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Research Article

Forward-Backward Splitting Methods for Accretive Operators in Banach Spaces

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Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two (possibly simpler) nonlinear operators. Most of the investigation on splitting methods is however carried out in the framework of Hilbert spaces. In this paper, we consider these methods in the setting of Banach spaces. We shall introduce two iterative forward-backward splitting methods with relaxations and errors to find zeros of the sum of two accretive operators in the Banach spaces. We shall prove the weak and strong convergence of these methods under mild conditions. We also discuss applications of these methods to variational inequalities, the split feasibility problem, and a constrained convex minimization problem.

1. Introduction

Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two (possibly simpler) nonlinear operators. Splitting methods for linear equations were introduced by Peaceman and Rachford [1] and Douglas and Rachford [2]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [3] and Lions and Mercier [4] (see also [5–7]). The central problem is to iteratively find a zero of the sum of two monotone operators A and B in a Hilbert space $\mathcal{H}$, namely, a solution to the inclusion problem

$$0 \in (A+B)x. \tag{1.1}$$

Many problems can be formulated as a problem of form (1.1). For instance, a stationary solution to the initial value problem of the evolution equation

$$\frac{\partial u}{\partial t} + Fu \ni 0, \quad u(0) = u_0 \tag{1.2}$$

can be recast as (1.1) when the governing maximal monotone F is of the form $F = A + B$ [4]. In optimization, it often needs [8] to solve a minimization problem of the form

$$\min_{x\in\mathcal{H}} f(x) + g(Tx), \tag{1.3}$$

where f, g are proper lower semicontinuous convex functions from $\mathcal{H}$ to the extended real line $\overline{\mathbb{R}} := (-\infty, \infty]$, and T is a bounded linear operator on $\mathcal{H}$. As a matter of fact, (1.3) is equivalent to (1.1) (assuming that f and $g \circ T$ have a common point of continuity) with $A := \partial f$ and $B := T^* \circ \partial g \circ T$. Here T^* is the adjoint of T and ∂f is the subdifferential operator of f in the sense of convex analysis. It is known [8, 9] that the minimization problem (1.3) is widely used in image recovery, signal processing, and machine learning.

A splitting method for (1.1) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A+B$. To solve (1.1), Lions and Mercier [4] introduced the nonlinear Peaceman-Rachford and Douglas-Rachford splitting iterative algorithms which generate a sequence $\{v_n\}$ by the recursion

$$v_{n+1} = \left(2J_\lambda^A - I\right)\left(2J_\lambda^B - I\right)v_n \tag{1.4}$$

and respectively, a sequence $\{v_n\}$ by the recursion

$$v_{n+1} = J_\lambda^A\left(2J_\lambda^B - I\right)v_n + \left(I - J_\lambda^B\right)v_n. \tag{1.5}$$

Here we use J_λ^T to denote the resolvent of a monotone operator T; that is, $J_\lambda^T = (I + \lambda T)^{-1}$.

The nonlinear Peaceman-Rachford algorithm (1.4) fails, in general, to converge (even in the weak topology in the infinite-dimensional setting). This is due to the fact that the generating operator $(2J_\lambda^A - I)(2J_\lambda^B - I)$ for the algorithm (1.4) is merely nonexpansive. However, the mean averages of $\{u_n\}$ can be weakly convergent [5]. The nonlinear Douglas-Rachford algorithm (1.5) always converges in the weak topology to a point u and $u = J_\lambda^B v$ is a solution to (1.1), since the generating operator $J_\lambda^A(2J_\lambda^B - I) + (I - J_\lambda^B)$ for this algorithm is firmly nonexpansive, namely, the operator is of the form $(I+T)/2$, where T is nonexpansive.

There is, however, little work in the existing literature on splitting methods for nonlinear operator equations in the setting of Banach spaces (though there was some work on finding a common zero of a finite family of accretive operators [10–12]).

The main difficulties are due to the fact that the inner product structure of a Hilbert space fails to be true in a Banach space. We shall in this paper use the technique of duality maps to carry out certain initiative investigations on splitting methods for accretive operators in Banach spaces. Namely, we will study splitting iterative methods for solving the inclusion problem (1.1), where A and B are accretive operators in a Banach space X.

We will consider the case where A is single-valued accretive and B is possibly multivalued m-accretive operators in a Banach space X and assume that the inclusion (1.1) has a solution. We introduce the following two iterative methods which we call Mann-type and respectively, Halpern-type forward-backward methods with errors and which generate a sequence $\{x_n\}$ by the recursions

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n\right), \tag{1.6}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left(J_{r_n}^B(x_n - r_n(Ax_n + \alpha_n)) + b_n\right), \tag{1.7}$$

where J_r^B is the resolvent of the operator B of order r (i.e., $J_r^B = (I + rB)^{-1}$), and $\{\alpha_n\}$ is a sequence in $(0,1]$. We will prove weak convergence of (1.6) and strong convergence of (1.7) to a solution to (1.1) in some class of Banach spaces which will be made clear in Section 3.

