Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

EXISTENCE AND ASYMPTOTIC BEHAVIOUR FOR STOCHASTIC HEAT EQUATIONS WITH MULTIPLICATIVE NOISE IN MATERIALS WITH MEMORY

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(Communicated by Aim Sciences)

Abstract. The existence and uniqueness of solutions for a stochastic reaction-diffusion equation with infinite delay is proved. Sufficient conditions ensuring stability of the zero solution are provided and a possibility of stabilization by noise of the deterministic counterpart of the model is studied.

1. **Introduction.** Our main goal in this paper is to establish the existence and uniqueness theorem and to analyze the long-time behaviour of a stochastic heat equation with memory subjected to multiplicative white noise. The starting point for our considerations is the following deterministic heat conduction model.

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ ($d \geq 1$). We denote by $u = u(x,t)$ the temperature at position $x \in \bar{\mathcal{O}}$ and time t. Following the theory developed by Coleman & Gurtin [9], Gurtin & Pipkin [17] and Nunziato [20], we assume that the density $e(x,t)$ of the internal energy and the heat flux $\phi(x,t)$ are related to the temperature and its gradient by constitutive relations:

$$e(x,t) = b_0 u(x,t), \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}}, \tag{1}$$

and

$$\phi(x,t) = -c_0 \nabla u(x,t) + \int_{-\infty}^{t} \gamma(t-s) \nabla u(x,s)\,ds, \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}}. \tag{2}$$

Here the constants $b_0 > 0$ and $c_0 > 0$ are called respectively heat capacity and thermal conduction. The heat flux relaxation function γ is assumed to be in $L^1(\mathbb{R}_+)$. A typical example is $\gamma(s) = \gamma_0 e^{-d_0 t}$, where $d_0 > 0$ and $\gamma_0 < 0$.

The energy balance for the system has the form

$$\partial_t e(x,t) = -\mathrm{div}\,\phi(x,t) + f(u(x,t),x,t), \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}}, \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 60H15, 35K40;

Key words and phrases. Stochastic heat equation, materials with memory, mean square exponential stability, stabilization.

Partially supported by Ministerio de Educación y Ciencia (Spain) and FEDER (European Community) grant MTM2005-01412.

where $f(u, x, t)$ is the energy supply which may depend on the temperature. Thus, after rescaling of constants, we arrive to the heat equation with memory of the form

$$\partial_t u(x,t) = \nu \Delta u(x,t) - \int_{-\infty}^{t} \gamma(t-s) \Delta u(x,s) ds + f(u(x,t), x, t), \tag{4}$$

where $t > 0$, $x \in \mathcal{O}$ and ν is a positive parameter. We also need to set some natural boundary conditions for $u(x,t)$.

This (deterministic) problem has been studied by many authors. For instance, existence and uniqueness of global solutions to general semilinear integro-differential equations have been analyzed in [1], two kind of equations are compared in [10], being the second a singular perturbation which approaches the first one when the memory kernel collapses into a Dirac mass; this kind of relaxation is also adopted in [13] to prove the existence of a robust family of exponential attractors, and the close behaviour of both problems, which contain, as particular cases, the Allen-Cahn and Cahn-Hiliard equations; trajectory and global (uniform) attractors for such general kind of problems are studied in [6, 14, 15, 16].

In this paper we are interested in the case when the function $f(u, x, t)$ describing the energy supply in (4) contains a stochastic term representing an environmental noise. More precisely we assume that

$$f(u, x, t) = -f(u) + h(t, u) + g(t, u) \partial_t W(x, t),$$

where $W(x,t)$ is a cylindrical Wiener process in $L^2(\mathcal{O})$, $\partial_t W$ is the generalized derivative with respect to t, and $f(v)$, $h(t, v)$ are appropriate (deterministic) functions and $g(t, \cdot)$ maps $L2(\mathcal{O})$ into the space of Hilbert-Schmidt operators in $L^2(\mathcal{O})$ (see Hypothesis 1 below). Thus, we arrive to a stochastic partial differential equation (SPDE for short) with memory of the form

$$u_t - \nu \Delta u + \int_{-\infty}^{t} \gamma(t-s) \Delta u(s) ds + f(u) = h(t,u) + g(t,u) \partial_t W(t), \tag{5}$$

in the bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the boundary condition

$$u(t, x) = 0 \quad \text{for} \quad x \in \partial \mathcal{O}. \tag{6}$$

We also need to equip (5) with the initial data:

$$u(t, x) = u_0(t, x) \quad \text{for} \quad t \le 0, \ x \in \mathcal{O}. \tag{7}$$

Note that we choose the homogenous Dirichlet boundary conditions (6) for the sake of definiteness only. In the same way we could treat Neumann or Robin boundary conditions, and even the non-homogenous case (by reducing to homogenous one). We also point out that we assume from the very beginning that our drift nonlinearity

$$\hat{f}(u, t) \equiv f(u) - h(t, u)$$

is splitted into two terms f and h. The first is coercive, the second is globally Lipschitz and may include spatial derivatives (see Hypothesis 1 below). We also note that linear and nonlinear versions of problem (5)-(7) in the case of an additive noise (i.e., $g(t, v)$ is a constant) have been considered in [7] and [2], respectively. Namely, in [7] the problem is considered in the case when $f \equiv h \equiv 0$, $g(u) = Q$ under a certain trace type conditions on the operator Q. In particular, it was shown that in the case of zero initial data the corresponding solution is a Gaussian process possessing some regularity properties. The paper [2] deals with the case$f \equiv h \equiv 0$, $g(u) = Q$, where Q^*Q is the trace class operator and is devoted to the proof of the existence of a pullback attractor in the case of exponentially fading memory.

The structure of the paper is as follows. In Section 2 we introduce notations and basic hypotheses. Then, in Section 3 we present our first result on existence and uniqueness which requires an integrability condition for the relaxation function γ only. In Section 4 we prove uniform exponential decaying of solutions in the case when the diffusion coefficient ν is large enough. In Section 5 we consider exponential decaying of solutions in the case of monotone exponentially fading relaxation function γ, but for a wider class of diffusion coefficients ν. Finally, under the same conditions concerning γ as in Section 5, we establish in Section 6 a result on possibility of stabilization of the deterministic problem (4) by noise.

2. **Notations and hypotheses.** Below we denote by $(\cdot,\cdot)$ and $|\cdot|$, respectively, the scalar product and norm in $L^2(\mathcal{O})$. We will also consider Sobolev spaces $H^s(\mathcal{O})$ and $H_0^s(\mathcal{O})$. We equip the space $H_0^1(\mathcal{O})$ with the scalar product

$$((u,v)) = \int_{\mathcal{O}} \nabla u(x)\cdot\nabla v(x)\,dx,$$

and denote by $\|\cdot\|$ the associate norm.

Let $W(t)$, $t\geq 0$, be an $L^2(\mathcal{O})$-valued cylindrical Wiener process. More precisely, we suppose given $\{\Omega,\mathcal{F},\mathbb{P}\}$ a complete probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of σ-subalgebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{\beta^j(t),\ t\geq 0,\ j=1,2,...\}$ be a sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on this probability space, and suppose that $\{e_j;\ j=1,2,...\}$ is an orthonormal basis of $L^2(\mathcal{O})$. Thus, we denote by $W(t),\ t\geq 0$, the cylindrical Wiener process with values in $L^2(\mathcal{O})$ defined formally as

$$W(t)=\sum_{j=1}^{\infty}\beta^j(t)e_j.$$

It is well known that this series does not converge in $L^2(\mathcal{O})$, but rather in any Hilbert space $\widetilde{K}$ such that $L^2(\mathcal{O})\subset\widetilde{K}$, and the injection of $L^2(\mathcal{O})$ into $\widetilde{K}$ is Hilbert-Schmidt (see e.g. [12]).

