BASES FOR PROJECTIVE MODULES IN $A_n(k)$

JESÚS GAGO-VARGAS

Abstract. Let $A_n(k)$ be the Weyl algebra, with k a field of characteristic zero. It is known that every projective finitely generated left module is free or isomorphic to a left ideal. Let M be a left submodule of a free module. In this paper we give an algorithm to compute the projective dimension of M. If M is projective and $\mathrm{rank}(M) \geq 2$ we give a procedure to find a basis.

Introduction.

The study of finitely generated projective modules over a ring is an interesting topic. We know that over polynomial rings they are free, as it was shown by Quillen and Suslin. There are several algorithmic versions of this theorem [Logar et al.(1992), Laubenbacher et al.(1997), Gago(2002)] that compute a basis from a system of generators. All of these procedures use Gröbner bases in polynomial rings. It is natural to extend these results to the Weyl Algebra $A_n(k)$, with k a field with characteristic zero. It is known that if a left finitely generated $A_n(k)$-module is projective and has rank greater or equal 2 then is free [Stafford(1978)]. Our goal is to give an algorithm to find a basis of these modules.

Projective modules in $A_n(k)$ are stably free [Stafford(1977)], so the first step is to find an isomorphism $P \oplus A_n(k)^s \simeq A_n(k)^t$ for some s, t. We develop this procedure in Section 1, together with an algorithm to compute the projective dimension of a module, that is valid for a broad class of rings. We note by $\mathrm{pdim}(M)$ the projective dimension of a module M. We require the computation of Gröbner bases in the ring and that every module has a finite free resolution. If M is projective we find a matrix that defines an isomorphism $M \oplus R^s \simeq R^t$. The starting point is a left R-module M defined by a system of generators in some R^m.

In Section 2 we follow the proof of [Stafford(1978)] with algorithmic tools to find a basis of a projective module. We develop, for completeness, the reference to [Swan(1968)] used in [Stafford(1978), Thm. 3.6(a)], to clarify where these computations are needed . We follow describing the minor changes to [Hillebrand et al.(2002)] to obtain two special generators of a left ideal, according to [Stafford(1978), Theorem 3.1]. Finally, we give an example of this procedure to build a basis of a projective module in $A_2(\mathbb{Q})$.

For all the computations we need an effective field k in the sense of [Cohen(1999)] to apply the Gröbner bases algorithm in $A_n(k)$. We have used in the examples $k = \mathbb{Q}$.

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1. Computing projective dimension.

Let R be a ring where it is possible to compute a finite free resolution of a left module, and we can determine if a right submodule of R^k is equal to R^k. Such a ring may be $k[x_1, \dots, x_n]$, $A_n(k)$ or more general rings like PBW algebras [Bueso et al.(1998)]. We make use of a characterization given in [Logar et al.(1992)], based on a finite free resolution of a module. The existence of a finite free resolution for a projective module M is equivalent for M to be stably free [McConnell et al.(1987)]. With the algorithm described in this section we test wether M is projective, and if the answer is yes we compute an isomorphism $M \oplus R^s \simeq R^t$ for some s, t. The procedure is by induction on the length of the resolution. We identify the homomorphisms with their matrices to simplify the notation.
Suppose

$$0 \to F_1 \overset{\alpha_1}{\to} F_0 \overset{\alpha_0}{\to} M \to 0$$

is a free resolution of M, with $\operatorname{rank}(F_i) = r_i$. If M is a projective module, this sequence splits, so there exists $\beta_1 : F_0 \to F_1$ such that $\beta_1 \alpha_1 = I_{r_1}$. We can compute this matrix from the rows of the matrix α_1: if we consider them as vectors of F_1, the right R-module generated must be equal to F_1. We express each vector of the canonical basis of F_1 as a linear combination of the rows of α_1, and with these coefficients we construct the matrix β_1. So we can give the isomorphism $F_1 \oplus \ker(\beta_1) \simeq F_0 \simeq F_1 \oplus M$ and a basis of $F_1 \oplus \ker(\beta_1)$.
Let

