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ATTRACTIVITY FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the long term dynamics of non-autonomous functional differential equations. Namely, we establish existence results on pullback attractors for non-linear neutral functional differential equations with time varying delays. The two main results differ in smoothness properties of delay functions.

1. **Introduction.** From an application point of view, differential equations relate state variables and their derivative(s); however, it is not always possible to formulate such a relation. In some cases, we can only establish connection between the rate of change of a difference operator and the state variables. In those situations, the difference operator uses the state variables at different time instants, and neutral functional differential equations (NFDE) are the appropriate mathematical models.

Neutral functional differential equations are of form

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{1}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous and maps bounded sets into bounded sets, where, for $r > 0$, $C = C([-r,0];\mathbb{R}^n)$ denotes the Banach space of continuous functions with the sup-norm. Furthermore, the difference operator $D : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous together with its first and second Fréchet derivatives with respect to its second variable; and the first and the second derivatives of D with respect to the second variable are continuous at zero. Lastly, for a given continuous function $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we denote by $x_t(\cdot)$ an element in C given by

$$x_t(\theta) = x(t+\theta),\ \theta \in [-r,0].$$

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When $x(\cdot)$ is a solution of (1), then $x_t(\cdot)$ is said to be the solution segment at time t.

When $D\phi = \phi(0)$ for all $\phi \in C$ then (1) becomes

$$\dot{x}(t) = f(t, x_t), \tag{2}$$

a delay differential equation. The knowledge about delay differential equations, in particular those of the form

$$\dot{x}(t) = f(t, x(t), x(t-1)),$$

has advanced substantially during the last half of a century, see [16, 21]. When the memory function on the right-hand side is more complicated because of the presence of delay distribution, our knowledge is not so advanced. Recently, stability results, existence results on periodic solutions to equations with distributed delays have been reported in [9, 10] and [11], respectively. The study of non-autonomous attracting sets of dynamical systems was initiated in [19, 20]. The notion of pullback attractors for non-autonomous dynamical systems was introduced [2, 12]. Furthermore, the asymptotic behaviour of non-autonomous ordinary differential equations is studied in [17]. Ideas for non-autonomous functional differential equations are presented in [1, 3, 4, 5]. Because of various possible reasons, the development of ideas for the family of equations complementary to (2) is much slower. Thus, every fact which might even be valid on a relatively small subset of the phase space C is of great value. For instance, [22] establishes existence of periodic solutions on BC, the Banach space (with the supremum norm) of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. We adumbrate that Theorem 3.2 of the present paper might suffer from similar limitation since some of our assumptions are fulfilled when solutions are bounded, although they can be satisfied also in other situations.

The present work focuses on difference operators of form

$$D\phi = \phi(0) - g(\phi(-\sigma))$$

where function g maps $\mathbb{R}^n$ into itself. In other words, the family of NFDEs that we consider here is

$$\frac{d}{dt}\left[x(t) - g(x(t-\sigma))\right] = F_0(t, x(t)) + \sum_{i=1}^{m} F_i(x(t - \rho_i(t))). \tag{3}$$

Here $\sigma > 0$, and $\rho_i : \mathbb{R} \to [0, h]$ are functions representing the variable delays of the model; additional restrictions are imposed on them in Sections 3 (3.1) and (3.2), as well as on the terms $F_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, \dots, m$. The inner product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. Throughout the paper, we assume that, for any continuous function $x : \mathbb{R} \to \mathbb{R}^n$, we have

(A_g): $|g(x(\tau-\sigma))|^2 \leq c^2|x(\tau)|^2 + \delta(\tau),\ 0 < c < 1,\ \tau \in \mathbb{R}$ such that $\delta(t) \in [0, M_\delta]$ for some $M_\delta \in \mathbb{R}^+$.

The aim of this work is to derive results on the asymptotic behaviour of solutions to NFDEs. Namely, it intends to extend the findings of [1] and [5] on the existence of pullback attractors for delay differential equations to NFDEs of form (3). The rest of the paper is organised as follows. Section 2 summarises the necessary theory of processes. In section 3, we state and prove our main results.