Let $0\leq t_0<t_1$ be given. For any separable Banach space X and $p\geq 1$, we denote by $M^p_{\mathcal{F}_t}(t_0,t_1;X)$ the space of all processes

$$\varphi\in L^p(\Omega\times(t_0,t_1),\mathbb{P}(d\omega)\times dt;X)$$

which are $\mathcal{F}_t$-progressively measurable. The space $M^p_{\mathcal{F}_t}(t_0,t_1;X)$ is a Banach subspace of $L^p(\Omega\times(t_0,t_1),\mathbb{P}(d\omega)\times dt;X)$.

We write $L^2_{\mathcal{F}_t}(\Omega;C([t_0,t_1];X))$ to denote the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes $\{\varphi(t);\ t_0\leq t\leq t_1\}$ satisfying

$$\mathbb{E}\left(\sup_{t_0\leq t\leq t_1}\|\varphi(t)\|_X^2\right)<\infty.$$

Let $\mathcal{L}^2(L^2(\mathcal{O}))$ be the separable Hilbert space of Hilbert-Schmidt operators in $L^2(\mathcal{O})$ equipped with the scalar product

$$\langle R,S\rangle_{\mathcal{L}^2(L^2(\mathcal{O}))}=\sum_{j=1}^{\infty}(Re_j,Se_j),\quad R,S\in\mathcal{L}^2(L^2(\mathcal{O})).$$

We denote by $\|\cdot\|_{\mathcal{L}^2(L^2(\mathcal{O}))}$ the corresponding norm.

For any process $\Psi \in M^2_{\mathcal{F}_t}(t_0,t_1;\mathcal{L}^2(L^2(\mathcal{O})))$ one can define the stochastic integral of Ψ with respect to the cylindrical Wiener process $W(t)$, denoted

$$\int_{t_0}^t \Psi(s)\,dW(s), \quad t_0 \le t \le t_1,$$

as the unique continuous $L^2(\mathcal{O})$-valued $\mathcal{F}_t$-martingale such that for all $v \in L^2(\mathcal{O})$,

$$(\int_{t_0}^t \Psi(s)\,dW(s), v) = \sum_{j=1}^{\infty} \int_{t_0}^t (\Psi(s)e_j, v)\,d\beta^j(s), \quad t_0 \le t \le t_1,$$

where the integral with respect to $\beta^j(s)$ is the Itô integral, and the series converges in $L^2(\Omega; C([t_0,t_1]))$. We refer to [12] for details and other properties of the stochastic integral defined in this way. In particular, we note that if $\Psi \in M^2_{\mathcal{F}_t}(t_0,t_1;\mathcal{L}^2(L^2(\mathcal{O})))$ and $\phi \in L^2(\Omega; L^\infty(t_0,t_1;L^2(\mathcal{O})))$ is $\mathcal{F}_t$-progressively measurable, then the series

$$\sum_{j=1}^{\infty} \int_{t_0}^t (\Psi(s)e_j, \phi(s))\,d\beta^j(s), \quad t_0 \le t \le t_1,$$

converges in $L^1(\Omega; C([t_0,t_1]))$, and defines a real valued continuous $\mathcal{F}_t$-martingale. We will use the notation

$$\int_{t_0}^t (\Psi(s)dW(s), \phi(s)) := \sum_{j=1}^{\infty} \int_{t_0}^t (\Psi(s)e_j, \phi(s))\,d\beta^j(s), \quad t_0 \le t \le t_1.$$

Our basic hypothesis is the following one.

Hypothesis 1. We impose the following assumptions:

- the kernel $\gamma(s)$ belongs to $L^1(\mathbb{R}_+)$;
- $f(v) \in C^1(\mathbb{R})$ possesses the property $f'(v) \ge 0$ for all $v \in \mathbb{R}$, $f(0) = 0$, and also satisfies the relations

$$vf(v) \ge a_0|v|^{p+1}, \quad |f'(v)| \le a_1\left(1 + |v|^{p-1}\right), \quad v \in \mathbb{R}, \tag{8}$$

  where $a_i > 0$ are constants and $p \ge 1$;
- The mappings $h : \mathbb{R}_+ \times H^s_0(\mathcal{O}) \to H^{s-1}(\mathcal{O})$ for some $0 \le s \le 1$ and $g : \mathbb{R}_+ \times L^2(\mathcal{O}) \to \mathcal{L}^2(L^2(\mathcal{O}))$ are measurable and uniform Lipschitz continuous (with respect to the functional variable), i.e., there exist constants $L_h > 0$ and $L_g > 0$ such that

$$\|h(t,u) - h(t,v)\|_{H^{s-1}(\mathcal{O})} \le L_h\|u - v\|_{H^s_0(\mathcal{O})}, \quad u, v \in H^s_0(\mathcal{O}), \tag{9a}$$

$$\|g(t,u) - g(t,v)\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))} \le L_g|u - v|^2, \quad u, v \in L^2(\mathcal{O}), \tag{9b}$$

  for almost all $t \in \mathbb{R}_+$. Moreover, we suppose that

$$h(\cdot,0) \in L^2(0,T;H^{s-1}(\mathcal{O})), \quad g(\cdot,0) \in L^2(0,T;\mathcal{L}^2(L^2(\mathcal{O}))), \tag{10}$$

  for every $T > 0$.

Typical examples of the terms f, h and g are the following:

- ⋆ $f(v) = P(v) + c \cdot v$, where $P(v)$ is a polynomial of an odd degree $p \ge 1$ such that $P(0) = 0$ and c is an appropriate constant (it is always possible to introduce the term cv artificially by adding $-cv$ in h);
- ⋆ $h(t,v) = \tilde{h}(v, \nabla v)$, where $\tilde{h} : \mathbb{R}^{d+1} \to \mathbb{R}$ is a globally Lipschitz function;

⋆ $g(t,v) = \tilde{g}(v)\cdot K$, where the function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and K is a Hilbert-Schmidt operator in $L^2(\mathcal{O})$.

3. **Existence and uniqueness of solution.** In this section we use a fixed point method in combination with an extension technique to prove the existence and uniqueness theorem of our system. For this reason, we consider problem (5)–(7) with initial data at arbitrary fixed moment $a \in \mathbb{R}_+$. Since we do not assume that the history data are smooth in time, it is also convenient to split the initial data into the history data defined on $(-\infty, a)$ and the initial value at $t = a$. Thus we consider the following problem

$$u_t - \nu\Delta u + \int_{-\infty}^{t} \gamma(t-s)\Delta u(s)ds + f(u) = h(t,u) + g(t,u)\partial_t W(t), \tag{11}$$

for $x \in \mathcal{O}$, $t > a$, with the boundary condition

$$u(t,x) = 0 \quad \text{for} \quad x \in \partial\mathcal{O},\ t > a, \tag{12}$$

and with initial data at the moment $t = a$

$$u(a,x) = u_0(x) \quad \text{and} \quad u(t,x) = u_-(t,x) \text{ for } t < a,\ x \in \mathcal{O}. \tag{13}$$

Below we assume that

$$u_0 \in L^2(\Omega, \mathcal{F}_a, \mathbb{P}; L^2(\mathcal{O})) \tag{14}$$

and $u_-(t,x)$ is a random function such that

$$\psi_a(t) := \int_{-\infty}^{a} \gamma(t-s)\Delta u_-(s)ds \in M^2_{\mathcal{F}_t}(a, a+T; H^{-1}(\mathcal{O})) \tag{15}$$

for every $T > 0$. We can state property (15) as an assumption concerning u_- in a more direct way. Indeed, we have that

$$\begin{aligned}
\int_a^{a+T} \|\psi_a(\tau)\|^2_{H^{-1}(\mathcal{O})} d\tau &\le \int_a^{a+T} \left[\int_{-\infty}^{a} |\gamma(\tau-r)|\|u_-(r)\|dr\right]^2 d\tau \\
&\le \int_a^{a+T} \int_{-\infty}^{a} |\gamma(\tau-r)|dr \int_{-\infty}^{a} |\gamma(\tau-r)|\|u_-(r)\|^2 dr d\tau \\
&\le \int_0^{\infty} |\gamma(r)|dr \int_{-\infty}^{a} \left(\int_a^{a+T} |\gamma(\tau-r)|d\tau\right) \|u_-(r)\|^2 dr.
\end{aligned}$$

