Global attractors for multivalued random dynamical systems generated by random differential inclusions with multiplicative noise*

T. Caraballo[1], J.A. Langa[1] and J. Valero[2]

[1] Departamento de Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla,
Apdo. de Correos 1160, 41080-Sevilla, Spain.
e-mails: caraball@cica.es ; langa@numer.us.es
[2] Universidad Cardenal Herrera CEU,
Comissari 3, 03203 Elche, Alicante, Spain.

Abstract

In this paper we consider a stochastic differential inclusion with multiplicative noise. It is shown that it generates a multivalued random dynamical system for which there also exists a global random attractor.

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1 Introduction

In our previous paper Caraballo et al. [7], we introduced the concept of multivalued random dynamical system (MRDS) by generalizing that of random dynamical systems (see Arnold [2]) in a suitable manner. We also developed a theory for the existence of global attractors for these

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multivalued semiflows which, in essence, seems the natural extension of the theory developed by Crauel and Flandoli [9]. Finally, the theory was applied to a class of stochastic differential inclusions with additive noise.

The main aim of this paper is to prove similar results to the ones in [7] but for another important family of stochastic differential inclusions, in fact, the perturbed ones by means of a multiplicative noise.

It is worth pointing out that the way in which we construct the MRDS in [7] is by making a suitable change of variable which transforms the problem into a family of inclusions depending on a parameter. In fact, when one deals with a general stochastic differential inclusion (see, e.g., Ahmed [1], Da Prato and Frankowska [10]), it is not always possible to make a change that conduces the problem to another equivalent one depending on a parameter. But there exists two important situations which permit us to do that: when the noise is additive or multiplicative. As the techniques to deal with these two problems are rather different, we aim to show in this paper how we can construct a MRDS and prove the existence of the global random attractor in the multiplicative case.

In Section 2, we collect the definitions and properties of the MRDS and also include the sufficient condition ensuring the existence of the global random attractor. In Section 3, we consider the multivalued semiflow generated by a differential inclusion perturbed by a multiplicative noise and prove that there exists the global random attractor. Finally, some applications are included in Section 4 to illustrate the theory.

2 Multivalued random dynamical systems and attractors

For the complete details and proofs of the results in this Section, the reader is referred to Caraballo et al. [7]. Let (X, d_X) be a complete and separable metric space with the Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$ a measure preserving group of transformations in Ω such that the map $(t, \omega) \mapsto \theta_t \omega$ is measurable and satisfying

$$\theta_{t+s} = \theta_t \circ \theta_s = \theta_s \circ \theta_t; \qquad \theta_0 = Id.$$

The parameter t takes values in $\mathbb{R}$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Definition 1 *A set valued map $G : \mathbb{R}^+ \times \Omega \times X \to C(X)$ ($C(X)$ denotes the set of non-empty closed subsets of X) is called a multivalued random dynamical system (MRDS) if is measurable (see Aubin and Frankowska [4], Definition 8.1.1) and satisfies*

i) $G(0, \omega) = Id$ *on* X;

ii) $G(t + s, \omega)x = G(t, \theta_s \omega)G(s, \omega)x,$ *for all* $t, s \in \mathbb{R}^+, x \in X, \omega \in \Omega$ *(perfect cocycle property).*

Remark 2 *When ii) holds identically (that is, on a set of measure one which does not depend either on t or s), we call G a perfect cocycle. We call G a crude cocycle if ii) holds for fixed s and all $t \in \mathbb{R}^+, x \in X$, $\mathbb{P}-a.s.$ (where the exceptional set N_s can depend on s). We call G a very crude cocycle if ii) holds for fixed $s, t \in \mathbb{R}^+$, for all $x \in X, \mathbb{P}-a.s.$ (where the exceptional set $N_{s,t}$ can depend on both s and t).*

Remark 3 *Throughout this paper all assertions about ω are assumed to hold on a θ_t invariant set of full measure, where this set does not depend on the time variable t. In order to avoid any confusion we shall write "for all $\omega \in \Omega$" instead of "for $\mathbb{P}$-a.a." when the time variable appears.*

Recall the definition of Hausdorff semi-distance between bounded sets of X.. For any $A, B \subset X$ bounded put $dist\,(A, B) = \sup_{y \in A} \inf_{x \in B} d_X\,(y, x)$.

Definition 4 *The MRDS G is said to be upper semicontinuous if for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$ it follows that given $x \in X$ and a neighbourhood of $G(t, \omega)x$, $\mathcal{O}(G(t, \omega)x)$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$ then $G(t, \omega)y \subset \mathcal{O}(G(t, \omega)x)$.*
On the other hand, G is called lower semicontinuous if for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$, given $x_n \to x$ $(n \to +\infty)$ and $y \in G(t, \omega)x$, there exists $y_n \in G(t, \omega)x_n$ such that $y_n \to y$.

It is said to be continuous if it is upper and lower semicontinuous.

Definition 5 *A closed random set D is a map $D : \Omega \to C(X)$, which is measurable. The measurability must be understood in the sense of Castaing and Valadier [8] for measurable multifunctions, that is, $\{D(\omega)\}_{\omega \in \Omega}$ is measurable if given $x \in X$ the map $\omega \in \Omega \mapsto dist(x, D(\omega))$ is measurable.*

A closed random set $D(\omega)$ is said to be negatively (resp. strictly) invariant for the MRDS G if

$$D(\theta_t \omega) \subset G(t, \omega)D(\omega) \ \ (\textit{resp. } D(\theta_t \omega) = G(t, \omega)D(\omega)), \quad \forall\ t \in \mathbb{R}^+, \omega \in \Omega.$$

Remark 6 *This concept of measurability and the previous one are equivalent (see Aubin and Frankowska [4, Theorem 8.3.1]).*

Let us assume the following conditions for the MRDS G:

(H1) There exists an absorbing random compact set $B(\omega)$, that is, for $\mathbb{P}-$almost all $\omega \in \Omega$ and every bounded set $D \subset X$, there exists $t_D(\omega)$ such that for all $t \geq t_D(\omega)$

$$G(t, \theta_{-t}\omega)D \subset B(\omega) \tag{1}$$

(H2) $G(t, \omega) : X \to C(X)$ is upper semicontinuous, for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Recall the definition of the limit set $\Lambda(D,\omega)=\Lambda_D(\omega)$ of a bounded subset $D\subset X$ as

$$\Lambda_D(\omega)=\cap_{T\geq 0}\overline{\cup_{t\geq T}G(t,\theta_{-t}\omega)D}. \tag{2}$$

The following Proposition gives some properties of the limit sets.

Proposition 7 *Assume conditions (H1) and (H2) hold. Then, for $\mathbb{P}-$almost all $\omega\in\Omega$ and every $D\subset X$ bounded, it follows:*

i) *$\Lambda_D(\omega)\subset B(\omega)$ is nonvoid and compact.*

ii) *$\Lambda_D(\omega)$ is negatively invariant, that is, $G(t,\omega)\Lambda_D(\omega)\supseteq\Lambda_D(\theta_t\omega)$ for all $t\in\mathbb{R}^+$. If G is lower semicontinuous, then $\Lambda_D(\omega)$ is strictly invariant.*

iii) *$\Lambda_D(\omega)$ attracts D,*

$$\lim_{t\to+\infty} dist(G(t,\theta_{-t}\omega)D,\Lambda_D(\omega))=0.$$

Definition 8 *The closed random set $\omega\mapsto\mathcal{A}(\omega)$ is called a global random attractor of the MRDS G if $\mathbb{P}-a.s.$*

i) *$G(t,\omega)\mathcal{A}(\omega)=\mathcal{A}(\theta_t\omega)$, for all $t\geq 0$, (that is, it is strictly invariant);*

ii) *for all bounded $D\subset X$, $\lim_{t\to+\infty}dist(G(t,\theta_{-t}\omega)D,\mathcal{A}(\omega))=0$;*

iii) *$\mathcal{A}(\omega)$ is compact.*

We now have the following theorem on the existence of random attractors for MRDS:

Theorem 9 *Let $(H1)-(H2)$ hold, the map $(t,\omega)\in\mathbb{R}^+\times\Omega\mapsto\overline{G(t,\omega)D}$ be measurable for all deterministic bounded sets $D\subset X$, and the map $(t,\omega,x)\in X\mapsto G(t,\omega)x$ have compact values. Then*

$$\mathcal{A}(\omega)=\overline{\cup_{D\subset X_{bounded}}\Lambda_D(\omega)}$$

is a global random attractor for G (measurable with respect to $\mathcal{F}$). It is unique and the minimal closed attracting set.