$$\mathcal{F} : 0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} F_{t-2} \xrightarrow{\alpha_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \dots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

be a finite free resolution of M with $\operatorname{rank}(F_i) = r_i$ and $t \geq 2$ (we take α_{-1} the null homomorphism). Again, if M is a projective module, then the short exact sequence

$$0 \to \ker(\alpha_0) \to F_0 \to M \to 0$$

splits, so $\ker(\alpha_0) = \operatorname{im}(\alpha_1)$ is projective. By induction, the modules $\operatorname{im}(\alpha_i)$, $i = 1, \dots, t$ are projective. In particular, $\operatorname{im}(\alpha_{t-1})$ is projective and the exact sequence

$$0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} \operatorname{im}(\alpha_{t-1}) \to 0$$

splits. Then there exists $\beta_t : F_{t-1} \to F_t$ such that $I_{r_t} = \beta_t \alpha_t$. The module $\ker(\beta_t)$ is projective, isomorphic to $\operatorname{im}(\alpha_{t-1})$ and we can compute the isomorphism $\ker(\beta_t) \oplus F_t \simeq F_{t-1}$. We consider the following sequence:

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \dots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

where

$$\widetilde{\alpha}_t(\mathbf{v}_t) = (\alpha_t(\mathbf{v}_t), \mathbf{0}), \qquad \widetilde{\alpha}_{t-1}(\mathbf{v}_{t-1}, \mathbf{v}_t) = (\alpha_{t-1}(\mathbf{v}_{t-1}), \mathbf{v}_t),$$
$$\widetilde{\alpha}_{t-2}(\mathbf{v}_{t-2}, \mathbf{v}_t) = \alpha_{t-2}(\mathbf{v}_{t-2})$$

Then it is an exact sequence and again the module $\operatorname{im}(\widetilde{\alpha}_{t-1})$ is projective. As before, the sequence

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} \operatorname{im}(\widetilde{\alpha}_{t-1}) \to 0 \tag{1}$$

splits and there exists $\widetilde{\beta}_t : F_{t-1} \oplus F_t \to F_t$ such that $I_{r_t} = \widetilde{\beta}_t \widetilde{\alpha}_t$. In this case,

$$\widetilde{\beta}_t = \begin{pmatrix} \beta_t & \theta \end{pmatrix}$$

where θ is the null matrix with order $r_t \times r_t$. Then $\widetilde{\beta}(\mathbf{v}_{t-1}, \mathbf{v}_t) = \beta_t(\mathbf{v}_{t-1})$, so $\ker(\widetilde{\beta}_t) = \ker(\beta_t) \oplus F_t \simeq F_{t-1}$. We can compute the isomorphism

$$\widetilde{\nu}_{t-1} : F_{t-1} \to \ker(\widetilde{\beta}_t).$$

Let

$$\widetilde{\gamma}_{t-1} = \widetilde{\alpha}_{t-1}\widetilde{\nu}_{t-1} : F_{t-1} \to F_{t-2} \oplus F_t. \tag{2}$$

Then the sequence

$$0 \to F_{t-1} \xrightarrow{\widetilde{\gamma}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \dots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

is exact. Because the sequence (1) splits, the homomorphism $\widetilde{\alpha}_{t-1}$ is an isomorphism between $\ker(\widetilde{\beta}_t)$ and $\operatorname{im}(\widetilde{\alpha}_{t-1})$, so $\widetilde{\gamma}_{t-1}$ is an isomorphism between F_{t-1} and $\operatorname{im}(\widetilde{\alpha}_{t-1}) = \ker(\widetilde{\alpha}_{t-2})$, and we have the exactness of the sequence (2). We apply again the process to $\widetilde{\gamma}_{t-1}$ to check the projectiveness of the module M.
We need the following result:

Theorem 1. *Let R be a ring and*

$$\mathcal{F} : \ldots \to F_d \to F_{d-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

a projective resolution. Let d be the smallest number such that $\{\operatorname{im} F_d \to F_{d-1}\}$ is projective. Then d does not depend on the resolution and $\operatorname{pdim}(M) = d$.