Therefore

$$\int_a^{a+T} \|\psi_a(\tau)\|^2_{H^{-1}(\mathcal{O})} d\tau \le (\gamma_\infty)^2 \int_{-\infty}^{a} \|u_-(r)\|^2 dr, \tag{16}$$

where

$$\gamma_\infty := \int_0^{\infty} |\gamma(r)|dr.$$

Thus (15) holds if we assume that

$$u_- \in L^2\left(\Omega \times (-\infty, a), \mathcal{F}_a \times \mathcal{B}(-\infty, a), \mathbb{P}(d\omega) \times dt; H^1_0(\mathcal{O})\right), \tag{17}$$

where $\mathcal{B}(-\infty, a)$ is the Borel σ-algebra on $(-\infty, a)$. We note that this property of the initial data looks as restrictive. However this is our pay off for a mild hypothesis concerning the relaxation function γ. If we assume (as in [6, 14, 15, 16]) that γ is exponentially fading, then condition (17) can be relaxed (see discussion in Section 5).

Definition 1. Let u_0 and u_- be given and satisfy (14) and (15). A solution to problem (11)–(13) on the interval $(a, a+T)$ is a stochastic process

$$u \in M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O})) \cap M^{p+1}_{\mathcal{F}_t}(a, a+T; L^{p+1}(\mathcal{O})),$$

such that

$$u \in L^2_{\mathcal{F}_t}(\Omega; C([a, a+T]; L^2(\mathcal{O})))$$

and the relation

$$u(t) + \int_a^t (-\nu\Delta u(r) + f(u(r)))\,dr + \int_a^t \left[\int_a^r \gamma(r-s)\Delta u(s)ds + \psi_a(r)\right] dr$$
$$= u_0 + \int_a^t h(r, u(r))\,dr + \int_a^t g(r, u(r))\,dW(r)$$

holds $\mathbb{P}$-a.s. in $\mathcal{V}^* := H^{-1}(\mathcal{O}) + L^{1+1/p}(\mathcal{O}) = \left[H^1_0(\mathcal{O}) \cap L^{p+1}(\mathcal{O})\right]^*$ for all $t \in [a, a+T]$, where $\psi_a(t)$ is given by (15).

Our main result is the following theorem.

Theorem 1. *Under Hypothesis 1, for each initial data u_0 and u_- satisfying* (14) *and* (15) *there exists a unique solution u to problem* (11) − (13) *in any interval $(a, a+T)$. This solution satisfies the energy balance equality*

$$\begin{aligned}
&|u(t)|^2 + 2\nu\int_a^t \|u(r)\|^2\,dr + 2\int_a^t (f(u(r)), u(r))\,dr \\
&+ 2\int_a^t (\int_a^r \gamma(r-s)\Delta u(s)ds + \psi_a(r), u(r))\,dr \\
&= |u_0|^2 + 2\int_a^t (h(r, u(r)), u(r))\,dr \\
&+ 2\int_a^t (g(r, u(r))\,dW(r), u(r)) + \int_a^t \|g(r, u(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr, \qquad \mathbb{P}-a.s.
\end{aligned} \tag{18}$$

for all $t \in [a, T+a]$.

Proof. We split the proof into two steps.

Step 1: local existence and uniqueness. At this step we prove the following assertion.

Proposition 1. *Under the assumptions of Theorem 1, there exists a constant $T^* > 0$, independent of the initial data u_0 and u_- and of the initial moment a, such that problem* (11) − (13) *possesses a unique solution on the interval $(a, a+T^*)$.*

Proof. We apply a fixed point argument.

Let $T > 0$ and $\widehat{u} \in M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O}))$ be given.

We consider the mapping

$$\mathfrak{T} : \widehat{u} \in M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O})) \mapsto u \in M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O}))$$

where u is the unique process $u(t)$ which belongs to

$$M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O})) \cap M^{p+1}_{\mathcal{F}_t}(a, a+T; L^{p+1}(\mathcal{O})) \cap L^2_{\mathcal{F}_t}(\Omega; C([a, a+T]; L^2(\mathcal{O})))$$

and solves the problem

$$u(t) + \int_a^t (-\nu\Delta u(r) + f(u(r)))\,dr + \int_a^t [\phi(\widehat{u}; r) + \psi_a(r)]\,dr$$
$$= u_0 + \int_a^t h(r, u(r))\,dr + \int_a^t g(r, u(r))\,dW(r) \quad \mathbb{P}\text{-a.s.} \tag{19}$$

in $H^{-1}(\mathcal{O}) + L^{1+1/p}(\mathcal{O})$ for all $t \in [a, a+T]$, where $\phi(\widehat{u};\tau)$ is defined by

$$\phi(\widehat{u};t) := \int_a^t \gamma(t-s)\Delta\widehat{u}(s)ds.$$

Observe that $\mathcal{T}$ is well defined, since for $\widehat{u} \in M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O}))$ one has that

$$\phi(\widehat{u};t) \in M^2_{\mathcal{F}_t}(a, a+T; H^{-1}(\mathcal{O})) \tag{20}$$

and

$$\int_a^{a+t} \|\phi(\widehat{u};\tau)\|^2_{H^{-1}(\mathcal{O})} d\tau \le \left[\int_0^t |\gamma(\tau)|d\tau\right]^2 \int_a^{a+t} \|\widehat{u}(\tau)\|^2 d\tau \tag{21}$$

for any $t \in [0, T]$. Indeed, it is sufficient to check (21). We obviously have that

$$\begin{aligned}\int_a^{a+t} \|\phi(\widehat{u};\tau)\|^2_{H^{-1}(\mathcal{O})} d\tau &\le \int_a^{a+t} \left[\int_a^\tau |\gamma(\tau-r)|\|\widehat{u}(r)\|dr\right]^2 d\tau \\ &\le \int_a^{a+t}\int_a^\tau |\gamma(\tau-r)|dr \int_a^\tau |\gamma(\tau-r)|\|\widehat{u}(r)\|^2 dr d\tau \\ &\le \int_0^t |\gamma(r)|dr \int_a^{a+t} \left(\int_r^{a+t} |\gamma(\tau-r)|d\tau\right) \|\widehat{u}(r)\|^2 dr,\end{aligned}$$

which implies (21).

Taking into account (15), (20) and (21), it is not difficult to see that the results in [21] (see also [3] or [18]) can be applied in order to obtain u from the desired class.

We prove now that there exists a $T^* > 0$ such that $\mathcal{T}$ is a contraction for every $T \le T^*$.