3 MRDS generated by a differential inclusion with multiplicative noise

Now, let X be a real separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Consider the following stochastic differential inclusion in Stratonovich's sense

$$\begin{cases}\dfrac{du}{dt}\in Au(t)+F(u(t))+\sigma u(t)\circ\dfrac{dw(t)}{dt},\ t\in(0,T),\\ u(0)=u_0,\end{cases} \tag{3}$$

where $\sigma \in \mathbb{R}$, $A : D(A) \subset X \to X$ is a linear operator and $w(t)$ is a two-sided, i.e. $t \in \mathbb{R}$, real Wiener processes with $w(0) = 0$. Although we could consider the more general case of a finite sum of the form $\sum_{i=1}^{n} \sigma_i u \circ \frac{dw_i(t)}{dt}$, we prefer to treat this case for the sake of clarity, bearing in mind that no new difficulties appear in dealing with this more general case.

Let us introduce the next conditions:

(A) The operator A is m-dissipative, i.e. $\forall y \in D(A)$ $\langle Ay, y \rangle \leq 0$, and $Im(A - \lambda I) = X$, $\forall \lambda > 0$.

(F1) $F : X \to C_v(X)$, where $C_v(X)$ is the set of all non-empty, bounded, closed, convex subsets of X.

(F2) The map F is Lipschitz on $\overline{D(A)}$, i.e. $\exists C \geq 0$ such that $\forall y_1, y_2 \in \overline{D(A)}$

$$dist_H(F(y_1), F(y_2)) \leq C \|y_1 - y_2\|,$$

where $dist_H(\cdot, \cdot)$ denotes the Hausdorff metric of bounded sets, i.e.

$$dist_H(A, B) = \max\{dist(A, B), dist(B, A)\}.$$

Let us consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) \mid \omega(0) = 0\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$, the Wiener measure $\mathbb{P}$, and the usual uniform convergence on bounded sets of $\mathbb{R}$. Recall that $w(t)(\omega) := \omega(t)$.

We make the change of variable $v(t) = e^{-\sigma\omega(t)} u(t)$. If we denote[1] $\alpha(t) = \alpha(t, \omega) = e^{-\sigma\omega(t)}$ then, the inclusion (3) turns into

$$\begin{cases} \dfrac{dv}{dt} \in Av(t) + \alpha(t) F\left(\alpha^{-1}(t) v(t)\right), \\ v(0) = v_0 = u_0. \end{cases} \tag{4}$$

We shall define the multivalued map $\widetilde{F} : [0, T] \times \Omega \times X \to C_v(X)$,

$$\widetilde{F}(t, \omega, x) = \alpha(t) F\left(\alpha^{-1}(t) x\right).$$

It is easy to obtain from $(F2)$ the existence of constants D_1, D_2 such that

$$\|F(x)\|^+ \leq D_1 + D_2 \|x\|,$$

where $\|F(x)\|^+ = \sup\limits_{y \in F(x)} \|y\|$. Hence,

$$\left\|\widetilde{F}(t, \omega, x)\right\|^+ \leq \alpha(t)\left(D_1 + D_2 \left\|\alpha^{-1}(t) x\right\|\right) = \alpha(t) D_1 + D_2 \|x\| = n(t, \omega, x).$$

It follows that $\widetilde{F}$ satisfies the next property:

[1] We will omit ω when no confusion is possible.

(F3) For any $x \in X$ there exists $n(\cdot) \in L_1(0,T)$ depending on x and ω such that

$$\left\|\widetilde{F}(t,\omega,x)\right\|^{+} \leq n(t), \text{ a.e. in } (0,T).$$

On the other hand, it is clear that $\widetilde{F}$ satisfies conditions $(F1)-(F2)$ for any fixed $t \in [0,T]$ and $\omega \in \Omega$, where the constant C does not depend on t or ω.

Consider also the equation

$$\begin{cases} \dfrac{dv(t)}{dt} = Av(t) + f(t), \\ v(0) = v_0, \end{cases} \tag{5}$$

where $f(\cdot) \in L_1([0,T], X)$.

Definition 10 *The function $u : [0,T] \to X$ is called a strong solution of problem (5) if:*

1. *$u(\cdot)$ is continuous on $[0,T]$ and $u(0) = u_0$;*

2. *$u(\cdot)$ is absolutely continuous on any compact subset of $(0,T)$ and almost everywhere (a.e.) differentiable on $(0,T)$;*

3. *$u(\cdot)$ satisfies (5) a.e. on $(0,T)$ (hence, $u(t) \in D(A)$, for a.a. $t \in (0,T)$).*

Definition 11 *The continuous function $v : [0,T] \to X$ is called an integral solution of problem (5) if:*

1. *$v(0) = v_0$;*

2. *$\forall \xi \in D(A)$,*

$$\|v(t) - \xi\|^2 \leq \|v(s) - \xi\|^2 + 2\int_s^t \langle f(\tau) + A\xi, v(\tau) - \xi\rangle \, d\tau, \ t \geq s. \tag{6}$$

It is well known (see Barbu [5, p.124]) that any strong solution of problem (5) is an integral solution.

Definition 12 *The function $v : [0,T] \times \Omega \to X$ is said to be an integral solution of problem (4) if for any $\omega \in \Omega$:*

1. *$v(\cdot) = v(\cdot, \omega) : [0,T] \to X$ is continuous.*

2. *$v(0) = v_0$;*

3. *For some selection $f \in L_1([0,T], X)$, $f(t) \in \widetilde{F}(t, \omega, v(t))$ a.e. on $(0,T)$, the inequality (6) holds.*

In what follows, we will omit ω if no confusion is possible.

If condition (A) holds and $f \in L_1([0,T],X)$, then $\forall v_0 \in \overline{D(A)}$ there exists a unique integral solution $v(\cdot)$ of (5) for each $T>0$ (see Barbu [5, p.124]). We shall denote this solution by $v(\cdot) = I(v_0)f(\cdot)$. Moreover, for any integral solutions $v_i(\cdot) = I(v_{i0})f_i(\cdot)$, $i=1,2$, the next inequality holds:

$$\|v_1(t)-v_2(t)\| \leq \|v_1(s)-v_2(s)\| + \int_s^t \|f_1(\tau)-f_2(\tau)\|\, d\tau,\ t \geq s. \tag{7}$$