The paper is organized as follows. In the next section we introduce the class of Banach spaces in which we shall study our splitting methods for solving (1.1). We also introduce the concept of accretive and m-accretive operators in a Banach space. In Section 3, we discuss the splitting algorithms (1.6) and (1.7) and prove their weak and strong convergence, respectively. In Section 4, we discuss applications of both algorithms (1.6) and (1.7) to variational inequalities, fixed points of pseudocontractions, convexly constrained minimization problems, the split feasibility problem, and linear inverse problems.

2. Preliminaries

Throughout the paper, X is a real Banach space with norm $\|\cdot\|$, distance d, and dual space X^*. The symbol $\langle x^*, x\rangle$ denotes the pairing between X^* and X, that is, $\langle x^*, x\rangle = x^*(x)$, the value of x^* at x. C will denote a nonempty closed convex subset of X, unless otherwise stated, and $\mathcal{B}_r$ the closed ball with center zero and radius r. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively, and $\omega_w(x_n)$ stands for the set of weak limit points of the sequence $\{x_n\}$.

The *modulus of convexity* of X is the function $\delta_X(\varepsilon) : (0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \tag{2.1}$$

Recall that X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$. Let $p > 1$. We say that X is *p-uniformly convex* if there exists a constant $c_p > 0$ so that $\delta_X(\varepsilon) \geq c_p\varepsilon^p$ for any $\varepsilon \in (0,2]$.

The *modulus of smoothness* of X is the function $\rho_X(\tau) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}. \tag{2.2}$$

Recall that X is called *uniformly smooth* if $\lim_{\tau\to 0}\rho_X(\tau)/\tau = 0$. Let $1 < q \leq 2$. We say that X is q-*uniformly smooth* if there is a $c_q > 0$ so that $\rho_X(\tau) \leq c_q\tau^q$ for $\tau > 0$. It is known that X is p-uniformly convex if and only if X^* is q-uniformly smooth, where $(1/p + 1/q = 1)$. For instance, L^p spaces are 2-uniformly convex and p-uniformly smooth if $1 < p \leq 2$, whereas p-uniformly convex and 2-uniformly smooth if $p \geq 2$.

The norm of X is said to be the Fréchet differentiable if, for each $x \in X$,

$$\lim_{\lambda\to 0}\frac{\|x+\lambda y\| - \|x\|}{\lambda} \tag{2.3}$$

exists and is attained uniformly for all y such that $\|y\| = 1$. It can be proved that X is uniformly smooth if the limit (2.3) exists and is attained uniformly for all (x, y) such that $\|x\| = \|y\| = 1$. So it is trivial that a uniformly smooth Banach space has a Fréchet differentiable norm.

The subdifferential of a proper convex function $f : X \to (-\infty, +\infty]$ is the set-valued operator $\partial f : X \to 2^X$ defined as

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) \leq f(y)\}. \tag{2.4}$$

If f is proper, convex, and lower semicontinuous, the subdifferential $\partial f(x) \neq \emptyset$ for any $x \in \operatorname{int}\mathfrak{D}(f)$, the interior of the domain of f. The *generalized duality mapping* $\mathcal{J}_p : X \to 2^{X^*}$ is defined by

$$\mathcal{J}_p(x) = \left\{ j(x) \in X^* : \langle j(x), x\rangle = \|x\|^p, \quad \|j(x)\| = \|x\|^{p-1} \right\}. \tag{2.5}$$

If $p = 2$, the corresponding duality mapping is called the normalized duality mapping and denoted by $\mathcal{J}$. It can be proved that, for any $x \in X$,

$$\mathcal{J}_p(x) = \partial\left(\frac{1}{p}\|x\|^p\right). \tag{2.6}$$

Thus we have the following subdifferential inequality, for any $x, y \in X$:

$$\|x+y\|^p \leq \|x\|^p + p\langle y, j(x+y)\rangle, \quad j(x+y) \in \mathcal{J}_p(x+y). \tag{2.7}$$

In particular, we have, for $x, y \in X$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad j(x+y) \in \mathcal{J}(x+y). \tag{2.8}$$

Some properties of the duality mappings are collected as follows.

Proposition 2.1 (see Cioranescu [13]). *Let* $1 < p < \infty$.

(i) *The Banach space X is smooth if and only if the duality mapping $\mathcal{J}_p$ is single valued.*

(ii) *The Banach space X is uniformly smooth if and only if the duality mapping J_p is single-valued and norm-to-norm uniformly continuous on bounded sets of X.*

Among the estimates satisfied by p-uniformly convex and p-uniformly smooth spaces, the following ones will come in handy.

