A comparison between two theories for multi-valued semiflows and their asymptotic behaviour

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Abstract. This paper presents a comparison between two abstract frameworks in which one can treat multi-valued semiflows and their asymptotic behaviour. We compare the theory developed by Ball [5] to treat equations whose solutions may not be unique, and that due to Melnik & Valero [25] tailored more for differential inclusions. Although they deal with different problems, the main ideas seem quite similar. We study their relationship in detail and point out some essential technical problems in trying to apply Ball's theory to differential inclusions.

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1. Introduction

The concept of the attractor has proved an extremely useful tool for studying the asymptotic behaviour of solutions of a wide variety of dynamical systems: deterministic systems in both the autonomous (e.g. Babin & Vishik [4], Hale [13], Ladyzhenskaya [22], Temam [31]) and non-autonomous cases (e.g. Chepyzhov & Vishik [9], Kloeden & Schmalfuss [20], Kloeden & Stonier [21]), and the stochastic flows generated by stochastic differential equations (e.g. Arnold [1], Crauel & Flandoli [11], Crauel et al. [10]). An important step in all these theories is the development of a general abstract framework in which to express the underlying dynamics of the problem (semiflows $S(t)$, processes $S(t,s)$, and cocycles $\varphi(t,\omega)$, respectively).

When one is interested in studying the asymptotic behaviour of solutions to multi-valued problems such as those arising in control theory and viability theory, or from equations without uniqueness or differential inclusions, we can still expect the attractor to be useful. The development of a theory of "multi-valued semiflows" is a necessary first step in the study of attractors for such problems.

In the 1960s and 1970s a number of papers appeared that treated this problem for differential inclusions on locally compact spaces (Bridgland [7], Bushaw [8], Kloeden [18], Roxin [27] & [26], Sell [28], Szegö & Treccani [30]; see Kloeden [19] for a review), while more recently multi-valued semiflows on general Banach spaces have been considered by various authors, often in the context of partial differential equations or inclusions (Ball [5], Barbu [6], Elmounir & Simondon [12], Kapustyan [17], Kapustian & Valero [16], Melnik & Valero [25]; for the use of multi-valued systems in numerical analysis see Lamba [23] and Lamba & Stuart [24]).

We will be mainly concerned with the recent works of Melnik & Valero [25] and Ball [5] as canonical examples of such theories that also discuss the way one might define an 'attractor' for such systems. We take the approach of Kloeden [18] as our prime 'historical' example.

In order to consider the attractors of partial differential inclusions, in [25] Melnik & Valero define a *multi-valued semiflow* as a multi-valued mapping $G_{MV} : \mathbb{R}^+ \times X \to 2^X$, where X is the phase space.

Differential equations without uniqueness are the main topic of [5], which presents a theory more in line with previous treatments. Here Ball defines the concept of a *generalized semiflow* G_B on the phase space X, essentially consisting of all possible solutions of the equation. Although he mainly works with this collection G_B (to some extent an advantage over the approach of [25] since the idea of an individual solution is a building block of the definition) he also considers the multi-valued map $T(t)u_0$ formed from the set of points reached in time t by elements of G_B (solutions) which began (at time zero) at u_0. The map $T(t)u_0$ has very similar properties to the multi-valued semiflow $G_{MV}(t, u_0)$ defined by Melnik & Valero.

Although Ball's paper deals primarily with equations without uniqueness, while that of Melnik & Valero concentrates on differential inclusions, in fact they deal with very similar problems in which the dynamics is governed by a collection of possible solutions through each initial condition.

We will see below in Proposition 2 that a generalized semiflow can be seen as a particular case of a multi-valued semiflow. A natural question is whether, given a multi-valued map $T : [0, +\infty) \times X \to P(X)$ (as in Melnik & Valero [25]), we can define a generalized semiflow G_B consisting of all "solutions" (with some appropriate properties). More generally speaking it is natural to ask whether the two theories are in fact distinct, and if so in what way.

The content of the paper is as follows. In Section 2 we recall Ball's definition of a generalized semiflow, and prove several properties of the multi-valued map $T(t)$ which arises naturally from his definition.

We compare his definition with the axiomatic approach adopted in the 1960s and 1970s. Then we give Melnik & Valero's definition of a multi-valued semiflow, and discuss the conditions required in order to construct a generalized semiflow from such a multi-valued semiflow. In Section 3 we give some canonical examples (an ODE and PDE without uniqueness, and an ordinary and a partial differential inclusion) and discuss when these give rise to generalized semiflows. We end by discussing how the existence of global attractors can be proved within the two theories, which was the starting point for our interest in the problem.

2. "Generalized" vs "multi-valued" semiflows

Let (X, ρ) be a complete metric space, and denote by 2^X, $P(X)$, $B(X)$, $C(X)$, $\mathrm{K}(X)$, and $C_v(X)$ the collections of all, nonempty, nonempty bounded, nonempty closed, nonempty compact, and nonempty closed convex subsets of X respectively. To measure the distance between sets we will use the Hausdorff metric d_H, defined as

$$d_H(B, C) = \max\{\mathrm{dist}(B, C), \mathrm{dist}(C, B)\} \tag{1}$$

where $\mathrm{dist}(B, C)$ is the Hausdorff semi-distance,

$$\mathrm{dist}(B, C) = \sup_{b\in B} \inf_{c\in C} \rho(b, c).$$

2.1. Generalized semiflows: definition

We now give Ball's definition of a generalized semiflow, including in addition the possibility of discrete time generalized semiflows. Note that the definition says nothing *a priori* about the continuity of the solutions $\varphi \in G_B$ in the case $\Gamma = \mathbb{R}$. However, we will restrict ourselves later to two classes of continuous solutions. In finite-dimensional problems (or more generally when X is locally compact) we will want to take φ continuous from $[0, \infty)$ into X, while in infinite-dimensional problems it is more useful to take φ continuous from $(0, \infty)$ into X (for more details see Section 2.5 below).

DEFINITION 1. *Let* Γ *be* $\mathbb{R}$ *or* $\mathbb{Z}$. *A* generalized semiflow G_B *on* X *is a family of maps* $\varphi : \Gamma_+ \to X$ *(called solutions) satisfying the following hypotheses:*

(H1) Existence: *for each* $z \in X$ *there is at least one* $\varphi \in G_B$ *with* $\varphi(0) = z$.

(H2) Translates of solutions are solutions: *if* $\varphi \in G_B$ *and* $\tau \in \Gamma_+$ *then* $\varphi^\tau \in G_B$ *where* $\varphi^\tau(t) := \varphi(t+\tau)$ *for all* $t \in \Gamma_+$.

(H3) Concatenation: *if* $\varphi, \psi \in G_B$ *and* $\psi(0) = \varphi(t)$ *for some* $t \in \Gamma_+$ *then* $\theta \in G_B$, *where for each* $\tau \in \Gamma_+$ *we define*

$$\theta(\tau) := \begin{cases} \varphi(\tau) & \text{for} \quad 0 \le \tau \le t, \\ \psi(\tau - t) & \text{for} \quad t < \tau. \end{cases}$$

(H4) Upper semicontinuity with respect to initial data: *if* $\{\varphi_n\} \subset G_B$ *with* $\varphi_n(0) \to z$, *then there exists a subsequence* $\{\varphi_\mu\}$ *of* $\{\varphi_n\}$ *and* $\varphi \in G_B$ *with* $\varphi(0) = z$ *such that* $\varphi_\mu(t) \to \varphi(t)$ *for each* $t \in \Gamma_+$.

If for each $z \in G_B$ *there is exactly one* $\varphi \in G_B$ *with* $\varphi(0) = z$, *then* G_B *is called a* semiflow.

This definition (including the initially perhaps unintuitive (H4)) arises naturally when one considers solutions of differential equations whose solutions are not unique (see also Sell [28]; it is easy to see (H1–4) when solutions are unique). Let us consider what is perhaps the simplest such problem (with $\Gamma = \mathbb{R}$):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \qquad y(0) = y_0, \tag{2}$$

where f is a bounded continuous function from $\mathbb{R}^n$ into $\mathbb{R}^n$. It is well known that there exists at least one solution of this problem for each initial condition, but there may exist more than one for a general continuous function f. However, the boundedness assumption does ensure that all solutions exist for all $t \in \mathbb{R}$.

