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NON–AUTONOMOUS AND RANDOM ATTRACTORS FOR DELAY RANDOM SEMILINEAR EQUATIONS WITHOUT UNIQUENESS

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Abstract. We first prove the existence and uniqueness of pullback and random attractors for abstract multi-valued non-autonomous and random dynamical systems. The standard assumption of compactness of these systems can be replaced by the assumption of asymptotic compactness. Then, we apply the abstract theory to handle a random reaction-diffusion equation with memory or delay terms which can be considered on the complete past defined by $\mathbb{R}^-$. In particular, we do not assume the uniqueness of solutions of these equations.

1. Introduction.

The intention of this article is to study the asymptotic behaviour of *multi-valued* non–autonomous and random dynamical systems. The long-time behaviour of these systems can be expressed by terms like pullback attractor and random attractor. The theories of these attractors are now well established as have been extensively developed over the last one and a half decades (see, e.g. Caraballo *et al.* [13], Cheban [17], Chueshov [19], Crauel and Flandoli [20], Flandoli and Schmalfuß [21], Kloeden [24], Kloeden and Schmalfuß [25], Robinson [28], Schmalfuß[30], amongst many others). Pullback and/or random attractors have

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proven to be appropriate concepts to describe the long–time behaviour of many dynamical systems arising in science, especially those exhibiting non-autonomous (see also Chepyzhov and Vishik [18]) and/or random features.

In this article we assume very weak assumptions on the dynamical systems under study. We deal with multi–valued systems connected with the fact that the standard assumption of compactness on this system is replaced by an asymptotic compactness hypothesis. We would also like to stress that, within a random set-up, it is more complicated to show that a multi–valued non–autonomous dynamical system is a random dynamical system. In this sense our article is a generalization of the results in Bates *et al.* [4] where asymptotically compact single valued systems are studied.

Therefore, we are interested in a general model which can cover several of the previously mentioned situations at the same time. In other words, our non-autonomous or random partial differential equations will cover non–linearities with very weak assumptions where non-uniqueness of solutions may happen as well as some hereditary (memory terms) properties, being the delay eventually infinite. Then one of the main difficulties in the random case is due to the fact that the *natural* phase space to be considered is not separable in general. On the other hand, solution operators are only asymptotically compact.

We also include in our theory several variants as those containing some hereditary characteristics as finite or bounded delays (see e.g. [14], [7], [6], [15], [26]) or others with non-uniqueness of solutions or modelled by differential inclusions (Caraballo *et al.* [10], [11], [12]).

Our first aim is to develop a joint theory for both multi-valued non-autonomous and random dynamical systems, pointing out the main differences between both frameworks. Needless to say that a partial differential equation coming from a stochastic partial differential equation with *additive white noise* after having performed a suitable transformation or change of variable, is non-autonomous. Thus, the theory of pullback attractors for non-autonomous dynamical systems can be applied to analyse the long–time behaviour. However, random or stochastic models usually need additional measurability properties in order to be well-posed. This introduces an additional and important difference into the analysis. However, dealing with differential equations with white noise terms would go beyond the content of this article. We refer to the forthcoming article [8] for more details.

Consequently, we have structured the content of the paper as follows. In Section 2 we include some preliminaries concerning the definitions of multi-valued non-autonomous dynamical systems (MNDS) and multi-valued random dynamical systems (MRDS) which turns to be an MNDS with an additional measurability property. We also prove a sufficient condition guaranteeing that an MNDS becomes an MRDS. Section 3 is devoted to prove a general result for the existence and uniqueness of pullback and random attractors for abstract MNDS. The crucial property ensuring this is the pullback asymptotic compactness. In Section 4, a rather general semilinear non-autonomous/random partial differential equation containing (eventually) infinite delays, and which may be related to a semilinear reaction-diffusion equation with memory and with random coefficients, is considered. We prove the existence of globally defined solutions and that these generate an MNDS which is to be studied in the next section. Indeed, in Section 5 we first prove the existence of a pullback attractor for the MNDS, and when the parameter space is a Polish space and has a probability structure, we are able to prove that the

MNDS is in fact an MRDS which possesses a random attractor. Some illustrative examples are finally included in Section 7.

2. **Preliminaries.**

In what follows we give some basic definitions for set–valued non–autonomous and random dynamical systems and formulate sufficient conditions for the existence of a pullback attractor for these systems which is a random set if the non–autonomous perturbation is a noise.

Non–autonomous dynamical systems are systems under influence of a non–autonomous perturbation. If this non–autonomous perturbation is a noise term then we have a random dynamical system. We are going to describe these systems in the following.

A pair (Ω, θ) where $\theta = (\theta_t)_{t\in\mathbb{R}}$ is a flow on Ω:

$$\begin{aligned}&\theta : \mathbb{R}\times\Omega \to \Omega\\ &\theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t\theta_\tau \quad \text{for } t,\, \tau \in \mathbb{R}\end{aligned}$$

is called a *non–autonomous perturbation*. As an example which describes typical non–autonomous perturbations we consider $\Omega = \mathbb{R}$ and $\theta_t\tau = t + \tau$ for $\tau = \omega \in \Omega$, $t \in \mathbb{R}$.

Let $\mathcal{P} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On this probability space we consider a *measurable* non–autonomous flow θ :

$$\theta : (\mathbb{R}\times\Omega, \mathcal{B}(\mathbb{R})\otimes\mathcal{F}) \to (\Omega, \mathcal{F}).$$

In addition, $\mathbb{P}$ is supposed to be ergodic with respect to θ, which means that every θ_t-invariant set has measure zero or one, $t \in \mathbb{R}$. Hence $\mathbb{P}$ is invariant with respect to θ_t. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ which is the model for a noise is called a *metric dynamical system*.

If we replace in the definition of a metric dynamical system the probability space $\mathcal{P}$ by its completion $\mathcal{P}^c := (\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ the above measurability property is not true in general, see Arnold [1] Appendix A. But for fixed $t \in \mathbb{R}$ we have that the mapping

$$\theta_t : (\Omega, \bar{\mathcal{F}}) \to (\Omega, \bar{\mathcal{F}})$$

is measurable.

We also mention the following well known ergodic theorem.

Theorem 2.1. *Let Y be a real random variable in L_1. Then*

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t Y(\theta_\tau\omega)d\tau = \mathbb{E}Y$$

on a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of measure one.

Outside this set of measure one we will replace the values of Y by $\mathbb{E}Y$ so that this version of Y has the above limit for *all* $\omega \in \Omega$.

From now on, let $X = (X, d_X)$ be a Polish space.

Let $D : \omega \to D(\omega) \in 2^X$ be a multi–valued mapping. The set of multi–functions $D : \omega \to D(\omega) \in 2^X$ with closed and non–empty images is denoted by $C(X)$**.** Let also denote by $P_f(X)$ the set of all non-empty closed subsets of the space X. Thus, it is equivalent to write that D is in $C(X)$, or $D : \Omega \to P_f(X)$.

Let $D:\omega\to D(\omega)$ be a multi–valued mapping in X over $\mathcal{P}$. Such a mapping is called a *random set* if

$$\omega\to\inf_{y\in D(\omega)} d_X(x,y)$$

is a random variable for every $x\in X$. It is well known that a mapping is a random set if and only if for every open set O in X the inverse image $\{\omega : D(\omega)\cap O\neq\emptyset\}$ is measurable, i.e., it belongs to $\mathcal{F}$ (see Hu and Papageorgiou [23, Proposition 2.1.4]).

Clearly, all this is also valid if we replace $\mathcal{P}$ by $\mathcal{P}^c$ and $\mathcal{F}$ by $\overline{\mathcal{F}}$. Further, along the paper, if we do not specify which probability space we are using ($\mathcal{P}$ or $\mathcal{P}^c$), it will mean that the result is valid for both cases.

It is also evident that if D is a random set with respect to $\mathcal{P}$, then it is also random with respect to $\mathcal{P}^c$.

We now formulate properties for random sets that will be needed in the following (see Castaing and Valadier [16] Chapter III and Hu and Papageorgiou [23] Chapter 2.2).

Lemma 2.2. *(i) Let $(D_n)_{n\in\mathbb{N}}$ be a family of random sets in $C(X)$. Then*

$$\overline{\bigcup_{n\in\mathbb{N}} D_n(\omega)}$$

is a random set in $C(X)$. If in addition $(D_n)_{n\in\mathbb{N}}$ is decreasing and every sequence (x_n) with $x_n\in D_n(\omega)$ is pre–compact, then

$$C(\omega):=\bigcap_{n\in\mathbb{N}} D_n(\omega)$$

is non–empty and measurable.

(ii) Let D be a random set in $C(X)$. Then, there exists a countable number of random variables Y_n, $n\in\mathbb{N}$, such that $Y_n(\omega)\in D(\omega)$ for all $\omega\in\Omega$ and

$$D(\omega)=\overline{\bigcup_{n\in\mathbb{N}} Y_n(\omega)}.$$

(iii) D is a random set with respect to $\mathcal{P}^c$ if and only if the graph of D

$$\mathrm{Gr}(D):=\{(\omega,x)\in\Omega\times X : x\in D(\omega)\}$$

belongs to $\overline{\mathcal{F}}\otimes\mathcal{B}(X)$.

We now introduce non–autonomous and random dynamical systems.

Definition 2.3. A multi–valued map $U:\mathbb{R}^+\times\Omega\times X\to P_f(X)$ is called a multi–valued non–autonomous dynamical system (MNDS) if

i) $U(0,\omega,\cdot)=\mathrm{id}_X$,
ii) $U(t+\tau,\omega,x)\subset U(t,\theta_\tau\omega,U(\tau,\omega,x))$ *(cocycle property) for all* $t,\tau\in\mathbb{R}^+,x\in X,\omega\in\Omega$.

It is called a strict MNDS if

iii) $U(t+\tau,\omega,x)=U(t,\theta_\tau\omega,U(\tau,\omega,x))$ *for all* $t,\tau\in\mathbb{R}^+,x\in X,\omega\in\Omega$.

An MNDS is called a multi–valued random dynamical system (MRDS) if the multi–valued mapping

$$(t,\omega,x)\to U(t,\omega,x)$$

is $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(X)$ measurable, i.e. $\{(t,\omega,x): U(t,\omega,x)\cap O\neq\emptyset\}\in\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(X)$ for every open set O of the topological space X.

For the above composition of multi–valued mappings we use that for any non–empty set $V\subset X$, $U(t,\omega,V)$ is defined by

$$U(t,\omega,V)=\bigcup_{x_0\in V}U(t,\omega,x_0).$$

We also note that the above measurability hypothesis is not standard at least for *single–valued* random dynamical system. However, for MRDS it is more difficult to derive measurability than for single valued systems.

We now introduce some topological properties of the MNDS U, but we first recall the definition of *Hausdorff semi-distance* of two non-empty sets A, B:

$$\mathrm{dist}_X(A,B)=\sup_{x\in A}\inf_{y\in B}d_X(x,y).$$

Definition 2.4. $U(t,\omega,\cdot)$ is called upper–semicontinuous at x_0 if for every neighborhood $\mathcal{U}$ of the set $U(t,\omega,x_0)$ there exists $\delta>0$ such that if $d_X(x_0,y)<\delta$ then

$$U(t,\omega,y)\in\mathcal{U}.$$

$U(t,\omega,\cdot)$ is called upper–semicontinuous if it is upper–semicontinuous at every x_0 in X.

Remark 1. (i) Note that if a mapping $U(t,\omega,\cdot)$ is upper–semicontinuous at x_0, then for all $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\mathrm{dist}_X(U(t,\omega,y),U(t,\omega,x_0))\leq\varepsilon,$$

for any y satisfying $d_X(y,x_0)\leq\delta(\varepsilon)$.

(ii) The converse is true when $U(t,\omega,x_0)$ is compact, see Aubin and Cellina [2].

It is not difficult to extend Definition 2.4 if we consider the upper–semicontinuity with respect to all variables assuming that Ω is a Polish space.

We now formulate a general condition ensuring that an MNDS defines an MRDS. We need the particular assumption that Ω is a Polish space and $\mathcal{F}$ the associated Borel–σ–algebra.