Proof. [Eisenbud(1995)], exercise A.3.13. □

Theorem 2. *The previous algorithm allows us to compute the projective dimension of a module.*

Proof. Let

$$0 \to F_n \xrightarrow{\alpha_n} F_{n-1} \xrightarrow{\alpha_{n-1}} \ldots \to F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

be a finite free resolution given by the procedure. Then $\operatorname{im}(\alpha_{n-1})$ is not projective, because the matrix α_n has not left inverse. We can suppose that M is not projective, otherwise we have had shortened the resolution. Then the sequence

$$0 \to \ker(\alpha_0) \to F_0 \to M \to 0$$

does not split, so $\operatorname{im}(\alpha_1) = \ker(\alpha_0)$ is not projective. In the same way, the short exact sequence

$$0 \to \ker(\alpha_1) \to F_1 \to \operatorname{im}(\alpha_1) \to 0$$

does not split and $\operatorname{im}(\alpha_2) = \ker(\alpha_1)$ is not projective. Then the modules

$$\operatorname{im}(\alpha_1), \operatorname{im}(\alpha_2), \ldots, \operatorname{im}(\alpha_{n-1})$$

are not projective and the module $\operatorname{im}(\alpha_n)$ is projective. Then the projective dimension of M is equal to n. □

Algorithm 1.1. *Projective dimension.*
Input: a left R-module M defined by its generators in R^r.
Output: Projective dimension of M and a minimal length free resolution. If $\operatorname{pdim}(M) =$ 0, *i.e. M is projective, the algorithm returns an isomorphism $M \oplus R^s \simeq R^t$.*

Let $\mathcal{F}$ be a finite free resolution of M:

$$0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} F_{t-2} \xrightarrow{\alpha_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \dots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

START:

if α_t *has no left inverse* **then**
pdim$(M) = t$. *STOP.*
else
let β_t *be a left inverse of* α_t.
end if
if $t = 1$ **then**
pdim$(M) = 0$ *and* $M \oplus F_1 \simeq \ker(\beta_1) \oplus F_1 \simeq F_0$. *STOP.*
else
compute the exact sequence

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \ldots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

and the matrix $\widetilde{\nu}_{t-1}$ *that gives the isomorphism* $\ker(\beta_t) \oplus F_t \simeq F_{t-1}$.
end if
Let $\widetilde{\gamma}_{t-1} = \widetilde{\alpha}_{t-1}\widetilde{\nu}_{t-1}$.
Let $\mathcal{F}$ *be the finite free resolution*

$$0 \to F_{t-1} \xrightarrow{\widetilde{\gamma}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \ldots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0.$$

go to *START.*

This algorithm has being programmed with *Macaulay 2* [Grayson et al.(1999)], using the routines for D-modules developed by A. Leykin and H. Tsai [Leykin et al.(2002)].

Example 1. Let $W = A_2(\mathbb{Q})$ and $I = W\langle x\partial_x - 1, x\partial_y, \partial_x^2, \partial_y^2\rangle$. We found a resolution of I of the form

$$0 \leftarrow I \overset{\widetilde{\alpha}_0}{\leftarrow} W^4 \overset{\widetilde{\gamma}_1}{\leftarrow} W^3 \leftarrow 0$$

where

$$\widetilde{\gamma}_1 = \begin{pmatrix} -\partial_x^2 & -x\partial_x + 1 & 0 \\ \partial_y & 0 & -x \\ 0 & \partial_y & \partial_x \\ -\partial_x & -x & 0 \end{pmatrix}.$$

The rows of the matrix $\widetilde{\gamma}_1$ do not generate W^3, because a Gröbner basis is given by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & \partial_y & \partial_x \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the ideal I is not projective, and its projective dimension is 1.