2. **Preliminaries.** To start with, recall that, for $r = \max\{\sigma, h\} > 0$, we denote by $C{=}C([-r, 0]; \mathbb{R}^n)$ the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with the usual $\|\phi\| = \sup_{s\in[-r,0]} |\phi(s)|$ supremum norm. The basic theory of neutral functional differential equations (see [8]) implies, under standard assumptions, the existence of the unique solution of (3) on $[s-r, \infty)$. That is, if an initial function $\phi \in C$ is prescribed at the initial time $s \in \mathbb{R}$ then there is an $x(\cdot; s, \phi)$ which satisfies (3) and, in addition, the initial condition $x_s(\cdot) = \phi$, in other words, $x_s(\theta){=}x(s{+}\theta){=}\phi(\theta)$ for all $\theta \in [-r, 0]$.

Now, we present the necessary background on the theory of processes. For more details on the topic we refer to [7]. The unique solution of the initial value problem associated to (2) defines the solution map $U(t, s) : C \ni \phi \mapsto x_t(\cdot; s, \phi) \in C$ for $t \geq s$, which is, in fact, a process (also called a two-parameters semigroup), i.e.

- $U(t, s) : C \to C$ is a continuous map for all $t \geq s$;
- $U(s, s) = id_C$, the identity on C, for all $s \in \mathbb{R}$,
- $U(t, s) = U(t, \tau)U(\tau, s)$ for $t \geq \tau \geq s$.

As in the autonomous case, we look for invariant attracting sets. First, we introduce the Hausdorff semi–distance between subsets A and B in a metric space (X, d) as

$$dist(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b).$$

Definition 2.1. Let U be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a (global) pullback attractor for U if, for all $s \in \mathbb{R}$, it satisfies

- $U(t, s)\mathcal{A}(s) = \mathcal{A}(t)$ for all $t \geq s$, and
- $\lim_{s\to\infty} dist(U(t, t - s)D, \mathcal{A}(t)) = 0$, for all bounded subsets D of X.

Definition 2.2. $\{B(t)\}_{t\in\mathbb{R}}$ is said to be absorbing with respect to the process U if, for $t \in \mathbb{R}$ and $D \subset X$ bounded, there exists $T_D(t) > 0$ such that for all $\tau \geq T_D(t)$

$$U(t, t - \tau)D \subset B(t).$$

The following results (see [13, 18]) shows that the existence of a family of compact absorbing sets implies the existence of a pullback attractor.

Theorem 2.3. *Let $U(t, s)$ be a process on a complete metric space X. If there exists a family $\{B(t)\}_{t\in\mathbb{R}}$ of compact absorbing sets then, there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ such that $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\subset X \\ bounded}} \Lambda_D(t)}$$

where

$$\Lambda_D(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{s\geq n} U(t, t - s)D}.$$

Theorem 2.4. *Suppose that $U(t, s)$ maps bounded sets into bounded sets and there exists a family $\{B(t)\}_{t\in\mathbb{R}}$ of bounded absorbing sets for U. Then there exists a pullback attractor for problem (3).*

We emphasize that it is possible to consider a more general definition of pullback attractor which attracts family of sets in a universe instead of only bounded sets (see [2],[15] for a detailed analysis of this theory). However, the present concept is enough for our interests.

3. **Main results.** Before formulating our main results, we include a lemma from [5] which will be useful in subsequent computations.

Lemma 3.1. *Let* $0 < \xi < 1$. *Then*

$$|u|^2 \leq \frac{1}{1-\xi}|u-v|^2 + \frac{1}{\xi}|v|^2$$

for any $u, v \in \mathbb{R}^n$.

3.1. **Continuously differentiable delay functions ρ_i.** In this section we impose conditions on the non-linearities of (3) as follows

(A1): $F_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and there exist $\alpha_0 > 0,\ \beta_0 \geq 0$ such that for any $\phi \in C$,

$$\langle F_0(t, \phi(0)), \phi(0) - g(\phi(-\sigma))\rangle \leq -\alpha_0|\phi(0) - g(\phi(-\sigma))|^2 + \beta_0,\ t \in \mathbb{R}.$$

(A2): $F_i,\ i = 1, \ldots, m$ is sublinear, i.e., there exist $k_i > 0,\ i = 1, \ldots, m$, such that

$$|F_i(x)|^2 \leq k_i^2(1 + |x|^2),\ x \in \mathbb{R}^n.$$

Remark 1. As it was already mentioned in the Introduction, **(A1)** might impose relatively strong limitations on the solutions of (3) and on function g. Namely, if g is continuous and the solutions of (3) are bounded then **(A1)** is satisfied. However, these properties are not necessary for satisfying **(A1)**.

