - 1} w_j P(x_{n-j}) \geq L \sum_{j=0}^{N_\Delta - 1} w_j x_{n-j} \geq \frac{K}{B_\Delta} \sum_{j=0}^{N_\Delta - 1} w_j = \frac{2K}{\Delta}.$$

Hence for $n \geq N_{\Delta,2}$ we have

$$\rho_\Delta\left(\mathbf{X}_n\right) = \left(1 - r\Delta\left(1 - \frac{\Delta}{K}\sum_{j=0}^{N_\Delta - 1} w_j P(x_{n-j})\right)\right)^{-1} \leq (1 + r\Delta)^{-1}$$

and consequently

$$x_{N_{\Delta,2}+j} \leq (1 + r\Delta)^{-j} x_{N_{\Delta,2}} \leq (1 + r\Delta)^{-j} R_\Delta < R_\Delta$$

for $j = 1, \ldots, N_{\Delta,3} - N_{\Delta,2}$, which is a contradiction. This completes the proof. □

Theorem 2. *Suppose that* $r\Delta < 1$. *Then the subset* $[0, R_\Delta]^{N_\Delta}$ *of* $\mathbb{R}_+^{N_\Delta}$ *is an absorbing set for the semi-dynamical system generated by the numerical scheme* (23) *in the cone* $\mathbb{R}_+^{N_\Delta}$.

Proof. Let $\mathbf{B}$ be a bounded subset of $\mathbb{R}_+^{N_\Delta}$. The Theorem asserts that there is an $N(\mathbf{B})$ such that

$$\mathbf{X}_n \in [0, R_\Delta]^{N_\Delta}, \qquad n \geq N(\mathbf{B}), \quad \mathbf{X}_0 \in \mathbf{B}.$$

This is certainly true from Lemma 3 if x_0 satisfies inequality (25). Therefore we need only to consider the case of $\mathbf{X}_0 \in \mathbf{B}$ with

$$x_0 > \frac{K}{LB_\Delta}.$$

If $[0, R_\Delta]^{N_\Delta}$ is not an absorbing set, then there exist initial vectors $\mathbf{X}_0^{(k)} \in \mathbf{B}$ with $x_0^{(k)} > \frac{K}{LB_\Delta}$ and a sequence $n_k \to \infty$ such that $x_{n_k}^{(k)} > R_\Delta$. Now $R_\Delta > \frac{K}{LB_\Delta}$ by definition, so for each k we must have

$$x_j^{(k)} > \frac{K}{LB_\Delta} \quad \text{for} \quad j = 0, 1, \ldots, n_k,$$

otherwise Lemma 3 would yield a contradiction. Consequently we have

$$\sum_{j=0}^{N_\Delta - 1} w_j P(x_{n-j}^{(k)}) \geq L \sum_{j=0}^{N_\Delta - 1} w_j x_{n-j}^{(k)} > \frac{K}{B_\Delta} \sum_{j=0}^{N_\Delta - 1} w_j = \frac{2K}{\Delta}$$

for all k and all $n \leq n_k$, from which it follows that

$$\rho_\Delta\left(\mathbf{X}_n\right) \leq (1 + r\Delta)^{-1}, \qquad \forall n \leq n_k.$$

Hence

$$x_j^{(k)} \leq (1 + r\Delta)^{-j + N_\Delta} x_0^{(k)} \qquad \text{for } j = 1, \ldots, n_k,$$

which contradicts (if n_k is large enough) the assumption that $x_{n_k}^{(k)} > R_\Delta$. □

6. **Existence of the numerical attractor and its upper semi-continuous convergence: Second main result.** When the step size is sufficiently small, i.e. satisfies (24), the mapping G_Δ in vector valued version (23) of the numerical scheme is well defined and continuous from the positive cone $\mathbb{R}_+^{N_\Delta}$ into itself and thus generates a discrete time semi-dynamical system on this cone, $\mathbb{R}_+^{N_\Delta}$. Such a restriction can be important in biological problems.