Let us denote by $D(y_0)$ the set of all classical solutions of (2) restricted to $\mathbb{R}_+$:

$$D(y_0) = \{\varphi \in C^1([0,\infty);\mathbb{R}^n) \quad \text{such that} \quad \varphi \quad \text{satisfies (2)}\},$$

and set

$$G = \bigcup_{y_0 \in \mathbb{R}^n} D(y_0).$$

Since the equation is autonomous, it is not difficult to check that G forms a generalized semiflow: (H1), (H2) and (H3) are obvious, and (H4) is also relatively straightforward to check. Indeed, the set of all solutions is equicontinuous since for any $\varphi \in G$

$$|\varphi(t) - \varphi(s)| \le \|f\|_\infty |t - s|.$$

So, if we consider a fixed bounded interval $I = [0, T]$ and a sequence of solutions φ_n with $\varphi_n(0)$ convergent, we obtain

$$|\varphi_n(t)| \leq |\varphi_n(0)| + T\|f\|_\infty, \qquad \text{for all} \qquad t \in [0, T],$$

and, consequently, uniformly bounded (on $[0, T]$) and equicontinuous. Thus we can apply the Arzelà-Ascoli Theorem to extract a subsequence that converges to a continuous function φ which is itself a solution of (2).

There are results due to Barbashin that are closely related to (H4); we will comment on these below in Section 2.6.

2.2. Generalized semiflows: properties

Let G_B be a generalized semiflow and let $E \subset X$. Define for $t \in \Gamma_+$:

$$T(t)E = \{\varphi(t) : \varphi \in G_B \text{ with } \varphi(0) \in E\} \tag{3}$$

We now state and prove various properties of the map $T(t)$ (cf. comments at start of Section 3 in Ball [5]).

PROPOSITION 2. *The map $T(t) : 2^X \to 2^X$ satisfies the following properties:*

(a) $\{T(t)\}_{t \in \Gamma_+}$ is a semigroup on 2^X, i.e. $T(0) = \mathrm{Id}_{2^X}$ and $T(t+s) = T(t)T(s)$ for all $t, s \in \Gamma_+$,

(b) $T(t)$ is monotone with respect to the partial order of set inclusion, i.e., $E \subset F$ implies $T(t)E \subset T(t)F$ for all $t \in \Gamma_+$,

(c) $T(t)x$ is compact for each $x \in X$, and

(d) if $\{K_n\}_{n \geq 1}$ is a sequence of compact subsets of X such that $\mathrm{dist}(K_n, K) \to 0$ as $n \to \infty$ then $\mathrm{dist}(T(t)K_n, T(t)K) \to 0$ for each $t \in \Gamma_+$.

Note that since $\mathrm{dist}(a, b) = \rho(a, b)$ for two points a and b, when G_B is a semiflow the result given in (d) reduces to

$$\rho(T(t)x_n, T(t)x) \to 0 \qquad \text{if} \qquad \rho(x_n, x) \to 0$$

as we would expect.

Proof. (a) The first part of this is trivial: $T(0)E = E$ for any $E \subset X$ because of (H1) and the definition of $T(t)E$. Now let us check that $T(t+s) = T(t)T(s)$. Fixing $E \subset X$, if $x \in T(t+s)E$ then there exist $x_0 \in E$ and a solution $\varphi \in G_B$ with $\varphi(0) = x_0$ and $\varphi(t+s) = x$. Using

(H2) we know that $\varphi^s \in G_B$, and since $\varphi^s(0) = \varphi(s) \in T(s)E$ and $\varphi^s(t) = x$, it follows that $T(t+s)E \subset T(t)T(s)E$. For the opposite inclusion we use (H3): if $x \in T(t)T(s)E$, then $x = \psi(t)$ with $\psi(0) \in T(s)E$ and this means that there exists another solution $\varphi \in G_B$ such that $\varphi(s) = \psi(0)$ with $\varphi(0) \in E$. If we define for each $\tau \in \Gamma_+$

$$\theta(\tau) = \begin{cases} \varphi(\tau) & \text{for} \quad 0 \leq \tau \leq s, \\ \psi(\tau - s) & \text{for} \quad s < \tau, \end{cases}$$

we have $\theta \in G_B$ with $\theta(0) \in E$ and $\theta(t+s) = \psi(t) = x$, and so $T(t)T(s)E \subset T(t+s)E$.

(b) It is clear that $T(t)$ is monotone by definition.

(c) If $y_n \in T(t)x$ then there exist solutions $\varphi_n \in G_B$ with $\varphi_n(0) = x$ and $\varphi_n(t) = y_n$. Since $\varphi_n(0) \to x$, by (H4) there is a subsequence $\varphi_{n'}$ and a $\varphi \in G_B$ such that (in particular) $\varphi_{n'}(t) \to \varphi(t)$, i.e. there exists a $\varphi \in G_B$ with $\varphi(0) = x$ and $\varphi(t) = y$. So $y_{n'} \to y \in T(t)x$. It follows that $T(t)x$ is compact.

(d) We consider a fixed value $t \in \Gamma_+$ and prove the result by contradiction. Assuming that $\mathrm{dist}(T(t)K_n, T(t)K) \not\to 0$, there exists an $\epsilon > 0$, a subsequence $\{K_{n'}\}$ and elements $a_{n'} \in T(t)K_{n'}$ such that

$$\mathrm{dist}(a_{n'}, T(t)K) > \epsilon \qquad \text{for all} \qquad n'. \tag{4}$$

But $a_{n'} = \varphi_{n'}(t)$ with $\varphi_{n'}(0) \in K_{n'}$, and so, since $\mathrm{dist}(K_n, K) \to 0$ with K compact, there exists a subsequence $\{\varphi_{n''}\}$ such that $\varphi_{n''}(0) \to z \in K$. It follows from (H4) that there exist a solution $\psi \in G_B$ and a subsequence $\{\varphi_{n'''}\}$ with $\varphi_{n'''}(t) \to \psi(t)$ and $\psi(0) = z \in K$. Thus $\psi(t) \in T(t)K$ which contradicts (4), proving the result. □

Property (d) is just the ε definition of upper semicontinuity, which is equivalent here to topological upper semicontinuity (u.s.c.) since $T(t)$ has compact values (see Aubin & Cellina [2] (pp. 41 & 45) for details).

2.3. Multi-valued semiflows

Melnik & Valero [25] define a multi-valued (semi)flow, or m-(semi)flow, as follows.

DEFINITION 3. *Let Γ be a nontrivial subgroup of $(\mathbb{R}, +)$. The set-valued map $G_{MV} : \Gamma \times X \to P(X)$ is said to be a multi-valued flow (or m-flow) if the following conditions are satisfied:*

(1) $G_{MV}(0, \cdot)$ is the identity map and

(2) defining for any subset B of X

$$G_{MV}(t, B) = \bigcup_{x \in B} G_{MV}(t, x)$$

we have $G_{MV}(t+s,x) \subset G_{MV}(t, G_{MV}(s,x))$ *for all* $t,s \in \Gamma$ *and for each* $x \in X$.

G_{MV} *is called an* m*-semiflow if we replace* Γ *by* $\Gamma_+ = \Gamma \cap \mathbb{R}_+$ *in the definition.*

A strict inclusion in an m-semiflow means (cf. Proposition 2) that it does not come from a set of solutions with the translation and concatenation properties. Since the translation of a solution is usually a solution, it is most likely that this strict inclusion will arise from a failure of the concatenation property, for example joining two C^1 functions in such a way that the resulting function is only C^0.

A more convincing (but non-autonomous) example arises in the 'general control systems' considered by Roxin [27]. Since one would expect the controls available to increase over time due to technological advances, we take $\{U_j\}_{j\geq 1}$ to be an increasing sequence of control sets, and denote the states attainable at time $n+1$ from a collection E of states at time n are

$$F_n(E) = \bigcup_{x\in E, u\in U_n} f(x;u).$$

If we define $G(n,m)E$ for $n \geq m$ by

$$G(n,m)E := \underbrace{(F_m \circ F_m \circ \ldots \circ F_m)}_{n-m\,\text{times}} E$$

then it is clear that in general

$$G(n+m,0)E \subset G(n+m,m)G(m,0)E,$$

since

$$U_0 \subset U_1 \Rightarrow (F_0 \circ F_0)(E) \subset F_1(F_0(E)).$$

Nevertheless, for all the applications considered in [25] the m-semiflow is constructed from selected solutions of various differential inclusions; since these satisfy both (H2) and (H3) equality holds in part (2) of Definition 3. However, the abstract definition of an m-semiflow contains no reference to solutions *per se*, so they have to be introduced as an (albeit natural) auxiliary concept. We reproduce here the definition of a *trajectory* from [25].

