Non–Linear Stochastic Partial Differential Equations with delays: Existence and uniqueness of solutions

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Clasificación A.M.S.: 60H, 35K.

1. Introduction

The main aim of this paper is to study stochastic PDE's with delay terms. In fact, we prove existence and uniqueness of solution (in Itô's sense) for a rather general type of stochastic PDEs with non linear monotone operators and with delays. We deal with the following stochastic parabolic equation:

$$(1) \qquad \begin{cases} dx(t) + [A(t,x(t)) + B(t,x(\tau(t))) + f(t)]\; dt = [C(t,x(\rho(t))) + g(t)]\, dw_t\,, & t>0 \\ x(0) = x_0\,, \end{cases}$$

where $A(t,.)$, $B(t,.)$, $C(t,.)$ are families of operators in Hilbert spaces, non linear eventually, and satisfying a monotonicity condition; w_t is a Hilbert valued Wiener process, and τ , ρ are delay functions.

When there are not delays ($\tau(t) = \rho(t) = 0$), the equation (1) has been studied: in the case $B = C = 0$, for A non linear, in Bensoussan [2] and Curtain [5], and for some type of non linear operators A, in Bensoussan–Temam [3] and Marcus [7]; in the case $C \neq 0$, $B = 0$, for linear A and C, in Balakrishann [1], for linear A and non linear C in Dawson [6], and for non linear monotone A and Lipschitz continuous C in Pardoux [8].

In the case with deviating arguments, Real [9] studies a rather general case when all of the operators are linear and there exists a term which is a non continuous martingale. However, we have not found in the literature the case we are going to analyze here.

We will adapt to our problem one of the most important method for solving non linear PDEs: the monotonicity method. Pardoux [8] also used an adaptation of that method for another type of non linear monotone equations: when $B = 0$ and without delays.

2. Statement of the problem and the main results

The theory of stochastic integrals in Hilbert spaces is well developed (see [8], for example).

We consider the classical pair of real separable Hilbert spaces V , H satisfying $V \hookrightarrow H$ (injection continuous and dense).

We will denote by $\|.\|$, $|.|$ and $\|.\|_*$ the norms in V , H and V' respectively; by $\langle .,. \rangle$ the duality product between V', V , and by $(.,.)$ the scalar product in H .

Let us fix $T > 0$ and, let w_t be a Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and taking values in the separable Hilbert space K , with incremental covariance operator W . Let $(\mathcal{F}_t)_{t\geq 0}$ be the σ-algebra generated by $\{w_s, 0 \leq s \leq t\}$, then w_t is a martingale relative to $(\mathcal{F}_t)_{t\geq 0}$.

As an abuse of notation, we also use $|.|$ for the norm in the linear continuous operator space $\mathcal{L}(K,H)$.

We denote by $I^p(0,T;V)$, for $p > 1$, the space of V–valued processes $(x(t))_{t\in[0,T]}$ (we will write $x(t)$ for short) measurable (from $[0,T]\times\Omega$ in V), and satisfying:

i) $x(t)$ is $\mathcal{F}_t$–measurable a.e. in t (in the sequel, we will write a.e.t.)

ii) $E\int_0^T |x_t|^p \, dt < +\infty$.

For short, we shall write $L^2(\Omega; C(-h,T;H))$ instead of $L^2(\Omega, \mathcal{F}, dP; C(-h,T;H))$.

Let $A(t,.): V \to V'$ be a family of non linear operators defined a.e.t., and let $p > 1$. We make the following hypotheses:

(a.1) Coercivity: $\exists\, \alpha > 0$, $\lambda \in R$: $2\langle A(t,x), x\rangle + \lambda |x|^2 \geq \alpha \|x\|^p$, $\forall x \in V$, a.e.t.

(a.2) Monotonicity: $2\langle A(t,x) - A(t,y), x-y\rangle + \lambda |x-y|^2 \geq 0$, $\forall x, y \in V$, a.e.t.

(a.3) Boundedness: $\exists\, \beta > 0$: $\|A(t,x)\|_* \leq \beta \|x\|^{p-1}$, $\forall x \in V$, a.e.t.

(a.4) Hemicontinuity: $\theta \in R \to \langle A(t, x+\theta y), z\rangle \in R$ is continuous $\forall x, y, z \in V$, a.e.t.

(a.5) Measurability: $t \in (0,T) \to A(t,x) \in V'$ is Lebesgue − measurable $\forall x \in V$, a.e.t.

Let $B(t,.): H \to H$ be a family of operators defined a.e.t., and satisfying:

(b.1) $B(t,0) = 0$

(b.2) Lipschitz condition: $\exists\, k_1$: $|B(t,x) - B(t,y)| \leq k_1 |x-y|$, $\forall x, y \in H$, a.e.t.