Let $\widehat{u}^1$, $\widehat{u}^2$ be given in $M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O}))$, and u^1, u^2 be the processes in

$$M^2_{\mathcal{F}_t}(a, a+T; H^1_0(\mathcal{O})) \cap M^{p+1}_{\mathcal{F}_t}(a, a+T; L^{p+1}(\mathcal{O})) \cap L^2_{\mathcal{F}_t}(\Omega; C([a, a+T]; L^2(\mathcal{O})))$$

which are solutions to (19) corresponding to $\widehat{u}^1$ and $\widehat{u}^2$ respectively. Then, applying Itô's formula to $\bar{u} = u^1 - u^2$, taking expectation, and denoting $\widehat{u} = \widehat{u}^1 - \widehat{u}^2$ we obtain

$$\begin{aligned}&\mathbb{E}|\bar{u}(t)|^2 + 2\nu\mathbb{E}\int_a^t \|\bar{u}(r)\|^2\, dr + 2\mathbb{E}\int_a^t (f(u^1(r)) - f(u^2(r)), \bar{u}(r))\, dr \\ &= -2\mathbb{E}\int_a^t \langle\phi(\widehat{u}, r), \bar{u}(r)\rangle\, dr - 2\mathbb{E}\int_a^t \langle h(r, u^1(r)) - h(r, u^2(r)), \bar{u}(r)\rangle\, dr \\ &\quad + \mathbb{E}\int_a^t \|g(r, u^1(r)) - g(r, u^2(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\, dr,\end{aligned} \tag{22}$$

for any $t \in [a, a+T]$. Using relation (21) we obtain

$$\left|\int_a^t \langle\phi(\widehat{u}, r), \bar{u}(r)\rangle\, dr\right| \le \frac{\nu}{2}\int_a^t \|\bar{u}(r)\|^2\, dr + \frac{(\gamma_T)^2}{2\nu}\int_a^{a+T} \|\widehat{u}(r)\|^2\, dr$$

for all $t \in [a, a+T]$, where $\gamma_T := \int_0^T |\gamma(\tau)|d\tau$. By Hypothesis 1 we have that

$$2\mathbb{E}\int_a^t (f(u^1(r)) - f(u^2(r)), \bar{u}(r))\, dr \ge 0;$$

$$
\begin{aligned}
2\mathbb{E}\int_a^t &\langle h(r,u^1(r)) - h(r,u^2(r)), \bar{u}(r)\rangle\, dr \\
&\le 2L_h\mathbb{E}\int_a^t \|\bar{u}(r)\|_{H^s(\mathcal{O})}\|\bar{u}(r)\|_{H^{1-s}(\mathcal{O})}\, dr \\
&\le 2c_s^2L_h^2\nu^{-1}\mathbb{E}\int_a^t |\bar{u}(r)|^2\, dr + \frac{\nu}{2}\mathbb{E}\int_a^t \|\bar{u}(r)\|^2\, dr, \qquad (23)
\end{aligned}
$$

where $c_s = b_s \cdot b_{1-s}$ and b_σ is the constant from the interpolation estimate

$$
\|u\|_{H^\sigma(\mathcal{O})} \le b_\sigma|u|^{1-\sigma}\|u\|^\sigma, \quad 0 \le \sigma \le 1
$$

($c_s = 1$ if either $s = 0$ or $s = 1$); and also

$$
\mathbb{E}\int_a^t \|g(r,u^1(r)) - g(r,u^2(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\, dr \le L_g\mathbb{E}\int_a^t |\bar{u}(r)|^2\, dr.
$$

Thus, we obtain from (22) that

$$
\begin{aligned}
\mathbb{E}|\bar{u}(t)|^2 &+ \nu E\int_a^t \|\bar{u}(r)\|^2\, dr \\
&\le \frac{(\gamma_T)^2}{\nu}\mathbb{E}\int_a^{a+T} \|\widehat{u}(r)\|^2\, dr + \left(2c_s^2\nu^{-1}L_h^2 + L_g\right)\int_a^t \mathbb{E}|\bar{u}(r)|^2\, dr \qquad (24)
\end{aligned}
$$

for $t \in [a, a+T]$. From this last inequality and applying Gronwall's Lemma we deduce that

$$
\mathbb{E}|\bar{u}(t)|^2 \le \frac{(\gamma_T)^2}{\nu}\exp\{(2c_s^2\nu^{-1}L_h^2 + L_g)T\}\mathbb{E}\int_a^{a+T} \|\widehat{u}(r)\|^2\, dr
$$

for any $t \in [a, a+T]$, and thus, from (24), one obtains

$$
\nu\mathbb{E}\int_a^{a+T} \|\bar{u}(r)\|^2\, dr \le C_T\mathbb{E}\int_a^{a+T} \|\widehat{u}(r)\|^2\, dr, \qquad (25)
$$

with

$$
C_T = \frac{(\gamma_T)^2}{\nu}\left[1 + (2c_s^2\nu^{-1}L_h^2 + L_g)T\exp\{(2c_s^2\nu^{-1}L_h^2 + L_g)T\}\right].
$$

Since $\gamma_T \to 0$ as $T \to 0$, it is clear from inequality (25), that there exists $T^* > 0$ such that for any $T \le T^*$, we have

$$
\mathbb{E}\int_a^{a+T} \|u^1(r) - u^2(r)\|^2\, dr \le \frac{1}{2}\mathbb{E}\int_a^{a+T} \|\widehat{u}^1(r) - \widehat{u}^2(r)\|^2\, dr,
$$

and thus the mapping $\mathcal{T} : \widehat{u} \mapsto u$ defined by (19) is a contraction in the space $M^2_{\mathcal{F}_t}(a, a+T; H_0^1(\mathcal{O}))$. Therefore, $\mathcal{T}$ possesses a unique fixed point u, $u = \mathcal{T}u$. From the definition of the mapping $\mathcal{T}$, the function u solves (19) with $\widehat{u} = u$, and hence u possesses the regularity required by Definition 1. □

Step 2: global existence. Now, we can complete the proof of Theorem 1. Let u^0 and u_- satisfy (14) and (15). We take $T^* > 0$ whose existence is ensured in Proposition 1. We thus have a solution u_1 in

$$
M^2_{\mathcal{F}_t}(a, a+T^*; H_0^1(\mathcal{O})) \cap M^{p+1}_{\mathcal{F}_t}(a, a+T^*; L^{p+1}(\mathcal{O})) \cap L^2_{\mathcal{F}_t}(\Omega; C([a, a+T^*]; L^2(\mathcal{O}))).
$$

Now we can define new initial data by the formulas

$$
u_0^a = u_1(a+T^*) \quad \text{and} \quad u_-^a(t) = \begin{cases} u_1(t), & \text{for } t \in (a, a+T^*); \\ u_-(t), & \text{for } t < a. \end{cases}
$$

It is clear that u_0^a and u_-^a satisfy conditions (14) and (15) with $a+T^*$ instead of a. Therefore, by Proposition 1, we have a solution u_2 on the interval $(a+T^*, a+2T^*)$. It is clear that the function defined as u_1 on $[a, a+T^*]$, and as u_2 on $[a+T^*, a+2T^*]$, is a solution to problem (11)–(13) on the interval $(a, a+2T^*)$. Proceeding inductively, in each step $n \geq 2$ we obtain a solution on any interval of the form $(a, a+nT^*)$.

The uniqueness of solution follows from the first step of the argument. The point is that a fixed point of $\mathcal{T}$ is unique.

Finally, taking into account the hypotheses on f, g and h, (14), (15), and the property (20) for $\widehat{u} = u$, we immediately obtain from the Itô formula (see [21] and also [18]) that, if u is the solution to problem (11)–(13) on the interval $(a, a+T)$, then it satisfies (18). □

4. **Stability properties of solutions.** In this section we prove exponential decaying of the solutions given by Theorem 1 when the diffusion coefficient ν is large enough. We also assume that there exist constants $C > 0$ and $\theta > 0$ such that

$$\|h(t,0)\|^2_{H^{-1}(\mathcal{O})} + \|g(t,0)\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))} \leq Ce^{-\theta t} \quad \text{for all } t \geq 0 \tag{26}$$

and

$$\gamma_\infty^{(\theta)} := \int_0^\infty |\gamma(\tau)| e^{\theta\tau} d\tau < \infty. \tag{27}$$

Theorem 2. *Under the assumptions in Theorem 1, suppose that* (26) *and* (27) *are satisfied. Moreover, assume that*

$$2\nu > 2\gamma_\infty^{(0)} + 2c_s L_h \lambda_1^{-1/2} + L_g \lambda_1^{-1}, \tag{28}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions and c_s is the constant in (23). *Then, there exist $C_1 > 0$, $C_2 > 0$ and $\varepsilon \in (0, \theta)$ such that for any*

$$u_0 \in L^2(\Omega, \mathcal{F}_a, \mathbb{P}; L^2(\mathcal{O})) \quad \textit{and} \quad u_- \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(-\infty, a; H_0^1(\mathcal{O}))),$$

the solution u to problem (11) − (13) *satisfies*

$$\mathbb{E}|u(t)|^2 \leq \left(\mathbb{E}|u_0|^2 + C_1\mathbb{E}|u_-|^2_{L^2(-\infty,a;H_0^1(\mathcal{O}))} + C_2\right) e^{-\varepsilon t} \quad \text{for all } t \geq a. \tag{29}$$