Theorem 3.2. *Assume that assumptions (A1) and (A2) are satisfied. Furthermore, suppose that each delay function* $\rho_i(\cdot)$ *is continuously differentiable with* $\rho_i'(t) \leq \rho_{i*} < 1$ *for all* $t \in \mathbb{R}$. *Then, if* $m^2\, k_i^2 < \alpha_0^2(1 - \rho_{i*})$, *for all* $i = 1, \ldots, m$ *then there exists a family of bounded absorbing sets,* $\{B(t)\}_{t\in\mathbb{R}}$, *and consequently, there exists a pullback attractor for this problem.*

Proof. Let $\lambda > 0$ be a constant to be determined later on, and denote (for the sake of simplicity) $\varepsilon = \frac{\alpha_0}{m}$. Denote $x(\tau) = x(\tau; t_0 - t, \psi),\ \tau \in [t_0 - t, t_0]$, for any $\psi \in C$ such that $\|\psi\| \leq d$. Then, by a suitable application of the Young inequality ($2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$), it follows that

$$
\begin{aligned}
\frac{d}{d\tau} e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 = {} & \lambda e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 \\
& + 2e^{\lambda\tau} \langle x(\tau) - g(x(\tau-\sigma)), F_0(\tau, x(\tau)) \rangle \\
& + 2e^{\lambda\tau} \sum_{i=1}^{m} \langle x(\tau) - g(x(\tau-\sigma)), F_i(x(\tau - \rho_i(\tau))) \rangle \\
\leq {} & (\lambda - 2\alpha_0) e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 \\
& + 2e^{\lambda\tau} \beta_0 + e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 \sum_{i=1}^{m} \varepsilon \\
& + e^{\lambda\tau} \sum_{i=1}^{m} \varepsilon^{-1} |F_i(x(\tau - \rho_i(\tau)))|^2 \\
\leq {} & (\lambda - \alpha_0)\, e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 \\
& + 2e^{\lambda\tau} \beta_0 + e^{\lambda\tau} \varepsilon^{-1} \sum_{i=1}^{m} k_i^2 \\
& + e^{\lambda\tau} \varepsilon^{-1} \sum_{i=1}^{m} k_i^2 |x(\tau - \rho_i(\tau))|^2.
\end{aligned}
$$

Using Lemma 1 with $u := x(\tau) - g(x(\tau-\sigma))$ and $v := g(x(\tau-\sigma))$ and assumption (A_g), we obtain

$$
\begin{aligned}
|x(\tau) - g(x(\tau-\sigma))|^2 \leq & \frac{|x(\tau)|^2}{1-c} + \frac{|g(x(\tau-\sigma))|^2}{c} \\
\leq & \frac{|x(\tau)|^2}{1-c} + c|x(\tau)|^2 + \frac{\delta(\tau)}{c} \\
\leq & \frac{|x(\tau)|^2}{1-c} + c|x(\tau)|^2 + \frac{M_\delta}{c}.
\end{aligned}
$$

Thus we derive that

$$
\begin{aligned}
\frac{d}{d\tau} e^{\lambda\tau} |x(\tau) - g(x(\tau-\sigma))|^2 \leq {} & (\lambda - \alpha_0)\, e^{\lambda\tau} \left(\frac{|x(\tau)|^2}{1-c} + c|x(\tau)|^2 \right) \\
& + \left(2\beta_0 + \varepsilon^{-1} \sum_{i=1}^{m} k_i^2 + (\lambda - \alpha_0) \frac{M_\delta}{c} \right) e^{\lambda\tau} \\
& + e^{\lambda\tau} \varepsilon^{-1} \sum_{i=1}^{m} k_i^2 |x(\tau - \rho_i(\tau))|^2.
\end{aligned}
$$

Integration on the interval $[t_0 - t, \tau]$ yields that

$$
\begin{aligned}
e^{\lambda\tau}|x(\tau)-&g(x(\tau-\sigma))|^2 - e^{\lambda(t_0-t)}|x(t_0-t) - g(x(t_0-t-\sigma))|^2 \qquad (4)\\
&\leq \left(c + \frac{1}{1-c}\right)(\lambda-\alpha_0)\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds \\
&\quad + \frac{\left((\lambda-\alpha_0)\frac{M_\delta}{c} + 2\beta_0 + \varepsilon^{-1}\sum_{i=1}^m k_i^2\right)}{\lambda}\left[e^{\lambda\tau} - e^{\lambda(t_0-t)}\right] \\
&\quad + \varepsilon^{-1}\sum_{i=1}^m k_i^2 \int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho_i(s))|^2 ds. \qquad (5)
\end{aligned}
$$