DEFINITION 4. *The map* $x(\cdot) : \Gamma_+ \to X$ *is said to be a* trajectory *of the* m*-semiflow* G_{MV} *corresponding to the initial condition* x_0 *if* $x(0) = x_0$ *and* $x(t+\tau) \in G_{MV}(t, x(\tau))$ *for every* $t, \tau \in \Gamma_+$.

For example, it is easy to check that (H2) implies that the solutions making up Ball's generalized semiflow are trajectories of the m-semiflow $T(t)$ defined in (3): $\varphi(t+\tau) = \varphi^\tau(t) \in T(t)\varphi(\tau)$.

If we require that $G_{MV}(t, B)$ in fact consists of a union of continuous trajectories of the m-semiflow G_{MV} then Melnik & Valero called G_{MV} a "time-continuous m-semiflow".

This notion of a trajectory, although used only by Melnik & Valero in their discussion of the connectedness of attractors, is extremely useful for us in comparing the two general frameworks and results from the previous literature. However, there are some distinctions between "trajectories" and "solutions", as we will now explain.

2.4. Solutions and trajectories.

As we previously mentioned, given a generalized semiflow G_B, it easily follows from Proposition 2 that we can construct an m-semiflow G_{MV} by setting

$$G_{MV}(t, x) = T(t)x \qquad t \in \mathbb{R}_+,\ x \in X. \tag{5}$$

Furthermore, we have a slightly stronger version of property *(2)* in the definition of an m-semiflow, since we automatically have the equality

$$G_{MV}(t+s, x) = G_{MV}(t, G_{MV}(s, x))$$

rather than an inclusion (what we will call a "strong m-semiflow"). We also know that $G_{MV}(t, x)$ is upper semicontinuous.

However, it is not immediately clear that this m-semiflow cannot have trajectories (in the sense of Definition 4) that are not solutions of the generalized semiflow G_B (cf. Szegö & Treccani [30, Obs. 5.2]).

We can rule out such spurious solutions using (H3) and (H4) when G_B consists of continuous functions (analogous result have been proved by Szegö & Treccani [30, Th. 5.1] and Barbashin, see [27]).

LEMMA 5. *Let G_B be a generalized semiflow consisting of functions that are continuous from J into X, where $J = (0, \infty)$ or $[0, \infty)$. Now let G_{MV} be the m-semiflow constructed from G_B by* (5)*. If $x(t)$ is a continuous trajectory (on J) of this m-semiflow, i.e.*

$$x(t+s) \in G_{MV}(t, x(s))$$

for all $t, s \in \mathbb{R}_+$ then $x \in G_B$.

Proof. Consider a sequence $\varphi_n \in G_B$ such that

$$\varphi_n(j2^{-n}) = x(j2^{-n}) \qquad \text{for all} \qquad j = 0, 1, 2, \ldots, n2^n.$$

The argument for the existence of such functions follows from that for φ_1: by the definition of $G_{MV}(1,\cdot)$ there exists a $\varphi \in G_B$ with $\varphi(0) = x(0)$ and $\varphi(1) = x(1)$. Similarly there exists a $\psi \in G_B$ with $\psi(0) = x(1)$ and $\psi(1) = x(2)$. So by concatenation there exists a $\varphi_1 \in G_B$ with $\varphi_1(0) = x(0)$, $\varphi_1(1) = x(1)$, $\varphi_1(2) = x(2)$. We can continue concatenating in this way to find a $\varphi_n \in G_B$ with $\varphi_n(j2^{-n}) = x(j2^{-n})$.

Now we consider the sequence $\{\varphi_n\} \subset G_B$. Since $\varphi_n(0) = x(0)$, by (H4) there is a $\varphi \in G_B$ and subsequence φ_μ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t > 0$. Since for any t of the form $t = j2^{-n}$ for some j and n the value of $\varphi_\mu(t)$ is always $x(t)$ for μ large enough, it follows from the continuity of φ and x that in fact $\varphi = x$. So $x \in G_B$. □

However, suppose rather that we construct an m-semiflow directly using a certain class of solutions of some model. In this case there is *a priori* no reason why there should not be limits of solutions of this m-semiflow that are not solutions themselves. This remark will be important in the applications of Section 3, where in each case we will have to check that the limit of solutions is still a solution: i.e. that convergence of a sequence of solutions φ_n to φ implies that φ is also a solution.

We would like to emphasise this point here, i.e. that in general one cannot expect that every set of solutions of a differential problem forms a generalized semiflow: we end this section with a simple example of an ordinary differential inclusion in which the set of all solutions does not satisfy (H4).

We take $X = [0, \infty)$, and define F by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \in [0, 1], \\ 1 & \text{if} \quad x > 1. \end{cases}$$

Consider the ordinary differential inclusion (in fact an ordinary differential equation)

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) \in F(x(t)), \tag{6}$$

and let G be the set of strong solutions of (12); writing $f_a(t) = a$ and $g_b(t) = b + t$ we have

$$G = \bigcup_{a \in [0,1]} f_a(\cdot) \cup \bigcup_{b \in (1,\infty)} g_b(\cdot).$$

It is obvious that G is an equicontinuous family and satisfies (H1), (H2) and (H3), but the more problematic (H4) does not hold: if we take a sequence $b_n \downarrow 1$ the solutions $g_{b_n}(t) = b_n + t$ converge to $g(t) = 1 + t$, but this is not a solution of (12). [This anomaly can be corrected by redefining $F(1)$ to be $\{0,1\}$, or the whole interval $[0,1]$, see Smirnov [29]. In this case the collection of all solutions, $G \cup \{g\}$, does give rise to a generalized semiflow.]

In the above example the initial m-semiflow did not have compact values, invalidating (H4) [recall Proposition 2]; this can happen for the solutions of a general differential inclusion $\dot{x} \in f(x)$ if f is not convex-valued: an example is given by Smirnov [29, p. 107].

2.5. Generalized semiflows from m-semiflows

Although it is easy to define an m-semiflow given a generalized semiflow, it is much harder (and not possible in general) to obtain a generalized semiflow $\mathcal{G}$ from a given m-semiflow G_{MV}. (Results of this section are related to those of Roxin [26], Bridgland [7] and Szegö & Treccani [30, Th. 5.1 & Obs. 5.2]; for similar ideas for differential equations without uniqueness see Sell [28].)

We start our discussion from an abstract point of view: if we are given an m-semiflow, what set of functions should we take? What properties should they satisfy? Looking at the definition of the set $T(t)E$ associated to a generalized semiflow G_B (it is the set of all points reached at time t by solutions which began in E), the first idea that comes to mind is to define

$$\mathcal{G} = \bigcup_{x \in X} \{\varphi : [0,\infty) \to X \mid \varphi(t) \in G_{MV}(t,x) \quad \text{for all} \quad t \in \mathbb{R}_+\}.$$

Since $G_{MV}(t,x)$ is non-empty, we can use the Axiom of Choice to find solutions which immediately satisfy (H1). However, in principle one is not able to ensure that these solutions should satisfy (H2). Indeed, suppose that $\varphi(\tau) \in G_{MV}(\tau,x)\ \forall \tau \in \Gamma_+$: then $\varphi(t+s_i) \in G_{MV}(t+s_i,x)$ with $i = 1,2$. But $\varphi^t(s_i) \in G_{MV}(s_i, G_{MV}(t,x))$ implies that there exist elements $x_i \in G_{MV}(t,x)$ with $\varphi^t(s_i) \in G_{MV}(s_i,x_i)$ and the problem is that x_1 does not have to be the same as x_2.