Lemma 2.5. *Let Ω be a Polish space and let $\mathcal{F}$ be the Borel–σ–algebra. Suppose that $(t,\omega,x)\mapsto U(t,\omega,x)$ is upper–semicontinuous. Then this mapping is measurable in the sense of Definition 2.3.*

Proof. Thanks to Proposition 1.2.5 in Hu and Papageorgiou [23], we have that for each closed subset $C\subset X$, the set

$$U^{-1}(C)=\{(t,\omega,x): U(t,\omega,x)\cap C\neq\emptyset\}$$

is closed, and thus is a Borel set in $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(X)=\mathcal{B}(\mathbb{R}^+\times\Omega\times X)$. This implies that the inverse of a closed set is measurable and then, by Theorem 2.2.4 in [23], the map is measurable. □

3. **Non–autonomous and random attractors for MNDS.** In this section we generalize the concept of pullback and random attractors to the case of an MNDS and prove a general result for the existence and uniqueness of attractors.

As a preparation we need the following definitions.

A multi–valued mapping D is said to be *negatively, strictly, or positively invariant* for the MNDS U if

$$D(\theta_t\omega) \underset{\supset}{\overset{\subset}{=}} U(t,\omega,D(\omega)) \quad \text{for } \omega\in\Omega,\ t\in\mathbb{R}^+.$$

Let $\mathcal{D}$ be the family of multi–valued mappings with values in $C(X)$. We say that a family $K\in\mathcal{D}$ is *pullback $\mathcal{D}$-attracting* if for every $D\in\mathcal{D}$

$$\lim_{t\to+\infty}\operatorname{dist}_X(U(t,\theta_{-t}\omega,D\left(\theta_{-t}\omega\right)),K(\omega))=0,\ \text{for all } \omega\in\Omega.$$

$B\in\mathcal{D}$ is said to be *pullback $\mathcal{D}$-absorbing* if for every $D\in\mathcal{D}$ there exists $T=T(\omega,D)>0$ such that

$$U(t,\theta_{-t}\omega,D\left(\theta_{-t}\omega\right))\subset B(\omega),\ \text{for all } t\geq T. \tag{1}$$

The following definition provides the main objective of this article. We have to introduce a particular set system (see Schmalfuß [31]): let $\mathcal{D}$ be a set of multi–valued mappings in $C(X)$ satisfying the inclusion closed property: suppose that $D\in\mathcal{D}$ and let D' be a multi–valued mapping in $C(X)$ such that $D'(\omega)\subset D(\omega)$ for $\omega\in\Omega$, then $D'\in\mathcal{D}$.

Definition 3.1. A family $A\in\mathcal{D}$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS U if it satisfies:

1. $A\left(\omega\right)$ is compact for any $\omega\in\Omega$;
2. A is pullback $\mathcal{D}$-attracting;
3. A is negatively invariant.

A is said to be a strict global pullback $\mathcal{D}$-attractor if the invariance property in the third item is *strict*.

A natural modification of this definition for MRDS is

Definition 3.2. Suppose U is an MRDS and suppose that the properties of Definition 3.1 are satisfied. In particular, we consider $\mathcal{D}$ to be a system of random sets. In addition, we suppose that A is a random set, with respect to $\mathcal{P}^c$. Then A is called a random global pullback $\mathcal{D}$-attractor.

Remark 2. (i) In contrast to the theory of random attractors for single valued random dynamical systems we have weaker assumptions on the measurability of A. Of course, it is desirable to obtain that A is a random set with respect to $\mathcal{P}$, but usually we need stronger assumptions in the applications to obtain this property.

(ii) For the last Definition 3.2 we assume that the system $\mathcal{D}$ from Definition 3.1 consists of random sets. So the inclusion property has to be checked only for random sets D'.

A consequence of the pullback convergence and invariance of $\mathbb{P}$ is that it reflexes the *forward* convergence to the attractor

$$\mathbb{P}-\lim_{t\to+\infty}\operatorname{dist}_X(U(t,\omega,D(\omega)),A(\theta_t\omega))=0$$

for all sets D such that $\omega \to U(t, \omega, D(\omega))$ is measurable for $t \geq 0$. Indeed, we have to replace in the formula for the pullback convergence ω by $\theta_t\omega$. However, this is only true in the weaker convergence in probability. There exist counterexamples which show that in general the forward convergence does not hold almost surely (see [1] page 488).

The main tool to prove the existence of an attractor is the pullback-omega-limit set for the MNDS U. For some multi–valued mappings D we define a pullback-omega-limit set as an ω–dependent set $\Lambda(D, \omega)$ given by

$$\Lambda(D, \omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} U(t, \theta_{-t}\omega, D(\theta_{-t}\omega))}.$$

This set is obviously closed, but in general it can be empty. It is not difficult to prove that $y \in \Lambda(D, \omega)$ if and only if there exist $t_n \to +\infty$ and $y_n \in U(t_n, \theta_{-t_n}\omega, D(\theta_{-t_n}\omega))$ such that

$$\lim_{n \to +\infty} y_n = y.$$

We then have the following lemma, which is a generalization of Theorem 6 and Lemma 8 in Caraballo *et al.* [10] to the case in which we consider the family $\mathcal{D}$ instead of the bounded sets of X.

Lemma 3.3. *Suppose that the MNDS $U(t, \omega, \cdot)$ is upper–semicontinuous for $t \geq 0$ and $\omega \in \Omega$. Let B be a multi–valued mapping such that the MNDS is asymptotically compact with respect to B i.e. for every sequence $t_n \to +\infty$, $\omega \in \Omega$ every sequence $y_n \in U(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$ is pre–compact.*

Then for $\omega \in \Omega$ the pullback–omega–limit set $\Lambda(B, \omega)$ is non–empty, compact, and

$$\lim_{t \to +\infty} \operatorname{dist}_X(U(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \Lambda(B, \omega)) = 0, \tag{2}$$

$$\Lambda(B, \theta_t\omega) \subset U(t, \omega, \Lambda(B, \omega)), \text{ for all } t \geq 0. \tag{3}$$

Proof. Take an arbitrary sequence $y_n \in U(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$ with $t_n \to +\infty$. Then, since U is pullback–asymptotically compact with respect to B, there exists a converging subsequence and its limit y belongs to $\Lambda(B, \omega)$, so that $\Lambda(B, \omega)$ is non–empty.

Let us prove the compactness of $\Lambda(B, \omega)$. For any sequence $\{y_n\} \subset \Lambda(B, \omega)$ there exist $t_n \to +\infty$ and $z_n \in U(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$, such that $d_X(y_n, z_n) < \frac{1}{n}$. Using again the pullback asymptotic compactness of U we obtain the existence of a converging subsequence $z_{n_k} \to z \in \Lambda(B, \omega)$. It follows that $y_{n_k} \to z$, so that $\Lambda(B, \omega)$ is compact.

The attracting property (2) is proved by contradiction. If this is not the case, then there exist $\varepsilon > 0$ and $y_n \in U(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$, $t_n \to +\infty$, for which

$$\operatorname{dist}_X(y_n, \Lambda(B, \omega)) > \varepsilon.$$

Again, since U is pullback–asymptotically compact with respect to B, it follows that (up to a subsequence) $y_n \to y \in \Lambda(B, \omega)$, which is not possible.

We prove now that (3) holds. If $y \in \Lambda(B, \theta_t\omega)$, then there exist sequences $y_n \in U(t_n, \theta_{-t_n}\theta_t\omega, x_n)$, $x_n \in B(\theta_{-t_n}\theta_t\omega)$, $t_n \to +\infty$, such that $y_n \to y$. For $t_n \geq t$, the composition property implies

$$U(t_n, \theta_{-t_n}\theta_t\omega, x_n) \subset U(t, \omega, U(t_n - t, \theta_{t-t_n}\omega, x_n)),$$

and then $y_n \in U(t,\omega,z_n)$, where $z_n \in U(t_n - t, \theta_{t-t_n}\omega, x_n)$. As before, accurate to a subsequence, $z_n \to z \in \Lambda(B,\omega)$. Since $x \mapsto U(t,\omega,x)$ is upper–semicontinuous with closed values, we have

$$y \in U(t,\omega,z) \subset U(t,\omega,\Lambda(B,\omega)).$$

□

This is the main theorem of this section:

Theorem 3.4. *Assume the hypotheses in Lemma 3.3. In addition, suppose that $B \in \mathcal{D}$ is pullback $\mathcal{D}$–absorbing. Then, the set A given by*

$$A(\omega) := \Lambda(B,\omega)$$

is a pullback $\mathcal{D}$-attractor. Furthermore, A is the unique element from $\mathcal{D}$ with these properties.

In addition, if U is a strict MNDS then A is strictly invariant.

Proof. We have to prove that

$$\lim_{t\to+\infty} \mathrm{dist}_X(U(t,\theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) = 0 \quad \text{for every } D \in \mathcal{D}. \tag{4}$$

Indeed by (2) for every $\varepsilon > 0, \omega \in \Omega$ there exists a $T(\omega,\varepsilon)$ such that for $t \geq T(\omega,\varepsilon)$

$$\mathrm{dist}_X(U(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), A(\omega)) < \varepsilon.$$

But for every $D \in \mathcal{D}$ we have that

$$U(T(\theta_{-t}\omega, D), \theta_{-t-T(\theta_{-t}\omega,D)}\omega, D(\theta_{-t-T(\theta_{-t}\omega,D)}\omega)) \subset B(\theta_{-t}\omega)$$

so that, by Definition 2.3,

$$\mathrm{dist}_X(U(t,\theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) < \varepsilon$$

for t large.

The third property from Definition 3.1 follows from (3).

Since

$$U(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for } t \geq T(\omega, B),$$

we have the relation $A(\omega) \subset B(\omega)$ for each $\omega \in \Omega$, so that $A \in \mathcal{D}$. But this shows that A is unique. Indeed suppose we have another pullback D–attractor A', then as

$$A'(\omega) \subset U(t,\theta_{-t}\omega, A'(\theta_{-t}\omega))$$

and

$$\lim_{t\to+\infty} \mathrm{dist}_X\,(U(t,\theta_{-t}\omega, A'(\theta_{-t}\omega)), A(\omega)) = 0,$$

we have that $A'(\omega) \subset A(\omega)$. Exchanging A and A' it follows that $A = A'$.

Finally, assume that U is a strict MNDS. Then A pullback attracts itself, so that

$$\begin{aligned} U(t,\omega,A(\omega)) &\subset U(t,\omega,U(\tau,\theta_{-\tau}\omega, A(\theta_{-\tau}\omega))) \\ &= U(t+\tau, \theta_{-\tau}\omega, A(\theta_{-\tau}\omega)) \\ &= U(t+\tau, \theta_{-t-\tau}\theta_t\omega, A(\theta_{-t-\tau}\theta_t\omega)), \end{aligned}$$

for any $\tau \geq 0$, and then for each $\varepsilon > 0$ there is $T(\varepsilon,\omega,\tau)$ such that

$$\mathrm{dist}_X\,(U(t,\omega,A(\omega)), A(\theta_t\omega)) < \varepsilon, \text{ if } t+\tau \geq T.$$

As $\varepsilon > 0$ is arbitrary we obtain that $U(t,\omega,A(\omega)) \subset A(\theta_t\omega)$, as required. □

With respect to the measurability of Definition 3.2 and the applications in Section 6 we suppose for the next lemma a complete probability space $\mathcal{P}^c$. However, the result is also valid for the space $\mathcal{P}$.

Lemma 3.5. *Under the assumptions in Theorem 3.4, let $\omega \to U(t,\omega,B(\omega))$ be a random set for $t \geq 0$. Assume also that $U(t,\omega,B(\omega))$ is closed for all $t \geq 0$ and $\omega \in \Omega$. Then the set A introduced in Theorem 3.4 is measurable.*

Proof. We introduce

$$C(\omega) = \bigcap_{\tau\in\mathbb{Z}^+} \overline{\bigcup_{t\geq\tau, t\in\mathbb{Z}^+} U(t,\theta_{-t}\omega, B(\theta_{-t}\omega))}$$

which is a random set by Lemma 2.2 (i).