Now we compute the integrals for the addends in the last sum.

$$
\begin{aligned}
\int_{t_0-t}^{\tau} & e^{\lambda s}|x(s-\rho_i(s))|^2 ds \\
&\leq \frac{1}{1-\rho_{i*}}\int_{t_0-t-h}^{\tau} e^{\lambda u+\lambda h}|x(u)|^2 du \\
&\leq \frac{e^{\lambda h}}{1-\rho_{i*}}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|x(u)|^2 du + \int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right] \\
&\leq \frac{e^{\lambda h}}{1-\rho_{i*}}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|\psi(u)|^2 du + \int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right] \\
&\leq \frac{d^2 e^{\lambda h}}{\lambda(1-\rho_{i*})}\left[e^{\lambda(t_0-t)} - e^{\lambda(t_0-t-h)}\right] \\
&\quad + \frac{e^{\lambda h}}{1-\rho_{i*}}\int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
e^{\lambda\tau}&|x(\tau) - g(x(\tau-\sigma))|^2 \\
&\leq e^{\lambda(t_0-t)}\left(\frac{d^2}{1-c} + d^2 c + \frac{M_\delta}{c}\right) \\
&\quad + \frac{(\lambda-\alpha_0)\frac{M_\delta}{c} + 2\beta_0 + \varepsilon^{-1}\sum_{i=1}^m k_i^2}{\lambda}\left[e^{\lambda\tau} - e^{\lambda(t_0-t)}\right] \\
&\quad + \frac{d^2 e^{\lambda h}}{\lambda}\left[e^{\lambda(t_0-t)} - e^{\lambda(t_0-t-h)}\right]\sum_{i=1}^m \frac{k_i^2\varepsilon^{-1}}{1-\rho_{i*}} \\
&\quad + \Bigg((\lambda-\alpha_0)\left(c + \frac{1}{1-c}\right) \\
&\qquad + e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m \frac{k_i^2}{1-\rho_{i*}}\Bigg)\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds.
\end{aligned}
$$

Now, observe that

$$\begin{aligned}\varepsilon^{-1}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}} &= \frac{m}{\alpha_0}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\\ &= \frac{1}{m\alpha_0}\sum_{i=1}^{m}\frac{m^2k_i^2}{1-\rho_{i*}}\\ &< \frac{1}{m\alpha_0}m\alpha_0^2\\ &= \alpha_0.\end{aligned}$$

Consequently, since $0<c<1$ we can choose a positive, but small enough, λ such that

$$(\lambda-\alpha_0)\left(c+\frac{1}{1-c}\right)+e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}<0.$$

This implies that

$$\begin{aligned}|x(\tau)-g(x(\tau-\sigma))|^2 \le &\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]e^{\lambda(t_0-t-\tau)}\\ &+\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}.\end{aligned}$$

Setting $\tau=t_0+\theta,\ \theta\in[-r,0]$ we obtain

$$\begin{aligned}&|x(t_0+\theta)-g(x(t_0+\theta-\sigma))|^2\\ &\quad\le\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]e^{-\lambda(t+\theta)}\\ &\quad+\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}.\end{aligned}$$

Since $(1-c)\,|u|^2-\frac{1-c}{c}\,|v|^2\le|u-v|^2$,

$$\begin{aligned}|x(t_0+\theta)|^2 \quad\le\quad &\frac{1}{1-c}\left\{\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_i^*}\right]e^{-\lambda(t+\theta)}\right.\\ &\left.+\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}\right\}+c\,|x(t_0+\theta)|^2+\frac{M_\delta}{c}.\end{aligned}$$

And so

$$\begin{aligned}|x(t_0+\theta)|^2 \quad\le\quad &\frac{1}{(1-c)^2}\left\{\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_i^*}\right]e^{-\lambda(t+\theta)}\right.\\ &\left.+\frac{(\lambda-\alpha_0)\frac{M}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}\right\}+\frac{M_\delta}{c(1-c)}.\end{aligned}$$