Although this problem can be overcome by considering the union over all subsets rather than all points in the above, i.e.

$$\mathcal{G} = \bigcup_{E \in P(X)} \{\varphi : [0,\infty) \to X \mid \varphi(t) \in G_{MV}(t,E) \quad \text{for all} \quad t \in \mathbb{R}_+\}$$

[for the given $\varphi(t) \in G_{MV}(t,E)$ it follows immediately that $\varphi^t(s_i) \subset G_{MV}(s_i, G_{MV}(t,E))$], now $\mathcal{G}$ cannot be expected to satisfy (H4) since this definition is far too free. In fact, (H3) is false too. Taking $E = X$, it is clear that $G_{MV}(t,X) \subseteq G_{MV}(s,X)$ if $t > s$ because

$$G_{MV}(s + (t-s), X) \subset G_{MV}(s, G_{MV}(t-s,X)).$$

Thus in general, with an inclusion (remember Proposition 2) we cannot find every element of $G_{MV}(\tau - t, X)$ in $G_{MV}(\tau, X)$ and therefore the concatenated function θ defined in (H3) does not belong to $\mathcal{G}$.

In order to proceed further, from the above discussions and at light of [5, Ths. 2.2, 2.3] we have to impose some assumptions on the set of trajectories (in the sense of Definition 4) of the m-semiflow G_{MV}.

(A1) For each $x_0 \in X$, there exists at least one trajectory that is continuous from $[0, \infty) \to X$.

(A2) For any sequence of continuous trajectories φ_n with $\varphi_n(0) \to z$ the sequence $\{\varphi_n\}$ is equicontinuous from I into X for any compact subinterval I of $[0, +\infty)$.

We note here the following useful result that allows us to use (A2) in finite-dimensional spaces in order to apply the Arzelà-Ascoli Theorem:

LEMMA 6. *Suppose that $F(t, \cdot) : X \to B(X)$ is an upper semi-continuous map for each $t \in \mathbb{R}_+$, and let $\{\varphi_n\}_{n \geq 0}$ be a sequence of trajectories with $\varphi_n(0) \to z$. Then φ_n is uniformly bounded on each bounded subinterval I of $\mathbb{R}_+$.*

Proof. Suppose not: then for each constant $M \geq 0$ there exist $t_M \in I$ and a trajectory φ_{μ_M} with $|\varphi_{\mu_M}(t_M)| > M$. So there exists a $t_* \in \bar{I}$ with $|\varphi_{\mu_M}(t_*)|$ increasing to ∞ when $M \to \infty$. On the other hand $F(t_*, z)$ is bounded, and as $F(t_*, \cdot)$ is u.s.c. the points $\varphi_{\mu_M}(t_*)$ must be in a neighbourhood of $F(t_*, z)$. But this is a bounded set, and we have obtained a contradiction. □

In infinite-dimensional spaces we will need to make slightly different assumptions: even in the single-valued case solutions need not be continuous on $[0, +\infty)$ if we are not in the appropriate space (e.g. solutions of the heat equation can significantly be more regular for $t > 0$ than the initial condition – see Ball [5, Ex. 2.2 & Th. 2.1] for further comments). By (A1$'$) and (A2$'$) we denote (A1) and (A2) with $[0, +\infty)$ replaced by $(0, +\infty)$, and with (A2$'$) including the additional assumption that for each fixed t the closure of

$$\bigcup_{n=1}^{\infty} \varphi_n(t)$$

is a compact subset of X.

We are now in a position to prove our main result. Note that we saw above that the $T(t)$ arising from a generalized semiflow must have compact values, so we impose this on G_{MV} from the beginning.

THEOREM 7. *Let $G_{MV} : \mathbb{R}_+ \times X \to K(X)$ be a strong m-semiflow such that $G_{MV}(t,\cdot)$ is upper semicontinuous and has compact values. If X is locally compact (resp. not locally compact) and (A1–2) (resp. (A1′–2′)) hold, then the collection of all trajectories of G_{MV} that are continuous from $J = [0,\infty)$ (resp. $(0,\infty)$) into X forms a generalized semiflow on X.*

Note that under the conditions of the theorem we obtain from G_{MV} a generalized semiflow $\mathcal{G}$ which generates as in (3) a map $T_{\mathcal{G}}(t)$ satisfying

$$T_{\mathcal{G}}(t)E \subseteq G_{MV}(t,E). \tag{7}$$

However, as discussed in Section 2.4, it may be the case that the construction of the theorem gives rise to elements of $\mathcal{G}$ that are not in fact "solutions" of the original problem.

Proof. We deal with properties (H1–4) in turn.

(H1) This follows immediately from (A1).

(H2) Let $x(t)$ be a trajectory, i.e. $x(t+\tau) \in G_{MV}(t, x(\tau))$ for all $t, \tau \in \mathbb{R}_+$; then, given $s \in \mathbb{R}_+$, x^s is a trajectory since

$$x^s(t+\tau) = x(t+\tau+s) \in G_{MV}(t, x(\tau+s)) = G_{MV}(t, x^s(\tau)).$$

(H3) Given two trajectories φ, ψ with $\varphi(t) = \psi(0)$ for some $t > 0$, we want to show that

$$\theta(\tau) = \begin{cases} \varphi(\tau) & \text{for } 0 \le \tau \le t \\ \psi(\tau - t) & \text{for } t < \tau \end{cases}$$

is a trajectory, i.e. that

$$\theta(t_1 + t_2) \in G_{MV}(t_1, \theta(t_2)) \qquad \text{for all } t_1,\ t_2 \in \mathbb{R}_+. \tag{8}$$

If $t_1 + t_2 \le t$ then θ coincides with φ and if $t_2 \ge t$ then θ coincides with ψ: in both cases (8) is trivial because φ and ψ are trajectories. Therefore, let us suppose that $t_1 + t_2 > t$ and $\bar{t} = t - t_2 > 0$, then

$$\begin{aligned} \theta(t_1+t_2) &= \psi(t_1+t_2-t) \\ &\in G_{MV}(t_1+t_2-t, \psi(0)) \\ &= G_{MV}(t_1+t_2-t, \varphi(t)) \\ &\subset G_{MV}(t_1+t_2-t, G_{MV}(\bar{t}, \varphi(t-\bar{t}))) \\ &= G_{MV}(t_1+t_2-t+\bar{t}, \varphi(t-\bar{t})). \end{aligned}$$

If the trajectories φ and ψ are continuous then clearly θ is.

(H4) Finally, we prove condition (H4). Suppose that there exists a sequence of trajectories $\{\varphi_n\}$ with $\varphi_n(0) \to z \in X$ as $n \to \infty$. Then, using (A2) we can apply the Arzelà-Ascoli Theorem to extract a subsequence $\varphi_{1,n}$ that converges uniformly on $[0,1]$ (in the finite case) or $[1,1]$ (in the infinite case) to a continuous function $\varphi(t)$. Another application of the Arzelà-Ascoli Theorem provides a further subsequence $\varphi_{2,n}$ which converges uniformly to φ on $[0,2]$ (or $[1/2,2]$). We continue in this way to find nested subsequences $\varphi_{j,n}$ converging to φ uniformly on $[0,n]$ (or $[1/n,n]$) and additionally defined as z in time zero. Finally $\varphi_\mu = \varphi_{\mu,\mu}$ converges uniformly to φ on any compact subinterval of J.

Now, we check that φ is a trajectory: i.e. that

$$\varphi(t+s) \in G_{MV}(t, \varphi(s)) \quad \text{for all} \quad t, s \in \mathbb{R}_+. \tag{9}$$

Indeed, we have that $\varphi_\mu(s) \to \varphi(s)$ and $\varphi_\mu(t+s) \to \varphi(t+s)$ and that $\varphi_\mu(t+s) \in G_{MV}(t, \varphi_\mu(s))$ because φ_μ are trajectories. Since $G_{MV}(t,\cdot)$ is u.s.c., given any neighbourhood N of $G_{MV}(t,\varphi(s))$ there exists $\mu_N \geq 1$ such that for all $\mu \geq \mu_N$, $G_{MV}(t, \varphi_\mu(s)) \subset N$. On the other hand $G_{MV}(t, \varphi(s))$ is closed and we can pass to the limit to obtain (9).

□

Without the equicontinuity property from condition (A2) or (A2′) in the above theorem we cannot obtain a subsequence satisfying (H4). However, we have already seen (in the discussion following the definition of a generalized semiflow) that this can arise naturally in applications, and we give two examples in Section 3 showing that this can also arise in the solutions of differential inclusions. It is possible to do something without (A2) in the case of discrete time, and this is discussed below.