2. Computing a basis.

Let k be a field of characteristic zero. Given a projective module over $A_n(k)$ with rank greater than 1, we are going to describe a procedure to compute a basis. We will need the standard Gröbner basis theory on $A_n(k)$ to make the computations. See for example [Castro(1987)] for a description of this algorithm. In [Hillebrand et al.(2002)] we found

Theorem 3. *Let* $\mathcal{R} = k(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$ *and* $I = \mathcal{R}\langle a, b, c\rangle$. *Then we can compute* $\tilde{a}, \tilde{b} \in \mathcal{R}$ *such that* $I = \mathcal{R}\langle a + \tilde{a}c, b + \tilde{b}c\rangle$.

As pointed out in [Hillebrand et al.(2002), Remark 3.15], the algorithm can be extended to $W = A_n(k) = k[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$. We need the following stronger result [Stafford(1978), Thm. 3.1]:

Theorem 4. *Let $I = W\langle a, b, c\rangle$ be a left W-ideal, and let $d_1, d_2 \in W - \{0\}$ be arbitrary elements. Then we can find $f_1, f_2 \in W$ such that*

$$I = W\langle a + d_1 f_1 c, b + d_2 f_2 c\rangle.$$

This can be accomplished with some minor changes to the proof of [Hillebrand et al.(2002), Lemma 3.10]. Following their notation, it is enough to take $g_1, g_2 \in W$ such that $h_1 d_1 g_1 + h_2 d_2 g_2 = 0$, and to apply [Hillebrand et al.(2002), Lemma 3.9] to $v = t d_2 g_2$. These changes appear in the proof of [Stafford(1978), Theorem 3.1]. The procedure is analogous for right ideals.

Definition 1. Let M be a left W-module and $\mathbf{v} \in M$. We say that $\mathbf{v}$ is unimodular in M if there exists $\varphi \in \mathrm{Hom}_W(M, W)$ such that $\varphi(\mathbf{v}) = 1$.

Remark 1. If $\mathbf{v}$ is a column vector in some W^m then $\mathbf{v}$ is unimodular if and only if the right ideal generated by its entries is equal to W. Through Gröbner bases, we can give the homomorphism that apply $\mathbf{v}$ in 1.

The following Lemma is a direct consequence of Theorem 4, and it will allow a 'cancellation' in some direct sums.

Lemma 1. [Stafford(1978), Lemma 3.5] *Let $M \subset W^m$ be a left W-module with* $\mathrm{rank}(M) \geq 2$ *and $\mathbf{a} \oplus t \in M \oplus W$ unimodular. Then there is an algorithm to find $\Phi \in \mathrm{Hom}_W(W, M)$ such that $\mathbf{a} + \Phi(t)$ is unimodular in M.*

Proof. Let $\mathbf{a}_1 \in M \subset W^m$ be a non zero element and consider $\Phi_1 : W^m \to W$ a projection such that $\Phi_1(\mathbf{a}_1) \neq 0$. Let $M_1 = M \cap \ker(\Phi_1)$, that we can compute by Gröbner bases. Then $\mathrm{rank}(M_1) = \mathrm{rank}(M) - 1 \geq 1$, so there exists $\mathbf{a}_2 \in M_1 - \mathbf{0}$. Let $\Phi_2 : W^m \to W$ be a projection such that $\Phi_2(\mathbf{a}_2) \neq 0$. If $\Phi_2(\mathbf{a}_1) \neq 0$ we can compute syzygies to get $r_1, r_2 \in W$ such that $\Phi_1(\mathbf{a}_1) r_1 + \Phi_2(\mathbf{a}_2) r_2 = 0$ and replace Φ_2 by the homomorphism $\Phi_1 r_1 + \Phi_2 r_2$. Then $\Phi_1(\mathbf{a}_2) = \Phi_2(\mathbf{a}_1) = 0$. Let $d_1 = \Phi_1(\mathbf{a}_1), d_2 = \Phi_2(\mathbf{a}_2)$ and consider the right ideal