We show that $A(\omega) = C(\omega)$. We just know from the construction in Theorem 3.4 that $A(\omega) \subset B(\omega)$. On account of the properties of A we observe that

$$C(\omega) \supset \bigcap_{\tau\in\mathbb{Z}^+} \overline{\bigcup_{t\geq\tau, t\in\mathbb{Z}^+} \underbrace{U(t,\theta_{-t}\omega, A(\theta_{-t}\omega))}_{\supset A(\omega)}} \supset A(\omega).$$

On the other hand we have that $C(\omega) \subset A(\omega)$ from

$$\overline{\bigcup_{t\geq\tau} U(t,\theta_{-t}\omega, B(\theta_{-t}\omega))} \supset \overline{\bigcup_{t\geq\tau, t\in\mathbb{Z}^+} U(t,\theta_{-t}\omega, B(\theta_{-t}\omega))}$$

and the fact that the family

$$\tau \in \mathbb{R} \to \overline{\bigcup_{t\geq\tau} U(t,\theta_{-t}\omega, B(\theta_{-t}\omega))}$$

is decreasing. □

4. **Mild solutions for abstract random evolution equations with non-uniqueness.** In this section we consider the following evolution equation

$$\frac{dy}{dt} = Ay + f(\theta_t\omega, y_t). \tag{5}$$

Here we suppose that A is the generator of a C_0 contraction semigroup $(S(t))_{t\geq 0}$ on a separable Banach space $(H, \|\cdot\|)$:

$$\|S(t)x\| \leq \|x\| e^{-\alpha t}, \quad \text{for some } \alpha > 0 \quad \text{and every } t \geq 0.$$

We need that the operators $S(t)$ for $t > 0$ are compact. The non–linear term f depends on ω and on a delay term

$$y_t(s) = \begin{cases} y(t+s) & \text{for } s \in [-t, 0] \\ x_0(s+t) & s < -t \end{cases}$$

where $t \geq 0$. Here x_0 is a given continuous function on $\mathbb{R}^-$ with values in H. According to the function x_0 we can equip (5) with an initial condition

$$y(t) = x_0(t), \qquad t \leq 0.$$

Before describing the assumptions on f, we first introduce the function space

$$C_\gamma = \{u \in C((-\infty,0]; H) : \lim_{\tau\to-\infty} u(\tau)\, e^{\gamma\tau} \text{ exists}\},$$

where $\gamma > \alpha$, and set $\|u\|_\gamma := \sup_{\tau\in(-\infty,0]} e^{\gamma\tau}\|u(\tau)\| < \infty$. This is a separable Banach space [22, p.15].

The main purpose of the article is to show the existence of a *random* attractor for the dynamical system generated by (5). However, we interpret this system at first as an MNDS which has a pullback attractor.

Suppose that $x_0 \in C_\gamma$. In what follows we assume that

$$f : \Omega \times C_\gamma \to H$$

satisfies:

a): the mapping
$$\omega \to f(\omega, y)$$
is $\mathcal{F}$ measurable for an arbitrarily fixed $y \in C_\gamma$,

b): the mapping
$$y \to f(\omega, y)$$
is continuous from C_γ into H for any fixed ω.

Assume that there exist two non–negative functions $c_i : \Omega \to \mathbb{R}$, $i = 1, 2$, which are measurable with respect to $\mathcal{F}$. Also, assume that

$$t \to c_1(\theta_t\omega)$$

is integrable with respect to every finite interval (a, b) and subexponentially growing for $t \to \pm\infty$ for $\omega \in \Omega$. This is the so called temperedness property if we have a random variable c_1. For c_2 we suppose that $\mathbb{E}c_2 < \infty$ (so that $c_2(\theta_t\omega)$ is locally integrable by the ergodic theorem) and also that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t c_2(\theta_\tau\omega)d\tau = \bar{c}_2.$$

By the ergodicity assumption and Theorem 2.1 we have that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t c_2(\theta_\tau\omega)d\tau = \mathbb{E}c_2 =: \bar{c}_2$$

on a $(\theta_t)_{t\in\mathbb{R}}$–invariant set of full measure. Let us replace outside this set (which has measure zero) the values of $c_2(\omega)$ by $\bar{c}_2$.

Suppose

$$\|f(\omega, y)\| \le c_1(\omega) + c_2(\omega)\|y\|_\gamma \quad \text{for } \omega \in \Omega \quad \text{and } y \in C_\gamma. \tag{6}$$

We emphasize that we do not assume that f is Lipschitz continuous in any sense.

We now prove that for every x_0 (5) has at least one solution. However, we will interpret the solution of (5) as a mild solution:

Definition 4.1. For $\omega \in \Omega$ and $x_0 \in C_\gamma$, a function $[0, T] \ni t \to y_t \in C_\gamma$ is called a mild solution of (5) with initial function x_0 if

$$y_t(s) = \begin{cases} S(t+s)x_0(0) + \int_0^{s+t} S(t+s-\tau)f(\theta_\tau\omega, y_\tau)d\tau & : \quad s \in [-t, 0] \\ x_0(s+t) & : \quad s < -t, \end{cases} \tag{7}$$

for all $t \in [0, T]$.

Note that in the last definition we express that the mild solution has the state space C_γ, not H. Alternatively, we can define a mild solution to (5) with state space H setting $s = 0$

$$y(t) = \begin{cases} S(t)x_0(0) + \int_0^t S(t-\tau)f(\theta_\tau\omega, y_\tau)d\tau & : \quad t \ge 0 \\ x_0(t) & : \quad t < 0. \end{cases}$$

We now introduce the following notation. Let $y \in C([0,T];H)$ with $y(0) = x_0(0)$ and $x_0 \in C_\gamma$. Then, for $\tau \in [0,T]$, we denote by $y \vee_\tau x_0$ the mapping from $\mathbb{R}^-$ to H defined by

$$y \vee_\tau x_0(s) := \begin{cases} y(\tau+s) & : \quad s \in (-\tau, 0] \\ x_0(\tau+s) & : \quad s \le -\tau. \end{cases}$$

We observe that for such function y the integral in (7) is well defined. Indeed, it is well known (see [22, p.15]) that the map $t \to y_t$ is continuous from $[0,T]$ into C_γ. Thus it is measurable. Then, since f is measurable w.r.t. the first variable and continuous w.r.t. the second variable, $(\tau,\omega) \to \theta_\tau\omega$ is also measurable and the spaces H, C_γ are separable, we obtain that the composition $\tau \to f(\theta_\tau\omega, y_\tau)$ is measurable (see [3, Lemma 8.2.3]). In a similar way we obtain then that $\tau \to S(t-\tau)f(\theta_\tau\omega, y_\tau)$ is measurable. Therefore, by (6) and the properties of the semigroup S, the integral in (7) exists.

Theorem 4.2. *Suppose that the above assumptions on S and f are satisfied. Then for every interval $[0,T]$ (5) possesses a solution in C_γ in the sense of Definition 4.1.*

Proof. We show the local existence of solutions of (7). The global existence then follows by Theorem 4.3 below. The proof follows Pazy [27] Theorem 6.2.1. Let us fix some $x_0 \in C_\gamma$, $\omega \in \Omega$.

Consider

$$B(R) = \{y \in C([0,T];H) : y(0) = x_0(0), \sup_{s\in[0,T]} \|x_0(0) - y(s)\| \le R\}.$$

$B(R)$ is a convex and bounded set in $C([0,T];H)$. For some sufficiently small $T > 0$ we introduce the mapping

$$\mathcal{T}_T : B(R) \to C([0,T];H),$$
$$\mathcal{T}_T(y)[t] := S(t)x_0(0) + \int_0^t S(t-\tau)f(\theta_\tau\omega, y \vee_\tau x_0)d\tau, \quad t \in [0,T].$$

We note that $\mathcal{T}_T(y) \in C([0,T];H)$ because $\tau \to \|f(\theta_\tau\omega, y \vee_\tau x_0)\| \in L_1([0,T])$. To see that the operator $\mathcal{T}_T$ maps $B(R)$ into itself, for appropriate R and T, we note that

$$\begin{aligned} \|f(\theta_r\omega, y \vee_r x_0)\| \le & c_1(\theta_r\omega) + c_2(\theta_r\omega) \sup_{\varrho\in[0,r]} e^{\gamma(\varrho-r)}\|y(\varrho)\| \\ & + c_2(\theta_r\omega) \sup_{\varrho\le -r} e^{\gamma\varrho}\|x_0(r+\varrho)\| \\ \le & c_1(\theta_r\omega) + c_2(\theta_r\omega) \sup_{\varrho\in[0,T]} \|y(\varrho)\| \\ & + c_2(\theta_r\omega)e^{-\gamma r} \sup_{\varrho\le -r} e^{\gamma(\varrho+r)}\|x_0(r+\varrho)\| \\ \le & c_1(\theta_r\omega) + c_2(\theta_r\omega) \sup_{\varrho\in[0,T]} \|y(\varrho)\| + c_2(\theta_r\omega)\|x_0\|_\gamma. \end{aligned} \tag{8}$$

The term $\sup_{\varrho\in[0,T]} \|y(\varrho)\|$ is bounded by $\|x_0\|_\gamma + R$. In addition, $\|S(t-\tau)x\| \le e^{-\alpha(t-\tau)}\|x\|$ so that, for small $T > 0$ (depending on ω), we have $\mathcal{T}_T(B(R)) \subset B(R)$.

On account of the continuity of $C_\gamma \ni y \to f(\omega, y)$ and (6) we obtain by Lebesgue's majorant theorem that $\mathcal{T}_T$ is continuous on $B(R)$ with the topology of $C([0,T];H)$.

To see that $\mathcal{T}_T$ is compact we first note that the sets

$$Z_t := \{z = \mathcal{T}_T(y)[t], y \in B(R)\}, \quad t \in [0,T]$$

are pre–compact. This is trivially true for $t = 0$. For $t > 0$ we introduce for sufficiently small $\varepsilon > 0$

$$\begin{aligned}\mathcal{T}_T^\varepsilon(y)[t] &= S(t)x_0(0) + \int_0^{t-\varepsilon} S(t-\tau)f(\theta_\tau\omega, y \vee_\tau x_0)d\tau \\ &= S(t)x_0(0) + S(\varepsilon)\int_0^{t-\varepsilon} S(t-\tau-\varepsilon)f(\theta_\tau\omega, y \vee_\tau x_0)d\tau.\end{aligned}$$

By (8) and the integrability conditions on c_1, c_2

$$\sup_{y\in B(R)} \| \int_0^{t-\varepsilon} S(t-\tau-\varepsilon)f(\theta_\tau, y \vee_\tau x_0)d\tau\|$$

is finite for appropriate $\varepsilon > 0$ so that $\mathcal{T}_T^\varepsilon(B(R))[t]$ is pre-compact by the compactness of $S(\varepsilon)$. Then for every $\varepsilon' > 0$ we have an $\varepsilon > 0$ such that

$$\|\mathcal{T}_T^\varepsilon(y)[t] - \mathcal{T}_T(y)[t]\| \le \int_{t-\varepsilon}^t e^{-\alpha(t-\tau)}\|f(\theta_\tau\omega, y \vee_\tau x_0)\|d\tau \le \varepsilon'$$

uniformly for $y \in B(R)$ so that Z_t is totally bounded, hence pre–compact.

To apply the Arzelà–Ascoli theorem we show that $\mathcal{T}_T(y)$, $y \in B(R)$, is equicontinuous. Notice that, for $t_2 > t_1 > 0$,

$$\begin{aligned}\|\mathcal{T}_T(y)[t_2] - \mathcal{T}_T(y)[t_1]\| \le &\|(S(t_2) - S(t_1))x_0(0)\| \\ &+ \int_0^{t_1} \|S(t_2-\tau) - S(t_1-\tau)\|\|f(\theta_\tau\omega, y \vee_\tau x_0)\|d\tau \\ &+ \int_{t_1}^{t_2} \|S(t_2-\tau)\|\|f(\theta_\tau\omega, y \vee_\tau x_0)\|d\tau.\end{aligned}$$

Since $S(t)$ is a compact operator for $t > 0$ we have that the mapping $t \to S(t)$ is norm-continuous for $t > 0$. Lebesgue's majorant theorem together with (8) imply the equicontinuity for $t > 0$. Similar arguments hold for $t_1 = t = 0$. Indeed, in the above formula the second term on the right hand side disappears for $t_1 = 0$.

The Schauder theorem gives the existence of a fixed point of $\mathcal{T}_T$ which is a local solution for (5).

To see that (7) has a solution for every $T > 0$ we refer to the following Theorem 4.3 and Remark 3. One consequence of this theorem is that explosions are impossible. □

The following theorem is needed to derive particular a priori estimates for the solution of (5).