One might hope to circumvent (A2) since property (H4) does not require uniform convergence of φ_μ to φ but only pointwise convergence[1]. However, the following result from Ball [5] shows that the structure of solutions (elements of G_B) means that pointwise convergence implies uniform convergence:

THEOREM 8. *(cf. [5]) Suppose that G_B is a generalized semiflow with each $\varphi \in G_B$ continuous from $(0,\infty)$ into X. Suppose that $\{\varphi_n\} \subset$*

[1] It seems an interesting problem to characterize those sequences of continuous functions f_n (say from $[0,1] \to \mathbb{R}$) that have a subsequence that converges only *pointwise* to another continuous function f: we have not been able to find any result along these lines in the literature.

G_B and $\varphi \in G_B$ with $\varphi_n(t) \to \varphi(t)$ for each $t > 0$. Then $\varphi_n \to \varphi$ uniformly on compact intervals of $(0, \infty)$.

Since the Arzelà-Ascoli Theorem is a characterization of uniformly convergent sequences of continuous functions, (A2) is thus necessary. In the finite-dimensional case the following result from Ball is stronger:

THEOREM 9. *(cf. [5]) Suppose that X is locally compact and that G_B is a generalized semiflow with each $\varphi \in G_B$ continuous from* $[0, \infty)$ *into X. Suppose that $\{\varphi_n\} \subset G_B$ and $\varphi \in G_B$ with $\varphi_n(t) \to \varphi(t)$ for each $t > 0$. Then in fact $\varphi_n \to \varphi$ uniformly for every compact subinterval of* $[0, \infty)$.

Taking into account the above comments and the facts that (H4) implies upper semicontinuity (this is easy to check by contradiction, as we saw in Proposition 2) and that $G(t, \cdot)$ has compact values, when dealing with continuous trajectories the assumptions of Theorem 7 are the minimal (optimal) set that will guarantee the construction of the desired generalized semiflow.

We can remove condition (A2) in Theorem 7 if we consider the case of discrete time.

PROPOSITION 10. *Let $G_{MV} : \mathbb{Z}_+ \times X \to \mathrm{K}(X)$ be a strong m-semiflow such that $G_{MV}(t, \cdot)$ is upper semicontinuous and compact-valued. Under assumption (A1) the collection $\mathcal{G}$ of all trajectories of G_{MV} (in the sense of Definition 4) forms a generalized semiflow on X.*

Proof. Properties (H1-3) follow as before. It only remains to check (H4). By assumption $\varphi_j(0)$ converges to z. Given an $\epsilon > 0$ the upper semicontinuity of $G_{MV}(1, \cdot)$ ensures that $G_{MV}(1, \varphi_j(0))$ is contained in the ϵ-neighbourhood of the compact set $G_{MV}(1, z)$ for all $j \geq j(\epsilon, 1)$. It follows that there exists a subsequence $\varphi_{j_n^1}$ such that $\varphi_{j_n^1}(0)$ and $\varphi_{j_n^1}(1)$ both converge (as $n \to \infty$).

Now we apply the same argument inductively to $G_{MV}(k+1, \varphi_{j_n^k}(0))$ to obtain a subsequence $\varphi_{j_n^{k+1}}$ of $\varphi_{j_n^k}$ such that $\varphi_{j_n^k}(t)$ converges for all $t = 0, \ldots, k$. Setting $\varphi_\mu = \varphi_{j_\mu^\mu}$ we obtain a subsequence such that $\varphi_\mu(t)$ converges for all $t \in \mathbb{Z}_+$. That the resulting limit is still a member of $\mathcal{G}$ follows as before. □

Of course, in this case clearly $T_{\mathcal{G}}(n)E = G_{MV}(n, E)$.

(We could try something similar in the case $\Gamma = \mathbb{R}_+$, but without (A2) the best that we could hope for would be to obtain a subsequence that converged for a countable subset of $\mathbb{R}_+$, e.g. $\mathbb{Q}_+$.)

2.6. Relation with 'general control systems'

We have already mentioned briefly the 'general control systems' developed by Roxin [27, 26]. His axioms, themselves a weakening of the requirements of Barbashin for 'general dynamical systems' (see [27]), were in turn weakened by Kloeden [18] to those for a 'general semidynamical system'. Recasting Kloeden's definition into a language similar to that used here, and restricting to the autonomous case, a general semidynamical system on a complete locally compact space E is a map $\Phi(t)x$ defined for all $t \geq 0$ and all $x \in E$ such that

(K1) $\Phi(t)x$ is a non-empty closed subset of E for each $x \in E$ and for all $t \geq 0$;

(K2) $\Phi(0)x = x$ for all $x \in E$;

(K3) $\Phi(t+s)x = \Phi(t)\Phi(s)x$ (where $\Phi(t)X = \cup_{x\in X}\Phi(t)x$);

(K4) $\Phi(t)x$ is continuous in t with respect to the Hausdorff metric for each fixed $x \in X$,

$$\lim_{s\to t} d_H(\Phi(t)x, \Phi(s)x) \to 0 \quad \text{as} \quad s \to t;$$

(K5) $\Phi(t)x$ is upper semicontinuous in (x, t), that is

$$\lim_{s\to t,\ y\to x} \operatorname{dist}(\Phi(s)y, \Phi(t)x) = 0.$$

Note that the assumptions here are stronger than (H1–4) and stronger than the requirements for an m-semiflow (we note in particular that (H4) is a consequence of axioms (K1–5), as shown by Barbashin (see [27, Th. 6.2])). The strongest restriction for the application of this theory is the requirement that the phase space is locally compact; given this the extra strength of the assumptions in (K1–5) pose few additional restrictions. Indeed, it is relatively easy to show that when the phase space is locally compact a weaker assumption on the continuity of trajectories allows us to recover (K4) and (K5) from (H1–4).

More precisely, we have the following result.

PROPOSITION 11. *Let X be locally compact and suppose that G_B is a generalized semiflow whose elements are continuous functions from $[0,\infty)$ into X. Then (K4) and (K5) hold.*

(If X is not locally compact then (K4) can only be obtained for all $t \in (0, \infty)$: compare Theorem 8 with Theorem 9.)

Proof. The continuity in (K4) consists of two parts,

$$\mathrm{dist}(T(t)x, T(s)x) \to 0 \quad \text{as} \quad s \to t, \quad t \in [0, \infty), \tag{10}$$

and

$$\mathrm{dist}(T(s)x, T(t)x) \to 0 \quad \text{as} \quad s \to t, \quad t \in [0, \infty). \tag{11}$$

Both can be proved by contradiction.

If (10) does not hold then there exist a constant $\varepsilon > 0$ and a sequence $s_n \to t$ with

$$\mathrm{dist}(T(t)x, T(s_n)x) \geq \varepsilon. \tag{12}$$

Since $T(t)x$ is compact we can find a $z_n \in T(t)x$ such that

$$\mathrm{dist}(z_n, T(s_n)x) = \mathrm{dist}(T(t)x, T(s_n)x),$$

and w.l.o.g. we suppose that $z_n \to z \in T(t)x$. Then

$$\mathrm{dist}(z, T(s_n)x) \geq \varepsilon.$$

To obtain a contradiction we now find elements $y_n \in T(s_n)x$ with $y_n \to z$. Observe that since $z \in T(t)x$ there is a solution $\varphi \in G_B$ such that $z = \varphi(t)$. Since solutions are continuous functions of time, $y_n = \varphi(s_n) \in T(s_n)x$ and $y_n \to z$. So (10) holds.

If (11) does not hold then there exist an $\varepsilon > 0$ and a sequence $s_n \to t$ such that

$$\mathrm{dist}(T(s_n)x, T(t)x) \geq \varepsilon.$$

Since each $T(s_n)x$ is compact we can find $z_n \in T(s_n)x$ such that

$$\mathrm{dist}(z_n, T(t)x) \geq \varepsilon.$$

Then there are elements $\varphi_n \in G_B$ such that $\varphi_n(0) = x$ and $z_n = \varphi_n(s_n)$. W.l.o.g. we can assume using (H4) that there is some $\varphi \in G_B$ with $\varphi(0) = x$ such that $\varphi_n(t) \to \varphi(t)$ for each $t \geq 0$.