$$I = \langle \Phi_1(\mathbf{a}), \Phi_2(\mathbf{a}), t\rangle W.$$

Then there exist $f_1, f_2 \in W$ such that

$$I = \langle \Phi_1(\mathbf{a}) + t f_1 d_1, \Phi_2(\mathbf{a}) + t f_2 d_2\rangle W.$$

Let $\Phi : W \to M$ be the homomorphism defined by $\Phi(1) = f_1 \mathbf{a}_1 + f_2 \mathbf{a}_2$. Then, as shown in [Stafford(1978), Lemma 3.5], $\mathbf{a} + \Phi(t)$ is unimodular, and we can compute $j \in \mathrm{Hom}_W(M, W)$ such that $j(\mathbf{a} + \Phi(t)) = 1$. □

Remark 2. The case $\mathbf{a} \neq \mathbf{0}$ is of special interest. In this case we can take $\mathbf{a}_1 = \mathbf{a}$ and obtain $\Phi_2(\mathbf{a}) = 0, d_1 = \Phi_1(\mathbf{a})$. We have to find f_1, f_2 such that

$$I = \langle d_1, 0, t\rangle W = \langle d_1 + t f_1 d_1, t f_2 d_2\rangle W.$$

Note that the problem is not to find two generators for the ideal I. We are looking for two special generators.

Proposition 1. [Swan(1968), Corollary 12.6] *Let $M \subset W^m$ be a left W-module with* $\mathrm{rank}(M) \geq 2$ *and $h : W \oplus N \to W \oplus M$ be an isomorphism with N a left W-module. Then $M \simeq N$.*

Proof. Let $h(1,\mathbf{0}) = (t_0, \mathbf{a}_0) \in W \oplus M$. The vector $(1,\mathbf{0})$ is unimodular so $(t_0,\mathbf{a}_0)$ too. Then we compute $\Phi : W \to M$ such that $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular in M and we get the homomorphism $j : M \to W$ with $j(\mathbf{a}'_0) = 1$. We consider the following homomorphisms:

$$\begin{array}{ll} g : W \oplus M \to W \oplus M, & g(t,\mathbf{a}) = (t, \mathbf{a} + \Phi(t)) \\ k : W \to W, & k(1) = t_0 \\ l : W \oplus M \to W \oplus M, & l(t,\mathbf{a}) = (t - (k \circ j)(\mathbf{a}), \mathbf{a}), \\ i : W \oplus N \to W \oplus M, & i = l \circ g \circ h \end{array}$$

Then i is isomorphism and $i(1,\mathbf{0}) = (0,\mathbf{a}'_0)$. We have $M = W\mathbf{a}'_0 \oplus \ker(j)$ and the following chain of isomorphisms

$$\begin{array}{l} N \simeq (W \oplus N)/W\mathbf{e}_1 \overset{i}{\to} (W \oplus M)/W\mathbf{a}'_0 = (W \oplus \ker(j) \oplus W\mathbf{a}'_0)/W\mathbf{a}'_0 \simeq \\ W \oplus \ker(j) \simeq W\mathbf{a}'_0 \oplus \ker(j) = M \end{array}$$