Theorem 2 says that the nonempty compact subset $[0, R_\Delta]^{N_\Delta}$ of $\mathbb{R}_+^{N_\Delta}$ is an absorbing set for the numerical scheme (23) in the cone $\mathbb{R}_+^{N_\Delta}$. By the theory of semi-dynamical systems, this dynamical system thus has a global attractor $\mathbf{A}_\Delta$ in $[0, R_\Delta]^{N_\Delta}$ (i.e. global with respect to the cone $\mathbb{R}_+^{N_\Delta}$).

To establish the upper semi-continuous convergence of the numerical attractor $\mathbf{A}_\Delta$ in $\mathbb{R}_+^{N_\Delta}$ to the attractor $\mathcal{A}_T$ of problem (P_T) in $C([-T,0];\mathbb{R}_+)$, we first need to embed the numerical attractor in the space $C([-T,0];\mathbb{R}_+)$, which we do by piecewise linear interpolation.

Recall that a global attractor consists of entire trajectories, so for any $\mathbf{X}_0$ in $\mathbf{A}_\Delta$, there exists an entire trajectory

$$\mathbf{X}_{n+1} = G_\Delta\left(\mathbf{X}_n\right), \qquad n \in \mathbb{Z},$$

through $\mathbf{X}_0$. Let $\chi_0 : [0,T] \to \mathbb{R}_+$ be the piecewise linear mapping interpolating the points x_{-N_Δ}, $x_{-N_\Delta+1}$, $\ldots$, x_{-1}, x_0, where the first point would be the first component of $\mathbf{X}_{-1}$ and the others are the components of $\mathbf{X}_0$. Obviously, $\chi_0 \in C([-T,0];\mathbb{R}_+)$. Finally, we represent $\mathbf{A}_\Delta$ in $C([-T,0];\mathbb{R}_+)$ through the subset

$$\mathcal{A}_T^{(\Delta)} := \left\{\chi_0 \in C([-T,0];\mathbb{R}_+) \,:\, \exists \mathbf{X}_0 \in \mathbf{A}_\Delta \text{ which } \chi_0 \text{ interpolates linearly}\right\}. \quad (26)$$

We then obtain the upper semi-continuous convergence of the numerical attractor in the following sense:

Theorem 3. *Consider problem* (P_T) *for a fixed* T. *Assume that* (2) *holds, that* P *is Lipschitz with Lipschitz constant* L_P, *and that* (20) *and* (24) *also hold. Then*

$$H^*_{C([-T,0];\mathbb{R})}\left(\mathcal{A}_T^{(\Delta)}, \mathcal{A}_T\right) \to 0 \quad \text{as} \quad \Delta \to 0,$$

where $\mathcal{A}_T$ *is the attractor of problem* (P_T) *and* $\mathcal{A}_T^{(\Delta)}$ *is the embedded numerical attractor defined by* (26).

Proof. We use a contradiction argument and suppose the opposite. Then, there exist $\varepsilon_0 > 0$ and a sequence $\Delta_n \to 0$ such that

$$H^*_{C([-T,0];\mathbb{R}_+)}\left(\mathcal{A}_T^{(\Delta_n)}, \mathcal{A}_T\right) > 3\varepsilon_0 \quad \forall n. \quad (27)$$

As in the continuous time result, we need a comparison of the solutions of the two problems, in this case of problem (P_T) and the numerical scheme (23), which is given by the global discretization error proved in the Appendix (see Lemma 4).

We first need to consider a time interval where apply our proof. By attraction, there exists a $T(\varepsilon_0) > 0$ (w.l.o.g. bigger than T) such that

$$\operatorname{dist}\left(G_T^+(t, B_{C([-T,0];\mathbb{R})}(0,\bar{R})), \mathcal{A}_T\right) < \varepsilon_0, \quad t \ge T(\varepsilon_0), \quad (28)$$

where $\bar{R} = 3\frac{K}{L}e^{rT}\left(\int_{-T}^0 w(s)\mathrm{d}s\right)^{-1}$ is an upper bound of all R_{Δ_n} (see Lemma 3). We pick Δ_n small enough so that $C_{T(\varepsilon_0)+1}\Delta_n^p \le \varepsilon_0$, where $C_{T(\varepsilon_0)+1} = \frac{L}{rC_2}e^{rC_2(T(\varepsilon_0)+1)}$

is the constant appearing in Lemma 4. For notational convenience we now write Δ instead Δ_n. Let also n^*_Δ be the first integer so that $n^*_\Delta \Delta =: T^* \in [T(\varepsilon_0), T(\varepsilon_0)+1)$.