If X is locally compact then Theorem 2.3 in [5] (Theorem 9 above) shows that this convergence is in fact uniform on compact subintervals of $[0, \infty)$. Thus for all $t \in [0, \infty)$ we can deduce that

$$z_n - \varphi(t) = [\varphi_n(s_n) - \varphi(s_n)] + [\varphi(s_n) - \varphi(t)] \to 0,$$

and so $z_n \to \varphi(t) \in T(t)x$ and we obtain (11).

The upper semicontinuity follows similarly via a contradiction argument. □

Since (10) and (11) combine to show that

$$d_H(T(t)x, T(s)x) \to 0 \quad \text{as} \quad s \to t,$$

in locally compact spaces the continuity of solutions on $[0, \infty)$ plus (H1–4) imply (K4) and (K5). Conversely Barbashin showed (see [27]) that under axioms (K1–5) all trajectories are continuous and forms a compact set provided initial time convergence holds (cf. our Lemma 5), so one can switch between Ball's theory and that of Kloeden in a consistent way. However, in infinite-dimensional Banach spaces the weaker assumptions of the newer versions of the theory are necessary.

We saw in Section 2.5 that a result like Barbashin's (a generalized semiflow from an m-flow) does not hold in general, assuming only the properties that Melnik & Valero's require for the 'attainability sets' $G_{MV}(t, x)$. In the next section we see what can be done with the set of solutions of particular example problems.

3. Applications

In this section we study some examples. In particular we want to check whether or not we can construct generalized semiflows that consist entirely of solutions rather than form the trajectories of the attainability map. In the light of Lemma 5, it is enough to check that the solutions are continuous and form a generalized semiflow by themselves. In this way we avoid the existence of spurious 'solutions' in the generalized semiflow.

We will see that this problem is significantly more involved for partial differential inclusions that for equations without uniqueness, since for inclusions each solution is associated with a different 'right-hand side' and selection theorems are necessary.

3.1. An ODE without uniqueness

We have already considered in Section 2.1 the simple example

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(0) = x_0,$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ a bounded continuous function, and shown that it gives rise to a generalized semiflow. Of course, we could also consider the equation from the point of view of m-semiflows, defining $G(t, x)$ precisely as in (5). In this case it is interesting to note that Kneser's Theorem on structure of $G(t, x)$ (see Theorem 4.1 in Hartman [14] for example) guarantees that it is closed. Since it is bounded it must also be compact (of course, this also follows from part (c) of Proposition 2).

3.2. A PDE without uniqueness

The main example that appears in Ball's paper is the 3D Navier-Stokes equations:

$$\begin{aligned} u_t + (u \cdot \nabla) u = \nu \Delta u - \nabla p + f, \\ \operatorname{div} u = 0, \end{aligned} \tag{13}$$

with boundary condition

$$u|_{\partial\Omega} = 0.$$

Consider the following spaces:

$$\begin{aligned} \mathcal{V} &= \{u \in C_0^\infty(\Omega)^3;\ \operatorname{div} u = 0\}, \\ H &= \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3, \\ V &= \{u \in H_0^1(\Omega)^3;\ \operatorname{div} u = 0\}. \end{aligned}$$

Let us denote by H_w the space H endowed with its weak topology. It is well known that given $u_0 \in H$, there exists at least one weak solution u to the problem (13) such that

$$u \in C([0,T]; H_w) \cap L^2(0,T;V), \quad \frac{\mathrm{d}u}{\mathrm{d}t} \in L^1(0,T;V') \quad \forall T > 0.$$

However it is not known whether this solution is unique, so the generalized semiflow framework is brought in to play. The collection G_{NS} of all such weak solutions clearly satisfies (H1) and (H3). What Ball shows in this paper (see Proposition 7.4 in [5]) is that G_{NS} is a generalized semiflow if and only if each weak solution is a continuous function from $(0,\infty)$ into H: this is currently an unproved hypothesis. Additional conditions needed to construct the global attractor are shown to be consequences of this same assumption (see Theorem 19 in Section 4 of this paper for details).

3.3. Two examples involving differential inclusions

3.3.1. *An ordinary differential inclusion*

As an illustrative example we recall briefly the case of a simple ordinary differential inclusion (see e.g. Roxin [26] or Aubin & Frankowska [3]).

Let us consider an $F : \mathbb{R}^n \to C_v(\mathbb{R}^n)$ such that $F(0)$ is bounded and F is globally Lipschitz,

$$\mathrm{d}_H(F(u), F(v)) \le L|u - v|.$$

Consequently, F has bounded convex values (this convexity is important to guarantee that the attainability map has closed values, cf. comments towards the end of section 2.4). We are going to show that it is possible to construct a generalized semiflow for the ordinary differential inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in F(u) \qquad u(0) = u_0. \tag{14}$$

Let us denote by G the set of strong or Carathéodory solutions to (14), that is all maps $u : [0, T] \to \mathbb{R}^n$ such that

(i) $u(0) = u_0$,

(ii) $u(\cdot)$ is continuous on $[0, T]$,

(iii) $u(\cdot)$ is absolutely continuous on any compact subinterval of $(0, T)$, and satisfies (14) a.e. on $(0, T)$.

In particular we require a measurable selection $h(x)$ with $h(x) \in F(x)$, so that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = h(u(t)).$$

The existence of a continuous (and not merely measurable) selection is guaranteed by the Chebyshev Selection Theorem (see Aubin & Cellina [2], p. 74), and then standard methods can be used to obtain the existence of a solution defined locally in time.

If we fix an interval $[0, T]$ then we can show that each solution and its corresponding selection are bounded, thereby obtaining solutions that are global in time. (The proof is standard and will be omitted.)

PROPOSITION 12. *Under the above assumptions, if u is a strong solution to (14) and $h(u)$ the corresponding selection then the following bounds hold:*

$$|u(t)|^2 \le \mathrm{e}^{(2L+1)t}|u(0)|^2 + \frac{C}{2L+1}(\mathrm{e}^{(2L+1)t} - 1)$$

and

$$|h(u(t))| \le C + L\left(\mathrm{e}^{(2L+1)t}|u(0)|^2 + \frac{C}{2L+1}(\mathrm{e}^{(2L+1)t} - 1)\right)^{1/2},$$

where $C = \operatorname{diam} F(0)$.

Since (H1–3) are straightforward for this example, we concentrate on (H4). Suppose we have a sequence of strong solutions u_n with converging initial data. The bound on $|h(u(t))|$ from the above proposition implies that the set of strong solutions of (14) are equicontinuous, which along with the bound on $|u(t)|$ enables the use of the Arzelà-Ascoli

Theorem to find a convergent subsequence. But, this is not enough on its own, since we also need to check that the limit is still a solution of the problem. This follows from an application of a selection convergence theorem from Aubin & Cellina [2], p. 60: any convergent subsequence $u_{n'} \to u$ with selectors $f_{n'}(t) \in F(u_{n'}(t))$ has a subsequence $f_{n''} \rightharpoonup f$ with $f(t) \in F(u(t))$.

Of course, the same conclusion arises for the non-autonomous case if we include an artificial time $s = t$ as an additional direction in the phase space.

3.4. A partial differential inclusion

We now turn to the more involved case of partial differential inclusions, which we describe within an abstract framework.

Let H be a Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. We consider the following evolution inclusion problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} \in -Ay + F(y), \qquad t \in [0,T], \tag{15}$$

$$y(0) = y_0 \in H, \tag{16}$$

where $A : D(A) \subset H \to H$ is a linear (unbounded) operator with $\mathrm{Im}(\mathrm{Id} + A) = H$ and compact inverse such that

$$(-Ax, x) \leq 0 \qquad \text{for all} \qquad x \in D(A),$$

(so $\overline{D(A)} = H$) and with e^{-At} an analytic semigroup. (This is a particular case of the theory developed by Melnik & Valero [25] which also covers multi-valued subdifferential operators $A : D(A) \subset X \to 2^X$ with X a Banach space.) We assume further that F is convex-valued, $F : H \to C_v(H)$, $F(0)$ is bounded, and F satisfies the global Lipschitz condition

$$\mathrm{d}_H(F(u), F(v)) \leq C_1|u - v|$$

(this is a strong assumption; in particular it implies that F has bounded values).