Proof. By assumption (28), we can take $\varepsilon > 0$ and $\alpha > 0$ such that $\varepsilon < \theta$, and

$$2\nu > 2\gamma_\infty^{(\varepsilon)} + 2c_s^2 L_h \lambda_1^{-1/2} + \alpha + [(1+\alpha)L_g + \varepsilon]\lambda_1^{-1}. \tag{30}$$

Applying Itô's formula to $e^{\varepsilon t}|u(t)|^2$ and taking expectation, we obtain

$$\begin{aligned}
& e^{\varepsilon t}\mathbb{E}|u(t)|^2 + 2\mathbb{E}\int_a^t e^{\varepsilon r}(\nu\|u(r)\|^2 + (f(u(r)), u(r)))\, dr \\
&\quad + 2\mathbb{E}\int_a^t e^{\varepsilon r}(\phi(u;r) + \psi_a(r), u(r))\, dr \\
&= \mathbb{E}|u_0|^2 + 2\mathbb{E}\int_a^t e^{\varepsilon r}(h(r,u(r)), u(r))\, dr \\
&\quad + \mathbb{E}\int_a^t e^{\varepsilon r}\|g(r,u(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\, dr + \varepsilon\mathbb{E}\int_a^t e^{\varepsilon r}|u(r)|^2\, dr
\end{aligned} \tag{31}$$

for all $t \geq a$, where $\psi_a(r)$ and $\phi(u;r)$ are given by (15) and (20) respectively. Arguing as in the proof of (21) and (16), one can see that

$$\left|2\mathbb{E}\int_a^t e^{\varepsilon r}(\phi(u;r)+\psi_a(r),u(r))\,dr\right| \leq (2\gamma_\infty^{(\varepsilon/2)}+\delta)\mathbb{E}\int_a^t e^{\varepsilon r}\|u(r)\|^2\,dr$$
$$+C(\delta,\varepsilon)\mathbb{E}|u_-|^2_{L^2(-\infty,a;H_0^1(\mathcal{O}))}$$

for any $\delta > 0$. On the other hand, by (9a), we have

$$\begin{aligned}
&2\mathbb{E}\int_a^t e^{\varepsilon r}\langle h(r,u(r)),u(r)\rangle\,dr \\
&\quad\leq 2c_sL_h\mathbb{E}\int_a^t e^{\varepsilon r}|u(r)|\|u(r)\|\,dr + 2\mathbb{E}\int_a^t e^{\varepsilon r}\|h(r,0)\|_{H^{-1}(\mathcal{O})}\|u(r)\|\,dr \\
&\quad\leq (2c_sL_h\lambda_1^{-1/2}+\alpha)\mathbb{E}\int_a^t e^{\varepsilon r}\|u(r)\|^2\,dr + \frac{1}{\alpha}\mathbb{E}\int_a^t e^{\varepsilon r}\|h(r,0)\|^2_{H^{-1}(\mathcal{O})}\,dr,
\end{aligned}$$

and by (9b),

$$\begin{aligned}
&\mathbb{E}\int_a^t e^{\varepsilon r}\|g(r,u(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr \\
&\quad\leq \frac{(1+\alpha)L_g}{\lambda_1}\mathbb{E}\int_a^t e^{\varepsilon r}\|u(r)\|^2\,dr + \left(1+\frac{1}{\alpha}\right)\mathbb{E}\int_a^t e^{\varepsilon r}\|g(r,0)\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr.
\end{aligned}$$

Now, taking into account (8), (26) and (30), we easily deduce from (31) that

$$e^{\varepsilon t}\mathbb{E}|u(t)|^2 \leq \mathbb{E}|u_0|^2 + C_1(\alpha,\varepsilon)\mathbb{E}|u_-|^2_{L^2(-\infty,a;H_0^1(\mathcal{O}))} + C_2(\alpha)\int_a^t e^{(\varepsilon-\theta)r}\,dr \tag{32}$$

for all $t \geq a$. This implies (29). □

From the above proof, it is easy to deduce the following result.

Corollary 1. *Under the assumptions of Theorem 2, if we also assume $h(t,0)\equiv 0$ and $g(t,0)\equiv 0$, then the zero solution is globally exponentially stable. Moreover, one can choose $C_2 = 0$ in (29).*

5. **A case study: exponentially fading memory.** In this section we present another approach to the study of stability properties of our problem. This approach goes back to an idea due to Dafermos [11].

We assume that the kernel $\gamma(s)$ belongs to $C^2(\mathbb{R}_+)$, $\lim_{s\to\infty}\gamma(s)=0$, and the function $\mu(s) := \gamma'(s)$ possesses the properties

$$\mu(s)\geq 0, \quad \mu'(s)+\delta\mu(s)\leq 0, \tag{33}$$

where δ is a positive constant. These hypotheses concerning $\gamma(s)$ imply that

$$0\leq \mu(s)\leq \mu(0)e^{-\delta s}, \quad s\in\mathbb{R}_+, \tag{34}$$

and

$$0\leq -\gamma(s)\leq \frac{\mu(0)}{\delta}e^{-\delta s}, \quad s\in\mathbb{R}_+. \tag{35}$$

It also follows from (33) that either (i) $\mu(s)>0$ for all $s\in\mathbb{R}_+$ or (ii) there exists $s_*>0$ such that $\mu(s)>0$ for $s\in[0,s_*)$ and $\mu(s)=0$ for all $s\geq s_*$. In the latter we have a retarded problem with *finite* delay and therefore we will concentrate mainly on the first case.

For the sake of simplicity we assume the initial time $a=0$.

Suppose that u_0 and u_- satisfy (14) and (17) with $a = 0$. Following the idea introduced by Dafermos [11] (see also [10, 13] and the survey [16]), we introduce the new variable

$$q(t; s, x) := \int_0^s u(t-\tau, x) d\tau = \int_{t-s}^t u(\tau, x) d\tau, \quad t, s \geq 0, \tag{36}$$

where $u(t)$ is the solution to (11)–(13) given by Theorem 1 for $t \geq 0$, and $u(t) = u_-(t)$ for $t < 0$. Observe that

$$\|q(t; s)\|^2 \leq s \int_{t-s}^t \|u(\tau)\|^2 d\tau,$$

and, consequently, by (34) and assumption (17)

$$\begin{aligned}\int_0^\infty \mu(s)\|q(t; s)\|^2\, ds &\leq \int_0^\infty s\mu(s) \int_{t-s}^t \|u(\tau)\|^2 d\tau\, ds \\ &\leq \mu(0) \int_0^\infty s e^{-\delta s} \int_{t-s}^t \|u(\tau)\|^2 d\tau\, ds \\ &= \frac{\mu(0)}{\delta^2} \int_{-\infty}^t [\delta(t-r) + 1] e^{-\delta(t-r)} \|u(r)\|^2\, dr < +\infty.\end{aligned} \tag{37}$$

Therefore, for all $t \geq 0$, the variable $q(t) := q(t, \cdot)$ belongs to the weighted Hilbert space

$$\mathcal{W}^1_\mu := L^2(\mathbb{R}_+, \mu(s) ds; H^1_0(\mathcal{O})),$$

which consists of $H^1_0(\mathcal{O})$-valued measurable functions $q(s)$ such that

$$\|q\|^2_{\mathcal{W}^1_\mu} := \int_0^\infty \mu(s)\|q(s)\|^2\, ds < +\infty.$$

In fact, it is not difficult to see that $q \in C(\mathbb{R}_+; \mathcal{W}^1_\mu)$. Moreover, integrating by parts, one can show that the pair (u, q) satisfies the equation

$$u_t - \nu\Delta u - \int_0^\infty \mu(s)\Delta q(t; s) ds + f(u) = h(t, u) + g(t, u)\partial_t W(t) \tag{38}$$

for $x \in \mathcal{O},\ t > 0$.