From (27) we deduce that there exists $\phi^{(\Delta)} \in \mathcal{A}_T^{(\Delta)}$ such that

$$\|\phi^{(\Delta)} - \psi\|_{C([-T,0];\mathbb{R})} \geq 3\varepsilon_0 \quad \forall \psi \in \mathcal{A}_T. \tag{29}$$

Denote $\Phi^{(\Delta)} \in \mathcal{A}_T^{(\Delta)}$ the element such that

$$\mathbf{X}_0 := (\Phi^{(\Delta)}(0), \Phi^{(\Delta)}(-\Delta), \ldots, \Phi^{(\Delta)}(-\Delta N_\Delta + 1))^\top$$

satisfies

$$G_\Delta^{n^*_\Delta}(\mathbf{X}_0) = \mathbf{X}_{n^*_\Delta} = (\phi^{(\Delta)}(0), \phi^{(\Delta)}(-\Delta), \ldots, \phi^{(\Delta)}(-\Delta N_\Delta + 1))^\top. \tag{30}$$

(This is possible by the strict invariance of $\mathbf{A}_\Delta$). Finally, let ϕ be the solution of the logistic problem (P_T) with initial value $\Phi^{(\Delta)}$, i.e. $\phi_t = G_T^+(t, \Phi^{(\Delta)})$.

Consider the numerical scheme on the interval $[0, T^*]$. Thanks to the global discretization error bound in Lemma 4 and the boundedness of the attractors (cf. Proposition 3, Corollary 1 and Theorem 2), we have

$$|x_n(t) - \phi(t)| \leq C_{T^*}\Delta^p \leq C_{T(\varepsilon_0)+1}\Delta^p \leq \varepsilon_0, \qquad t \in [0, T^*].$$

where $x_n(t)$ is the piecewise linear interpolation function on $[0, T^*]$ of the numerical iterations x_n with $n = 0, 1, \ldots, n^*_\Delta$. In particular, by (30) the last $N_\Delta + 1$ nodes give rise to $\phi^{(\Delta)}$. This combines with (28) to give (recalling that Δ is actually Δ_n)

$$\left|\phi^{(\Delta_n)} - \psi\right| < 2\varepsilon_0 \quad \text{for some } \psi \in \mathcal{A}_T,$$

which contradicts (29). □

Then, as an immediate consequence of Theorems 1 and 3, we obtain the following result, which roughly speaking reads as an upper semi-continuous convergence of the attractors $\mathcal{A}_T^{(\Delta)}$ towards any segment in $[-T, 0]$ of the attractor $\mathcal{A}_\infty$ as $\Delta \to 0$.

Corollary 3. *Under the assumptions of Theorems* 1 *and* 3*, the embedded numerical attractors* $\mathcal{A}_T^{(\Delta)}$ *converge upper semi-continuously to the truncated set in time* $[-T, 0]$ *of the elements in the attractor* $\mathcal{A}_\infty$ *of problem* (P_∞) *as* $\Delta \to 0$.

Remark 6. *Steady state solutions are easily identified in the logistic model and its approximations. Specifically, the constant solutions equal to* 0 *and the points* $\bar{x}_\infty$ *such that* $P(\bar{x}_\infty)\|w\|_{L^1} = K$ *are always steady state solutions of* G_∞^+ *(at least, provided that* $P(0) \leq K\|w\|_{L^1}^{-1}$*). Similarly, the constant solutions equal to* 0 *and* $\bar{x}_T$ *with* $P(\bar{x}_T)\|w\|_{L^1(-T,0)} = K$ *are steady state solutions of* (P_T)*, while* 0 *and* $\bar{x}_\Delta$ *with* $P(\bar{x}_\Delta) = K\left(\Delta \sum_{j=0}^{N_\Delta - 1} w_j\right)^{-1}$ *are steady states of the numerical scheme. Note that the points (and thus the corresponding constant functions) of the approximate systems converge continuously to their counterparts for* G_∞^+ *as* $T \to +\infty$ *and* $\Delta \to 0$.