As with ordinary differential inclusions, a strong solution $y(\cdot)$ of (15)-(16) is a continuous function on $[0,T]$ with $y(0) = y_0$, $y(\cdot)$ absolutely continuous on any compact subinterval of $(0,T)$, such that (15) holds a.e. on $(0,T)$. However, it is also useful to define a weaker notion of solution which we term (following the single-valued case) a mild solution.

DEFINITION 13. *The map $y : [0,T] \to H$ is called a mild solution of (15)-(16) if it is continuous, $y(0) = y_0$, and there exists a selection*

$f \in L^1([0,T];H)$ of F (that is $f(t) \in F(y(t))$ a.e. on $[0,T]$), such that y satisfies

$$y(t) = \mathrm{e}^{-At} y_0 + \int_0^t \mathrm{e}^{-A(t-s)} f(s)\,\mathrm{d}s \tag{17}$$

for a.e. $t \in [0,T]$ (i.e. $y(t)$ is a mild solution of $\mathrm{d}y/\mathrm{d}t = -Ay + f(t)$).

Before showing how we can define a generalized semiflow using this definition, we will discuss its relationship to that given in Melnik & Valero's paper. There they make use of the much more general theory of partial differential inclusions in which the linear operator A can also be multi-valued (full details of this can be found in Chapter III of Barbu [6]). In order to deal with the fact that A is multi-valued one needs to introduce a generalisation of the notion of a mild solution, which they term an "integral solution". In our setting, an integral solution of (17) is a continuous function $y : [0,T] \to H$ with $y(0) = y_0$ such that for all $u \in D(A)$

$$|y(t) - u|^2 \le |y(s) - u|^2 + 2\int_s^t (f(\tau) + Au, y(\tau) - u)\,\mathrm{d}\tau, \quad t \ge s. \tag{18}$$

An integral solution of (15)-(16) is a function $y(t)$ for which there exists a selection $f \in L^1([0,T];H)$ of F such that y is an integral solution of (17).

The two notions of solutions can be shown to be equivalent for our example: first, note that any strong solution is an integral solution (this is straightforward). Now we show that any integral solution is also a mild solution (the argument is due to Valero, personal communication): take a sequence of strong solutions $u_n = I(u_0^n)f_n$ with $f_n \to f$ and $u_0^n \to u_0$ in $L^1(0,T;H)$ and H respectively (the existence of such sequence is guaranteed by Corollary 2.2 in Barbu [6], Chapter III). Then $u_n \to u$ in $C([0,T];H)$. The contraction property of e^{-At} leads to

$$|\mathrm{e}^{-At}u_0^n - \mathrm{e}^{-At}u_0| \le |u_0^n - u_0| \to 0,$$
$$\int_0^t |\mathrm{e}^{-A(t-s)}f_n(s) - \mathrm{e}^{-A(t-s)}f(s)|\,\mathrm{d}s \le \int_0^t |f_n(s) - f(s)|\,\mathrm{d}s \to 0.$$

Now, since the strong solutions u_n are mild solutions, we can take limits in order to show that integral solutions of the problem (15) are mild solutions as well:

$$\begin{array}{ccccc} u_n(t) & = & \mathrm{e}^{-At}u_0^n + & \int_0^t \mathrm{e}^{-A(t-s)} f_n(s)\,\mathrm{d}s \\ \downarrow & & \downarrow & \downarrow \\ u(t) & = & \mathrm{e}^{-At}u_0 + & \int_0^t \mathrm{e}^{-A(t-s)} f(s)\,\mathrm{d}s. \end{array}$$

Since there is always a unique integral solution and a unique mild solution of (17) it follows that the definitions are equivalent.

Under our assumptions, and using Definition 13, there exists at least one mild solution of (15)-(16), cf. [25], and we take G to be the set of all such mild solutions. For this collection (A1) is easily seen to be true, and we now show that both parts of assumption (A2′) hold. Although we will not apply Theorem 7, rather checking (H4) directly, we will still need (A1) and (A2′) in order to do this.

We first check the equicontinuity property (recall that we only require equicontinuity on compact subintervals of $(0,\infty)$).

PROPOSITION 14. *For each $0<\theta<1$ all solutions of (15) satisfy*

$$|u(t)-u(s)| \le K_{[\epsilon,T]}(\theta,|u_0|)\,[|t-s|^\theta + |t-s|] \qquad \textit{for all} \qquad t,s\in[\epsilon,T].$$

To prove this proposition, we need the following lemma from Henry [15]:

LEMMA 15. *We have the following two estimates: there exists $\lambda>0$ such that for any $0\le\alpha<1$*

$$\|A^\alpha \mathrm{e}^{-At}\|_{\mathrm{op}} \le C_\alpha t^{-\alpha}\mathrm{e}^{-\lambda t} \tag{19}$$

and, for any $0<\alpha<1$

$$|(\mathrm{e}^{-At}-I)x| \le C'_\alpha t^\alpha |A^\alpha x|. \tag{20}$$

The first observation is that the strong condition on F gives a uniform bound on all solutions.

LEMMA 16. *If $u(t)$ is a solution of (15) and $f(t)\in F(u(t))$ then*

$$|f(t)| \le M(T,|u_0|) \qquad \textit{for all} \qquad t\in[0,T].$$

Proof. Note that the Lipschitz assumption on F implies that

$$|F(u)| \le C_1|u| + \mathrm{diam}[F(0)] \equiv C_0 + C_1|u|.$$

In particular, since every solution of (15) satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(t)$$

with $f(t)\in F(u(t))$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + |A^{1/2}u|^2 \le C_0|u| + C_1|u|^2$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 \le C_0^2 + (1+2C_1)|u|^2,$$

from which the result follows. □

We can now prove the equicontinuity result.

Proof. Essentially we follow the work in Henry [15]. We consider $u(t+h)-u(t)$ with $t \geq \epsilon$; using the integral expression for the solution, we have

$$\begin{aligned} u(t+h)-u(t) \ = \ & (\mathrm{e}^{-Ah}-I)\mathrm{e}^{-At}x_0 + \int_0^t (\mathrm{e}^{-Ah}-I)\mathrm{e}^{-A(t-s)}f(s)\,\mathrm{d}s \\ & + \int_t^{t+h} \mathrm{e}^{-A(t+h-s)}f(s)\,\mathrm{d}s, \end{aligned}$$

and so

$$\begin{aligned} & |u(t+h)-u(t)| \leq \\ & C'_\theta h^\theta |A^\theta \mathrm{e}^{-At}x_0| + \int_0^t C'_\theta h^\theta |A^\theta \mathrm{e}^{-A(t-s)}f(s)|\,\mathrm{d}s + \int_t^{t+h} |f(s)|\,\mathrm{d}s \\ & \leq C'_\theta h^\theta C_\theta \mathrm{e}^{-\lambda t} t^{-\theta}|x_0| + C'_\theta h^\theta C_\theta \int_0^t (t-s)^{-\theta}|f(s)|\,\mathrm{d}s + hM(T,|u_0|) \\ & \leq K(\theta,|x_0|,\epsilon,T)h^\theta + hM(T,|u_0|) \end{aligned}$$

which provides the equicontinuity property needed for (A2). □

As for the second part of (A2) (compactness) we show that the solutions are bounded in $D(A^\alpha)$, which is compactly embedded in H. The proof is almost straightforward, since

$$\begin{aligned} |A^\alpha u(t)| \ \leq \ & \|A^\alpha \mathrm{e}^{-At}\|_{\mathrm{op}}|u_0| + \int_0^t \|A^\alpha \mathrm{e}^{-A(t-s)}\|_{\mathrm{op}} M(|u_0|,s)\,\mathrm{d}s \\ \leq \ & C_\alpha t^{-\alpha}\mathrm{e}^{-\lambda t}|u_0| + C_\alpha M(|u_0|,t)\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}s \\ \leq \ & K(t,|u_0|). \end{aligned}$$

We now need to check that sequences of solutions have a subsequence that converges to a limit that is itself a solution. First use the Arzelà-Ascoli to find a (diagonal) subsequence such that u_n converges uniformly to u on any compact subinterval $(0,\infty)$ while (relabeling the same) $f_n \rightharpoonup f$ weakly in $L^1((0,T);H)$. Then u is a mild solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(t).$$

However, while (A2′) only requires convergence on compact subintervals of the open interval $(0,\infty)$ (which we have just shown), we require that $u(t)$ is a mild solution of (15), and by Definition 13 this requires in addition that $u(t)$ should be continuous on the interval $[0,\infty)$ and satisfies (17). But these properties follow since, passing to the limit in the expressions for u_n, $u(t)$ satisfies

$$u(t) = \mathrm{e}^{-At} z + \int_0^t \mathrm{e}^{-A(t-s)} f(s)\,\mathrm{d}s,$$

from whence $u(t) \to z$ as $t \to 0$ and the function is indeed continuous on $[0, \infty)$. It remains to check that $f(t) \in F(u(t))$, but this follows once more using the same selection convergence theorem from Aubin & Cellina [2], p. 60, that we used in the ordinary differential inclusion case.