Lemma 2.2 (see Xu [14]). *Let $1 < p < \infty$, $q \in (1, 2]$, $r > 0$ be given.*

(i) *If X is uniformly convex, then there exists a continuous, strictly increasing and convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + \lambda\|y\|^p - W_p(\lambda)\varphi(\|x-y\|), \quad x, y \in \mathcal{B}_r,\ 0 \leq \lambda \leq 1, \tag{2.9}$$

where $W_p(\lambda) = \lambda^p(1-\lambda) + (1-\lambda)\lambda^p$.

(ii) *If X is q-uniformly smooth, then there exists a constant $\kappa_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle \mathcal{J}_q(x), y\rangle + \kappa_q\|y\|^q, \quad x, y \in X. \tag{2.10}$$

The best constant κ_q satisfying (2.10) *will be called the q-uniform smoothness coefficient of X. For instance [14], for $2 \leq p < \infty$, L^p is 2-uniformly smooth with $\kappa_2 = p - 1$, and for $1 < p \leq 2$, L^p is p-uniformly smooth with $\kappa_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$, where t_p is the unique solution to the equation*

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \quad 0 < t < 1. \tag{2.11}$$

In a Banach space X with the Fréchet differentiable norm, there exists a function $h : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to 0} h(t)/t = 0$ and for all $x, u \in X$

$$\frac{1}{2}\|x\|^2 + \langle u, \mathcal{J}(x)\rangle \leq \frac{1}{2}\|x + u\|^2 \leq \frac{1}{2}\|x\|^2 + \langle u, \mathcal{J}(x)\rangle + h(\|u\|). \tag{2.12}$$

Recall that $T : C \to C$ is a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. From now on, $\mathrm{Fix}(T)$ denotes the fixed point set of T. The following lemma claims that the demiclosedness principle for nonexpansive mappings holds in uniformly convex Banach spaces.

Lemma 2.3 (see Browder [15]). *Let C be a nonempty closed convex subset of a uniformly convex space X and T a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

A set-valued operator $A : X \to 2^X$, with domain $\mathfrak{D}(A)$ and range $\mathcal{R}(A)$, is said to be *accretive* if, for all $t > 0$ and every $x, y \in \mathfrak{D}(A)$,

$$\|x - y\| \leq \|x - y + t(u - v)\|, \quad u \in Ax,\ v \in Ay. \tag{2.13}$$

It follows from Lemma 1.1 of Kato [16] that A is accretive if and only if, for each $x, y \in \mathfrak{D}(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0, \quad u \in Ax,\ v \in Ay. \tag{2.14}$$

An accretive operator A is said to be *m-accretive* if the range $\mathcal{R}(I + \lambda A) = X$ for some $\lambda > 0$. It can be shown that an accretive operator A is m-accretive if and only if $\mathcal{R}(I + \lambda A) = X$ for all $\lambda > 0$.

Given $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is *α-inverse strongly accretive* (α-isa) of order q if, for each $x, y \in \mathfrak{D}(A)$, there exists $j_q(x - y) \in \mathcal{J}_q(x - y)$ such that

$$\langle u - v, j_q(x - y) \rangle \geq \alpha \|u - v\|^q, \quad u \in Ax, \; v \in Ay. \tag{2.15}$$

When $q = 2$, we simply say α-isa, instead of α-isa of order 2; that is, T is α-isa if, for each $x, y \in \mathfrak{D}(A)$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that

$$\langle u - v, j(x - y) \rangle \geq \alpha \|u - v\|^2, \quad u \in Ax, \; v \in Ay. \tag{2.16}$$

Given a subset K of C and a mapping $T : C \to K$, recall that T is a retraction of C onto K if $Tx = x$ for all $x \in K$. We say that T is sunny if, for each $x \in C$ and $t \geq 0$, we have

$$T(tx + (1 - t)Tx) = Tx, \tag{2.17}$$

whenever $tx + (1 - t)Tx \in C$.

The first result regarding the existence of sunny nonexpansive retractions onto the fixed point set of a nonexpansive mapping is due to Bruck.

Theorem 2.4 (see Bruck [17]). *If X is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set* $\mathrm{Fix}(T)$*, then there exists a sunny nonexpansive retraction of C onto* $\mathrm{Fix}(T)$.

The following technical lemma regarding convergence of real sequences will be used when we discuss convergence of algorithms (1.6) and (1.7) in the next section.

Lemma 2.5 (see [18, 19]). *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+$, $\{\alpha_n\} \subset (0, 1)$, and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n, \quad \forall n \geq 0. \tag{2.18}$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(i) If $b_n \leq \alpha_n M$ where $M \geq 0$, then $\{a_n\}$ is a bounded sequence.

(ii) If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \to \infty} b_n/\alpha_n \leq 0$, then $\lim a_n = 0$.