If (A), $(F1)-(F3)$ hold, then $\forall v_0 \in \overline{D(A)}$ there exists at least one integral solution of (4) for each $T>0$ (see Tolstonogov [13], Theorem 3.1). Moreover, for any $z(\cdot) = I(z_0)g(\cdot)$, $g(\cdot) \in L_1([0,T],X)$, and any $v_0 \in \overline{D(A)}$ there exists an integral solution $v(\cdot) = I(v_0)f(\cdot)$ of (4) such that

$$\|v(t)-z(t)\| \leq \xi(t),\ \forall t \in [0,T], \tag{8}$$

$$\|f(t)-g(t)\| \leq \rho(t) + 2C\xi(t),\ \text{a.e. on } (0,T), \tag{9}$$

where

$$\rho(t) = 2dist\left(g(t), \widetilde{F}(t,\omega,z(t))\right),$$

$$\xi(t) = \|v_0 - z_0\| \exp(2Ct) + \int_0^t \exp(2C(t-s))\rho(s)ds.$$

Since $T>0$ is arbitrary, each solution can be extended on $[0,\infty)$. Let us denote by $\mathcal{D}(v_0,\omega)$ the set of all integral solutions of (4) such that $v(0)=v_0$. We define the maps $G : \mathbb{R}^+ \times \Omega \times \overline{D(A)} \to P(\overline{D(A)})$, $\theta_s : \Omega \to \Omega$ as follows

$$G(t,\omega)v_0 = \{\alpha^{-1}(t)v(t) \mid v(\cdot) \in \mathcal{D}(v_0,\omega)\},$$

$$\theta_s\omega = \omega\,(s+\cdot) - \omega\,(s) \in \Omega.$$

Proposition 13 *Let (A), $(F1),(F2)$ hold. Then G satisfies the cocycle property*

$$G\,(t_1+s,\omega)\,x = G\,(t_1,\theta_s\omega)\,G\,(s,\omega)\,x,\ \forall t_1,s \geq 0, x \in X, \omega \in \Omega.$$

Proof. First let $y \in G\,(t_1+s,\omega)\,x$. Then $y = y\,(t_1+s) = \alpha^{-1}(t_1+s)v\,(t_1+s)$, where $v\,(\cdot) \in \mathcal{D}\,(x,\omega)$. It is clear that $y\,(s) \in G\,(s,\omega)\,x$. We have to prove that $y \in G\,(t_1,\theta_s\omega)\,y\,(s)$. Define $z\,(t) = \alpha^{-1}(s)\alpha(t+s)y\,(t+s)$, $\forall t \geq 0$, $g\,(t) = \alpha^{-1}(s)f\,(t+s)$, *a.e.* $t>0$, where $\alpha(t)y\,(t) = v\,(t) = I\,(x)\,f\,(t)$. For any $r \leq t$, $\xi \in D\,(A)$ we obtain

$$\begin{aligned}
\|z\,(t)-\xi\|^2 &= \left\|\alpha^{-1}(s)v\,(t+s)-\xi\right\|^2 = \alpha^{-2}(s)\,\|v\,(t+s)-\alpha(s)\xi\|^2 \\
&\leq \alpha^{-2}(s)\,\|v\,(r+s)-\alpha(s)\xi\|^2 \\
&\quad +2\alpha^{-2}(s)\textstyle\int_r^t \langle f\,(\tau+s)+\alpha(s)A\xi, v\,(\tau+s)-\alpha(s)\xi\rangle\, d\tau \\
&\leq \|z\,(r)-\xi\|^2 \\
&\quad +2\textstyle\int_r^t \langle g\,(\tau)+A\xi, z\,(\tau)-\xi\rangle\, d\tau.
\end{aligned}$$

On the other hand,

$$\begin{aligned} g(t) &\in \alpha^{-1}(s)\alpha(t+s)F\left(\alpha^{-1}(t+s)v(t+s)\right) \\ &= \alpha^{-1}(s)\alpha(t+s)F\left(y(t+s)\right) \\ &= \alpha^{-1}(s)\alpha(t+s)F\left(\alpha(s)\alpha^{-1}(t+s)z(t)\right), \end{aligned}$$

and, as[2] $\alpha^{-1}(s)\alpha(t+s) = e^{\sigma(\omega(s)-\omega(t+s))} = e^{-\sigma(\theta_s\omega)(t)}$, then

$$g(t) \in e^{-\sigma(\theta_s\omega)(t)}F(e^{\sigma(\theta_s\omega)(t)}z(t)) = \widetilde{F}\left(t, \theta_s\omega, z(t)\right)..$$

Therefore, since $z(0) = y(s)$, it follows that $z(\cdot) \in \mathcal{D}(y(s), \theta_s\omega)$. Since $y = y(t_1+s) = \alpha(s)\alpha^{-1}(t_1+s)z(t_1) = e^{\sigma(\theta_s\omega)(t_1)}z(t_1)$, we get $y \in G(t_1, \theta_s\omega)\, y(s)$. Hence,

$$G(t_1+s, \omega)\, x \subset G(t_1, \theta_s\omega)\, G(s, \omega)\, x.$$

Conversely, let $y \in G(t_1, \theta_s\omega)\, G(s, \omega)\, x$. Then there exist $v_1(\cdot) \in \mathcal{D}(x, \omega)$ and $v_2(\cdot) \in \mathcal{D}(y_1(s), \theta_s\omega)$, $y_1(s) = \alpha^{-1}(s)v_1(s)$, such that $y = e^{\sigma(\theta_s\omega)(t_1)}v_2(t_1)$. Let

$$z(t) = \begin{cases} v_1(t), & \text{if } 0 \le t \le s, \\ \alpha(s)v_2(t-s), & \text{if } s \le t, \end{cases}$$

$$f(t) = \begin{cases} f_1(t), & \text{if } 0 \le t \le s, \\ \alpha(s)f_2(t-s), & \text{if } s \le t, \end{cases}$$

where $v_1(\cdot) = I(x)\, f_1(\cdot)$, $v_2(\cdot) = I(y_1(s))\, f_2(\cdot)$. We have to check that for a.a. $t \in (0, T)$, $f(t) \in \widetilde{F}(t, \omega, z(t)) = \alpha(t)F\left(\alpha^{-1}(t)z(t)\right)$. If $t \le s$ it is obvious that $f(t) \in \alpha(t)F\left(\alpha^{-1}(t)z(t)\right)$. If $t \ge s$ we have

$$\begin{aligned} f(t) &= \alpha(s)f_2(t-s) \\ &\in \alpha(s)e^{-\sigma(\theta_s\omega)(t-s)}F\left(e^{\sigma(\theta_s\omega)(t-s)}v_2(t-s)\right) \\ &= \alpha(s)e^{-\sigma(\omega(t)-\omega(s))}F(e^{\sigma(\omega(t)-\omega(s))}v_2(t-s)) \\ &= \alpha(t)F(\alpha^{-1}(t)z(t)) \\ &= \widetilde{F}(t, \omega, z(t)). \end{aligned}$$

It remains to prove that $z(\cdot)$ satisfies (6) for any $r \le t$. If $t \le s$ the inequality is evident. If

[2] Notice that $\alpha(t+s, \omega) = \alpha(t, \theta_s\omega)\alpha(s, \omega)$.