Appendix: Error bound for the numerical scheme. We now establish a global discretization error bound for the numerical scheme (23) applied to the problem (P_T) with a fixed finite delay $T > 0$. The exact values of the constants C_1, …, C_{12} which appear below are not essential for the theoretical results of this paper.

Let $x(t)$ be the solution of the finite delay equation corresponding to the initial value $\psi \in C([-T,0];\mathbb{R})$. Then

$$x(t) = x(n\Delta) + r\int_{n\Delta}^{t} x(s)F(s)\,\mathrm{d}s$$

for $t \geq t_n = n\Delta$ with $n = 0, 1, 2, \ldots$, where

$$F(s) = 1 - \frac{1}{K}\int_{-T}^{0} w(\rho)P(x(s+\rho))\mathrm{d}\,\rho = 1 - \frac{1}{K}\int_{s-T}^{s} w(s-\rho)P(x(\rho))\mathrm{d}\,\rho.$$

The corresponding numerical solution x_n of the numercial scheme (23) with initial data $x_{-j} = \psi(-j\Delta)$ for $j = 0, 1, \ldots, N_\Delta - 1$, reads

$$x_{n+1} = x_n + r x_{n+1} F_\Delta(t_n)\,\Delta$$

where

$$F_\Delta(t_n) = 1 - \frac{1}{K}\sum_{j=0}^{N_\Delta - 1} w_j \;\; P\left(x_{n-j}\right)\,\Delta.$$

We consider the linear interpolation function $x_n(t)$ constructed from the numerical scheme and compare this with the solution $x(t)$ of the delay differential equation.

Lemma 4. *Under the assumptions of Theorem* 3 *the linear interpolation function* $x_n(t)$ *of the iterates* $\{x_n\}_n \geq 0$ *of the numerical scheme* (22) *defined by*

$$x_n(t) := x_n + r\int_{n\Delta}^{t} x_{n+1}F_\Delta(t_n)\,\mathrm{d}s, \qquad t \in [n\Delta, (n+1)\Delta],$$

converges to the solution $x(t)$ *uniformly on any finite interval* $[0, T^*]$ *with at least order* p*, i.e. with the following error bound:*

$$|x(t) - x_n(t)| \leq \frac{L}{rC_2}\,\mathrm{e}^{rC_2T^*}\,\Delta^p \qquad \forall t \in [0, T^*].$$

Proof. Consider the difference of both functions $x(t)$ and $x_n(t)$:

$$\begin{aligned}
& |x(t) - x_n(t)| \\
\leq\; & |x(n\Delta) - x_n| + r\int_{n\Delta}^{t} |x(s)F(s) - x_{n+1}F_\Delta(t_n)|\,\mathrm{d}s \\
\leq\; & |x(n\Delta) - x_n| + r\int_{n\Delta}^{t} |x(s)|\,|F(s) - F_\Delta(t_n)|\,\mathrm{d}s + r\int_{n\Delta}^{t} |x(s) - x_{n+1}|\,|F_\Delta(t_n)|\,\mathrm{d}s \\
\leq\; & |x(n\Delta) - x_n| + rC_1\int_{n\Delta}^{t} |F(s) - F_\Delta(t_n)|\,\mathrm{d}s + rC_2\int_{n\Delta}^{t} |x(s) - x_{n+1}|\,\mathrm{d}s \\
\leq\; & |x(n\Delta) - x_n| + rC_1\int_{n\Delta}^{t} |F(s) - F_\Delta(t_n)|\,\mathrm{d}s + rC_2\int_{n\Delta}^{t} |x(s) - x_n(s)|\,\mathrm{d}s \\
& \quad + rC_2\int_{n\Delta}^{t} |x_n(s) - x_{n+1}|\,\mathrm{d}s
\end{aligned}$$

where constants C_1 and C_2 come from uniform bounds on $|x(t)|$, $|F(t)|$ and $|x_n|$, $|x_{n+1}|$, $|F_\Delta(t_n)|$ on the time interval $[0, T^*]$ under consideration. (By continuity the solutions starting in a common bounded set are bounded over any finite time interval).