3. Splitting Methods for Accretive Operators

In this section we assume that X is a real Banach space and C is a nonempty closed subset of X. We also assume that A is a single-valued and α-isa operator for some $\alpha > 0$ and B is an m-accretive operator in X, with $\mathfrak{D}(A) \supset C$ and $\mathfrak{D}(B) \supset C$. Moreover, we always use J_r to denote the resolvent of B of order $r > 0$; that is,

$$J_r \equiv J_r^B = (I + rB)^{-1}. \tag{3.1}$$

It is known that the m-accretiveness of B implies that J_r is single valued, defined on the entire X, and firmly nonexpansive; that is,

$$\|J_r x - J_r y\| \leq \|s(x-y) + (1-s)(J_r x - J_r y)\|, \quad x, y \in X, \ 0 \leq s \leq 1. \tag{3.2}$$

Below we fix the following notation:

$$T_r := J_r(I - rA) = (I + rB)^{-1}(I - rA). \tag{3.3}$$

Lemma 3.1. *For $r > 0$, $\mathrm{Fix}(T_r) = (A+B)^{-1}(0)$.*

Proof. From the definition of T_r, it follows that

$$\begin{aligned} x = T_r x &\Longleftrightarrow x = (I + rB)^{-1}(x - rAx) \\ &\Longleftrightarrow x - rAx \in x + rBx \\ &\Longleftrightarrow 0 \in Ax + Bx. \end{aligned} \tag{3.4}$$

□

This lemma alludes to the fact that in order to solve the inclusion problem (1.1), it suffices to find a fixed point of T_r. Since T_r is already "split," an iterative algorithm for T_r corresponds to a splitting algorithm for (1.1). However, to guarantee convergence (weak or strong) of an iterative algorithm for T_r, we need good metric properties of T_r such as nonexpansivity. To this end, we need geometric conditions on the underlying space X (see Lemma 3.3).

Lemma 3.2. *Given $0 < s \leq r$ and $x \in X$, there holds the relation*

$$\|x - T_s x\| \leq 2\|x - T_r x\|. \tag{3.5}$$

Proof. Note that $((x - T_r x)/r) - Ax \in B(T_r x)$. By the accretivity of B, we have $j_{s,r} \in \mathcal{J}(T_s x - T_r x)$ such that

$$\left\langle \frac{x - T_s x}{s} - \frac{x - T_r x}{r}, j_{s,r} \right\rangle \geq 0. \tag{3.6}$$

It turns out that

$$\begin{aligned} \|T_s x - T_r x\|^2 &\leq \frac{r-s}{r}\langle x - T_r x, j_{s,r}\rangle \\ &\leq \left|1 - \frac{s}{r}\right| \|x - T_r x\| \|T_s x - T_r x\|. \end{aligned} \tag{3.7}$$

This along with the triangle inequality yields that

$$\begin{aligned}\|x - T_s x\| &\leq \|x - T_r x\| + \|T_r x - T_s x\| \\ &\leq \|x - T_r x\| + \left|1 - \frac{s}{r}\right| \|x - T_r x\| \\ &\leq 2\|x - T_r x\|.\end{aligned} \tag{3.8}$$

□

We notice that though the resolvent of an accretive operator is always firmly nonexpansive in a general Banach space, firm nonexpansiveness is however insufficient to estimate useful bounds which are required to prove convergence of iterative algorithms for solving nonlinear equations governed by accretive operations. To overcome this difficulty, we need to impose additional properties on the underlying Banach space X. Lemma 3.3 below establishes a sharper estimate than nonexpansiveness of the mapping T_r, which is useful for us to prove the weak and strong convergence of algorithms (1.6) and (1.7).

Lemma 3.3. *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q in X. Then, given $s > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in \mathcal{B}_s$,*

$$\begin{aligned}\|T_r x - T_r y\|^q \leq \|x - y\|^q - r\left(\alpha q - r^{q-1}\kappa_q\right) \|Ax - Ay\|^q \\ - \phi_q\left(\|(I - J_r)(I - rA)x - (I - J_r)(I - rA)y\|\right),\end{aligned} \tag{3.9}$$

where κ_q is the q-uniform smoothness coefficient of X (see Lemma 2.2).

Proof. Put $\hat{x} = x - rAx$ and $\hat{y} = y - rAy$. Since $(\hat{x} - J_r\hat{x})/r \in B(J_r\hat{x})$, it follows from the accretiveness of B that

$$\begin{aligned}\|J_r\hat{x} - J_r\hat{y}\| &\leq \left\|(J_r\hat{x} - J_r\hat{y}) + \frac{r}{2}\left(\frac{\hat{x} - J_r\hat{x}}{r} - \frac{\hat{y} - J_r\hat{y}}{r}\right)\right\| \\ &= \left\|\frac{1}{2}(\hat{x} - \hat{y}) + \frac{1}{2}(J_r\hat{x} - J_r\hat{y})\right\|.\end{aligned} \tag{3.10}$$