Theorem 4.3. *Let y_t be any mild solution of (7) on $[0,T)$, $T \in \mathbb{R}^+ \cup \{+\infty\}$with a initial function $x_0 \in C_\gamma$. Then y_t satisfies the inequality*

$$\|y_t\|_\gamma \le e^{-\alpha t + \int_0^t c_2(\theta_\tau\omega)d\tau}\|x_0\|_\gamma + \int_0^t e^{-\alpha(t-\tau) + \int_\tau^t c_2(\theta_q\omega)dq} c_1(\theta_\tau\omega)d\tau. \qquad (9)$$

Proof. We have

$$\|y_t\|_\gamma \leq \max\left(\sup_{s\leq -t}\|x_0(s+t)\|e^{\gamma s}, \sup_{s\in[-t,0]}\|S(t+s)x_0(0)\|e^{\gamma s}\right.$$
$$\left. + \sup_{s\in[-t,0]}\|\int_0^{s+t} S(s+t-\tau)f(\theta_\tau\omega, y_\tau)d\tau\|e^{\gamma s}\right).$$

The first term on the right hand side of the last inequality is equal to

$$\sup_{s\leq 0}\|x_0(s)\|e^{\gamma(s-t)} = e^{-\gamma t}\|x_0\|_\gamma.$$

For the second term we have the estimate

$$\sup_{s\in[-t,0]}\|S(s+t)x_0(0)\|e^{\gamma s} \leq \sup_{s\in[-t,0]} e^{-\alpha(s+t)}\|x_0(0)\|e^{\gamma s}$$
$$\leq e^{-\alpha t}\sup_{s\in[-t,0]} e^{(-\alpha+\gamma)s}\|x_0(0)\| \leq e^{-\alpha t}\|x_0(0)\|.$$

The third term can be estimated as follows

$$\sup_{s\in[-t,0]}\|\int_0^{s+t} S(t+s-\tau)f(\theta_\tau\omega, y_\tau)d\tau\|e^{\gamma s}$$
$$\leq \sup_{s\in[-t,0]}\int_0^{s+t} e^{\alpha(-t-s+\tau)}(c_1(\theta_\tau\omega)+c_2(\theta_\tau\omega)\|y_\tau\|_\gamma)d\tau e^{\gamma s}$$
$$\leq \int_0^t e^{-\alpha(t-\tau)}c_1(\theta_\tau\omega)d\tau + \int_0^t e^{-\alpha(t-\tau)}c_2(\theta_\tau\omega)\|y_\tau\|_\gamma d\tau.$$

Collecting all these estimates we have that

$$\|y_t\|_\gamma \leq \max\left(e^{-\gamma t}\|x_0\|_\gamma, e^{-\alpha t}\|x_0(0)\| + \int_0^t e^{-\alpha(t-\tau)}c_1(\theta_\tau\omega)d\tau\right.$$
$$\left. + \int_0^t e^{-\alpha(t-\tau)}c_2(\theta_\tau\omega)\|y_\tau\|_\gamma d\tau\right)$$
$$\leq e^{-\alpha t}\|x_0\|_\gamma + \int_0^t e^{-\alpha(t-\tau)}(c_1(\theta_\tau\omega)+c_2(\theta_\tau\omega)\|y_\tau\|_\gamma)d\tau.$$

We obtain the desired inequality by the Gronwall lemma. □

Remark 3. A consequence of this theorem is that, in case of a *finite* maximal interval of existence $[0, t_{max})$ of a solution, there are no explosions:

$$\limsup_{t\uparrow t_{max}}\|y_t\|_\gamma < \infty.$$

But the case of such a finite interval carrying a bounded solution can be excluded similar to Pazy [27] Theorem 6.2.2 applying (6). Hence for every $x_0 \in C_\gamma$, $\omega \in \Omega$ every solution of (5) is global.

5. **Pullback attractors for the equation with infinite delay.** Along this section we assume the same conditions on S and f given at the beginning of Section 4.

We define the multi–valued mapping $U(t, \omega, x_0)$ to be the set of mild solutions (7) in the sense of Definition 4.1 at time $t \geq 0$, that is, $U(t, \omega, x_0) = \cup\, y_t$, where the union is taken within the set of mild solutions $[0, +\infty) \ni t \to y_t \in C_\gamma$ such that $y_0 = x_0$. We stress here that we know from the last section that every local solution can be extended to a global solution.

Lemma 5.1. *The map U is a strict MNDS. In particular, for any fixed $t \geq 0$ we have $U(t,\omega,D(\omega)) \in C(C_\gamma)$ if $D \in C(C_\gamma)$.*

Proof. Let $z \in U(t+\tau,\omega,x_0)$. Then there exists a solution y of (7) such that $z = y_{t+\tau}$. Define the function u as $u_t = y_{t+\tau}$ for $t \geq 0$. Hence $u_0 = y_\tau$. The function u solves (7) with ω replaced by $\theta_\tau\omega$, and $x_0 = y_\tau$. Indeed, for $s \in [-t,0]$ we have

$$\begin{aligned}
u_t(s) =& y_{t+\tau}(s) = S(t+\tau+s)x_0(0) + \int_0^{s+t+\tau} S(t+\tau+s-r)f(\theta_r\omega,y_r)dr \\
=& S(t+s)\left(S(\tau)x_0(0) + \int_0^\tau S(\tau-r)f(\theta_r\omega,y_r)dr\right) \\
&+ \int_\tau^{s+t+\tau} S(t+\tau+s-r)f(\theta_r\omega,y_r)dr \\
=& S(t+s)y_\tau(0) + \int_0^{s+t} S(t+s-v)f(\theta_{v+\tau}\omega,u_v)dv.
\end{aligned}$$

It is clear that $u_t(s) = y_\tau(s+t)$ for $s < -t$. Thus

$$z \in U(t,\theta_\tau\omega,y_\tau) \subset U(t,\theta_\tau\omega,U(\tau,\omega,x_0)).$$

Since z is arbitrary we obtain $U(t+\tau,\omega,x_0) \subset U(t,\theta_\tau\omega,U(\tau,\omega,x_0))$.

Now let $z \in U(t,\theta_\tau\omega,U(\tau,\omega,x_0))$. Then there exist y^1 solving (7) and y^2 solving (7) (with ω replaced by $\theta_\tau\omega$) and y_τ^1 such that $y_0^2 = y_\tau^1$ and $y_t^2 = z$. Define the function

$$y_t = \begin{cases} y_t^1, \text{ if } 0 \leq t \leq \tau \\ y_{t-\tau}^2, \text{ if } \tau \leq t, \end{cases}$$

which is a solution of (7). Indeed, for $t \leq \tau$ the equality $y_t = y_t^1$ implies immediately that $y(\cdot)$ is a mild solution. If $t \geq \tau$, then for $s \in [-t+\tau,0]$ we have

$$\begin{aligned}
y_t(s) &= y_{t-\tau}^2(s) = S(t-\tau+s)y_\tau^1(0) + \int_0^{s+t-\tau} S(t-\tau+s-r)f(\theta_{r+\tau}\omega,y_r^2)dr \\
&= S(t-\tau+s)\left(S(\tau)x_0(0) + \int_0^\tau S(\tau-r)f\left(\theta_r\omega,y_r^1\right)dr\right) \\
&+ \int_\tau^{s+t} S(t+s-r)f(\theta_r\omega,y_{r-\tau}^2)dr \\
&= S(t+s)x_0(0) + \int_0^{s+t} S(t+s-r)f(\theta_r\omega,y_r)dr.
\end{aligned}$$

Also, for $s \in [-t,-t+\tau]$ we get

$$\begin{aligned}
y_t(s) = y_{t-\tau}^2(s) = y_\tau^1(s+t-\tau) = S(s+t)x_0(0) + \int_0^{s+t} S(s+t-r)f\left(\theta_r\omega,y_r^1\right)dr \\
= S(s+t)x_0(0) + \int_0^{s+t} S(s+t-r)f(\theta_r\omega,y_r)dr.
\end{aligned}$$

Finally, for $s < -t$ it is clear that $y_t(s) = y_{t-\tau}^2(s) = y_\tau^1(s+t-\tau) = x_0(s+t)$.

Hence, $z = y_t^2 = y_{t+\tau} \in U(t+\tau,\omega,x_0)$. Since z is arbitrary, $U(t,\theta_\tau\omega,U(\tau,\omega,x_0)) \subset U(t+\tau,\omega,x_0)$.

We also note that $U(t,\omega,D)$ belongs to $C(C_\gamma)$ if $D\in C(C_\gamma)$ where the proof follows by the continuity of $C_\gamma\ni y\to f(\omega,y)$, (6) and Lebesgue's majorant theorem. $\square$

In the sequel let us consider the system $\mathcal{D}$ given by the multi–valued mapping D in $C(C_\gamma)$ with $D(\omega)\subset B_{C_\gamma}(0,\varrho(\omega))$, the closed ball with center zero and radius ϱ, which is supposed to have a subexponential growth:

$$\lim_{t\to\pm\infty}\frac{\log^+\varrho(\theta_t\omega)}{t}=0\quad\text{for }\omega\in\Omega.$$

$\mathcal{D}$ is called the family of subexponentially growing multi–functions in $C(C_\gamma)$.

Of course, the property on $\mathcal{D}$ given in Definition 3.1 holds.

Lemma 5.2. *For the function c_2 defined at the beginning of Section 4 suppose that*

$$\mathbb{E}c_2=\bar c_2<\alpha. \tag{10}$$

Then the ball $B(\omega)$ in C_γ with center zero and random (w.r.t. $\mathcal{F}$) radius

$$R(\omega)=2\int_{-\infty}^0 e^{\alpha\tau+\int_\tau^0 c_2(\theta_s\omega)ds}c_1(\theta_\tau\omega)d\tau \tag{11}$$

is contained in $\mathcal{D}$. In addition, B is pullback $\mathcal{D}$–absorbing in the sense of (1) and we have that

$$U(t,\omega,B(\omega))\subset B(\theta_t\omega)$$

for $t\ge 0$ and $\omega\in\Omega$.

Proof. We note that R is well defined and $t\to R(\theta_t\omega)$ is subexponentially growing what follows for instance by Caraballo *et al.* [9]. To see the pullback absorption we replace in the formula in Theorem 4.3 for all $t\ge 0$ the parameter ω by $\theta_{-t}\omega$. We then note that

$$e^{-\alpha t+\int_{-t}^0 c_2(\theta_\tau\omega)d\tau}$$

tends to zero exponentially fast for $t\to+\infty$ thanks to our assumption on c_2 and the ergodic Theorem 2.1 with the modification of c_2 on a set of measure zero if c_2 is a random variable. In addition, we can write the second integral in (9) with $\theta_{-t}\omega$ instead of ω as

$$\int_{-t}^0 e^{\alpha\tau+\int_\tau^0 c_2(\theta_s\omega)ds}c_1(\theta_\tau\omega)d\tau.$$

The conclusion then follows for $t\to+\infty$.

The forward invariance follows then easily if we replace $\|x_0\|_\gamma$ by $R(\omega)$ in (9). $\square$

Remark 4. We note that $\mathbb{R}\ni t\mapsto R(\theta_t\omega)$ is continuous because this function solves the differential equation

$$\frac{dr}{dt}=(-\alpha+c_2(\theta_t\omega))r+2c_1(\theta_t\omega),\ r(0)=R(\omega).$$

We now study qualitative properties of U. For that we apply the results from the last section by setting $X=C_\gamma$.

Lemma 5.3. *For fixed $t\ge 0$ and $\omega\in\Omega$, the mapping $x_0\to U(t,\omega,x_0)$ is upper–semicontinuous.*

Proof. In the case that $U(t,\omega,\cdot)$ is not upper–semicontinuous then there exist a neighborhood $M_{t,\omega}$ of $U(t,\omega,x_0)$, a sequence $\{x_0^n : n\in\mathbb{N}\}$, $x_0^n\to x_0$ with convergence in C_γ and elements $y_t^n\in U(t,\omega,x_n)\notin M_{t,\omega}$. We show that $\lim_{n'\to+\infty} y_t^{n'} =: y_0$ for some subsequence (n') in $\mathbb{N}$, which is an element in $U(t,\omega,x_0)$. This is a contradiction. To see that y_t^n is relatively compact we apply the Arzelà–Ascoli theorem. By the properties of the sequence x_0^n (which is pre–compact in C_γ) it is sufficiently to show that $y_t^n(s)$, $s\in[-t,0]$ is pre–compact. We note that by Theorem 4.3 the set

$$\{y_t^n : n\in\mathbb{N}\}$$

is bounded in C_γ because $\{x_0^n : n\in\mathbb{N}\}$ is bounded in C_γ. Hence

$$\sup_{n\in\mathbb{N},s\in[-t,0]}\|y_t^n(s)\|<\infty \tag{12}$$

The same argument as in the proof of Theorem 4.2 yields the relative compactness of $Z(s):=\{y_t^n(s) : n\in\mathbb{N}\}$. In particular, $\{S(t+s)x_0^n(0) : n\in\mathbb{N}\}$ is pre–compact. Similarly, we can apply the equicontinuity argument of Theorem 4.2. Indeed, to see the equicontinuity of $\{y_t^n : n\in\mathbb{N}\}$ at $s\in(-t,0]$ we still use the fact that $r\to S(r)$ is continuous in norm for $r>0$.
Following the idea of the proof of Theorem 4.2 to see equicontinuity at $s=-t$ we have to study the following equation with $r:=s+t>0$

$$y_t^n(s)=y^n(r)=S(r)x_0^n(0)+\int_0^r S(r-\tau)f(\theta_\tau\omega,y_\tau^n)d\tau.$$

Note that the norm of the integral on the right hand side is small uniformly with respect to n if r is small applying (6) and (12).