Now we have

$$|x_n(s) - x_{n+1}| \leq r\,|x_{n+1}|\,|F_\Delta(t_n)|\,((n+1)\Delta - s) \leq rC_3\,((n+1)\Delta - s),$$

so

$$rC_2 \int_{n\Delta}^{t} |x_n(s) - x_{n+1}|\,\mathrm{d}s \leq r^2 C_4\,\Delta^2,$$

from which it follows that

$$\begin{aligned} |x(t) - x_n(t)| \quad &\leq \quad |x(n\Delta) - x_n| + rC_1 \int_{n\Delta}^{t} |F(s) - F_\Delta(t_n)|\,\mathrm{d}s \\ &\quad + rC_2 \int_{n\Delta}^{t} |x(s) - x_n(s)|\,\mathrm{d}s + r^2 C_4\,\Delta^2. \end{aligned} \tag{31}$$

In addition

$$\begin{aligned} |F(s) - F_\Delta(t_n)| \quad &\leq \quad \frac{1}{K} \sum_{j=0}^{N_\Delta - 1} \int_{-(j+1)\Delta}^{-j\Delta} |w(\rho)P(x(s+\rho)) - w_j P(x_{n-j})|\,\mathrm{d}\rho \\ &\leq \quad \frac{1}{K} \sum_{j=0}^{N_\Delta - 1} \int_{-(j+1)\Delta}^{-j\Delta} \big(|w(\rho) - w_j|\,P(x(s+\rho)) \\ &\qquad + w_j\,|P(x(s+\rho)) - P(x_{n-j})|\big)\,\mathrm{d}\rho \\ &\leq \quad \sum_{j=0}^{N_\Delta - 1} \left(C_5 \int_{-(j+1)\Delta}^{-j\Delta} |w(\rho) - w_j|\,\mathrm{d}\rho \right. \\ &\qquad \left. + \frac{\bar{w} L_P}{K} \int_{-(j+1)\Delta}^{-j\Delta} |x(s+\rho) - x_{n-j}|\,\mathrm{d}\rho \right), \end{aligned} \tag{32}$$

where the first bound C_5 in last inequality comes from the boundedness of $P(x(s+\rho))$, and L_P is the Lipschitz constant for P.

Observe that the first term at the end of (32) can be controlled by $C_5 N_\Delta M \Delta^{2p+1}$ thanks to assumption (20), and the second term can be treated as follows (recall that $s \in [n\Delta, t] \subset [n\Delta, (n+1)\Delta])$:

$$\begin{aligned}
&\sum_{j=0}^{N_\Delta-1} \int_{-(j+1)\Delta}^{-j\Delta} |x(s+\rho) - x_{n-j}| \,\mathrm{d}\rho \\
= &\sum_{j=0}^{N_\Delta-1} \int_{s-(j+1)\Delta}^{s-j\Delta} |x(r) - x_{n-j}| \,\mathrm{d}r \\
\leq &\sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(r) - x_{n-j}| \,\mathrm{d}r \\
\leq &\sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} \left(|x(r) - x_{n-j}(r)| + |x_{n-j}(r) - x_{n-j}|\right) \mathrm{d}r \\
\leq &\sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(r) - x_{n-j}(r)| \,\mathrm{d}r + C_6 N_\Delta \Delta^2.
\end{aligned}$$

Now, taking into account the last inequality in (32) and that $N_\Delta \Delta = T$, we deduce

$$|F(s) - F_\Delta(t_n)| \quad \leq \quad C_7 \left(\Delta^{2p} + \Delta\right) + \frac{\overline{w} L_P}{K} \sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(\rho) - x_{n-j}(\rho)| \,\mathrm{d}\rho.$$

Thus, putting this into (31) we have

$$\begin{aligned}
&|x(t) - x_n(t)| \\
\leq \;& |x(n\Delta) - x_n| + rC_2 \int_{n\Delta}^{t} |x(s) - x_n(s)| \,\mathrm{d}s + r^2 C_4 \,\Delta^2 \\
&+ rC_1 \int_{n\Delta}^{t} \left(C_7 \left(\Delta^{2p} + \Delta\right) + \frac{\overline{w} L_P}{K} \sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(\rho) - x_{n-j}(\rho)| \,\mathrm{d}\rho \right) \mathrm{d}s \\
\leq \;& |x(n\Delta) - x_n| + rC_2 \int_{n\Delta}^{t} |x(s) - x_n(s)| \,\mathrm{d}s \\
&+ C_8 \,\Delta \sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(\rho) - x_{n-j}(\rho)| \,\mathrm{d}\rho + C_9 \left(\Delta^{2p} + \Delta\right) \Delta
\end{aligned}$$