Since $x, y \in \mathcal{B}_s$, by the accretivity of A it is easy to show that there exists $t > 0$ such that $\hat{x} - \hat{y} \in \mathcal{B}_t$; hence, $J_r\hat{x} - J_r\hat{y} \in \mathcal{B}_t$ for J_r is nonexpansive. Now since X is uniformly convex, we can use Lemma 2.2 to find a continuous, strictly increasing and convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = 0$, satisfying

$$\begin{aligned}\left\|\frac{1}{2}(\hat{x} - \hat{y}) + \frac{1}{2}(J_r\hat{x} - J_r\hat{y})\right\|^q &\leq \frac{1}{2}\|\hat{x} - \hat{y}\|^q + \frac{1}{2}\|J_r\hat{x} - J_r\hat{y}\|^q \\ &\quad - W_q\left(\frac{1}{2}\right)\varphi\left(\|(I - J_r)\hat{x} - (I - J_r)\hat{y}\|\right) \\ &\leq \|\hat{x} - \hat{y}\|^q - \frac{1}{2^q}\varphi\left(\|(I - J_r)\hat{x} - (I - J_r)\hat{y}\|\right),\end{aligned} \tag{3.11}$$

where the last inequality follows from the nonexpansivity of the resolvent J_r. Letting $\phi_q = \varphi/2^q$ and combining (3.10) and (3.11) yield

$$\|T_r x - T_r y\|^q \leq \|\hat{x} - \hat{y}\|^q - \phi_q(\|(I - J_r)\hat{x} - (I - J_r)\hat{y}\|). \tag{3.12}$$

On the other hand, since X is also q-uniformly smooth and A is α-isa of order q, we derive that

$$\begin{aligned}\|\hat{x} - \hat{y}\|^q &= \|(x - y) - r(Ax - Ay)\|^q \\ &\leq \|x - y\|^q + \kappa_q r^q \|Ax - Ay\|^q - rq\langle Ax - Ay, \mathcal{J}_q(x - y)\rangle \\ &\leq \|x - y\|^q - r\left(\alpha q - r^{q-1}\kappa_q\right)\|Ax - Ay\|^q.\end{aligned} \tag{3.13}$$

Finally the required inequality (3.9) follows from (3.12) and (3.13). □

Remark 3.4. Note that from Lemma 3.3 one deduces that, under the same conditions, if $r \leq (\alpha q/\kappa_q)^{1/(q-1)}$, then the mapping T_r is nonexpansive.

3.1. Weak Convergence

Mann's iterative method [20] is a widely used method for finding a fixed point of nonexpansive mappings [21]. We have proved that a splitting method for solving (1.1) can, under certain conditions, be reduced to a method for finding a fixed point of a nonexpansive mapping. It is therefore the purpose of this subsection to introduce and prove its weak convergence of a Mann-type forward-backward method with errors in a uniformly convex and q-uniformly smooth Banach space. (See [22] for a similar treatment of the proximal point algorithm [23, 24] for finding zeros of monotone operators in the Hilbert space setting.) To this end we need a lemma about the uniqueness of weak cluster points of a sequence, whose proof, included here, follows the idea presented in [21, 25].

Lemma 3.5. *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm, and let $\{T_n\}$ be a sequence of nonexpansive self-mappings on C with a nonempty common fixed point set F. If $x_0 \in C$ and $x_{n+1} := T_n x_n + e_n$, where $\sum_{n=1}^{\infty}\|e_n\| < \infty$, then $\langle z_1 - z_2, J(y_1 - y_2)\rangle = 0$ for all $y_1, y_2 \in F$ and all z_1, z_2 weak limit points of $\{x_n\}$.*

Proof. We first claim that the sequence $\{x_n\}$ is bounded. As a matter of fact, for each fixed $p \in F$ and any $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - p\| &= \|T_n x_n - T_n p + e_n\| \\ &\leq \|x_n - p\| + \|e_n\|.\end{aligned} \tag{3.14}$$

As $\sum_{n=1}^{\infty}\|e_n\| < \infty$, we can apply Lemma 2.5 to find that $\lim_{n\to\infty}\|x_n - p\|$ exists. In particular, $\{x_n\}$ is bounded.

Let us next prove that, for every $y_1, y_2 \in F$ and $0 < t < 1$, the limit

$$\lim_{n\to\infty}\|tx_n + (1-t)y_1 - y_2\| \tag{3.15}$$

exists. To see this, we set $S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n$ which is nonexpansive. It is to see that we can rewrite $\{x_n\}$ in the manner

$$x_{n+m} = S_{n,m}x_n + c_{n,m}, \quad n, m \geq 1, \tag{3.16}$$

where

$$\begin{aligned} c_{n,m} = {} & T_{n+m-1}(T_{n+m-2}(\cdots T_{n-1}(T_n x_n + e_n) + e_{n-1}\cdots) + e_{n+m-2}) \\ & + e_{n+m-1} - S_{n,m}x_n. \end{aligned} \tag{3.17}$$