Now, observe that the partial derivatives of q are $q_s(t; s) = u(t-s)$, and $q_t(t; s) = u(t) - u(t-s)$. Using the properties on the function μ it is easy to see that q_t and q_s belong to $L^2(0, T; \mathcal{W}^1_\mu)$ for all $T > 0$. Then, we have

$$\begin{aligned}&-\int_0^\infty \mu(s)\langle \Delta q(t; s), u(t)\rangle ds = \int_0^\infty \mu(s)((q(t; s), q_t(t; s) + q_s(t; s)))ds \\ &= \frac{1}{2}\frac{d}{dt}\|q(t)\|^2_{\mathcal{W}^1_\mu} + \frac{1}{2}\int_0^\infty \mu(s)\partial_s(\|q(t; s)\|^2) ds.\end{aligned} \tag{39}$$

But, taking into account that $q(t, 0) = 0,\ \lim_{s\to\infty} \mu(s)\|q(t, s)\|^2 = 0$, and integrating by parts in the last term in (39), we obtain

$$\begin{aligned}&-\int_0^\infty \mu(s)\langle \Delta q(t; s), u(t)\rangle ds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)\|q(t; s)\|^2 ds - \frac{1}{2}\int_0^\infty \mu'(s)\|q(t; s)\|^2 ds,\end{aligned} \tag{40}$$

for all $t \geq 0$. This equality and (33) yield that

$$\frac{d}{dt}(e^{\varepsilon t}\|q(t)\|^2_{\mathcal{W}^1_\mu}) + (\delta-\varepsilon)e^{\varepsilon t}\|q(t)\|^2_{\mathcal{W}^1_\mu} \leq 2e^{\varepsilon t}\int_0^\infty \mu(s)((q(t,s),u(t)))\,ds, \quad (41)$$

for any $\varepsilon \in \mathbb{R}$ and $t \geq 0$. This relation and Itô's formula allow us to prove the following assertion.

Theorem 3. *In addition to the assumptions in Theorem 1, suppose that* (26) *and* (33) *are satisfied, and moreover*

$$2\nu > 2c_s L_h \lambda_1^{-1/2} + L_g\lambda_1^{-1}, \quad (42)$$

where, as in Theorem 2, $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions and c_s is the constant in (23). *Then, there exists $C_1 > 0$, $C_2 > 0$ and $\varepsilon \in (0,\theta)$ such that for any*

$$u_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};L^2(\mathcal{O})) \quad \textit{and} \quad u_- \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};L^2(-\infty,0;H^1_0(\mathcal{O}))),$$

the solution u to problem (11) − (13) *(with $a = 0$) satisfies*

$$\mathbb{E}|u(t)|^2 + \mathbb{E}\|q(t)\|^2_{\mathcal{W}^1_\mu} \leq (\mathbb{E}|u_0|^2 + C_1\mathbb{E}\|u_-\|^2_{L^2(-\infty,0;H^1_0(\mathcal{O}))} + C_2)e^{-\varepsilon t} \quad (43)$$

for all $t \geq 0$, where $q(t) \equiv q(t;s,x)$ is given by (36). *Moreover, $C_2 = 0$ in* (43) *provided $h(t,0) \equiv 0$ and $g(t,0) \equiv 0$.*

Remark 1. If we compare Theorems 2 and 3, we see that, in contrast with (28), the condition in (42) does not contain the term depending on the relaxation kernel γ. This reflects that the function γ has the 'right' sign in the case of Theorem 3.

Proof. Thanks to assumption (42), we can take $\varepsilon > 0$ and $\alpha > 0$ such that $\varepsilon < \min\{\delta,\theta\}$, where δ and θ are parameters from (33) and (26) respectively, and

$$2\nu > 2c_s L_h\lambda_1^{-1/2} + \alpha + [(1+\alpha)L_g + \varepsilon]\lambda_1^{-1}. \quad (44)$$

Let u be a solution to (11)–(13) with $a = 0$. Using the energy balance equality in (18) one can see

$$\begin{aligned}
&e^{\varepsilon t}\mathbb{E}|u(t)|^2 + 2\mathbb{E}\int_0^t e^{\varepsilon r}(\nu\|u(r)\|^2 + (f(u(r)),u(r)))\,dr \\
&+ 2\mathbb{E}\int_0^t\int_0^\infty \mu(s)e^{\varepsilon r}((q(r,s),u(r)))\,ds\,dr \\
&= \mathbb{E}|u_0|^2 + 2E\int_0^t e^{\varepsilon r}(h(r,u(r)),u(r))\,dr \\
&+ \mathbb{E}\int_0^t e^{\varepsilon r}\|g(r,u(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr + \varepsilon\mathbb{E}\int_0^t e^{\varepsilon r}|u(r)|^2\,dr, \quad t \geq 0.
\end{aligned} \quad (45)$$

Now, taking into account (8), (9), (41) and that $\varepsilon < \delta$, it follows from (45) that

$$\begin{aligned}
&e^{\varepsilon t}(\mathbb{E}|u(t)|^2 + \mathbb{E}\|q(t)\|^2_{\mathcal{W}^1_\mu}) + 2\nu\mathbb{E}\int_0^t e^{\varepsilon r}\|u(r)\|^2\,dr \\
&\leq \mathbb{E}|u_0|^2 + \mathbb{E}\|q(0)\|^2_{\mathcal{W}^1_\mu} + 2c_sL_h\mathbb{E}\int_0^t e^{\varepsilon r}|u(r)|\|u(r)\|,dr \\
&+ 2\mathbb{E}\int_0^t e^{\varepsilon r}\|h(r,0)\|_{H^{-1}(\mathcal{O})}\|u(r)\|\,dr + (1+\alpha)L_g\mathbb{E}\int_0^t e^{\varepsilon r}|u(r)|^2\,dr \\
&+ (1+1/\alpha)\mathbb{E}\int_0^t e^{\varepsilon r}\|g(r,0)\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr + \varepsilon\mathbb{E}\int_0^t e^{\varepsilon r}|u(r)|^2\,dr, \quad t \geq 0.
\end{aligned} \quad (46)$$

From (26), (44) and (46) we have now

$$e^{\varepsilon t}(\mathbb{E}|u(t)|^2 + \mathbb{E}\|q(t)\|^2_{\mathcal{W}^1_\mu}) \le \mathbb{E}|u_0|^2 + \mathbb{E}\|q(0)\|^2_{\mathcal{W}^1_\mu} + C(1+1/\alpha)\int_0^t e^{(\varepsilon-\theta)r}\,dr \quad (47)$$

for $t \ge 0$. Observe that from (37) we obtain that

$$\|q(0)\|^2_{\mathcal{W}^1_\mu} \le \frac{\mu(0)}{\delta^2}\|u_-\|^2_{L^2(-\infty,0;H^1_0(\mathcal{O}))} \quad \mathbb{P}-\text{a.s.} \quad (48)$$

Taking into account that $\varepsilon < \theta$, then (43) follows from (47) and (48). □

As a consequence of Theorem 3, one can also obtain pathwise exponential decay of the solutions.