Then u is a mild solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(t).$$

Therefore we have shown (H4), and the set of all mild solutions of (15) forms a generalized semiflow.

4. Relations between the two theories of attractors

Our interest in these two theories arose from their application to the investigation of the long-time behaviour of solutions. Indeed, both the papers in which the two abstract frameworks discussed here were developed generalise the notion of global attractor from single-valued dynamical systems to multi-valued evolutions (whether they come from systems without uniqueness or from differential inclusions). We now discuss the analogies and differences between the attractor results provided by both theories.

An initial remark is that both frameworks approach the theory of attractors in a similar way, Melnik & Valero giving all their definitions using the m-semiflow G_{MV}, and Ball using the equivalent $T(t)$ derived from the collection G_B of solutions (see (3)). In what follows we use $G(t, \cdot)$ for either $G_{MV}(t, \cdot)$ or $T(t)\cdot$ when no confusion can arise.

The important concepts are as follows:

DEFINITION 17.

(a) *It is said that A* attracts *B if* $\lim_{t\to\infty} \mathit{dist}(G(t, B), A) = 0$.

(b) *The semiflow G is called* eventually bounded *if for any bounded set B, there exists a sufficiently large constant $\tau = \tau(B)$ such that $\gamma_\tau^+(B)$ is bounded, where, as usual in dynamical systems, $\gamma_\tau^+(B)$ denotes the set of all points reached at any time greater than τ by solutions beginning in B: $\cup_{t\geq\tau} G(t, B)$.*

(c) *The* ω-limit set of M *is defined as the set of limits of all converging sequences* $\{\xi_n\}$ *where* $\xi_n \in G(t_n, M)$. *As in the single-valued case this is the same as the intersection of all* $\gamma_t^+(M)$ *with* $t \in \Gamma_+$.

(d) *The semiflow is called* point dissipative *if there is a bounded set* B_0 *such that all solutions are attracted by* B_0. *(The solutions will be "absorbed" by any neighbourhood of* B_0.*)*

(e) *The semiflow is* asymptotically upper semicompact *if for any bounded set* B *such that for some* $T(B) \in \Gamma_+$, $\gamma_{T(B)}^+(B) \in B(X)$, *any sequence* $\xi_n \in G(t_n, B)$ *with* $t_n \to \infty$ *is precompact in* X.

(f) *The semiflow is* asymptotically compact *if for any sequence of solutions* φ_n *with* $\{\varphi_n(0)\}$ *bounded, and any sequence* $t_n \to \infty$, *the set* $\{\varphi_n(t_n)\}$ *is precompact.*

Obviously asymptotically compact is equivalent to asymptotically upper semicompact plus eventually bounded.

The definition of "an attractor" in both papers is similar, and is essentially a compact, invariant set that attracts all bounded sets. However, in the light of certain applications (see Remark 4 in [25]) Melnik & Valero initially require the attractor only to be negatively semi-invariant, i.e. $\mathcal{A} \subset G(t, \mathcal{A})$ for all $t \in \Gamma_+$:

THEOREM 18. *(cf. [25], Theorem 3 and Remark 8) Let* G_{MV} *be a pointwise dissipative and asymptotically upper semicompact* m*-semiflow, and suppose that* $G_{MV}(t, \cdot) : X \to P(X)$ *has closed graph. If for any bounded* B *there exists a* $T(B)$ *such that* $\gamma_{T(B)}^+(B)$ *is bounded, then* G_{MV} *has a compact global attractor* $\mathcal{A}$ *which is minimal among all the closed sets attracting each* $B \in B(X)$.

We note here that the condition that G_{MV} has closed graph is automatically satisfied if $G_{MV}(t, \cdot) : X \to C(X)$ is upper semicontinuous for any $t \in \Gamma_+$ (see Aubin & Cellina [2], for example).

The negatively semi-invariant attractor of Theorem 18 becomes fully invariant whenever we have

$$G_{MV}(t_1 + t_2, x) = G_{MV}(t_1, G_{MV}(t_2, x)) \qquad \text{for all} \qquad x \in X$$

(what we called a strong m-semiflow).

The result from [5] is, of course, similar.

THEOREM 19. *(cf. [5], Theorem 3.3) A generalized semiflow* G_B *has a global attractor if and only if* G_B *is point dissipative and asymptotically compact. The global attractor* $\mathcal{A}$ *is unique and is given by*

$$\mathcal{A} = \bigcup_{B \in B(X)} \omega(B).$$

Furthermore, $\mathcal{A}$ is the maximal compact invariant subset of X.

Observe that although Theorem 18 seems to be stated under sufficient conditions (while those in Theorem 19 are also necessary), in fact it is easy to show that the three assumptions of eventual boundedness, pointwise dissipativity, and asymptotic upper-semicompactness are equivalent to the existence of a compact global attractor $\mathcal{A}$. We saw in Section 2 that when dealing with generalized semiflows of continuous solutions, (H2) and (H3) enable the construction of a strong m-semiflow, and it is clear that the two notions of attractor agree in this case.

The only apparently significant difference between the two results is that Melnik & Valero ask for the m-application $G_{MV}(t,\cdot)$ to have closed values and be upper semicontinuous or satisfy the closed graph condition. Both the requirement of closed values and of having a closed graph are consequences of the strong property (H4). Moreover, the m-semiflow arising from a generalized semiflow is ε upper semicontinuous and the m-application has compact values (Proposition 2), so the m-semiflow is upper semicontinuous.

Indeed, given a problem (P) whose solutions form a generalized semiflow, to check that

$$T(t)x = \{\varphi(t,x) \quad | \quad \varphi \text{ solution of } (P),\, \varphi(0) = x\}$$

is closed consider a sequence of points $x_n = \varphi_n(t)$ with $\varphi_n(0) = x$ converging to a point $\bar{x} \in X$. Trivially (H4) gives a solution φ such that $\varphi(0) = x$ and $\varphi_\mu(s) \to \varphi(s)$ for all $s \in \Gamma_+$ and so $\bar{x} \in G(t,x)$.

To check that $T(t)(\cdot)$ has closed graph, let $\{(x_n, y_n)\}$ be a sequence converging to (x,y) with $(x_n, y_n) \in \operatorname{Graph} T(t)(\cdot)$. Then there exists a sequence of solutions $\{\varphi_n\}$ with $\varphi_n(0) = x_n$ and $\varphi_n(t) = y_n$, but then by (H4) it is possible to extract a subsequence and a solution φ of (P) with $\varphi(0) = x$ and $\lim_{\mu\to\infty} \varphi_\mu(s) = \varphi(s)$ for all $s \in \Gamma_+$. So $\varphi(t) = y$ and $(x,y) \in \operatorname{Graph} T(t)(\cdot)$.

Conclusion

We have considered the relationship between Ball's generalized semiflows, Melnik & Valero's m-semiflows, and the more historical approach to general control systems developed in the 70s by Barbashin, Bushaw, Roxin and Kloeden among others. Although (under a continuity assumption on individual solutions) Ball's theory and that of Kloeden coincide when the phase space is locally compact, the generality of the more recent theories is necessary in order to treat the infinite-dimensional case.

Although the abstract theories of Ball and Melnik & Valero have many points of similarity, they have essential differences. The main obstacle to a smooth passage from one to the other is the upper semicontinuity property (H4), which might be related to the subtle distinction between 'trajectories' and 'solutions'. When (H4) can be shown to hold we believe that the extra structure available in Ball's formulation (which includes the notion of a solution into the definition) makes this more attractive, while Melnik & Valero's approach has the undoubted advantage of greater generality.

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