To see the equicontinuity of the functions formed by the first expression on the right hand side we have to show that for every $\varepsilon>0$ there exists a $\delta>0$ such that for $n\in\mathbb{N}$ and $r\le\delta$ we have that $\|S(r)x_0^n(0)-x_0^n(0)\|\le\varepsilon$. If not, there would exist $\varepsilon>0$, sequences $n\to+\infty, r_n\to 0$ such that $\|S(r_n)x_0^n(0)-x_0^n(0)\|\ge\varepsilon$. Choosing n sufficiently large such that for r in $[0,t]$ the estimate

$$\|S(r)(x_0^n(0)-x_0(0))\|\le\frac{\varepsilon}{4}$$

holds, we then have that

$$\begin{aligned}\|S(r_n)x_0^n(0)-x_0^n(0)\|\le&\|S(r_n)(x_0^n(0)-x_0(0))\|+\|S(r_n)x_0(0)-x_0(0)\|\\&+\|x_0^n(0)-x_0(0)\|\le\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\varepsilon\end{aligned}$$

for large n. This is a contradiction. □

Now we prove the following compactness conclusion for the MNDS U.

Lemma 5.4. *Assume that (10) holds. The multi–valued dynamical system U is pullback asymptotically compact with respect to B defined in Lemma 5.2.*

Proof. Let z_t be the unique (mild) solution of

$$\frac{dz}{dt}=Az,\qquad z_0=x_0\in C_\gamma \tag{13}$$

given by

$$z_t(s)=\begin{cases} S(t+s)x_0(0) & : \quad s\in[-t,0]\\ x_0(s+t) & : \quad s\le -t\end{cases}$$

so that

$$\|z_t\|_\gamma \le \max(e^{-\alpha t}\|x_0(0)\|, e^{-\gamma t}\|x_0\|_\gamma) \le e^{-\alpha t}\|x_0\|_\gamma. \tag{14}$$

Let y_τ be a solution of (5) with initial function x_0 so that $y_\tau \in U(\tau,\theta_{-t}\omega,x_0)$. Then there exists $u_\tau \in C_\gamma$ such that $y_\tau = z_\tau + u_\tau$ where u_τ is a mild solution of

$$\frac{du}{d\tau} = Au + f(\theta_{\tau-t}\omega, y_\tau), \qquad u_0(s) = 0 \quad \text{for } s \le 0. \tag{15}$$

Let $t_n \to \infty$ and $x_0^n \in B(\theta_{-t_n}\omega)$. The solution to this initial function of (5), with $\theta_{-t_n}\omega$ instead of ω, is denoted by y_τ^n. According to Lemma 5.2, $y_\tau^n \in B(\theta_{-t_n+\tau}\omega)$, hence $\|y_\tau^n\|_\gamma \le R(\theta_{-t_n+\tau}\omega)$. Let u^n be the solution of (15) with $t = t_n$ which can be written as

$$u^n(t_n+s) = \int_0^{t_n+s} S(t_n+s-\tau)f(\theta_{-t_n+\tau}\omega, y_\tau^n)d\tau, \qquad s \in [-t_n, 0].$$

Similar to the proof of Theorem 4.2 and the above calculations we can find an estimate of $\|u^n(t_n+s_1) - u^n(t_n+s_2)\|$, $-T \le s_1 < s_2 \le 0$ for an arbitrary $T > 0$ which gives us the equicontinuity of $\{u^n(t_n+\cdot) : n \in \mathbb{N}\}$ on $[-T,0]$. In addition, we are also able to prove the pre–compactness of $\{u^n(t_n+s) : n \in \mathbb{N}\}$ for $s \in [-T,0]$. By the Arzelà–Ascoli theorem there exist a subsequence $\{n'\}$ and a function $\psi : \mathbb{R}^- \to H$ which is the uniform limit of $u^{n'}(t_{n'}+\cdot)$ on every interval $[-T,0]$.

The following a priori estimate holds

$$\begin{aligned} \|u^n(t_n+s)\| &\le \int_{-\infty}^0 e^{-\alpha(s-\tau)}(2c_1(\theta_\tau\omega) + c_2(\theta_\tau\omega)R(\theta_\tau\omega))d\tau \\ &= e^{-\alpha s}R(\omega), \quad s \in \mathbb{R}^-. \end{aligned} \tag{16}$$

The last inequality follows from Remark 4 including the existence of the integral where the properties of c_1, c_2 and the continuity of $t \to R(\theta_t\omega)$ are needed. From this inequality we can derive

$$\|u^n(t_n+s)\|e^{\gamma s} \le \int_{-\infty}^0 e^{\alpha\tau}(2c_1(\theta_\tau\omega) + c_2(\theta_\tau\omega)R(\theta_\tau\omega))d\tau, \quad s \in \mathbb{R}^-. \tag{17}$$

From (17),

$$\sup_{s\in[-T,0]} \|u^{n'}_{t_{n'}}(s)\|e^{\gamma s} \le R(\omega)$$

and then

$$\lim_{n'\to\infty} \sup_{s\in[-T,0]} \|u^{n'}_{t_{n'}}(s)\|e^{\gamma s} = \sup_{s\in[-T,0]} \|\psi(s)\|e^{\gamma s} \le R(\omega)$$

for every $T > 0$. Hence

$$\sup_{T>0} \sup_{s\in[-T,0]} \|\psi(s)\|e^{\gamma s} \le R(\omega)$$

that is, not only does ψ belong to C_γ but also $\|\psi\|_\gamma \le R(\omega)$.

In addition $u^{n'}_{t_{n'}}(\cdot)$ converges to ψ in C_γ. To prove that we have to check that for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\sup_{s\le 0} \|u^{n'}_{t_{n'}}(s) - \psi(s)\|e^{\gamma s} \le \varepsilon \qquad \text{for all } n' \ge N(\varepsilon). \tag{18}$$

Let us now consider $\gamma > \alpha$. For every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$e^{-(\gamma-\alpha)T_\varepsilon}R(\omega) \le \frac{\varepsilon}{2}.$$

Since the convergence of $u^{n'}_{t_{n'}}(\cdot)$ to ψ holds in compact intervals, in order to prove (18) we only need to check that

$$\sup_{s\le -T_\varepsilon}\|u^{n'}_{t_{n'}}(s)-\psi(s)\|e^{\gamma s}\le\varepsilon\qquad\text{for all } n'\ge N(\varepsilon).$$

But thanks to (16)

$$\|u^{n'}_{t_{n'}}(s)\|e^{\gamma s}\le e^{(\gamma-\alpha)s}R(\omega)\quad\text{for } s\le 0,$$

which together with the choice of T_ε implies

$$\sup_{s\le -T_\varepsilon}\|u^{n'}_{t_{n'}}(s)\|e^{\gamma s}\le\frac{\varepsilon}{2}.$$

Moreover,

$$\sup_{s\in(-T,-T_\varepsilon]}\|\psi(s)\|e^{\gamma s}\le\lim_{n'\to\infty}\sup_{s\in[-T,-T_\varepsilon]}\|u^{n'}_{t_{n'}}(s)\|e^{\gamma s}\le\frac{\varepsilon}{2}$$

for every $T>T_\varepsilon$. Hence the convergence of $\{u^{n'}_{t_{n'}}(\cdot)\}$ to ψ is in C_γ.

We then have

$$y^n_t=u^n_t+z^n_t$$

where z^n_t is the solution of (13) with initial function x^n_0. Since $x^n_0\in B(\theta_{-t_n}\omega)$ it follows by (14)

$$\lim_{t_n\to+\infty}\|z^n_{t_n}\|_\gamma=0$$

so that we have found the convergence of $y^{n'}_{t_{n'}}$ to ψ in C_γ which is the conclusion of the lemma. □

According to Theorem 3.4 summarizing Lemmas 5.3, 5.2, 5.4 we have

Theorem 5.5. *Suppose (10). Then the MNDS generated by (5) has a pullback $\mathcal{D}$–attractor A in $C(C_\gamma)$.*

Corollary 1. *Suppose that $\gamma'>\alpha$ such that the assumptions on f given at the beginning of Section 4 are satisfied with respect to γ'. Then there exists a pullback $\mathcal{D}_{\gamma'}$–attractor $A_{\gamma'}$ where $\mathcal{D}_{\gamma'}$ consists of the subexponentially growing multi–functions in $C(C_{\gamma'})$. By the embedding*

$$\|u\|_{\gamma'}\le\|u\|_\gamma,\quad\textit{for } u\in C_\gamma,$$

for $\gamma'>\gamma>\alpha$ there exists a $\mathcal{D}_\gamma$–pullback attractor A_γ such that $A_\gamma(\omega)\subset A_{\gamma'}(\omega)$.

Remark 5. It is really interesting to stress out the relationship that there exists between the uniqueness of pullback attractors A_γ and the systems of attracted sets $\mathcal{D}_\gamma$. Observe that from the Definition 3.1 every pullback $\mathcal{D}_\gamma$-attractor is an invariant set. According to the Corollary 1, the $\mathcal{D}_{\gamma'}$-attractor $A_{\gamma'}$ attracts the infinite number of $\mathcal{D}_\gamma$-attractors A_γ, for $\gamma'>\gamma>\alpha$, since $A_\gamma\in\mathcal{D}_\gamma\subset\mathcal{D}_{\gamma'}$. However, for γ' there exists a unique attractor $A_{\gamma'}$. Indeed, A_γ does not have to attract the elements from $\mathcal{D}_{\gamma'}$.

6. **Random attractors for equations with infinite delay.** As before, along this section we assume the same conditions on S and f given at the beginning of Section 4.

We now apply the results proved in the previous sections to show the existence of a random attractor for (5).

From now on in this section we suppose that Ω can be equipped with a metric providing a Polish space. $\mathcal{F}$ is defined to be the Borel–σ–algebra of Ω. Finally θ_t is supposed to be continuous on Ω for $t \in \mathbb{R}$. We assume also that the map

$$\Omega \times C_\gamma \ni (\omega, y) \mapsto f(\omega, y) \in H \tag{19}$$

is continuous.

We have to prove that the MNDS generated by (5) is an MRDS.

Theorem 6.1. *Assume condition (19) and also that for every $\omega_0 \in \Omega$ and $t_0 \in \mathbb{R}$ there exists a neighborhood $V = V(\omega_0, t_0)$ such that for some $\mu > 1$*

$$\int_0^t \left(c_1 (\theta_\tau \omega)^\mu + c_2 (\theta_\tau \omega)^\mu\right) d\tau \le C(\omega_0, t_0) < \infty, \quad \text{for all } (\omega, t) \in V. \tag{20}$$

Then the mapping

$$(t, \omega, x_0) \to U(t, \omega, x_0)$$

is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(C_\gamma)$ measurable.