By nonexpansivity, we have that

$$\|c_{n,m}\| \leq \sum_{k=n}^{n+m-1} \|e_k\|, \tag{3.18}$$

and the summability of $\{e_n\}$ implies that

$$\lim_{n,m\to\infty} \|c_{n,m}\| = 0. \tag{3.19}$$

Set

$$\begin{aligned} a_n &= \left\| t x_n + (1-t)y_1 - y_2 \right\|, \\ d_{n,m} &= \left\| S_{n,m}\big(t x_n + (1-t)y_1\big) - \big(t S_{n,m}x_n + (1-t)y_1\big) \right\|. \end{aligned} \tag{3.20}$$

Let K be a closed bounded convex subset of X containing $\{x_n\}$ and $\{y_1, y_2\}$. A result of Bruck [26] assures the existence of a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$g\big(\left\| U(tx + (1-t)y) - (tUx + (1-t)Uy) \right\|\big) \leq \left\| x - y \right\| - \left\| Ux - Uy \right\| \tag{3.21}$$

for all $U : K \to X$ nonexpansive, $x, y \in K$ and $0 \leq t \leq 1$. Applying (3.21) to each $S_{n,m}$, we obtain

$$\begin{aligned} \mathrm{g}(d_{n,m}) &\leq \left\| x_n - y_1 \right\| - \left\| S_{n,m}x_n - S_{n,m}y_1 \right\| \\ &= \left\| x_n - y_1 \right\| - \left\| x_{n+m} - y_1 - c_{n,m} \right\| \\ &\leq \left\| x_n - y_1 \right\| - \left\| x_{n+m} - y_1 \right\| + \|c_{n,m}\|. \end{aligned} \tag{3.22}$$

Now since $\lim_{n\to\infty}\|x_n - y_1\|$ exists, (3.19) and (3.22) together imply that

$$\lim_{n,m\to\infty} d_{n,m} = 0. \tag{3.23}$$

Furthermore, we have

$$a_{n+m} \leq a_n + d_{n,m} + \|c_{n,m}\|. \tag{3.24}$$

After taking first $\limsup_{m\to\infty}$ and then $\liminf_{n\to\infty}$ in (3.24) and using (3.19) and (3.23), we get

$$\limsup_{m\to\infty} a_m \leq \liminf_{n\to\infty} a_n + \lim_{n,m\to\infty}(d_{n,m} + \|c_{n,m}\|) = \liminf_{n\to\infty} a_n. \tag{3.25}$$

Hence the limit (3.15) exists.

If we replace now x and u in (2.12) with $y_1 - y_2$ and $t(x_n - y_1)$, respectively, we arrive at

$$\begin{aligned} &\frac{1}{2}\|y_1 - y_2\|^2 + t\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle \\ &\quad\leq \frac{1}{2}\|tx_n + (1-t)y_1 - y_2\|^2 \\ &\quad\leq \frac{1}{2}\|y_1 - y_2\|^2 + t\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle + h(t\|x_n - y_1\|). \end{aligned} \tag{3.26}$$

Since the $\lim_{n\to\infty}\|x_n - y_1\|$ exists, we deduce that

$$\begin{aligned} &\frac{1}{2}\|y_1 - y_2\|^2 + t\limsup_{n\to\infty}\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle \\ &\quad\leq \lim_{n\to\infty}\frac{1}{2}\|tx_n + (1-t)y_1 - y_2\|^2 \\ &\quad\leq \frac{1}{2}\|y_1 - y_2\|^2 + t\liminf_{n\to\infty}\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle + o(t), \end{aligned} \tag{3.27}$$

where $\lim_{t\to 0} o(t)/t = 0$. Consequently, we deduce that

$$\limsup_{n\to\infty}\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle \leq \liminf_{n\to\infty}\langle x_{\mathrm{n}} - y_1, \mathcal{J}(y_1 - y_2)\rangle + \frac{o(t)}{t}. \tag{3.28}$$

Setting t tend to 0, we conclude that $\lim_{n\to\infty}\langle x_n - y_1, \mathcal{J}(y_1 - y_2)\rangle$ exists. Therefore, for any two weak limit points z_1 and z_2 of $\{x_n\}$, $\langle z_1 - y_1, \mathcal{J}(y_1 - y_2)\rangle = \langle z_2 - y_1, \mathcal{J}(y_1 - y_2)\rangle$; that is, $\langle z_1 - z_2, \mathcal{J}(y_1 - y_2)\rangle = 0$. □

Theorem 3.6. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Assume that $S = (A+B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the perturbed iterative scheme*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n), \tag{3.29}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\{a_n\}, \{b_n\} \subset X$, $\{\alpha_n\} \subset (0,1]$, *and* $\{r_n\} \subset (0,+\infty)$. *Assume that*

(i) $\sum_{n=1}^{\infty} \|a_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|b_n\| < \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n$;

(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ *converges weakly to some* $x \in S$.