Theorem 4. *Under the assumptions in Theorem 3, there exists $\widetilde{\varepsilon} > 0$ satisfying that, for each*

$$u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathcal{O})) \quad \textit{and} \quad u_- \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(-\infty, 0; H^1_0(\mathcal{O}))),$$

there exists a random time $T(u_0, u_-, \omega) \ge 0$ such that the solution u to problem (11) − (13) *satisfies* $\mathbb{P}$*-a.s.*

$$|u(t,\omega)|^2 + \|q(t,\omega)\|^2_{\mathcal{W}^1_\mu} \le e^{-\widetilde{\varepsilon}t}, \quad \textit{for all } t \ge T(u_0, u_-, \omega). \quad (49)$$

Proof. We take $\alpha > 0$ such that $2\nu > 2c_sL_h\lambda_1^{-1/2} + \alpha + (1+\alpha)L_g\lambda_1^{-1}$. Then, using Itô's formula and (41) with $\varepsilon = 0$, we have for any integer $N \ge 0$,

$$\begin{aligned} &|u(t)|^2 + \|q(t)\|^2_{\mathcal{W}^1_\mu} \\ &\quad \le |u(N)|^2 + \|q(N)\|^2_{\mathcal{W}^1_\mu} + \frac{1}{\alpha}\int_N^{N+1}\|h(r,0)\|^2_{H^{-1}(\mathcal{O})}\,dr \\ &\qquad + \left(1+\frac{1}{\alpha}\right)\int_N^{N+1}\|g(r,0)\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr \\ &\qquad + 2\sup_{N\le\bar{t}\le N+1}\int_N^{\bar{t}}(g(r,u(r))dW(r),u(r)) \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (50)$$

for all $N \le t \le N+1$. Burkholder-Davis-Gundy's inequality implies now

$$\begin{aligned} &2\mathbb{E}\left\{\sup_{N\le\bar{t}\le N+1}\int_N^{\bar{t}}(g(r,u(r))dW(r),u(r))\right\} \\ &\quad \le \frac{1}{2}\mathbb{E}\left\{\sup_{N\le\bar{t}\le N+1}|u(\bar{t})|^2\right\} + 18\mathbb{E}\int_N^{N+1}\|g(r,u(r))\|^2_{\mathcal{L}^2(L^2(\mathcal{O}))}\,dr, \end{aligned}$$

and thanks to Theorem 3 and (50), there exists a constant $\widetilde{C} > 0$, depending on u_0 and u_-, but not on N, such that

$$\mathbb{E}\left\{\sup_{N\le t\le N+1}(|u(t)|^2 + \|q(t)\|^2_{\mathcal{W}^1_\mu})\right\} \le \widetilde{C}e^{-\varepsilon N} \quad \text{for all } N \ge 0, \quad (51)$$

with the constant $\varepsilon \in (0,\theta)$ that appears in (43). Now, (49) can be obtained from (51) in a standard way applying Chebyshev's inequality and Borel-Cantelli's lemma (see, e.g., [4] for a similar situation for a 2D Navier-Stokes model). □

6. **Eventual stabilization by noise.** In addition to the assumptions in Theorem 3, we assume in this section that

- $h(t,0)=0$ and $g(t,0)=0$ for all $t\geq 0$,
- $h:\mathbb{R}_+\times L^2(\mathcal{O})\to L^2(\mathcal{O})$, and there exists $\widetilde{L}_h>0$ such that

$$|h(t,u)-h(t,v)|\leq \widetilde{L}_h|u-v| \quad \text{for all } t\geq 0 \text{ and any } u,v\in L2(\mathcal{O}), \tag{52}$$

and consider the question of stabilizing the null solution of the deterministic problem

$$u_t-\nu\Delta u+\int_{-\infty}^t \gamma(t-s)\Delta u(s)ds+f(u)=h(t,u), \quad x\in\mathcal{O},\ t>0, \tag{53}$$

by adding a suitable multiplicative noise.

It follows from (52) that one can take $L_h=\lambda_1^{-1/2}\widetilde{L}_h$. Thus, when $\nu>\widetilde{L}_h\lambda_1^{-1}$, Theorem 3 ensures that the zero solution of (53) is exponentially stable in $L^2(\mathcal{O})$.

Our goal in this section is to show that we can choose a very simple stochastic perturbation (i.e. a suitable function g and a Wiener process W) such that the zero solution of (11) − (13) becomes exponentially stable for a wider range of values of the diffusion parameter ν (i.e. for some of those satisfying $\nu\leq\widetilde{L}_h\lambda_1^{-1}$). This means that noise stabilizes the system.

To construct this stochastic perturbation we denote by $\beta(t)$ a standard real $\mathcal{F}_t$-Wiener process, and for any $\sigma\in\mathbb{R}$ consider the problem

$$u_t-\nu\Delta u+\int_{-\infty}^t \gamma(t-s)\Delta u(s)ds+f(u)=h(t,u)+\sigma u\partial_t\beta(t), \quad x\in\mathcal{O},\ t>0, \tag{54}$$

in the bounded domain $\mathcal{O}$ with the boundary condition

$$u(t,x)=0,\ x\in\partial\mathcal{O},\ t>0, \tag{55}$$

and the initial conditions

$$u(0,x)=u_0(x),\ u(t,x)=u_-(t,x),\ t<0,\ x\in\mathcal{O}. \tag{56}$$

Obviously, Theorem 1 can be applied to this situation to ensure existence and uniqueness of solution to (54)-(56).

Theorem 5. *Assume that f satisfies the assumptions in Hypothesis 1 and that $\gamma\in C^2(\mathbb{R}_+)$ satisfies the assumptions in Section 5 (see (33)-(35)). Moreover suppose that h satisfies (52) and $h(t,0)=0$ for all $t\geq 0$. Let u be a solution to problem (54)–(56) with the initial data*

$$u_0\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};L^2(\mathcal{O})) \quad \text{and} \quad u_-\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};L^2(-\infty,0;H_0^1(\mathcal{O}))). \tag{57}$$

Then, the top Lyapunov exponent $\Lambda(u_0,u_-;\omega)$ defined by the formula

$$\Lambda(u_0,u_-;\omega):=\limsup_{t\to\infty}\frac{1}{t}\log\left(|u(t,\omega)|^2+\|q(t,\omega)\|^2_{\mathcal{W}^1_\mu}\right) \qquad \mathbb{P}-a.s., \tag{58}$$

where q is given by (36), satisfies the relation

$$\Lambda(u_0,u_-;\omega)\leq 2\left(\widetilde{L}_h-\nu\lambda_1\right)_+-G(\sigma,\delta) \quad \mathbb{P}\text{-}a.s. \tag{59}$$

Here $s_+:=(|s|+s)/2$ and

$$G(\sigma,\delta):=\min_{0\leq s\leq 1}\left\{2\sigma^2s^2-(\delta+\sigma^2)s+\delta\right\}, \tag{60}$$

where δ is the parameter in (33).

As a consequence of Theorem 5 we have the following assertion on stability of the zero solution to (54).

Corollary 2. *In addition to the hypotheses of Theorem 5 we assume that*

$$0<\sigma^2\le \delta/3 \quad \textit{and} \quad \nu\lambda_1>\widetilde{L}_h-\sigma^2/2. \tag{61}$$

Then, the zero solution to (54) – (55) *is pathwise exponentially stable. Namely, there exists a deterministic number* $\widetilde{\varepsilon}>0$ *such that for any initial data* u_0 *and* u_- *satisfying* (57) *there exists a random variable* $T(\omega)\equiv T(\omega;u_0,u_-)>0$ *such that, for almost all* $\omega\in\Omega$,

$$|u(t,\omega)|^2+\|q(t,\omega)\|^2_{\mathcal{W}^1_\mu}\le e^{-\widetilde{\varepsilon}t} \quad \textit{for all } t\ge T(\omega), \tag{62}$$

where u *is the solution to problem* (54)–(56) *and* q *is given by* (36).