Thus

$$\sup_{\theta\in[-r,0]}|x(t_0+\theta))|^2 \le \frac{1}{(1-c)^2}\left\{\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^m\frac{k_i^2}{1-\rho_{i*}}\right]e^{-\lambda t+\lambda r}\right.$$
$$\left.+\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^m k_i^2}{\lambda}\right\}+\frac{M_\delta}{c(1-c)}$$
$$\le 1+\frac{1}{(1-c)^2}\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^m k_i^2}{\lambda}+\frac{M_\delta}{c(1-c)}$$

provided that

$$t\ge T_D=\lambda^{-1}\log\frac{1}{(1-c)^2}\left\{\left[\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}+\frac{d^2e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^m\frac{k_i^2}{1-\rho_{i*}}\right]e^{\lambda r}\right\}.$$

Consequently, the family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ given by $B(t):=B$, for all $t\in\mathbb{R}$, where B denotes the ball in $C([-r,0];\mathbb{R}^n)$ centred at zero with radius $R=1+\frac{1}{(1-c)^2}\frac{(\lambda-\alpha_0)\frac{M_\delta}{c}+2\beta_0+\varepsilon^{-1}\sum_{i=1}^m k_i^2}{\lambda}+\frac{M_\delta}{c(1-c)}$, is absorbing. On the other hand, as the associated process maps bounded sets of $C([-r,0];\mathbb{R}^n)$ into bounded sets, then Theorem 2.4 (see also Theorem 4.1 in [4]) ensures the existence of the pullback attractor. □

3.2. **Measurable delay functions.** In the previous section, the differentiability of the delay functions played an important role. Now we prove a similar result on the existence of pullback attractor when the aforementioned differentiability condition is relaxed. Our analysis will be carried out assuming that the delay functions $\rho_i(\cdot)$ are only measurable. But we would like to point out that there exists another technique which can be used when the variable delays are continuous, the so-called Razumikhin method (see, for instance [14]). This will be analysed in a subsequent paper.

In this section, we assume that our non-delay term satisfies a non-autonomous dissipativity condition, i.e. we impose the following assumption on F_0:

(A1'): $F_0:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is continuous and there exist $\alpha_0>0$, and a non-negative measurable function $\beta(\cdot)$ such that

$$\langle F_0(t,\phi(0)),\phi(0)-g(\phi(-\sigma))\rangle\le-\alpha_0|\phi(0)-g(\phi(-\sigma))|^2+\beta(t),\ t\in\mathbb{R},x\in\mathbb{R}^n. \quad (6)$$

As for F_i, we assume Lipschitz continuity, i.e.,

(A2'): There exists $L_i>0,\ i=1,\dots,m$ such that for any $x,y\in\mathbb{R}^n$

$$|F_i(x)-F_i(y)|\le L_i|x-y|,$$

and $F_i(0)=0$.

Furthermore, we shall use Lemma 3.1 of [6] formulated as follows.

Lemma 3.3. *Let $y:[t_0-t,\infty)\to\mathbb{R}^+$ be a function such that, there are constants $\gamma>0,\ \kappa,\kappa'>0$ with $\frac{\kappa'}{\gamma}<1$ and*

$$y(t)\le\begin{cases}\kappa e^{-\gamma t}+\kappa'\int_{t_0}^t e^{-\gamma(t-s)}\sup\limits_{\theta\in[-r,0]}y(s+\theta)ds,\ t>t_0,\\ \kappa e^{-\gamma t},\ t\in[t_0-r,t_0].\end{cases}$$

Then

$$y(t)\le\kappa e^{-\mu t},\ t\ge t_0-r \quad (7)$$

where μ is the positive solution of $\frac{\kappa'}{\gamma-\mu}e^{\mu r} = 1$.

Now we can establish our main result in this section.

Theorem 3.4. *Assume that assumptions (A1') and (A2') are satisfied and that there exists $\lambda \in (0, \alpha_0)$ such that*

$$\int_{-\infty}^{t} e^{\lambda s}\beta(s)\, ds < +\infty, \quad \forall t \in \mathbb{R}. \tag{8}$$

Then, if $\rho_i,\ i = 1, \ldots, m$, is measurable, and

$$\lambda(1-c)^2\alpha_0 \geq 2m\sum_{i=1}^{m} L_i^2,$$

there exists a family of bounded absorbing sets, $\{B(t)\}_{t\in\mathbb{R}}$, and consequently, there exists a pullback attractor for the process generated by (3).