Proof. Write $T_n = (I + r_n B)^{-1}(I - r_n A)$. Notice that we can write

$$J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n = T_n x_n + e_n, \tag{3.30}$$

where $e_n = J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n - T_n x_n$. Then the iterative formula (3.29) turns into the form

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(T_n x_n + e_n). \tag{3.31}$$

Thus, by nonexpansivity of J_{r_n},

$$\begin{aligned} \|e_n\| &\le \|J_{r_n}(x_n - r_n(Ax_n + a_n)) - T_n x_n\| \|b_n\| \\ &\le r_n\|a_n\| + \|b_n\|. \end{aligned} \tag{3.32}$$

Therefore, condition (i) implies

$$\sum_{n=1}^{\infty} \|e_n\| < \infty. \tag{3.33}$$

Take $z \in S$ to deduce that, as $S = \mathrm{Fix}(T_n)$ and T_n is nonexpansive,

$$\begin{aligned} \|x_{n+1} - z\| &\le (1 - \alpha_n)\|x_n - z\| + \alpha_n\|T_n x_n - T_n z + e_n\| \\ &\le \|x_n - z\| + \alpha_n\|e_n\|. \end{aligned} \tag{3.34}$$

Due to (3.33), Lemma 2.5 is applicable and we get that $\lim_{n\to\infty}\|x_n - z\|$ exists; in particular, $\{x_n\}$ is bounded. Let $M > 0$ be such that $\|x_n\| < M$, for all $n \in \mathbb{N}$, and let $s = q(M + \|z\|)^{q-1}$. By (2.7) and Lemma 3.3, we have

$$\begin{aligned} \|x_{n+1} - z\|^q &\le \|(1 - \alpha_n)(x_n - z) + \alpha_n(T_n x_n - z) + \alpha_n e_n\|^q \\ &\le \|(1 - \alpha_n)(x_n - z) + \alpha_n(T_n x_n - z)\|^q + \alpha_n\langle e_n, \mathcal{J}(x_{n+1} - z)\rangle \\ &\le (1 - \alpha_n)\|x_n - z\|^q + \alpha_n\|T_n x_n - T_n z\|^q + \alpha_n q\|e_n\|\|x_{n+1} - z\|^{q-1} \\ &\le \|x_n - z\|^q - \alpha_n r_n\left(q\alpha - r_n^{q-1}\kappa_q\right)\|Ax_n - Az\|^q \\ &\quad - \phi_q(\|x_n - r_n Ax_n - T_n x_n + r_n Az\|) + s\|e_n\|. \end{aligned} \tag{3.35}$$

From (3.35), assumptions (ii) and (iii), and (3.33), it turns out that

$$\lim_{n\to\infty} \|Ax_n - Az\|^q + \|x_n - r_n Ax_n - T_n x_n + r_n Az\| = 0. \tag{3.36}$$

Consequently,

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.37}$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $\varepsilon > 0$ such that $r_n \geq \varepsilon$ for all $n \geq 0$. Then, by Lemma 3.2,

$$\lim_{n\to\infty} \|T_\varepsilon x_n - x_n\| \leq 2 \lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.38}$$

By Lemmas 3.3 and 3.1, T_ε is nonexpansive and $\mathrm{Fix}(T_\varepsilon) = S \neq \emptyset$. We can therefore make use of Lemma 2.3 to assure that

$$\omega_w(x_n) \subset S. \tag{3.39}$$

Finally we set $U_n = (1-\alpha_n)I + \alpha_n T_n$ and rewrite scheme (3.31) as

$$x_{n+1} = U_n x_n + e'_n, \tag{3.40}$$

where the sequence $\{e'_n\}$ satisfies $\sum_{n=1}^{\infty} \|e'_n\| < \infty$. Since $\{U_n\}$ is a sequence of nonexpansive mappings with S as its nonempty common fixed point set, and since the space X is uniformly convex with a Fréchet differentiable norm, we can apply Lemma 3.5 together with (3.39) to assert that the sequence $\{x_n\}$ has exactly one weak limit point; it is therefore weakly convergent. □

3.2. Strong Convergence

Halpern's method [27] is another iterative method for finding a fixed point of nonexpansive mappings. This method has been extensively studied in the literature [28–30] (see also the recent survey [31]). In this section we aim to introduce and prove the strong convergence of a Halpern-type forward-backward method with errors in uniformly convex and q-uniformly smooth Banach spaces. This result turns out to be new even in the setting of Hilbert spaces.

Theorem 3.7. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Assume that $S = (A+B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n), \tag{3.41}$$

where $u \in X$, $J_{r_n} = (I + r_n B)^{-1}$, $\{a_n\}, \{b_n\} \subset X$, $\{\alpha_n\} \subset (0,1]$, *and* $\{r_n\} \subset (0,+\infty)$. *Assume the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \|a_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|b_n\| < \infty$, *or* $\lim_{n\to\infty}\|a_n\|/\alpha_n = \lim_{n\to\infty}\|b_n\|/\alpha_n = 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ *converges in norm to* $z = Q(u)$, *where* Q *is the sunny nonexpansive retraction of* X *onto* S.