$r < s < t$ we get

$$\begin{aligned}
\|z(t)-\xi\|^2 &= \|\alpha(s)v_2(t-s)-\xi\|^2 = \alpha^2(s)\|v_2(t-s)-\alpha^{-1}(s)\xi\|^2 \\
&\le \alpha^2(s)\left\|v_2(0)-\alpha^{-1}(s)\xi\right\|^2 \\
&\quad +2\alpha^2(s)\int_s^t \left\langle f_2(\tau-s)+\alpha^{-1}(s)A\xi, v_2(\tau-s)-\alpha^{-1}(s)\xi\right\rangle d\tau \\
&= \|v_1(s)-\xi\|^2 \\
&\quad +2\alpha^2(s)\int_s^t \left\langle f_2(\tau-s)+\alpha^{-1}(s)A\xi, v_2(\tau-s)-\alpha^{-1}(s)\xi\right\rangle d\tau \\
&\le \|v_1(r)-\xi\|^2 \\
&\quad +2\int_r^s \langle f_1(\tau)+A\xi, v_1(\tau)-\xi\rangle d\tau \\
&\quad +2\int_s^t \langle \alpha(s)f_2(\tau-s)+A\xi, \alpha(s)v_2(\tau-s)-\xi\rangle d\tau \\
&= \|z(r)-\xi\|^2 + 2\int_r^t \langle f(\tau)+A\xi, z(\tau)-\xi\rangle d\tau.
\end{aligned}$$

Finally, the case $s \le r$ is similar. Therefore, $z(\cdot) \in \mathcal{D}(x,\omega)$ and

$$y = y(t_1+s) = \alpha^{-1}(t_1+s)z(t_1+s) \in G(t_1+s,\omega)x.$$

Hence,

$$G(t_1,\theta_s\omega)G(s,\omega) \subset G(t_1+s,\omega)x,$$

and the proof is complete. ■

Let $C([0,T],X)$, $0 < T \le \infty$, be the Banach space of continuous functions from $[0,T]$ into X. By $\pi_T : C([0,T'],X) \to C([0,T],X)$, $T' > T$, we denote the projection operator

$$\pi_T(y(\cdot)) = \{\widetilde{y}(\cdot) \in C([0,T],X) \mid \widetilde{y}(s) = y(s), \forall t \in [0,T]\}.$$

Lemma 14 *Let $(A),(F1),(F2)$ hold. Then for any $v_0 \in \overline{D(A)}, \omega \in \Omega$ the set $\pi_T(\mathcal{D}(v_0,\omega))$ is bounded in $C([0,T],X)$. Consequently, for each $t \ge 0, v_0 \in \overline{D(A)}, \omega \in \Omega$ the set $G(t,\omega)v_0$ is bounded.*

Proof. We have seen before (see $F3$) that there exist constants $D_1, D_2 \ge 0$ such that $\forall x \in \overline{D(A)}$, $\forall y \in \widetilde{F}(t,\omega,x)$,

$$\|y\| \le \alpha(t)D_1 + D_2\|x\|. \tag{10}$$

We take a fixed $T > 0$. We shall denote by $z(\cdot) \in C([0,T],X)$ the unique integral solution of the equation

$$\begin{cases} \dfrac{dz}{dt} = Az(t),\ 0 \le t \le T, \\ z(0) = v_0. \end{cases}$$

Let $r_0 = \max\{\|z(t)\|, 0 \le t \le T\}$ and let $r(\cdot) \in C([0,T])$ be the solution to the equation

$$\begin{cases} r'(t) = D_1 + D_2 r(t),\ 0 \le t \le T, \\ r(0) = r_0. \end{cases}$$

Consider an arbitrary solution $v(\cdot) \in \mathcal{D}(v_0, \omega)$, $v(\cdot) = I(v_0)f(\cdot)$. Then it follows from (7) and (10) that

$$\begin{aligned} \|v(t)\| &\leq \|z(t)\| + \textstyle\int_0^t \|f(\tau)\| \, d\tau \\ &\leq r_0 + \textstyle\int_0^t (\alpha(\tau) D_1 + D_2 \|v(\tau)\|) d\tau, \; \forall t \leq T. \end{aligned}$$

Let $K(\omega)$ be such that $\sup_{t \in [0,T]} \{\alpha(t)\} \leq K(\omega)$.Then

$$\|v(t)\| \leq r_0 + D_1 K(\omega) t + D_2 \int_0^t \|v(\tau)\| \, d\tau.$$

Using the Gronwall Lemma we have that for any $t \in [0, T]$ the next inequalities are satisfied:

$$\begin{cases} \|v(t)\| \leq (r_0 + \frac{\widetilde{D}_1}{D_2}) \exp(D_2 t) - \frac{\widetilde{D}_1}{D_2} = r(t, \omega), \text{ if } D_2 \neq 0, \\ \|v(t)\| \leq r_0 + \widetilde{D}_1 t, \text{ if } D_2 = 0, \end{cases} \tag{11}$$

where $\widetilde{D}_1 = D_1 K(\omega)$. Hence, $\pi_T \mathcal{D}(v_0, \omega)$ is bounded in $C([0, T], X)$, $\forall T \geq 0$, $\forall v_0 \in \overline{D(A)}, \forall \omega \in \Omega$. It is obvious from the definition of G that the set $G(t, \omega)v_0$ is bounded for each $t \geq 0$, $v_0 \in \overline{D(A)}, \omega \in \Omega$. ■

For $T > 0$ and bounded $B \subset \overline{D(A)}$, let us denote $\mathcal{D}(B, \omega) = \cup_{x \in B} \mathcal{D}(x, \omega)$ and

$$M(B, \omega, T) = \{f(\cdot) \in L_1([0, T], X) \mid v(\cdot) = I(x)f(\cdot), v(\cdot) \in \pi_T \mathcal{D}(B, \omega)\}.$$

Lemma 15 *Let $(A), (F1), (F2)$ hold. Then for any $T > 0, \omega \in \Omega$ and any bounded set $B \subset \overline{D(A)}$ the sets $M(B, \omega, T)$ and $\pi_T \mathcal{D}(B, \omega)$ are bounded in $L_\infty([0, T], X)$ and $C([0, T], X)$, respectively.*

Proof. Let $x \in B$, $T > 0$ be arbitrary. In view of Lemma 14, there exists $K_1(\omega) > 0$ such that for any $v(\cdot) \in \mathcal{D}(x, \omega)$,

$$\|v(t)\| \leq K_1(\omega), \forall t \in [0, T].$$

We take an arbitrary $u(\cdot) \in \mathcal{D}(B, \omega)$, $u(0) = y \in B$. Then in view of (8) there exists $v(\cdot) \in \mathcal{D}(x, \omega)$ such that

$$\|v(t) - u(t)\| \leq \exp(2CT) \|x - y\|, \text{ on } [0, T].$$

Hence

$$\|u(t)\| \leq \|v(t)\| + \exp(2CT) \|x - y\| \leq K_1(\omega) + \exp(2CT) K_2, \text{ on } [0, T],$$

where $K_1(\omega)$, K_2 depend on B. We have proved that $\pi_T \mathcal{D}(B, \omega)$ is bounded in $C([0, T], X)$.

Further, we must prove that $M(B, \omega, T)$ is bounded in $L_\infty([0, T], X)$. Let $f(\cdot) \in M(B, \omega, T)$ be arbitrary. Then, there exist $x \in B$, $x(\cdot) \in \mathcal{D}(x, \omega)$, such that $x(\cdot) = I(x)f(\cdot)$, $f(t) \in \widetilde{F}(t, \omega, x(t))$, a.e. on $(0, T)$. In view of property $(F3)$

$$\|f(t)\| \leq \alpha(t) D_1 + D_2 \|x(t)\|, \text{ a.e. on } (0, T).$$

Since $\pi_T \mathcal{D}(B,\omega)$ is bounded in $C([0,T],X)$, we obtain the required result. ∎

This semi-distance $dist\,(\cdot,\cdot)$, defined on the Hilbert space X, has the following useful properties, which are easy to check:

1. $\forall A_i, B_i \subset X, i = 1,2,$
$$dist\,(A_1 + B_1, A_2 + B_2) \leq dist\,(A_1, A_2) + dist\,(B_1, B_2)\,.$$

2. $\forall A, B, C \subset X,$
$$dist\,(A,B) \leq dist\,(A,C) + dist\,(C,B)\,..$$

3. $\forall A \subset X, \alpha, \beta \in \mathbb{R},$
$$dist\,(\alpha A, \beta A) \leq |\alpha - \beta|\,\|A\|^{+}\,.$$

4. $\forall A, B \subset X, \alpha \in \mathbb{R},$
$$dist\,(\alpha A, \alpha B) = |\alpha|\,dist\,(a,B)\,.$$

Proposition 16 *Let* (A), $(F1), (F2)$ *hold. For any* $T > 0$, $\omega_n \to \omega_0$, $u_0^n \to u_0$

$$dist\,(G\,(t,\omega_0)\,u_0, G\,(t,\omega_n)\,u_0^n) \to 0, \;\; as\ n \to \infty,$$

uniformly with respect to $t \in [0,T]$.