We need a form of Gronwall inequality for

$$\begin{aligned}
|x(t) - x_n(t)| \quad \leq \quad & |x(n\Delta) - x_n| + rC_2 \int_{t_n}^{t} |x(s) - x_n(s)| \,\mathrm{d}s \\
&+ C_8 \,\Delta \sum_{j=0}^{N_\Delta-1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(s) - x_{n-j}(s)| \,\mathrm{d}s + C_9 \left(\Delta^{2p} + \Delta\right) \Delta.
\end{aligned} \tag{33}$$

To solve this we can suppose that the initial value $\xi \in C([-T, 0]; \mathbb{R})$ is in fact Lipschitz continuous with uniform constant L. This is not a great restriction as the solutions become Lipschitz after the delay time has elapsed and we are dealing with solutions in the attractor for this comparison study. We proceed inductively, first treating the case $n = 0$. We have

$$|x(s) - x_{n-j}(s)| \leq L\Delta \qquad \forall s \in [t_{n-j-1}, t_{n-j+1}], \tag{34}$$

for $j = 1, \ldots, N_\Delta$, when $n = 0$.

Then, splitting in (33) the sum term as

$$\sum_{j=1}^{N_\Delta - 1} \int_{t_{n-j-1}}^{t_{n-j+1}} |x(s) - x_{n-j}(s)|\, \mathrm{d}s + \int_{t_{n-1}}^{t_n} |x(s) - x_{n-j}(s)|\, \mathrm{d}s + \int_{t_n}^{t_{n+1}} |x(s) - x_{n-j}(s)|\, \mathrm{d}s$$

and using (34) and the bound $\int_{t_n}^{t_{n+1}} |x(s) - x_{n-j}(s)|\, \mathrm{d}s \leq C_{10}\Delta$, for $n = 0$ the inequality (33) yields

$$\begin{aligned} |x(t) - x_0(t)| &\leq rC_2 \int_0^t |x(s) - x_n(s)|\, \mathrm{d}s + 2C_8\, TL\, \Delta^2 + C_{11}\left(\Delta^{2p} + \Delta\right)\Delta \\ &= C_{12}(\Delta^{2p} + \Delta)\Delta + rC_2 \int_0^t |x(s) - x_0(s)|\, \mathrm{d}s \end{aligned} \tag{35}$$

for $0 \leq t \leq \Delta$ (since $x(0) = x_0$ the initial term disappears here). The Gronwall inequality yields

$$|x(t) - x_0(t)| \leq C_{12}\left(\Delta^{2p} + \Delta\right)\Delta\, \mathrm{e}^{rC_2\Delta}. \tag{36}$$

We pick $\Delta > 0$ small enough so that

$$C_{12}\left(\Delta^{2p} + \Delta\right)\Delta \mathrm{e}^{rC_2\Delta} \leq L\Delta^{1+p},$$

which is possible if $p < 1$. Obviously $L\Delta^{1+p} \leq L\Delta^1$, so

$$|x(\Delta) - x_0(\Delta)| \leq L\Delta^{1+p} \leq L\Delta.$$

We repeat the argument for $n = 1$, the main difference now being that we have to include the "initial condition" at time t_1 in the inequality (33). This leads to a similar expression to (35), namely

$$|x(t) - x_1(t)| \leq |x(\Delta) - x_1| + C_{12}\left(\Delta^{2p} + \Delta\right)\Delta + rC_2 \int_\Delta^t |x(s) - x_1(s)|\, \mathrm{d}s,$$

so

$$|x(t) - x_1(t)| \leq L\Delta^{1+p} + L\Delta^{1+p}\mathrm{e}^{-rC_2\Delta} + rC_2 \int_\Delta^t |x(s) - x_1(s)|\, \mathrm{d}s$$

for $\Delta \leq t \leq 2\Delta$. The Gronwall inequality then gives

$$|x(t) - x_1(t)| \leq L\Delta^{1+p}\left(1 + \mathrm{e}^{-rC_2\Delta}\right)\mathrm{e}^{rC_2\Delta} = L\Delta^{1+p}\left(\mathrm{e}^{rC_2\Delta} + 1\right).$$