Proof. According to Lemma 2.5 we show that the above mapping is upper–semicontinuous. We could follow exactly the proof of Lemma 5.3 except that we do not have fixed ω, t. However, condition (20) gives us some uniformity with respect to ω. Let $t_n \to t_0$, $x_n \to x_0$ in C_γ and x_n, $x_0 \in C_\gamma$, $\omega_n \to \omega_0$ and $\xi_n \in U(t_n, \omega_n, x_n)$. Choose T with $t_n \le T$. We consider y^n to be

$$y^n(t) = S(t)x_n(0) + \int_0^t S(t-\tau) f(\theta_\tau \omega_n, y^n \vee_\tau x_n) d\tau, \; t \le T, \tag{21}$$

for $t \ge 0$ and $y^n(t) = x_n(t)$ for $t < 0$. This is a mild solution of (5) which satisfies $y^n_{t_n} = y^n \vee_{t_n} x_n = \xi_n$. Using condition (20) and the Arzelà–Ascoli theorem we obtain, in a similar way as in Lemma 5.3, that y^n is pre–compact in $C([0,T], H)$ for every $T > 0$. Then there exist a subsequence (n') and a limit point $y^0 \in C([0,T]; H)$ with $y^{n'} \to y^0$ uniformly on $[0,T]$. We extend y^0 by x_0 for $t \le 0$. This new function is continuous at zero which follows from (21) for $t = 0$. Hence, on account of the uniform convergence of x_n and y^n, we have that for a given $\varepsilon > 0$

$$\sup_{t \in [-T,0]} \|y^{n'} \vee_{t_{n'}} x_{n'}(t) - y^0 \vee_{t_0} x_0(t)\| e^{\gamma t} < \frac{\varepsilon}{4}.$$

On the other hand we have

$$\begin{aligned} \sup_{t \le -T} \|x_{n'}(t + t_{n'}) - x_0(t + t_0)\| e^{\gamma t} \le & \sup_{t \le -T} \|x_{n'}(t + t_{n'}) - x_0(t + t_{n'})\| e^{\gamma t} \\ & + \sup_{t \le -T} \|x_0(t + t_{n'}) - x_0(t + t_0)\| e^{\gamma t}. \end{aligned}$$

Since the first term on the right hand side is bounded by $\|x_{n'} - x_0\|_{C_\gamma}$, we can make this term be less than $\varepsilon/4$ for large n'. Since $x_0 \in C_\gamma$, we know that there exists $\lim_{\tau\to-\infty} x_0(\tau)\, e^{\gamma\tau} = x \in H$. Then we can choose a $T_0 > T$ such that

$$\begin{aligned}\sup_{t\le -T_0} &\|x_0(t+t_{n'}) - x_0(t+t_0)\| e^{\gamma t}\\ &\le \sup_{t\le -T_0}\left(\|x_0(t+t_{n'}) - x\| e^{\gamma t} + \|x - x_0(t+t_0)\| e^{\gamma t}\right)\\ &< \frac{\varepsilon}{4}.\end{aligned}$$

But on the compact interval $[-T_0, -T]$ we have that

$$\sup_{t\in[-T_0,-T]} \|x_0(t+t_{n'}) - x_0(t+t_0)\| e^{\gamma t} < \frac{\varepsilon}{4}$$

for n' large. This gives the convergence of $\xi_{n'}$ in C_γ to ξ_0. Then $\xi_0 = y^0 \vee_{t_0} x_0 \in C_\gamma$.

Condition (19) gives for every τ

$$f(\theta_\tau\omega_{n'}, y^{n'} \vee_\tau x_{n'}) \to f(\theta_\tau\omega_0, y^0 \vee_\tau x_0).$$

To see that y^0 satisfies

$$y^0(t) = S(t)x_0(0) + \int_0^t S(t-\tau) f(\theta_\tau\omega_0, y^0 \vee_\tau x_0) d\tau,\ 0 \le t \le t_0,$$

we mention that the integrands of (21) are bounded by the function

$$e^{\alpha\tau}(c_1(\theta_\tau\omega_{n'}) + c_2(\theta_\tau\omega_{n'})M), \qquad M := \sup_{n'} \sup_{\tau\in[0,T]} \|y_\tau^{n'}\|_\gamma < \infty.$$

The uniform integrability condition (20) together with Vitali's convergence theorem for finite measures give us the convergence of the above integrals. Hence $\xi_0 \in U(t, \omega_0, x_0)$ so that U is upper–semicontinuous. □

The following lemma is needed to prove the measurability of the pullback attractor.

Lemma 6.2. *In addition to (19), (10) and (20), assume that the mapping $\omega \mapsto R(\omega)$ defined in (11) is the radius of a ball in C_γ such that*

$$\limsup_{\omega\to\omega_0} R(\omega) \le R(\omega_0) \qquad \text{for } \omega_0 \in \Omega. \tag{22}$$

Then the multi–function $\omega \to U(t, \omega, B(\omega)) \subset C_\gamma$ is $\bar{\mathcal{F}}$ measurable for $t \ge 0$. In addition $U(t, \omega, B(\omega)) \in C(C_\gamma)$.

Proof. According to Lemma 2.2(iii) we show that for fixed t the graph $\mathrm{Gr}(U(t, \omega, B(\omega)))$ is closed. Suppose that

$$\lim_{n\to\infty} (\omega_n, y^n) = (\omega_0, y^0) \quad \text{in } \Omega \times C_\gamma \quad \text{where } y^n \in U(t, \omega_n, B(\omega_n)).$$

We have

$$y^n(s) = x_0^n(s+t) \quad \text{for } s \le -t, \quad x_0^n \in B(\omega_n).$$

Then $x_0^n \to x_0^0$ in C_γ. On account of the properties of R for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for $n \ge n_0$ we have $\|x_0^n\|_\gamma \le R(\omega_0) + \varepsilon$ and then $x_0^0 \in B_{C_\gamma}(0, R(\omega_0) + \varepsilon)$. This holds for every $\varepsilon > 0$ so that $x_0^0 \in B(\omega_0)$. Now we have

$$\begin{aligned}&\lim_{n\to\infty} \|x_0^n(0) - x_0^0(0)\| = 0,\\ &\lim_{n\to\infty} \|\int_0^t S(t-\tau)(f(\theta_\tau\omega_n, y_\tau^n) - f(\theta_\tau\omega_0, y_\tau^0)) d\tau\| = 0, \qquad \text{for } t \ge 0,\end{aligned}$$

so that y^0 is a mild solution of (5) in C_γ. For the convergence we use a similar argument as in the proof of Theorem 6.1. Since $x_0^0 \in B(\omega_0)$ we conclude that $y^0 \in U(t,\omega_0,B(\omega_0))$. Thus the graph of $U(t,\omega,B(\omega))$ is closed in $\Omega\times C_\gamma$. Then $\mathrm{Gr}(U(t,\cdot,B(\cdot)))\in\bar{\mathcal{F}}\otimes\mathcal{B}(C_\gamma)$, so that $U(t,\omega,B(\omega))$ is $\bar{\mathcal{F}}$ measurable. We need here that C_γ is separable.

The second statement follows similarly setting $\omega_n=\omega_0$. □

This is the main theorem of the section.

Theorem 6.3. *Suppose that the assumptions of Lemma 6.2 hold. Then the pullback attractor A introduced in Theorem 5.5 is $\bar{\mathcal{F}}$ measurable with respect to images in the space C_γ, that is, A is a random attractor.*

Proof. It is a consequence of Lemmas 6.2, 3.5 and 5.1. □

In the following we formulate conditions ensuring the assumptions in Lemma 6.2 on $R(\omega)$.

Lemma 6.4. *Let (10) hold. Suppose that there exists $\mu>1$ such that for every $\omega_0\in\Omega$*

$$\int_\tau^0 c_2(\theta_s\omega)^\mu ds \le c(\tau,\omega_0) \quad \textit{for every } \tau<0, \tag{23}$$

$$\int_{-\infty}^0 e^{\mu(\alpha-\rho)\tau+\mu\int_\tau^0 c_2(\theta_s\omega)ds}c_1(\theta_\tau\omega)^\mu d\tau \le c(\omega_0), \tag{24}$$

for ω in some neighborhood V of ω_0 and for some $\rho>0$ such that $\alpha-\rho>\bar{c}_2$. The mappings $\omega\to c_1(\omega)$, $\omega\to c_2(\omega)$ are assumed to be continuous. Then $\omega\to R(\omega)$ is continuous.

Proof. We rewrite (11) as

$$2\int_{-\infty}^0 e^{(\alpha-\rho)\tau+\int_\tau^0 c_2(\theta_s\omega)ds}c_1(\theta_\tau\omega)e^{\rho\tau}d\tau$$

so that $e^{\rho\tau}d\tau$ is a finite measure on $\mathcal{B}(\mathbb{R}^-)$. Then the second inequality in the assumptions ensures that the integrand is uniformly integrable for $\omega\in V$. The first inequality ensures

$$\lim_{n\to\infty}\int_\tau^0 c_2(\theta_s\omega_n)ds=\int_\tau^0 c_2(\theta_s\omega_0)ds \quad \text{for } \tau<0.$$

Vitali's convergence theorem about uniformly integrable functions for finite measures gives the conclusion. □

For Lemma 6.2 and Theorem 6.3 we need that the underlying probability space is complete. In the following we would like to avoid this assumption. The price we have to pay are stronger conditions on c_1, c_2.

Theorem 6.5. *Suppose that conditions (19), (10) hold, so that there exists a $\mathcal{D}-$ pullback attractor A in C_γ. Assume that for some $\mu>1$*

$$\int_0^t \left(c_1(\theta_\tau\omega)^\mu + c_2(\theta_\tau\omega)^\mu\right)d\tau \le C(t)<\infty, \quad \textit{for all } \omega\in\Omega, \tag{25}$$

and also that the radius of the absorbing set $R(\omega)$ is uniformly bounded on $\omega\in\Omega$. Then:

1. $\cup_{\omega\in\Omega} A(\omega)$ *is pre–compact in* C_γ.
2. *The map* $\omega \to A(\omega)$ *is upper–semicontinuous.*
3. *The map* $\omega \to A(\omega)$ *is* $\mathcal{F}$ *measurable in* C_γ.

Proof. Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence in $\cup_{\omega\in\Omega} A(\omega)$. We know from the negative invariance of A and $A \subset B$ that there exists $x_0^n \in A(\theta_{-t_n}\omega_n) \subset B(\theta_{-t_n}\omega_n)$ and $\xi_n = y_{t_n}^n \in U(t_n, \theta_{-t_n}\omega_n, x_0^n)$, where y_τ^n is the corresponding mild solution. According to Lemma 5.2 and the assumption on $R(\omega)$ we have that $\|y_\tau^n\|_\gamma \le R(\theta_{-t_n+\tau}\omega_n) \le R$. Now put $y_\tau^n = z_\tau^n + u_\tau^n$, where

$$u^n(t_n + s) = \int_0^{t_n+s} S(t_n + s - \tau) f(\theta_{-t_n+\tau}\omega_n, y_\tau^n) \, d\tau, \ \ s \in [-t_n, 0].$$

With this ansatz, repeating the same arguments of Theorem 6.1 (but using (25) instead of (20)) and Lemma 5.4 we obtain that $\{\xi_n : n \in \mathbb{N}\}$ contains a convergent subsequence.

Denote $K = \overline{\cup_{\omega\in\Omega} A(\omega)}^{C_\gamma}$. Further, for $\omega_n \to \omega$ we have

$$\begin{aligned} &\mathrm{dist}_{C_\gamma}(A(\omega_n), A(\omega)) \le dist_{C_\gamma}(U(t, \theta_{-t}\omega_n, A(\omega_n)), A(\omega)) \\ &\qquad \le \mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega_n, K), U(t, \theta_{-t}\omega, K)) + \mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega, K), A(\omega)). \end{aligned}$$

Let $\varepsilon > 0$. By the pullback attraction property there exists $T(\varepsilon, K, \omega)$ such that

$$\mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega, K), A(\omega)) < \varepsilon, \text{ if } t \ge T.$$

Fix such a t. Then $dist_{C_\gamma}(U(t, \theta_{-t}\omega_n, K), U(t, \theta_{-t}\omega, K)) \to 0$ as $n \to \infty$. In other case there would exist $\delta > 0$ and $x_n \in K$, such that

$$\mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega_n, x_n), U(t, \theta_{-t}\omega, K)) > \delta \text{ for all } n.$$

Noting that $x_n \to x_0 \in K$ (up to a subsequence), the upper semicontinuity of $(\omega, x) \to U(t, \omega, x)$ (see the proof of Theorem 6.1 and (25)) implies that

$$\begin{aligned} &\mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega_n, x_n), U(t, \theta_{-t}\omega, K)) \\ &\le \mathrm{dist}_{C_\gamma}(U(t, \theta_{-t}\omega_n, x_n), U(t, \theta_{-t}\omega, x_0)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which is a contradiction.

Hence, $\mathrm{dist}_{C_\gamma}(A(\omega_n), A(\omega)) \to 0$ as $n \to \infty$. Since $\omega \to A(\omega)$ has compact values, the upper semicontinuity follows by Remark 1. The measurability then follows from Lemma 2.5. □

7. **Examples.** To illustrate our theory we consider the following situation.