Proof. Let $z \in G\,(t,\omega_0)\,u_0$, $t \in [0,T]$, be arbitrary. Then $z = \alpha_0^{-1}\,(t)\,z\,(t)$, where $z\,(\cdot) = I\,(u_0)\,f_0\,(\cdot)$, $\alpha_0\,(t) = e^{-\sigma\omega_0(t)}$, $f_0\,(\tau) \in \alpha_0\,(\tau)\,F\left(\alpha_0^{-1}\,(\tau)\,z\,(\tau)\right) = \widetilde{F}(\tau,\omega_0,z(\tau))$, a.e. on $(0,T)$.

Consider now the sequences $\omega_n \to \omega_0$, $u_0^n \to u_0$. Denote

$$\begin{aligned} \rho_n\,(\tau) &= 2dist\left(f_0\,(\tau)\,, \widetilde{F}(\tau,\omega_n,z(\tau))\right) \\ &= 2dist\left(f_0\,(\tau)\,, \alpha_n\,(\tau)\,F\left(\alpha_n^{-1}\,(\tau)\,z\,(\tau)\right)\right). \end{aligned}$$

In view of (8) there exist solutions $v_n\,(\cdot) = I\,(u_0^n)\,f_n\,(\cdot)\,,\ f\,(\tau) \in \widetilde{F}(\tau,\omega_n,v_n(\tau))$, such that

$$\|v_n\,(t) - z\,(t)\| \leq \|u_0^n - u_0\|\exp(2Ct) + \int_0^t \exp(2C(t-s))\rho_n(s)ds \text{ on } [0,T]\,.$$

Taking into account that the functions $\alpha_0\,(\cdot)\,,\ \alpha_0^{-1}\,(\cdot)\,, \alpha_n\,(\cdot)\,,\ \alpha_n^{-1}\,(\cdot)$ are uniformly bounded on $C\,([0,T]\,,\mathbb{R}_+)\,,$ the convergence $\alpha_n \to \alpha_0, \alpha_n^{-1} \to \alpha_0^{-1}$ in $C\,([0,T]\,,\mathbb{R}_+)\,,$ $F2 - F3$, Lemma 14 and the properties of the semi-distance $dist\,(\cdot,\cdot)$ cited above, we have

$$\begin{aligned}
\rho_n(\tau) &\le 2dist\left(\widetilde{F}(\tau,\omega_0,z(\tau)),\widetilde{F}(\tau,\omega_n,z(\tau))\right)\\
&= 2dist\left(\alpha_0(\tau)F\left(\alpha_0^{-1}(\tau)z(\tau)\right),\alpha_n(\tau)F\left(\alpha_n^{-1}(\tau)z(\tau)\right)\right)\\
&\le 2dist\left(\alpha_0(\tau)F\left(\alpha_0^{-1}(\tau)z(\tau)\right),\alpha_n(\tau)F\left(\alpha_0^{-1}(\tau)z(\tau)\right)\right)\\
&\quad +2dist\left(\alpha_n(\tau)F\left(\alpha_0^{-1}(\tau)z(\tau)\right),\alpha_n(\tau)F\left(\alpha_n^{-1}(\tau)z(\tau)\right)\right)\\
&\le 2\left|\alpha_0(\tau)-\alpha_n(\tau)\right|\left\|F\left(\alpha_0^{-1}(\tau)z(\tau)\right)\right\|^+\\
&\quad +2C\alpha_n(\tau)\left\|z(\tau)\right\|\left|\alpha_0^{-1}(\tau)-\alpha_n^{-1}(\tau)\right|\\
&\le 2\left(D_1+D_2\alpha_0^{-1}(\tau)\left\|z(\tau)\right\|\right)\left|\alpha_0(\tau)-\alpha_n(\tau)\right|\\
&\quad +2C\alpha_n(\tau)\left\|z(\tau)\right\|\left|\alpha_0^{-1}(\tau)-\alpha_n^{-1}(\tau)\right|\\
&\le K_1\left|\alpha_0(\tau)-\alpha_n(\tau)\right|+K_2\left|\alpha_0^{-1}(\tau)-\alpha_n^{-1}(\tau)\right|,
\end{aligned}$$

so that $\rho_n(\tau)\to 0$, as $n\to\infty$, uniformly on $[0,T]$. We note that K_1,K_2 depend only on u_0,ω,ω_n and T.

Therefore, for any $\varepsilon>0$ we can choose $N>0$ (which does not depend on either t or $z\in G(t,\omega)\,u_0$) such that $\forall n\ge N,\forall\tau\in[0,T]$

$$\rho_n(\tau)\le\frac{\varepsilon}{2T\exp(2CT)},\quad \left\|u_0^n-u_0\right\|\le\frac{\varepsilon}{2\exp(2CT)}.$$

Hence, $\left\|v_n(t)-z(t)\right\|\le\varepsilon$, $\forall n\ge N,\forall t\in[0,T]$.

Further, $v_n=\alpha_n^{-1}(t)\,v_n(t)\in G(t,\omega_n)\,u_0^n$ and

$$\begin{aligned}
\left\|v_n-z\right\| &= \left\|\alpha_n^{-1}(t)\,v_n(t)-\alpha_0^{-1}(t)\,z(t)\right\|\\
&\le \left\|\alpha_0^{-1}(t)\left(v_n(t)-z(t)\right)\right\|+\left\|\left(\alpha_n^{-1}(t)-\alpha_0^{-1}(t)\right)v_n(t)\right\|\\
&\le R_1\varepsilon+R_2\varepsilon.
\end{aligned}$$

Since N does not depend on either t or $z\in G(t,\omega)\,u_0$, we obtain the statement of the proposition. ■

Lemma 17 *Let $(A),(F1),(F2)$ hold and the semigroup $S(t,\cdot)$ generated by the operator A be compact. Then for any $T>0$, $\omega\in\Omega$ and $u_0\in\overline{D(A)}$ the set $\pi_T\mathcal{D}(u_0,\omega)$ is compact and the semiflow G has compact values.*

Proof. From Tolstonogov [13, Theorem 3.4], it follows that for each $v_0\in\overline{D(A)}$, $T>0$ the set $\pi_T\mathcal{D}(u_0,\omega)$ is compact in $C([0,T],X)$. Hence, the set

$$K=\{y\in X: y=v(T),\,v(\cdot)\in\pi_T\mathcal{D}(u_0,\omega)\}$$

is compact, so that $G(T,\omega)\,u_0=\alpha^{-1}(T)\,K$, as the continuous image of a compact set, is compact as well. ■

Proposition 18 *Let* $(A), (F1), (F2)$ *hold and the semigroup* $S(t,\cdot)$ *generated by the operator* A *be compact. Then for any* $t_n \to t$, $\omega_n \to \omega_0$, $u_0^n \to u_0$ *one has*

$$dist\left(G\left(t,\omega_0\right)u_0, G\left(t_n,\omega_n\right)u_0^n\right) \to 0, \text{ as } n \to \infty.$$

Hence, the map $(t,\omega,u_0) \mapsto G(t,\omega)u_0$ *is lower semicontinuous.*