We can show inductively that we have

$$|x(t) - x_n(t)| \leq L\left(1 + \mathrm{e}^{rC_2\Delta} + \ldots + \mathrm{e}^{nrC_2\Delta}\right)\Delta^{1+p} \leq \frac{L}{rC_2}\,\mathrm{e}^{rC_2T^*}\,\Delta^p$$

for $n\Delta \leq t \leq (n+1)\Delta$, where $n = 0, 1, \ldots, \lfloor T^*/\Delta \rfloor$, so we have convergence with at least order p over the interval $[0, T^*]$, i.e.

$$|x(t) - x_n(t)| \leq \frac{L}{rC_2}\,\mathrm{e}^{rC_2T^*}\,\Delta^p$$

for $n = 0, 1, \ldots, \lfloor T^*/\Delta \rfloor$. The proof is finished. □

Remark 7. *Knowledge of the exact order of the numerical scheme* (22) *is not essential for our results in this paper. We note that if* $p \geq 1/2$, *then we can bound the expression in* (36) *by* $C_{13}\Delta^2$ *instead of* $L\Delta^{p+1}$ *and it follows that the numerical scheme has order* 1 *instead of order* p. *This is the best we can expect from the numerical scheme* (22) *which is a composite of the implicit Euler scheme and the rectangle rule for evaluating the integral. Higher order methods here should result in a higher order numerical scheme provided the weighting function* w *in the integral term is sufficiently smooth.*

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REFERENCES

[1] A. Bellen and M. Zennaro, *Numerical Methods for delay Differential Equations,* Oxford University Press, Oxford, 2003.

[2] T. Caraballo and J.A. Langa, On the upper semicontinuity of cocycle attractors for nonautonomous and random dynamical systems, Dynamics of Continuous, Discrete and Impulsive Systems, **10** (2003), 491–514.

[3] T. Caraballo, J. A. Langa and J. Robinson, Attractors for differential equations with variable delays, J. Math. Anal. Appl. **260** (2001), 421–438.

[4] T. Caraballo, P. Marín-Rubio and J. Valero, Autonomous and non-autonomous attractors for differential equations with delays, J. Diff. Eqns. **208** (2005), no. 1, 9-41.

[5] T. Caraballo, P. Marín-Rubio and J. Valero, Attractors for differential equations with unbounded delays. Submitted.

[6] B.M. Garay, A brief survey on the numerical dynamics for functional differential equations, in *Modeling and Computations in Dynamical Systems*, Editors: E.J. Doedel, G. Domokos and I.G. Kevrekidis, World Scientific Publ., Singapore, 2006; pp. 33–46.

[7] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. **21** (1978), 11-41.

[8] J. K. Hale and S. M. V. Lunel, *Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, New York: Springer-Verlag, 1993.

[9] G. Hines, Upper semicontinuity of the attractor with respect to parameter dependent delays, J. Diff. Eqns. **123** (1995), 56–92.

[10] K. Ito and F. Kappel, Approximation of infinite delay and Volterra type equations, Numer. Math. **54** (1989), 405–444.

[11] P. Kloeden, Upper semi continuity of attractors of retarded delay differential equations in the delay, Bulletin Aus. Math. Soc.. **73** (2006), 299–306.

[12] P.E. Kloeden and J. Lorenz, Stable attracting sets in dynamical systems and in their one-step discretizations, SIAM J. Numer. Anal. **23** (1986), 986–995.

[13] Li Desheng and P. Kloeden, Equi-attraction and the continuous dependence of attractors on parameters, Glasg. Math. J. **46** (2004), 131–141.

[14] V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, Set-Valued Anal. **6** (1998), 83-111.

[15] A. M. Stuart and A. R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge Universite Press, 1996.

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