The isomorphism is defined as follows. Take $\mathbf{v}_1, \ldots, \mathbf{v}_r$ a set of generators of N. Let $i(0,\mathbf{v}_i) = (\alpha_i, \mathbf{u}_i)$ where $\alpha_i \in W, \mathbf{u}_i \in M$. The map $(W \oplus M)/W\mathbf{a}'_0 \to W \oplus \ker(j)$ works taking an element of $W \oplus M$, decomposes the component in M as a sum $\mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in W\mathbf{a}'_0, \mathbf{w} \in \ker(j)$ and takes $\mathbf{w}$. For this step note that if $\mathbf{u} \in M$ and $\lambda = j(\mathbf{u})$ then $\mathbf{u} = (\lambda\mathbf{a}'_0) + (\mathbf{u} - \lambda\mathbf{a}'_0)$ is the desired decomposition. □

Remark 3. When the module N is of the form W^s, then M is isomorphic to a free module, so it has a basis. Such a basis is the image of $\mathbf{e}_i, i = 1, \ldots, s$.

Algorithm 2.1. *Computing a basis.*
Input: an isomorphism $W^t \overset{h}{\simeq} W^s \oplus M$*, with* $t - s \geq 2$.
Output: a basis of the module M*.*

START:
if $s = 0$ **then**
  $\{ h(\mathbf{e}_1,), \ldots, h(\mathbf{e}_t) \}$ *is a basis.*
  STOP.
end if
Let $h(1,\mathbf{0}) = (t_0, \mathbf{a}_0)$*, with* $t_0 \in W, \mathbf{a}_0 \in W^{s-1} \oplus M$.
Compute $\Phi : W \to W^{s-1} \oplus M$ *such that* $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ *is unimodular.*
Compute $j : W^{s-1} \oplus M \to W$ *such that* $j(\mathbf{a}'_0) = 1$.
Let $i : W \oplus W^{t-1} \to W \oplus (W^{s-1} \oplus M)$ *as defined in Prop. 1.*
Let $h : W^{t-1} \to W^{s-1} \oplus M$ *the isomorphism defined by*

$$h(\mathbf{e}_i) = \alpha_i\mathbf{a}'_0 + \mathbf{u}_i - \lambda_i\mathbf{a}'_0$$

where $i(0,\mathbf{e}_i) = (\alpha_i, \mathbf{u}_i), \alpha_i \in W, \mathbf{u}_i \in W^{s-1} \oplus M, \lambda_i = j(\mathbf{u}_i)$.
go to *START*

As in the previous section, this algorithm has been programmed with *Macaulay2*.

Example 2. Let $W = A_2(\mathbb{Q})$, and $\mathbf{f} = \begin{pmatrix} x\partial_y & xy & \partial_x \end{pmatrix}$. Then $P = \ker \mathbf{f}$ is a projective module, because $\mathbf{f}$ is a unimodular row. Let

$$\beta = \begin{pmatrix} -y\partial_x \\ \partial_x\partial_y \\ -x \end{pmatrix}.$$

Then $\mathbf{f}\cdot\beta = 1$, and $\operatorname{im}\beta \oplus P = W^3$. The isomorphism $h : W \oplus W^2 \to W \oplus P$ is given by the matrix

$$h = \begin{pmatrix} x\partial_y & xy & \partial_x \\ xy\partial_x\partial_y + x\partial_x + 1 & xy^2\partial_x & y\partial_x^2 \\ -x\partial_x\partial_y^2 & -xy\partial_x\partial_y + 1 & -\partial_x^2\partial_y \\ x^2\partial_y & x^2y & x\partial_x + 2 \end{pmatrix}.$$

Then

$$t_0 = x\partial_y, \mathbf{a}_0 = \begin{pmatrix} xy\partial_x\partial_y + x\partial_x + 1 \\ -x\partial_x\partial_y^2 \\ x^2\partial_y \end{pmatrix}.$$