Proof. Take T such that $t_n < T, t < T$, $\forall n$. In view of Lemma 17 the set $\pi_T\mathcal{D}(u_0,\omega_0)$ is equicontinuous. Then for any $\varepsilon > 0$ there exist $N > 0$ such that $\forall n > N, \forall v(\cdot) \in \pi_T\mathcal{D}(u_0,\omega_0)$,

$$\|v(t_n) - v(t)\| < \varepsilon.$$

The continuity of $\alpha^{-1}(\cdot)$ and the compactness of $\pi_T\mathcal{D}(u_0,\omega_0)$ implies the existence of constants $R_1, R_2 > 0$ such that

$$\begin{aligned} \left\|\alpha^{-1}(t_n)v(t_n) - \alpha^{-1}(t)v(t)\right\| &\le \left\|\alpha^{-1}(t)\left(v(t_n) - v(t)\right)\right\| \\ &\le \left\|\left(\alpha^{-1}(t_n) - \alpha^{-1}(t)\right)v(t_n)\right\| \\ &\le R_1\varepsilon + R_2\varepsilon. \end{aligned}$$

Hence,

$$dist\left(G\left(t,\omega_0\right)u_0, G\left(t_n,\omega_0\right)u_0\right) \to 0, \text{ as } n \to \infty.$$

Further, the last inequality and Proposition 16 imply that for any $\varepsilon > 0$ there exist $N > 0$ such that $\forall n > N$

$$\begin{aligned} dist\left(G\left(t,\omega_0\right)u_0, G\left(t_n,\omega_n\right)u_0^n\right) &\le dist\left(G\left(t,\omega_0\right)u_0, G\left(t_n,\omega\right)u_0\right) \\ &\quad + dist\left(G\left(t_n,\omega\right)u_0, G\left(t_n,\omega_n\right)u_0^n\right) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

which completes the proof. ■

Theorem 19 *Let* (A), $(F1), (F2)$ *hold and the semigroup* $S(t,\cdot)$ *generated by the operator* A *be compact. Then* G *generates a MRDS.*

Proof. Proposition 13 and Lemma 17 imply that the cocycle property is satisfied and that G has compact values. It remains to prove that the multivalued map $G : \mathbb{R}^+ \times \Omega \times \overline{D(A)} \to C(\overline{D(A)})$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}\left(\overline{D(A)}\right)$. Since this σ-algebra contains all open sets of the complete separable metric space $\mathbb{R}^+ \times \Omega \times \overline{D(A)}$ and G has closed values, the lower semicontinuity of G proved in Proposition 18 implies the measurability of the semiflow. Indeed, the map G is measurable if and only if the inverse image of any open set $\mathcal{O} \subset \overline{D(A)}$

$$G^{-1}(\mathcal{O}) = \left\{(t,\omega,x) \in \mathbb{R}_+ \times \Omega \times \overline{D(A)} : G(t,\omega)x \cap \mathcal{O} \neq \emptyset\right\}$$

is measurable (see Aubin and Frankowska, [4], Theorem 8.3.1.). Since the map G lower semicontinuous, the inverse image of any open set is open and then measurable (see Aubin and Frankowska [4, p.40]). ■

3.1 Existence of the global random attractor

In order to obtain the existence of a compact absorbing set we need more regularity of the integral solutions. Namely, we shall suppose that each integral solution of (4) is, in fact, a strong solution of (5).

Proposition 20 *Let* (A), $(F1), (F2)$ *hold. Suppose that each integral solution of (4),* $v(\cdot) = I(u_0) f(\cdot)$ *is a strong solution of (5). Let there exist constants* $\delta > 0$, $M \geq 0$ *such that* $\forall u \in D(A)$, $y \in F(u)$,

$$\langle y, u\rangle \leq (-\delta+\varepsilon)\|u\|^2 + M, \tag{12}$$

where $\varepsilon \geq 0$ *is the biggest constant such that*

$$\langle Au, u\rangle \leq -\varepsilon \|u\|^2, \forall u \in D(A). \tag{13}$$

Then there exists a random radius $r(\omega) > 0$ *such that for* $\mathbb{P}-$ *almost all* $\omega \in \Omega$ *and any bounded set* $B \subset \overline{D(A)}$ *we can find* $T(B) = T(B, \omega) \geq 1$ *for which*

$$\|G(-1+t_0, \theta_{-t_0}\omega) u_0\|^+ \leq r(\theta_{-1}\omega), \ \forall t_0 \geq T(B), \forall u_0 \in B.$$

Remark 21 *We observe that, since* A *is m-dissipative,* $\langle Au, u\rangle \leq 0, \forall u \in D(A)$.

Proof. We note that for any $y \in G(r, \theta_s\omega) u_0$, $y = \alpha^{-1}(r, \theta_s\omega) v(r)$, being $v(\cdot) = I(u_0) f(\cdot)$ an integral solution of

$$\begin{cases} \dfrac{dv}{dt} \in Av(t) + \alpha(t, \theta_s\omega) F(\alpha^{-1}(t, \theta_s\omega) v(t)), \\ v(0) = u_0, \end{cases}$$

which is a strong solution of (5) with

$$f(t) \in \alpha(t, \theta_s\omega) F(\alpha^{-1}(t, \theta_s\omega) v(t)), \text{ a.e. in } (0, T).$$

After the change of variable $z(t) = \alpha(s, \omega) v(t)$, we obtain that $y = \alpha^{-1}(r+s, \omega) z(t)$, being $z(\cdot)$ the integral solution (in fact, a strong one) of the problem

$$\begin{cases} \dfrac{dz}{dt} = Az(t) + g(t), \\ z(0) = \alpha(s, \omega) u_0, \end{cases} \tag{14}$$

where $g(t) = \alpha(t+s, \omega) h(t)$, and $h(t) \in F\left(\alpha^{-1}(t+s, \omega) z(t)\right)$, a.e. in $(0, T)$.

In our case $s = -t_0$, $r = -1 + t_0$. Multiplying (14) by $z(t)$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z(t)\|^2 &= \langle Az(t), z(t)\rangle + \langle g(t), z(t)\rangle \\ &= \langle Az(t), z(t)\rangle + \langle \alpha(t-t_0, \omega) h(t), z(t)\rangle \\ &= \langle Az(t), z(t)\rangle \\ &\quad + \alpha^2(t-t_0, \omega) \left\langle h(t), \alpha^{-1}(t-t_0, \omega) z(t)\right\rangle. \end{aligned}$$

Now by (12), (13) we get

$$\frac{d}{dt}\|z(t)\|^2 \le -2\varepsilon \|z(t)\|^2 + 2(\varepsilon - \delta)\|z(t)\|^2 + 2M\alpha^2(t-t_0,\omega)$$

and thus,

$$\frac{d}{dt}\|z(t)\|^2 \le -2\delta \|z(t)\|^2 + 2M\alpha^2(t-t_0,\omega). \tag{15}$$

Multiplying (15) by $\exp(2\delta t)$ and integrating over $(0,-1+t_0)$ we obtain

$$\begin{aligned}\|z(-1+t_0)\|^2 \le\ & \exp(-2\delta(-1+t_0))\|z(0)\|^2 \\ & +2M\exp(-2\delta(-1+t_0))\textstyle\int_0^{-1+t_0}\exp(2\delta s)\,\alpha^2(s-t_0,\omega)ds\end{aligned}$$

and then, by the change $s-t_0=\tau$,

$$\begin{aligned}\|z(-1+t_0)\|^2 \le\ & \exp(-2\delta(-1+t_0))\|z(0)\|^2 \\ & +2M\exp(2\delta)\textstyle\int_{-\infty}^{-1}\exp(2\delta\tau)\,\alpha^2(\tau,\omega)d\tau.\end{aligned}$$

Now, by standard arguments (see, e.g., Crauel and Flandoli [9]) it easily follows that the mapping $\tau \mapsto \exp(2\delta\tau)\,\alpha^2(\tau,\omega)$ is pathwise integrable over $(-\infty,-1]$, and similarly $\exp(2\delta\tau)\,\alpha^2(\tau,\omega) \to 0$ as $\tau \to -\infty$, $\mathbb{P}$-a.s.