Assume $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Let us set $H = L^2(\mathcal{O})$, denote by $|| \cdot ||$ the norm in H , and let $-A$ be the Laplace operator Δ with homogeneous Dirichlet conditions. It is worth remembering that

$$\begin{aligned} D(A) &= \{u \in H_0^1(\mathcal{O}) : Au \in L^2(\mathcal{O})\}, \\ \alpha &= \inf\{||\nabla u||_{L^2(\mathcal{O})} : u \in H_0^1(\mathcal{O}), ||u|| = 1\}. \end{aligned}$$

Here $H_0^1(\mathcal{O})$ is the Sobolev space of functions in H with generalized derivatives in H which are zero on the boundary $\partial\mathcal{O}$. Then A is the generator of a C_0 contraction semigroup $(S(t))_{t\ge0}$ on H that actually satisfies

$$||S(t)\psi|| \le e^{-\alpha t}||\psi||, \text{ for every } t \ge 0 \text{ and } \psi \in H,$$

being $\alpha > 0$ the first eigenvalue of A in $H_0^1(\mathcal{O})$. Moreover, $S(t)$, $t > 0$, are compact operators on H.

Assume that Ω is a Polish space on which the mappings θ_t are continuous and let $\mathcal{F}$ be the Borel–σ–algebra of Ω.

7.1. **Example 1.** Let $l : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that for a fixed real number $a \in \mathbb{R}$ the mapping $l(\cdot, a)$ is measurable, and for fixed $\omega \in \Omega$, the mapping $a \in \mathbb{R} \longmapsto l(\omega, a) \in \mathbb{R}$ is continuous. Suppose also that there exist two non-negative tempered random variables $\widetilde{c}_1, \widetilde{c}_2$ (w.r.t. $\mathcal{F}$) and that $\widetilde{c}_1$ is such that $t \longmapsto \widetilde{c}_1(\theta_t \omega)$ is integrable with respect to every finite interval (t_1, t_2) and $\omega \in \Omega$. In addition suppose that

$$|l(\omega, a)| \le \widetilde{c}_1(\omega) + \widetilde{c}_2(\omega)|a|, \ \omega \in \Omega, a \in \mathbb{R},$$

where $|\cdot|$ denotes the absolute value in $\mathbb{R}$.

Let $\varrho : \Omega \to \mathbb{R}^-$ be a random variable (w.r.t. $\mathcal{F}$) in general unbounded, $\gamma > \alpha$ and consider $f : \Omega \times C_\gamma \to H$ given by

$$f(\omega, \xi)(x) = l(\omega, \xi(\varrho(\omega))(x)), \text{ for } \omega \in \Omega, \ \xi \in C_\gamma \text{ and } x \in \mathcal{O}.$$

It is easy to see that the function f is well-defined. Moreover,

$$\begin{aligned}
||f(\omega, \xi)|| &= \left(\int_{\mathcal{O}} |f(\omega, \xi)(x)|^2 dx \right)^{1/2} \\
&\le \left(\int_{\mathcal{O}} \left((1+\delta)\widetilde{c}_1(\omega)^2 + (1+\delta^{-1})\widetilde{c}_2(\omega)^2 |\xi(\varrho(\omega))(x)|^2 \right) dx \right)^{1/2} \\
&\le (1+\delta)^{1/2} \widetilde{c}_1(\omega) |\mathcal{O}|^{1/2} + (1+\delta^{-1})^{1/2} \widetilde{c}_2(\omega) e^{-\gamma\varrho(\omega)} e^{\gamma\varrho(\omega)} ||\xi(\varrho(\omega))|| \\
&\le (1+\delta)^{1/2} \widetilde{c}_1(\omega) |\mathcal{O}|^{1/2} + (1+\delta^{-1})^{1/2} \widetilde{c}_2(\omega) e^{-\gamma\varrho(\omega)} ||\xi||_\gamma,
\end{aligned}$$

where $\delta > 0$ and $|\mathcal{O}|$ denotes the Lebesgue measure of $\mathcal{O}$.

Let us define the Nemitskii operator $J : \Omega \times H \to H$ by $J(\omega, y)(x) = l(\omega, y(x))$, for $x \in \mathcal{O}$.

Lemma 7.1. *$\omega \to J(\omega, y)$ is measurable for all $y \in H$ and $y \to J(\omega, y)$ is continuous for all $\omega \in \Omega$.*

Proof. Take first a constant function $y(x) \equiv u \in \mathbb{R}$. The map $\omega \mapsto l(\omega, u)$ is measurable by assumption. Define the map $G : \Omega \to H$ by $G(\omega)(x) = l(\omega, u)$, for all $x \in \mathcal{O}$. We claim that $G(\omega)$ is measurable. Indeed, for any $v \in H$ we have

$$(G(\omega), v) = \int_{\mathcal{O}} l(\omega, u) v(x) dx = l(\omega, u) \int_{\mathcal{O}} v(x) dx = l(\omega, u) v_0.$$

Since the last map is measurable and the space H is separable, $G(\omega)$ is a measurable map by Pettis' theorem (see [32, p.131]). The equality $G(\omega) = J(\omega, y)$ is obvious.

Further, let y be a step function, that is,

$$y(x) = \begin{cases} u_1, \text{ if } x \in \mathcal{O}_1, \\ \vdots \\ u_m, \text{ if } x \in \mathcal{O}_m, \end{cases}$$

where $\mathcal{O} = \cup_{i=1}^m \mathcal{O}_i$ and $\mathcal{O}_i \cap \mathcal{O}_j = \varnothing$ for $i \neq j$. For each u_i we can take the measurable map $\omega \to l(\omega, u_i)$. Define the map $G : \Omega \to H$ by

$$G(\omega)(x) = \begin{cases} l(\omega, u_1), \text{ if } x \in \mathcal{O}_1, \\ \vdots \\ l(\omega, u_m), \text{ if } x \in \mathcal{O}_m. \end{cases}$$

We claim that $G(\omega)$ is measurable. Indeed, for any $v \in H$ we have

$$(G(\omega), v) = \sum_{i=1}^{m} \int_{\mathcal{O}_i} l(\omega, u_i)\, v(x)\, dx = \sum_{i=1}^{m} l(\omega, u_i)\, v_i,$$

so that $G(\omega)$ is measurable and again the equality $G(\omega) = J(\omega, y)$ is obvious.

Finally, take a sequence of step functions y_n converging to y in H. Passing to a subsequence it holds:

$$y_n(x) \to y(x) \quad \text{for a.e. } x \in \mathcal{O}, \tag{26}$$

$$\text{there exists } h \in H \text{ such that } |y_n(x)| \le h(x) \quad \text{for a.e. } x \in \mathcal{O}, \tag{27}$$

(see Brezis [5], Th. IV.9). We know that the maps $G_n(\omega) = J(\omega, y_n)$ are measurable. It is clear from (26)-(27) that (passing to a subsequence) for all $\omega \in \Omega$

$$J(\omega, y_n)(x) = l(\omega, y_n(x)) \to l(\omega, y(x)) = J(\omega, y)(x), \text{ for a.e. } x \in \mathcal{O},$$

and

$$|l(\omega, y_n(x))| \le \widetilde{c}_1(\omega) + \widetilde{c}_2(\omega)|h(x)|, \text{ for a.e. } x \in \mathcal{O}.$$

Thus, it follows by Lebesgue's theorem that $J(\omega, y_n) \to J(\omega, y)$ in H, for every $\omega \in \Omega$. Then $\omega \to J(\omega, y)$ is measurable.

In a similar way as in the previous lines one can prove that if $y_n \to y$ in H, then $J(\omega, y_n) \to J(\omega, y)$ in H. From this property the continuity of $y \to J(\omega, y)$ follows. □

Further, the map $\xi \to f(\omega, \xi)$ is continuous. Indeed, suppose that $\xi_n \to \xi$ in C_γ. Note then that $\xi_n(\varrho(\omega)) \to \xi(\varrho(\omega))$ in H, so that arguing as in the proof of Lemma 7.1 we obtain $f(\omega, \xi_n) \to f(\omega, \xi)$, and then the continuity follows.

Since J is a Caratheodory map and $f(\omega, \xi) = J(\omega, \xi(\varrho(\omega)))$, it follows from [3, Lemma 8.2.3] that $\omega \to f(\omega, \xi)$ is measurable.

Let us define

$$c_1(\omega) := (1+\delta)^{1/2} |\mathcal{O}|^{1/2} \widetilde{c}_1(\omega),$$
$$c_2(\omega) := (1+\delta^{-1})^{1/2} e^{-\gamma \varrho(\omega)} \widetilde{c}_2(\omega),$$

and assume that

$$\mathbb{E} e^{-\gamma \varrho(\omega)} \widetilde{c}_2(\omega) < \frac{\alpha}{(1+\delta^{-1})^{1/2}}.$$

Then, it is straightforward to check that all the assumptions on f in Sections 4 and 5 are fulfilled. The only condition we need to check is the temperedness of c_2 :

$$\lim_{t \to \pm\infty} \frac{\log^+ \left(e^{-\gamma \varrho(\theta_t \omega)} \widetilde{c}_2(\theta_t \omega) \right)}{|t|} \le \gamma \lim_{t \to \pm\infty} \frac{-\varrho(\theta_t \omega)}{|t|} + \lim_{t \to \pm\infty} \frac{\log^+ (\widetilde{c}_2(\theta_t \omega))}{|t|}$$
$$= \gamma \lim_{t \to \pm\infty} \frac{-\varrho(\theta_t \omega)}{|t|}.$$

The last term is equal to zero if $-\varrho$ has a sublinear growth θ–almost surely, for which, as a consequence of the Birkhoff ergodic theorem, a sufficient condition is that

$$\mathbb{E} \sup_{t \in [0,1]} (-\varrho(\theta_t \omega)) < \infty$$

(see Arnold [1], Page 165).

As a particular case, we can choose for ϱ the negative absolute value of a one–dimensional stationary Ornstein-Uhlenbeck process, that is, the unique stationary solution z^* of the stochastic differential equation

$$dz = -zdt + dW,$$

since z^* has continuous trajectories and satisfies

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{|t|} = 0,$$

which follows by the Burkholder inequality (see Caraballo *et al.* [9]).

Notice that the analysis we have just done ensures the existence of a pullback attractor A for the corresponding equation (5). We are now interested in checking that A is also a random attractor. Let then the maps $(\omega, a) \to l(\omega, a)$, $\omega \to \varrho(\omega)$ be continuous and let the maps $\widetilde{c}_1, \widetilde{c}_2$ be locally bounded. We can prove that the map $\Omega \times C_\gamma \ni (\omega, \xi) \mapsto f(\omega, \xi) \in H$ is continuous. Suppose $\omega_n \to \omega$, $\xi_n \to \xi$ in C_γ. Note then that $\xi_n(\varrho(\omega_n)) \to \xi(\varrho(\omega))$ in H, so that arguing as in the proof of Lemma 7.1 we obtain that (up to a subsequence)

$$f(\omega_n,\xi_n)(x) = l(\omega_n, \xi_n(\varrho(\omega_n))(x)) \to l(\omega, \xi(\varrho(\omega))(x)) = f(\omega,\xi)(x), \text{ for a.e. } x \in \mathcal{O},$$

and

$$|f(\omega_n,\xi_n)(x)| \le \widetilde{c}_1(\omega_n) + \widetilde{c}_2(\omega_n)|h(x)| \le \widetilde{C}_1(\omega) + \widetilde{C}_2(\omega)|h(x)|,$$

for some $h \in H$, which is a majorant for $f(\omega_n, \xi_n)$. Thanks to Lebesgue's majorant theorem we obtain $f(\omega_n, \xi_n) \to f(\omega, \xi)$, and then we obtain the required continuity, so that condition (19).

Next we will prove that under suitable assumptions we can apply Theorem 6.1 to have an MRDS. Assume that $\widetilde{c}_i(\omega)$ are uniformly bounded in the following sense: $\widetilde{c}_1(\omega) \le C_1$, $\widetilde{c}_2(\omega)e^{-\gamma\varrho(\omega)} \le C_2$, for all $\omega \in \Omega$, with

$$C_2 < \frac{\alpha}{(1+\delta^{-1})^{1/2}}.$$

Then, it follows immediately that $\widetilde{c}_1(\omega), \widetilde{c}_2(\omega)$ satisfy (20). Thus for any $t_0 \in \mathbb{R}$ and $\omega_0 \in \Omega$ there exists a neighborhood $V(t_0, \omega_0)$ such that

$$\int_0^t (c_1(\theta_\tau\omega))^\mu + (c_2(\theta_\tau\omega))^\mu) d\tau$$

$$= \int_0^t ((1+\delta)^{1/2}|\mathcal{O}|^{1/2}\widetilde{c}_1(\theta_\tau\omega))^\mu + ((1+\delta^{-1})^{1/2}e^{-\gamma\varrho(\theta_\tau\omega)}\widetilde{c}_2(\theta_\tau\omega))^\mu) d\tau \le C(t_0, \omega_0),$$

for any $(t, \omega) \in V$. Therefore, we have an MRDS.