Proof. Let us consider the number $\lambda \in (0, \alpha_0)$ from (8), pick $\varepsilon = \dfrac{2\sum_{i=1}^{m} L_i^2}{\lambda(1-c)^2}$, and denote $x(\tau) = x(\tau; t_0 - t, \psi),\ \tau \in [t_0 - t, t_0]$, for any $\psi \in C$ such that $\|\psi\| \leq d$, and $t_0 \in \mathbb{R}$. Then, applying again the Young inequality in the delay terms below, we obtain

$$\begin{aligned}
\frac{d}{d\tau}e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 &= \lambda e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 \\
&\quad + 2e^{\lambda\tau}\langle x(\tau) - g(x(\tau-\sigma)), F_0(\tau, x(\tau))\rangle \\
&\quad + 2e^{\lambda\tau}\sum_{i=1}^{m}\langle x(\tau) - g(x(\tau-\sigma)), F_i(x(\tau-\rho_i(\tau)))\rangle \\
&\leq (\lambda - 2\alpha_0)e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 \\
&\quad + 2e^{\lambda\tau}\beta(\tau) + e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 m\varepsilon \\
&\quad + e^{\lambda\tau}\varepsilon^{-1}\sum_{i=1}^{m}|F_i(x(\tau-\rho_i(\tau)))|^2 \\
&\leq (\lambda - \alpha_0)e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 + 2e^{\lambda\tau}\beta(\tau) \\
&\quad + e^{\lambda\tau}\varepsilon^{-1}\sum_{i=1}^{m}L_i^2|x(\tau-\rho_i(\tau))|^2 \\
&\leq 2e^{\lambda\tau}\beta(\tau) + e^{\lambda\tau}\varepsilon^{-1}\sum_{i=1}^{m}L_i^2|x(\tau-\rho_i(\tau))|^2.
\end{aligned}$$

Integration on the interval $[t_0 - t, \tau]$ yields that

$$\begin{aligned}
&e^{\lambda\tau}|x(\tau) - g(x(\tau-\sigma))|^2 - e^{\lambda(t_0-t)}|x(t_0-t) - g(x(t_0-t-\sigma))|^2 \\
&\quad \leq 2\int_{t_0-t}^{\tau} e^{\lambda s}\beta(s)ds + \varepsilon^{-1}\sum_{i=1}^{m}L_i^2\int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho_i(s))|^2 ds.
\end{aligned} \tag{9}$$

The integrand in the last sum can be estimated

$$\int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho_i(s))|^2 ds \leq \int_{t_0-t}^{\tau} e^{\lambda s}\sup_{\theta\in[-r,0]}|x(s+\theta)|^2 ds$$

Thus we have

$$
\begin{aligned}
&|x(\tau)-g(x(\tau-\sigma))|^2 \qquad (10)\\
&\quad\le e^{\lambda(t_0-t-\tau)}\left(\frac{|x(t_0-t)|^2}{1-c}+c|x(t_0-t)|^2+\frac{M_\delta}{c}\right)\\
&\qquad + e^{-\lambda\tau}\left(2\int_{t_0-t}^{\tau} e^{\lambda s}\beta(s)ds+\varepsilon^{-1}\sum_{i=1}^m L_i^2\int_{t_0-t}^{\tau} e^{\lambda s}\sup_{\theta\in[-r,0]}|x(s+\theta)|^2 ds\right).
\end{aligned}
$$