We take

$$r_1^2(\theta_{-1}\omega) = 1 + 2M\int_{-\infty}^{-1}\exp(-\delta(-1-\tau))\,\alpha^2(\tau,\omega)d\tau,$$

$$r(\theta_{-1}\omega) = \alpha^{-1}(-1,\omega)r_1(\theta_{-1}\omega).$$

The radius $r(\theta_{-1}\omega)$ is $\mathbb{P}$-a.s. finite, because of the above considerations. For a bounded set B and almost all $\omega \in \Omega$, we choose $T(B) = T(B,\omega) \ge 1$ such that

$$\exp(-2\delta(-1+t_0))\,\alpha^2(-t_0,\omega)\|u_0\|^2 \le 1,\ \forall t_0 \ge T(B),\forall u_0 \in B.$$

Since $y = \alpha^{-1}(-1,\omega)z(-1+t_0)$ we have

$$\|y\| \le \left\|\alpha^{-1}(-1,\omega)z(-1+t_0)\right\| \le r(\theta_{-1}\omega),$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$ and any $y \in G(-1+t_0,\theta_{-t_0}\omega)\,u_0$, $u_0 \in B$. ■

Theorem 22 *Let the conditions of Proposition 20 hold, the semigroup $S(t,\cdot)$ generated by the operator A be compact and the multivalued map $G(1,\omega)$ be compact (that is, it maps bounded sets into precompact ones). Then, G has the minimal global random attractor $\mathcal{A}(\omega)$. Moreover, it is measurable with respect to $\mathcal{F}$.*

Proof. First, since $x \in X \mapsto G(t,\omega)x$ lower semicontinuous, the map $(t,\omega) \mapsto \overline{G(t,\omega)D}$ is measurable for all deterministic bounded sets $D \subset X$, $t \ge 0$ and $\mathbb{P}-$a.s. (see Remark 7 in [7]). On the other hand, it follows from Lemma 17 that G has compact values.

Let us define the random ball

$$B\left(r\left(\theta_{-1}\omega\right)\right)=\left\{u\in\overline{D\left(A\right)}\mid\|u\|\leq r\left(\theta_{-1}\omega\right)\right\},$$

where $r\left(\theta_{-1}\omega\right)$ is taken from Proposition 20. Let

$$K\left(\omega\right)=\overline{G\left(1,\theta_{-1}\omega\right)B\left(r\left(\theta_{-1}\omega\right)\right)}.$$

The set $K\left(\omega\right)$ is $\mathbb{P}$-a.s. compact, since the operator $G\left(1,\theta_{-1}\omega\right)$ is compact. It follows from Proposition 20 that for any bounded nonrandom set B and $\mathbb{P}$-a.a $\omega\in\Omega$ there exists $T\left(B\right)=T\left(B,\omega\right)\geq1$ such that $\forall t_0\geq T\left(B\right)$,

$$G\left(t_0,\theta_{-t_0}\omega\right)B\subset G\left(1,\theta_{-1}\omega\right)G\left(-1+t_0,\theta_{-t_0}\omega\right)B\subset K\left(\omega\right).$$

Therefore, G has the compact random absorbing set $K\left(\omega\right)$ and we can apply Theorem 9 to ensure the existence of the global random attractor for the multivalued random semiflow. ■

4 Applications

4.1 The case of a subdifferential map

In order to check when the operator $G\left(1,\omega\right)$ is compact we shall consider the case where the operator $-A=\partial\varphi$ is the subdifferential of a proper lower semicontinuous function $\varphi:X\rightarrow(-\infty,+\infty]$ (being $\partial\varphi$ in our case a linear operator). Inclusion (3) turns into

$$\left\{\begin{array}{c}\dfrac{du}{dt}\in-\partial\varphi(u)+F(u)+\sigma u\circ\dfrac{dw\left(t\right)}{dt},\ t\in[0,T],\\ u(0)=u_0,\end{array}\right.\tag{16}$$

where $F:X\rightarrow2^X$ is a multivalued map, satisfying $(F1)-(F2)$, $\overline{D(A)}=\overline{D(\partial\varphi)}$. It is well known (see Barbu [5, p.54 and 71]) that $-\partial\varphi$ is an m-dissipative operator. Moreover, $\overline{D(\varphi)}=\overline{D(\partial\varphi)}$. Then, $\partial\varphi$ generates a nonlinear semigroup of operators $S(t,\cdot):\overline{D(\varphi)}\rightarrow\overline{D(\varphi)}$ and the differential inclusion (16) gives the multivalued map $G\left(t,\omega\right):\overline{D(\varphi)}\rightarrow2^{\overline{D(\varphi)}}$. It is known (see Haraux [11, p.1398]) that the semigroup S is compact if the following condition is satisfied:

(H) The level sets

$$M_R=\{u\in D(\varphi)\mid\|u\|\leq R,\ \varphi(u)\leq R\}$$

are compact in X for any $R>0$.

If (H) holds, it follows from Lemma 17 that G has compact values. Hence, $G(t,\omega):\overline{D(\varphi)}\rightarrow K(\overline{D(\varphi)})$, where $K(\overline{D(\varphi)})$ is the set of all nonempty compact subsets of $\overline{D(\varphi)}$. Theorem 19 implies that G generates a MRDS.

Further we shall remind the next regularity result for solutions of the inclusion

$$\begin{cases} \dfrac{dv}{dt} \in -\partial\varphi(v) + f(t), \ t \in [0,T], \\ \qquad v(0) = v_0 \in \overline{D(\varphi)}. \end{cases} \tag{17}$$

Proposition 23 *(Brezis [6, p.82] and Barbu [5, p.189]) For any $f(\cdot) \in L_2([0,T],X)$, $v_0 \in \overline{D(\varphi)}$, there exists a unique strong solution of inclusion (17) such that*

$$v(\cdot) \in C([0,T],X), \ \sqrt{t}\frac{dv}{dt} \in L_2([0,T],X),$$

$$\varphi(v(\cdot)) \in C((0,T]), \ \varphi(v(\cdot)) \in L_1([0,T]),$$

and $\varphi(v(t))$ is absolutely continuous on $[\delta,T]$, $\forall\delta > 0$. Moreover,

$$\left|\frac{dv}{dt}\right|^2 + \frac{d}{dt}\varphi(v(t)) = \left\langle f, \frac{dv}{dt}\right\rangle, \text{ a.e. on } (0,T). \tag{18}$$

If $u_0 \in D(\varphi)$ then $\dfrac{du}{dt} \in L_2([0,T],X)$ and $\varphi(u)$ is absolutely continuous on $[0,T]$.

We note the next important consequence of the preceding proposition. Let us take an arbitrary integral solution of inclusion (4), $v(\cdot) \in \mathcal{D}(v_0,\omega)$, $v(\cdot) = I(v_0)f(\cdot)$. It follows from Lemma 15 that $f(\cdot) \in L_2([0,T],X)$. Then, since the solution of (17) is unique for any $v_0 \in \overline{D(\varphi)}$, it follows that $v(\cdot)$ is a strong solution of (17).

Further, Proposition 23 and Lemma 15 allow us to prove an important property of the map G.