(b.3) Measurability: $t \in (0,T) \to B(t,x) \in H$ is Lebesgue–measurable, $\forall x \in V$.

And let $C(t,.): H \to \mathcal{L}(K,H)$ be another family defined a.e.t. and verifying:

(c.1) $C(t,0) = 0$

(c.2) Lipschitz condition: $\exists\, k_2$: $|C(t,x) - C(t,y)| \leq k_2 |x-y|$, $\forall x, y \in H$, a.e.t.

(c.3) Measurability: $t \in (0,T) \to C(t,x) \in \mathcal{L}(K,H)$ is Lebesgue–measurable $\forall x \in H$.

We also consider two measurable functions (of delay) $\rho, \tau : [0,T] \to [0,T]$, such that

$$0 \leq \rho(t), \tau(t) \leq t, \quad \forall t \in [0,T]. \qquad (\rho.\tau)$$

For f, g we suppose that

$$f \in I^2(0,T;H),\, g \in I^2(0,T;\mathcal{L}(K,H)). \qquad \text{(f.g)}$$

And finally, we are given an initial value $x_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$.

Now, we state the following problem:

$$(PC) \qquad \begin{cases} \textit{To find a process } x \in I^p(0,T;V) \cap L^2(\Omega; C(0,T;H)) \textit{ such that :} \\ x(t) + \int_0^t [A(s,x(s)) + B(s, x(\tau(s))) + f(s)]\, ds \\ \qquad = x_0 + \int_0^t [C(s, x(\rho(s))) + g(s)]\, dw_s, \quad P-\text{a.s.}, \ \forall t \in [0,T]. \end{cases}$$

The main result we prove is the following theorem

Theorem 1

Assume the precedent conditions. Then, there exists a unique solution of (PC) *in* $I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$.

Proof. (See [4]) Uniqueness follows from Ito's formula and Gronwall's inequality. For the existence, we consider the equations

$$(*) \qquad x^1(t) + \int_0^t \left[A(s, x^1(s)) + \frac{\lambda}{2} x^1(s)\right] ds + \int_0^t f(s)\, ds = x_0 + \int_0^t g(s)\, dw_s$$

$$(**) \qquad x^{n+1}(t) + \int_0^t \left[A(s, x^{n+1}(s)) + \frac{\lambda}{2} x^{n+1}(s)\right] ds + \int_0^t B(s, x^n(\tau(s)))\, ds + \int_0^t f(s)\, ds$$
$$= x_0 + \int_0^t \frac{\lambda}{2} x^n(s)\, ds + \int_0^t C(s, x^n(\rho(s)))\, dw_s + \int_0^t g(s)\, dw_s, \ \forall n = 1, 2, 3, ...$$

and we prove that there exists a sequence of solutions for $(*) - (**)$, $\{x^n\}_{n \geq 1} \subset I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$.

Last, we prove that the sequence $\{x^n\}$ is convergent in $I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$, and the limit process is the solution of (PC). ∎

Remark 1.– We observe that theorem 1 also holds when V is a separable and reflexive Banach space with $V \hookrightarrow H$.

Remark 2.– We note that theorem 1 holds when ρ, τ take negative values.

Theorem 2

Assume the hypotheses in theorem 1*, but changing* $(\rho.\tau)$ *by the following:*

$$\exists\, h > 0 \text{ such that } \quad -h \le \tau(t), \rho(t) \le t\,, \quad \forall t \in [0,T]\,,$$

and let ψ *be a process such that* $\psi \in I^p(-h,0;V) \cap L^2(\Omega; C(-h,0;H))$ *(where these spaces are defined in the obvious manner, setting* $\mathcal{F}_t = \mathcal{F}_0$, $\forall t \in [-h,0]$*). Then, there exists a unique process* $x \in I^p(-h,T;V) \cap L^2(\Omega; C(-h,T;H))$ *such that,*

$$(PC)' \qquad \begin{cases} x(t) + \int_0^t [A(s,x(s)) + B(s,x(\tau(s))) + f(s)]\; ds \\ \qquad = \psi(0) + \int_0^t [C(s,x(\rho(s))) + g(s)]\; dw_s\,, \quad P - \text{a.s.}, \quad \forall t \in [0,T], \\ x(t) = \psi(t)\,, \quad t \in (-h,0] \end{cases}$$

Proof. See Caraballo [4] ∎

Remark 3.– Some examples are given in Caraballo [4] in order to justify the results.

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To appear in Collectanea Mathematica