Proof. Notice that $G(\sigma,\delta)=\sigma^2$ in this case. Therefore, by Theorem 5, there exists $\widetilde{\varepsilon}>0$ such that

$$\Lambda(u_0,u_-;\omega)\le 2\left(\widetilde{L}_h-\nu\lambda_1\right)_+-\sigma^2\le-\widetilde{\varepsilon}<0 \quad \mathbb{P}\text{-a.s.}$$

under condition (61). This implies (62) by the definition in (58). □

Proof of Theorem 5. Let u_0 and u_- be given. The idea is to apply Itô's formula to $\log(|u(t)|^2+\|q(t)\|^2_{\mathcal{W}^1_\mu})$. However, we cannot proceed directly because of regularity and blow-up reasons. Let us recall briefly how to avoid this difficulty, reasoning in the same way as in the proof of Theorem 4.3 in [5].

It is not difficult to see that, given two pairs of initial data (u_0^1,u_-^1) and (u_0^2,u_-^2), the corresponding solutions u_1 and u_2 satisfy $u_1|_\Gamma=u_2|_\Gamma$ $\mathbb{P}$-a.s., where $\Gamma=\{\omega\in\Omega\ :\ u_0^1(\omega)=u_0^2(\omega),\ u_-^1(\omega)=u_-^2(\omega)\}$. Therefore, setting

$$\begin{aligned}\Omega_0&:=\{\omega\in\Omega\ :\ u_0(\omega)=0,\ u_-(\omega)=0\}\\&=\{\omega\in\Omega\ :|\ u_0(\omega)|^2+\|q(0,\omega)\|^2_{\mathcal{W}^1_\mu}=0\},\end{aligned}$$

it is immediate that the solution u to (54)–(56) with data (u_0,u_-) satisfies $u|_{\Omega_0}=0$ $\mathbb{P}$-a.s., and consequently we must prove (59) for the set $\Omega\setminus\Omega_0$. Thus, changing the values of u_0 on Ω_0, we can suppose without loss of generality that

$$|u_0(\omega)|^2+\|q(0,\omega)\|^2_{\mathcal{W}^1_\mu}>0 \quad \text{for all } t\ge 0,\ \mathbb{P}\text{-a.s.}. \tag{63}$$

Observe that, as it is easy to see, this assumption implies $\|q(t,\omega)\|_{\mathcal{W}^1_\mu}>0$ for all $t>0$, $\mathbb{P}$-a.s. and, in particular,

$$|u(t,\omega)|^2+\|q(t,\omega)\|^2_{\mathcal{W}^1_\mu}>0 \quad \text{for all } t\ge 0,\ \mathbb{P}\text{-a.s.}. \tag{64}$$

Now, under the assumption (63), we consider the sequence of stopping times

$$\tau_n=\inf\{t\ge 0:\ |u(t)|^2+\|q(t)\|^2_{\mathcal{W}^1_\mu}\le 1/n\}.$$

Thanks to (64), $\lim_{n\to+\infty}\tau_n(\omega)=+\infty$ $\mathbb{P}-$a.s..

Let us fix $\phi_n\in C^2(\mathbb{R})$ such that $\phi_n(r)=\log r$ for all $r\ge 1/n$, and denote (u^n,q^n) the processes defined by $u^n(t)=u(t\wedge\tau_n)$ and $q^n(t)=q(t\wedge\tau_n)$.

Then, applying Itô's formula to $\phi_n(|u^n(t)|^2+\|q^n(t)\|^2_{\mathcal{W}^1_\mu})$, and taking into account (40), we easily obtain

$$
\begin{aligned}
&\log(|u^n(t)|^2 + \|q^n(t)\|^2_{\mathcal{W}^1_\mu}) + 2\nu \int_0^{t\wedge\tau_n} \frac{\|u^n(r)\|^2}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, dr \\
&- \int_0^{t\wedge\tau_n} \frac{1}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}} \int_0^\infty \mu'(s)\|q^n(r;s)\|^2\, ds\, dr \\
&+ 2\int_0^{t\wedge\tau_n} \frac{(f(u^n(r)), u^n(r))}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, dr \\
&= \log(|u_0|^2 + \|q(0)\|^2_{\mathcal{W}^1_\mu}) + 2\int_0^{t\wedge\tau_n} \frac{\langle h(r, u^n(r)), u^n(r)\rangle}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, dr \\
&+ \int_0^{t\wedge\tau_n} \frac{\sigma^2|u^n(r)|^2}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, dr \\
&- 2\int_0^{t\wedge\tau_n} \frac{\sigma^2|u^n(r)|^4}{(|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu})^2}\, dr \\
&+ 2\int_0^{t\wedge\tau_n} \frac{\sigma|u^n(r)|^2}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, d\beta(r) \quad \text{for all } t \ge 0,
\end{aligned}
$$

and, consequently, from (8), (33) and (52) we obtain

$$
\begin{aligned}
\log(|u^n(t)|^2 + \|q^n(t)\|^2_{\mathcal{W}^1_\mu}) \le \log(|u_0|^2 + \|q(0)\|^2_{\mathcal{W}^1_\mu}) &+ 2t\cdot\left(\widetilde{L}_h - \nu\lambda_1\right)_+ \\
&+ \int_0^{t\wedge\tau_n} \frac{\sigma^2|u^n(r)|^2 - \delta\|q^n(r)\|^2_{\mathcal{W}^1_\mu}}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, dr \\
&- \int_0^{t\wedge\tau_n} \frac{2\sigma^2|u^n(r)|^4}{(|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu})^2}\, dr \qquad (65)\\
&+ \int_0^{t\wedge\tau_n} \frac{2\sigma|u^n(r)|^2}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}}\, d\beta(r) \quad \text{for all } t \ge 0.
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
&\frac{\sigma^2|u^n(r)|^2 - \delta\|q^n(r)\|^2_{\mathcal{W}^1_\mu}}{|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu}} - \frac{2\sigma^2|u^n(r)|^4}{(|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu})^2} \\
&= \frac{-\sigma^2|u^n(r)|^4 + (\sigma^2 - \delta)|u^n(r)|^2\|q^n(r)\|^2_{\mathcal{W}^1_\mu} - \delta\|q^n(r)\|^4_{\mathcal{W}^1_\mu}}{(|u^n(r)|^2 + \|q^n(r)\|^2_{\mathcal{W}^1_\mu})^2} \\
&\le -\inf_{x\ge 0, y>0}\left\{\frac{\sigma^2x^2 + (\delta - \sigma^2)xy + \delta y^2}{(x+y)^2}\right\} = -G(\sigma, \delta),
\end{aligned}
$$

where $G(\sigma, \delta)$ is given by (60). Thus, when $n \to \infty$, we easily obtain from (65) that

$$
\begin{aligned}
&\log\left(|u(t)|^2 + \|q(t)\|^2_{\mathcal{W}^1_\mu}\right) \le \log\left(|u_0|^2 + \|q(0)\|^2_{\mathcal{W}^1_\mu}\right) \\
&+ \left[2\left(\widetilde{L}_h - \nu\lambda_1\right)_+ - G(\sigma, \delta)\right] t + \int_0^t \frac{2\sigma|u(r)|^2}{|u(r)|^2 + \|q(r)\|^2_{\mathcal{W}^1_\mu}}\, d\beta(r), \qquad (66)
\end{aligned}
$$

for all $t \geq 0$, $\mathbb{P}$-a.s.. Now, observe that the stochastic integral

$$M_t := \int_0^t \frac{|u(r)|^2}{|u(r)|^2 + \|q(r)\|^2_{\mathcal{W}^1_\mu}} \, d\beta(r), \quad t \geq 0,$$

is a square integrable continuous real martingale with associated increasing process

$$\langle M \rangle_t = \int_0^t \frac{|u(r)|^4}{(|u(r)|^2 + \|q(r)\|^2_{\mathcal{W}^1_\mu})^2} \, dr, \quad t \geq 0,$$

satisfying

$$\limsup_{t \to \infty} \frac{\langle M \rangle_t}{t} \leq 1 < \infty,$$

and consequently, by the law of large numbers for local martingales (see for example [19] Theorem 3.4 Chapter 1) we have that $\lim_{t\to\infty} t^{-1} M_t = 0$ $\mathbb{P}$-a.s. Therefore (59) follows from (66). □

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