Now, using again Lemma 3.1, we deduce

$$
\begin{aligned}
|x(\tau)|^2 &\le \frac{1}{1-c}|x(\tau)-g(x(\tau-\sigma))|^2+\frac{1}{c}|g(x(\tau-\sigma))|^2\\
&\le \frac{1}{1-c}\left(e^{\lambda(t_0-t)}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)+2\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{-\lambda\tau}\\
&\quad + ((1-c)\varepsilon)^{-1}\sum_{i=1}^m L_i^2\int_{t_0-t}^{\tau} e^{-\lambda(\tau-s)}\sup_{\theta\in[-r,0]}|x(s+\theta)|^2 ds\\
&\quad + c|x(\tau)|^2+\frac{M_\delta}{c}. \qquad (11)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
|x(\tau)|^2 &\le \frac{1}{(1-c)^2}|x(\tau)-g(x(\tau-\sigma))|^2+\frac{1}{c(1-c)}|g(x(\tau-\sigma))|^2\\
&\le \frac{1}{(1-c)^2}\left(e^{\lambda(t_0-t)}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)+2\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{-\lambda\tau}\\
&\quad + ((1-c)^2\varepsilon)^{-1}\sum_{i=1}^m L_i^2\int_{t_0-t}^{\tau} e^{-\lambda(\tau-s)}\sup_{\theta\in[-r,0]}|x(s+\theta)|^2 ds\\
&\quad + \frac{M_\delta}{c(1-c)}\\
&\le \frac{1}{(1-c)^2}\left(e^{\lambda(t_0-t)}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)+2\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{-\lambda\tau}\\
&\quad + ((1-c)^2\varepsilon)^{-1}\sum_{i=1}^m L_i^2\int_{t_0-t}^{\tau} e^{-\lambda(\tau-s)}\sup_{\theta\in[-r,0]}|x(s+\theta)|^2 ds\\
&\quad + \frac{M_\delta e^{\lambda t_0}}{c(1-c)}e^{-\lambda\tau}. \qquad (12)
\end{aligned}
$$

Let

$$
\kappa=\frac{e^{\lambda(t_0-t)}}{(1-c)^2}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)+\frac{2}{(1-c)^2}\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds+\frac{M_\delta e^{\lambda t_0}}{c(1-c)},
$$

$$
\kappa'=\frac{\varepsilon^{-1}}{(1-c)^2}\sum_{i=1}^m L_i^2
$$

and $\gamma=\lambda$. Then, taking into account that we have chosen $\varepsilon=\dfrac{2\sum_{i=1}^m L_i^2}{\lambda(1-c)^2}$, it follows that

$$
\frac{\kappa'}{\gamma}=\frac{\varepsilon^{-1}(1-c)^{-2}\sum_{i=1}^m L_i^2}{\lambda}=\frac{1}{2}<1.
$$

Thus, Lemma 3.3 implies that

$$\begin{aligned}|x(\tau)|^2 &\le \kappa e^{-\mu\tau}\\ &= \left(\frac{e^{\lambda(t_0-t)}}{(1-c)^2}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)+\frac{2}{(1-c)^2}\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\right.\\ &\quad \left.+\frac{M_\delta e^{\lambda t_0}}{c(1-c)}\right)e^{-\mu\tau}, \qquad (13)\end{aligned}$$

where μ is the solution of the next equation which belongs to the interval $(0,\lambda)$:

$$\frac{\kappa'}{\lambda-\mu}e^{\mu r}=1.$$

Thus we obtain that

$$\begin{aligned}&\sup_{\theta\in[-r,0]}|x(t_0+\theta)|^2\\ &\le \frac{e^{\mu r}}{(1-c)^2}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)e^{(\lambda-\mu)t_0}e^{-\lambda t}+\frac{2e^{\mu r}}{(1-c)^2}\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\\ &\quad+\frac{M_\delta e^{\lambda t_0}}{c(1-c)}e^{\mu r}\\ &\le \left(1+\frac{M_\delta e^{\lambda t_0}}{c(1-c)}+\frac{2}{(1-c)^2}\int_{-\infty}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{\mu r}\end{aligned}$$

provided

$$t\ge T_D=\lambda^{-1}\log\left(\frac{1}{(1-c)^2}\left(\frac{d^2}{1-c}+d^2c+\frac{M_\delta}{c}\right)e^{(\lambda-\mu)t_0}\right).$$

Consequently, the family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ in $C([-r,0];\mathbb{R}^n)$ given by $B(t):=B(0;\varrho(t))$, for all $t\in\mathbb{R}$, where $B(0;\varrho(t))$ denotes the ball centred at zero with radius $\varrho(t)=\left(1+\frac{M_\delta e^{\lambda t}}{c(1-c)}+\frac{2}{(1-c)^2}\int_{-\infty}^{t} e^{\lambda s}\beta(s)ds\right)e^{\mu r}$, is absorbing. Taking into account again that the associated process maps bounded sets into bounded sets of $C([-r,0];\mathbb{R}^n)$, the existence of the pullback attractor is ensured again by Theorem 2.4. □

4. **Conclusion.** In this paper we presented two novel results on the asymptotic behaviour of solutions to non-linear functional differential equations. Namely, we presented two results on the existence of pullback attractor for neutral functional differential equations with multiple delays.

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