To prove the measurability of the pullback attractor we apply Theorem 6.5. Indeed, thanks to our assumptions on $\widetilde{c}_i(\omega)$, condition (25) holds and

$$\begin{aligned} R(\omega) &= 2(1+\delta)^{1/2}|\mathcal{O}|^{1/2} \int_{-\infty}^0 e^{\alpha\tau + (1+\delta^{-1})^{1/2}\int_\tau^0 e^{-\gamma\varrho(\theta_r\omega)}\widetilde{c}_2(\theta_r\omega)dr}\widetilde{c}_1(\theta_\tau\omega)d\tau \\ &\le 2(1+\delta)^{1/2}|\mathcal{O}|^{1/2}C_1 \int_{-\infty}^0 e^{(\alpha-(1+\delta^{-1})^{1/2}C_2)\tau}d\tau \\ &= \frac{2(1+\delta)^{1/2}|\mathcal{O}|^{1/2}C_1}{\alpha-(1+\delta^{-1})^{1/2}C_2}, \end{aligned} \tag{28}$$

so that the radius $R(\omega)$ is uniformly bounded.

Hence, these assumptions ensure that Theorem 6.5 holds, so that the attractor is measurable with respect to $\mathcal{F}$ and we do not need to consider a complete probability space.

Finally, we note that using additional continuity assumptions on the maps $\omega \mapsto \widetilde{c}_i(\omega)$, $i = 1, 2$, we could prove that the radius $R(\omega)$ is continuous, which in turn would imply condition (22). Hence, we could apply Theorem 6.3. However, this is not necessary, since we have obtained that the attractor is measurable with respect to $\mathcal{F}$ with weaker assumptions, but with more restrictive assumptions on $\widetilde{c}_i(\omega)$.

We would like to say that trivially we can set $\varrho \equiv 0$, situation in which we have a standard, non–delay non–linearity.

7.2. **Example 2.** In this second example we consider the same $\mathcal{O}$, H, A and $\gamma > \alpha$ as in the previous example.

Let $g : H \to H$ be a continuous function such that

$$||g(u)|| \leq d_1 + d_2||u||, \text{ for all } x \in H,$$

where d_1, d_2 are positive real constants. Let us consider $\sigma : \Omega \times \mathbb{R}^- \to \mathbb{R}^+$ such that for fixed real number $a \in \mathbb{R}^-$ the mapping $\sigma(\cdot, a)$ is measurable, and that for fixed $\omega \in \Omega$, $(\omega, s) \longmapsto \sigma(\omega, s)$ is regular enough so that the integral

$$\widetilde{c}_2(\omega) := \int_{-\infty}^{0} \sigma(\omega, s) e^{-\gamma s} ds$$

is finite for $\omega \in \Omega$, being $\widetilde{c}_2$ a tempered random variable (w.r.t. $\mathcal{F}$) with $\mathbb{E}\widetilde{c}_2 < \frac{\alpha}{d_2}$. Hence the function $f : \Omega \times C_\gamma \to H$ given by

$$f(\omega, \xi) = \int_{-\infty}^{0} \sigma(\omega, s) g(\xi(s)) ds,$$

is well-defined, for $\omega \in \Omega$, $\xi \in C_\gamma$. Moreover, defining $\widetilde{c}_1(\omega) := \int_{-\infty}^{0} \sigma(\omega, s) ds$ we have

$$\begin{aligned} ||f(\omega, \xi)|| &\leq d_1 \int_{-\infty}^{0} \sigma(\omega, s) ds + d_2 \int_{-\infty}^{0} \sigma(\omega, s) e^{-\gamma s} e^{\gamma s} ||\xi(s)|| ds \\ &= d_1 \widetilde{c}_1(\omega) + d_2 \widetilde{c}_2(\omega) ||\xi||_\gamma =: c_1(\omega) + c_2(\omega) ||\xi||_\gamma. \end{aligned}$$

It is clear that the map $\omega \to f(\omega, \xi)$ is measurable for any fixed $\xi \in C_\gamma$.

We shall prove now that for any fixed ω the map $\xi \to f(\omega, \xi)$ is continuous from C_γ into H. Suppose $\xi_n \to \xi$ in C_γ. Note that

$$\sigma(\omega, s) g(\xi_n(s)) \to \sigma(\omega, s) g(\xi(s)), \text{ for all } s \leq 0,$$

and for any $M \geq 0$

$$\|\sigma(\omega, s) g(\xi_n(s)) - \sigma(\omega, s) g(\xi(s))\| \leq \sigma(\omega, s) C(M), \text{ for any } s \in [-M, 0].$$

Then using Lebesgue's majorant theorem we obtain

$$\int_{-M}^{0} \|\sigma(\omega, s) g(\xi_n(s)) - \sigma(\omega, s) g(\xi(s))\| ds \to 0, \text{ for any } M > 0.$$

Also, for any $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ such that

$$\begin{aligned}
&\int_{-\infty}^{-M} \|\sigma(\omega,s)g(\xi_n(s)) - \sigma(\omega,s)g(\xi(s))\|\, ds \\
&\leq \int_{-\infty}^{-M} (\sigma(\omega,s)(d_1 + d_2||\xi_n(s)||) + \sigma(\omega,s)(d_1 + d_2||\xi(s)||))\, ds \\
&\leq 2d_1 \int_{-\infty}^{-M} \sigma(\omega,s)ds + d_2 \left(\sup_n \|\xi_n\|_{C_\gamma} + \|\xi\|_{C_\gamma}\right) \int_{-\infty}^{-M} e^{-\gamma s}\sigma(\omega,s)ds \leq \varepsilon.
\end{aligned}$$

Hence, for any $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that

$$\|f(\omega_n, \xi_n) - f(\omega, \xi)\| \leq 2\varepsilon, \text{ if } n \geq N,$$

and the continuity follows.

Since $\widetilde{c}_1(\omega) \leq \widetilde{c}_2(\omega)$ for all $\omega \in \Omega$, thanks to the hypothesis on $\widetilde{c}_2$, we know that both c_i are well-defined and are positive tempered random variables. Moreover, f satisfies condition (6). We assume also that $t \mapsto \widetilde{c}_1(\theta_t\omega)$ are integrable with respect to every finite interval (t_1, t_2) and $\omega \in \Omega$. By the ergodic theorem the same is true for $\widetilde{c}_2$. It is clear that (10) is satisfied.

Up to now, we have guaranteed that the following delayed random evolution equation

$$\frac{dy}{dt} = -\Delta y + f(\theta_t\omega, y_t)$$

generates an MNDS which has a pullback attractor A. Now, in order to prove that this MNDS is also an MRDS which has as random attractor A, we want to apply the results in the last section. For that, we assume that $\Omega \times \mathbb{R}^- \ni (\omega, s) \mapsto \sigma(\omega, s) \geq 0$ is continuous and satisfies

$$\sigma(\omega, s) \leq D(\omega)\, e^{\delta s},$$

where $\delta > \gamma$ and $\omega \mapsto D(\omega)$ is a bounded function. In particular, we suppose

$$D(\omega) \leq C := \frac{\alpha(\delta - \gamma)}{2d_2}, \text{ for all } \omega \in \Omega. \tag{29}$$

We start proving condition (19): the map $\Omega \times C_\gamma \ni (\omega, \xi) \mapsto f(\omega, \xi) \in H$ is continuous. Suppose then $\omega_n \to \omega$, $\xi_n \to \xi$ in C_γ. Note that

$$\sigma(\omega_n, s)g(\xi_n(s)) \to \sigma(\omega, s)g(\xi(s)), \text{ for all } s \leq 0,$$

and for any $M \geq 0$

$$\|\sigma(\omega_n, s)g(\xi_n(s)) - \sigma(\omega, s)g(\xi(s))\| \leq C(M), \text{ for any } s \in [-M, 0].$$

Then using Lebesgue's majorant theorem we obtain

$$\int_{-M}^{0} \|\sigma(\omega_n, s)g(\xi_n(s)) - \sigma(\omega, s)g(\xi(s))\|\, ds \to 0, \text{ for any } M > 0.$$

Also, for any $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ such that

$$\begin{aligned}
&\int_{-\infty}^{-M} \|\sigma(\omega_n, s)g(\xi_n(s)) - \sigma(\omega, s)g(\xi(s))\| \, ds \\
&\le \int_{-\infty}^{-M} \left(\sigma(\omega_n, s)(d_1 + d_2||\xi_n(s)||) + \sigma(\omega, s)(d_1 + d_2||\xi(s)||)\right) ds \\
&\le d_1 \int_{-\infty}^{-M} (D(\omega_n) + D(\omega)) e^{\delta s} ds \\
&+ d_2 \int_{-\infty}^{-M} \left(D(\omega_n) e^{(\delta-\gamma)s} e^{\gamma s} \|\xi_n(s)\| + D(\omega) e^{(\delta-\gamma)s} e^{\gamma s} \|\xi(s)\| \right) ds \\
&\le C_1 \int_{-\infty}^{-M} e^{\delta s} ds + C_2 \left(\|\xi_n\|_\gamma + \|\xi\|_\gamma \right) \int_{-\infty}^{-M} e^{(\delta-\gamma)s} ds \le \varepsilon.
\end{aligned}$$

Hence, for any $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that

$$\|f(\omega_n, \xi_n) - f(\omega, \xi)\| \le 2\varepsilon, \text{ if } n \ge N.$$

We now aim to prove condition (20). Consider $\mu > 1$. Taking into account that $\delta > \gamma$ and the definitions of $c_1(\omega)$ and $c_2(\omega)$, we obtain

$$\begin{aligned}
\int_0^t (c_1(\theta_\tau\omega)^\mu + c_2(\theta_\tau\omega)^\mu) d\tau &\le \int_0^t \int_{-\infty}^0 D(\theta_\tau\omega)^\mu e^{\mu\delta s}(d_1^\mu + d_2^\mu e^{-\mu\gamma s}) ds d\tau \\
&=: C(t, \mu, \delta, \gamma) < \infty, \quad \text{for all } \omega \in \Omega,
\end{aligned}$$

so the condition for Theorem 6.1 holds, and therefore we have an MRDS. We note that the stronger condition (25) also holds.

In order to prove that A is also a random attractor for this MRDS we first notice that $R(\omega)$ is defined by

$$R(\omega) = 2 \int_{-\infty}^0 e^{\alpha\tau + \int_\tau^0 \left(d_2 \int_{-\infty}^0 \sigma(\theta_s\omega, r) e^{-\gamma r} dr \right) ds} \left(d_1 \int_{-\infty}^0 \sigma(\theta_\tau\omega, s) ds \right) d\tau.$$

Moreover

$$\begin{aligned}
R(\omega) &\le 2 \int_{-\infty}^0 e^{\alpha\tau + C d_2 \int_\tau^0 \left(\int_{-\infty}^0 e^{(\delta-\gamma)r} dr \right) ds} C d_1 \left(\int_{-\infty}^0 e^{\delta s} ds \right) d\tau \\
&\le \frac{2Cd_1}{\delta} \int_{-\infty}^0 e^{\left(\alpha - \frac{Cd_2}{\delta-\gamma}\right)\tau} d\tau = \frac{2Cd_1}{\delta\left(\alpha - \frac{Cd_2}{\delta-\gamma}\right)} = \frac{4Cd_1}{\delta\alpha},
\end{aligned}$$

so that $R(\omega)$ is uniformly bounded on $\omega \in \Omega$, and thus the conditions for Theorem 6.5 hold. We have obtained that A is measurable with respect to $\mathcal{F}$.

As in the previous example, with additional continuity conditions on $\widetilde{c}_1$, $\widetilde{c}_2$ we could prove the continuity of the radius.

It is worth mentioning that the case of distributed finite delay can be considered in the present example, that is, we could have considered a function $f : \Omega \times C_\gamma \to H$ defined by

$$f(\omega, \xi) = \int_{-h}^0 \sigma_1(\omega, s) g(\xi(s)) ds + \int_{-\infty}^0 \sigma_2(\omega, s) g(\xi(s)) ds,$$

with σ_1, σ_2 satisfying similar conditions that σ at the beginning of this example. However, the finite delay term would not contribute significantly to this example, since it can be embedded into the infinite delay term.

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