Theorem 24 *Let property (H) hold. Then for any bounded $B \subset X$, any $T > 0$ and $\omega \in \Omega$, there exists $R(\omega) > 0$ such that $G(T,\omega)B \subset M_{R(\omega)}$.*

Proof. In view of Lemma 15, the set $M(B,\omega,T)$ is bounded in $L_\infty([0,T],X)$ and then it is bounded in $L_2([0,T],X)$ for any bounded $B \subset X, T > 0$, $\omega \in \Omega$. We take an arbitrary $v(\cdot) \in \mathcal{D}(B,\omega)$, $v(\cdot) = I(v_0)f(\cdot)$, $v_0 \in B$. Consider first that $v_0 \in D(\varphi)$. It follows from equality (18) that

$$t\left|\frac{dv}{dt}\right|^2 + t\frac{d}{dt}\varphi(v) = t\left\langle f, \frac{dv}{dt}\right\rangle, \text{ a.e. on } (0,T).$$

Hence,

$$\int_0^T t\left|\frac{dv}{dt}\right|^2 dt + T\varphi(v(T)) = \int_0^T t\left\langle f, \frac{dv}{dt}\right\rangle dt + \int_0^T \varphi(v(t))dt,$$

and then,

$$T\varphi(v(T)) \le \frac{1}{2}\int_0^T t\left|\frac{dv}{dt}\right|^2 dt + T\varphi(v(T)) \le$$

$$\le \frac{1}{2}\int_0^T t\left\|f(t)\right\|^2 dt + \int_0^T \varphi(v(t))dt. \tag{19}$$

On the other hand, there is no loss of generality in assuming that $\min\{\varphi(v) : v \in X\} = \varphi(x_0) = 0$. Indeed, let $x_0 \in D(\partial\varphi)$, $v_0 \in \partial\varphi(x_0)$. If we introduce the new function $\widetilde{\varphi}(v) = \varphi(v) - \varphi(x_0) - (y_0, v - x_0)$, then the equation

$$\frac{dv}{dt} + \partial\varphi(v) \ni f(t)$$

is equivalent to

$$\frac{dv}{dt} + \partial\widetilde{\varphi}(v) \ni f(t) - y_0 = \widetilde{f}(t)$$

and $\min\{\widetilde{\varphi}(v) : v \in X\} = \widetilde{\varphi}(x_0) = 0$. It is clear that $\widetilde{\varphi}$ satisfies also property (H).

Hence, since $f(t) - \dfrac{dv(t)}{dt} \in \partial\varphi(v(t))$ a.e. on $(0,T)$, we have

$$\varphi(v(t)) \le \left\langle f(t) - \frac{dv(t)}{dt}, v(t) - x_0 \right\rangle.$$

Integrating over $(0,T)$ we get

$$\int_0^T \varphi(v(t))dt \le \frac{1}{2}\|v(0) - x_0\|^2 - \frac{1}{2}\|v(T) - x_0\|^2 + \int_0^T \|f(t)\|\,\|v(t) - x_0\|\,dt \le$$

$$\le \frac{1}{2}\|v(0) - x_0\|^2 + \int_0^T \|f(t)\|\,\|v(t) - x_0\|\,dt.$$

Since $0 \in -\partial\varphi(x_0)$, it follows from inequality (7) that

$$\|v(t) - x_0\| \le \|v(0) - x_0\| + \int_0^T \|f(\tau)\|\,d\tau,\ 0 \le t \le T.$$

It follows from the last two inequalities and Lemma 15 that

$$\int_0^T \varphi(v(t))dt \le \left(\|v(0) - x_0\| + \int_0^T \|f(t)\|\,dt\right)^2 \le D(\omega) < \infty. \tag{20}$$

Since the set $M(B,\omega,T)$ is bounded in $L_\infty([0,T],X)$ for any bounded set B, $D(\omega)$ does not depend on $v(\cdot) \in \mathcal{D}(B,\omega)$. Using (20) in relation (19) we obtain that, for any $T > 0$, there exists $K(\omega) > 0$ such that $\varphi(v(T)) \le K(\omega)$.

Now let $v_0 \in B$ be arbitrary. We can assume without loss of generality that B is open and then there exists a sequence $v_0^n \to v_0$, where $v_0^n \in D(\varphi)$, $v_0^n \in B$. In view of (8), for each v_0^n there exists an integral solution of (4), $v^n(\cdot)$ such that

$$\|v^n(t) - v(t)\| \le \|v_0^n - v_0\| \exp(2Ct),\ \forall t \in [0,T].$$

Since $\varphi(v^n(T)) \le K(\omega)$, $\forall n$, and φ is lower semicontinuous, we get

$$\varphi(v(T)) \le \liminf_{n\to\infty} \varphi(v^n(T)) \le K(\omega).$$

On the other hand, by Lemma 15, $\mathcal{D}(B,\omega)$ is bounded in $C([0,T],X)$. Hence,

$$\|v(T)\| \le L(\omega) < \infty,\ \forall v(\cdot) \in \mathcal{D}(B,\omega).$$

Therefore,

$$G(T,\omega)B \subset M_{R(\omega)},$$

where $R(\omega) = \max\{K(\omega)\alpha^{-1}(T,\omega), L(\omega)\alpha^{-1}(T,\omega)\}$. It follows from (H) that $G(T,\omega)B$ is precompact in X. ■

Corollary 25 *Let property (H) hold. Then, for any $T > 0$ and $\omega \in \Omega$, $G(T,\omega)$ is compact.*

Proof. We must prove that for every bounded set B, any $T > 0$ and $\omega \in \Omega$, the set $G(T,\omega)B$ is precompact in X. But this fact follows immediately from (H) and the previous Theorem. ■

Theorem 26 *Let $(F1)-(F2)$, (H) and (12) be satisfied. Then, G has the minimal global invariant random attractor $\mathcal{A}(\omega)$, which is measurable with respect to $\mathcal{F}$.*

Proof. It is a consequence of Theorem 22 and Corollary 25. ■

4.2 Reaction-diffusion inclusions

Let $f : \mathbb{R} \to C_v(\mathbb{R})$ be a multivalued map. Assume that f is Lipschitz, i.e. $\exists C \geq 0$ such that $\forall x, z \in \mathbb{R}$

$$dist_H(f(x), f(z)) \leq C\|x - z\|. \tag{21}$$

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\mathcal{O}$. Consider the stochastic inclusion

$$\begin{cases} \frac{\partial u}{\partial t} \in \Delta u + f(u) + h + \sigma u \circ \frac{dw(t)}{dt}, \text{ on } \mathcal{O} \times (0,T), \\ u = 0, \text{ on } \partial\mathcal{O} \times (0,T), \\ u(x,0) = u_0(x) \text{ on } \mathcal{O}, \end{cases} \tag{22}$$

where $h(\cdot) \in L_2(\mathcal{O})$. Define the operators $A : D(A) \to X$, $F : X \to 2^X$, $X = L_2(\mathcal{O})$,

$$Au = \Delta u,$$

$$F(u) = \{y \in X : y(x) \in f(u(x)) + h(x)\},$$

with $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. The map $-A$ is the subdifferential of a proper, convex, lower semicontinuous function φ and the map F satisfies $(F1)-(F2)$. Moreover, condition (H) is satisfied and $\overline{D(\varphi)} = X$ (see Melnik and Valero [12], Section 3.2.2.). Hence, (22) is a particular case of (16).

We shall assume that there exist $M_1 \geq 0$, $\delta > 0$ such that $\forall s \in \mathbb{R}$, $\forall z \in f(s)$,

$$zs \leq (\lambda_1 - 2\delta)|s|^2 + M_1, \tag{23}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

We obtain the following theorem:

Theorem 27 *Let (21), (23) hold. Then, the MRDS generated by (22) has the minimal global invariant random attractor $\mathcal{A}(\omega)$, which is measurable with respect to $\mathcal{F}$.*

Proof. We have to check that (12) holds. In our case $\varepsilon = \lambda_1$. In view of (23) for any $y \in D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, $\xi \in F(y)$,

$$(\xi, y) \leq (\lambda_1 - 2\delta) \|y\|^2 + M_1\mu(\mathcal{O}) + \langle h, y \rangle$$

$$\leq (\lambda_1 - \delta) \|y\|^2 + M,$$

for some $M \geq 0$, so that (12) holds. The statement follows from Theorem 26. ■

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