We must find $\Phi : W \to P$ such that $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular. Let $\Phi_1 : P \to W$ be the projection over the first component and $\mathbf{a}_2 \in P \cap \ker(\Phi_1)$ not null. For example,

$$\mathbf{a}_2 = \begin{pmatrix} 0 \\ \partial_x^2\partial_y \\ -xy\partial_x\partial_y - x\partial_x - 2y\partial_y - 2 \end{pmatrix}$$

and let $\Phi_2 : W \to P$ be the projection over the second component. Because $\Phi_2(\mathbf{a}_0) \neq 0$, we have to compute $r_1, r_2 \in W$ such that $\Phi_1(\mathbf{a}_0)r_1 + \Phi_2(\mathbf{a}_2)r_2 = 0$. In this case, we get

$$r_1 = -\partial_x^2\partial_y, r_2 = xy\partial_x\partial_y - 2y\partial_y + 1,$$

and following the notation of the proof of Lemma 1,

$$d_1 = xy\partial_x\partial_y + x\partial_x + 1, d_2 = xy\partial_x^3\partial_y^2 + x\partial_x^3\partial_y + \partial_x^2 dy.$$

We have to find $f_1, f_2 \in W$ such that $\langle d_1, t_0\rangle W = \langle d_1 + t_0 f_1 d_1, t_0 f_2 d_2\rangle W$. Applying the modified procedure of [Hillebrand et al.(2002)], we find

$$f_1 = 0, f_2 = x + y.$$

Let $\Phi : W \to P$ be the morphism defined by $\Phi(1) = (x+y)\mathbf{a}_2$. Then $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular and we can compute the morphism $j : P \to W$ such that $j(\mathbf{a}'_0) = 1$. The output is too large to be included here, but has the form

$j =$
$(-\frac{2}{63}x^2y^7\partial_x^4\partial_y^5 - \frac{2}{63}xy^8\partial_x^4\partial_y^5 + \frac{5}{126}x^3y^6\partial_x^3\partial_y^6 + \ldots - \frac{433}{9}x\partial_x + 17x\partial_y + 1,$
$\frac{2}{63}xy^8\partial_x^3\partial_y^4 - \frac{5}{126}x^2y^7\partial_x^2\partial_y^5 + \frac{10}{63}xy^8\partial_x^2\partial_y^5 + + \ldots + \frac{5}{3}xy - \frac{137}{6}y^2, 0)$

Also we can build the matrices associated to the other morphisms

$$g = \begin{pmatrix} 1 & \mathbf{0} \\ \Phi & I_3 \end{pmatrix}, k = (x\partial_y), l = \begin{pmatrix} 1 & -k\cdot j \\ \mathbf{0} & I_3 \end{pmatrix},$$
$$i = l\cdot g\cdot h = \begin{pmatrix} 0 & \alpha_2 & \alpha_3 \\ \mathbf{a}'_0 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$$

where
$\mathbf{u}_2 = (xy^2\partial_x, x^2y\partial_x^2\partial_y + xy^2\partial_x^2\partial_y - xy\partial_x\partial_y + 1,$
$-x^3y^2\partial_x\partial_y - x^2y^3\partial_x\partial_y - x^3y\partial_x - x^2y^2\partial_x - 2x^2y^2\partial_y - 2xy^3\partial_y - x^2y - 2xy^2)^t,$
$\mathbf{u}_3 = (y\partial_x^2, x\partial_x^3\partial_y + y\partial_x^3\partial_y,$
$-x^2y\partial_x^2\partial_y - xy^2\partial_x^2\partial_y - x^2\partial_x^2 - xy\partial_x^2 - 4xy\partial_x\partial_y - 3y^2\partial_x\partial_y - 3x\partial_x - 3y\partial_x - 2y\partial_y)^t$

Then

$$\mathbf{w}_1 = (\alpha_2 - \lambda_2)\mathbf{a}'_0 + \mathbf{u}_2, \mathbf{w}_2 = (\alpha_3 - \lambda_3)\mathbf{a}'_0 + \mathbf{u}_3$$

is a basis of P, where $\lambda_i = j(\mathbf{u}_i), i = 2, 3$.

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Departamento de Álgebra, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, ESPAÑA
E-mail address: `gago@us.es`