Proof. Let $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S whose existence is ensured by Theorem 2.4. Let (y_n) be a sequence generated by

$$y_{n+1} = \alpha_n u + (1-\alpha_n) T_n y_n, \tag{3.42}$$

where we abbreviate $T_n := J_{r_n}(I - r_n A)$. Hence to show the desired result, it suffices to prove that $y_n \to z$. Indeed, since J_{r_n} and $I - r_n A$ are both nonexpansive, it follows that

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &\le (1-\alpha_n)\left\|J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n - J_{r_n}(y_n - r_n A y_n)\right\| \\ &\le (1-\alpha_n)\left\|(I - r_n A)x_n - (I - r_n A)y_n - r_n a_n\right\| + \|b_n\| \\ &= (1-\alpha_n)\left\|x_n - y_n\right\| + L(\|a_n\| + \|b_n\|),\end{aligned} \tag{3.43}$$

where $L := \max(1, (\alpha q/\kappa_q)^{1/(q-1)})$. According to condition (i), we can apply Lemma 2.5(ii) to conclude that $\|x_n - y_n\| \to 0$ as $n \to \infty$. □

We next show $y_n \to z$. Indeed, since $S = \mathrm{Fix}(T_n)$ and T_n is nonexpansive, we have

$$\begin{aligned}\|y_{n+1} - z\| &\le \alpha_n\|u - z\| + (1-\alpha_n)\|T_n y_n - T_n z\| \\ &\le \alpha_n\|u - z\| + (1-\alpha_n)\|y_n - z\|.\end{aligned} \tag{3.44}$$

Hence, we can apply Lemma 2.5(i) to claim that $\{y_n\}$ is bounded.

Using the inequality (2.7) with $p = q$, we derive that

$$\begin{aligned}\|y_{n+1} - z\|^q &= \|\alpha_n(u - z) + (1-\alpha_n)T_n y_n - z\|^q \\ &\le (1-\alpha_n)^q\|T_n y_n - z\|^q + q\alpha_n\langle u - z, \mathcal{J}_q(y_{n+1} - z)\rangle.\end{aligned} \tag{3.45}$$

By condition (iii), we have some $\delta > 0$ such that

$$1 - \alpha_n \ge \delta, \qquad (1-\alpha_n) r_n\left(\alpha q - r_n^{q-1}\kappa_q\right) \ge \delta, \tag{3.46}$$

for all $n \in \mathbb{N}$. Hence, by Lemma 3.3 we get from (3.45) that

$$\begin{aligned} \|y_{n+1} - z\|^q \le (1 - \alpha_n)\|y_n - z\|^q - \delta\phi_q(\|y_n - r_n A y_n - T_n y_n + r_n A z\|) \\ - \delta\|Ay_n - Az\|^q + q\alpha_n\langle u - z, \mathcal{J}_q(y_{n+1} - z)\rangle. \end{aligned} \tag{3.47}$$

Let us define $s_n = \|y_n - z\|^q$ for all $n \ge 0$. Depending on the asymptotic behavior of the sequence $\{s_n\}$ we distinguish two cases.

Case 1. Suppose that there exists $N \in \mathbb{N}$ such that the sequence $\{s_n\}_{n \ge N}$ is nonincreasing; thus, $\lim_{n\to\infty} s_n$ exists. Since $\alpha_n \to 0$ and $\|e_n\| \to 0$, it follows immediately from (3.47) that

$$\lim_{n\to\infty} \|Ay_n - Az\|^q + \|y_n - r_n A y_n - T_n y_n + r_n A z\| = 0. \tag{3.48}$$

Consequently,

$$\lim_{n\to\infty} \|T_n y_n - y_n\| = 0. \tag{3.49}$$

By condition (iii), there exists $\varepsilon > 0$ such that $r_n \ge \varepsilon$ for all $n \ge 0$. Then, by Lemma 3.2, we get

$$\lim_{n\to\infty} \|T_\varepsilon y_n - y_n\| \le \lim_{n\to\infty} \|T_n y_n - y_n\| = 0. \tag{3.50}$$

The demiclosedness principle (i.e., Lemma 2.3) implies that

$$\omega_w(y_n) \subset S. \tag{3.51}$$

Note that from inequality (3.47) we deduce that

$$s_{n+1} \le (1 - \alpha_n)s_n + q\alpha_n\langle u - z, \mathcal{J}_q(y_{n+1} - z)\rangle. \tag{3.52}$$

Next we prove that

$$\limsup_{n\to\infty} \langle u - z, \mathcal{J}_q(y_n - z)\rangle \le 0. \tag{3.53}$$

Equivalently (should $\|y_n - z\| \nrightarrow 0$), we need to prove that

$$\limsup_{n\to\infty} \langle u - z, \mathcal{J}(y_n - z)\rangle \le 0. \tag{3.54}$$

To this end, let z_t satisfy $z_t = tu + T_\varepsilon z_t$. By Reich's theorem [32], we get $z_t \to Q_S u = z$ as $t \to 0$. Using subdifferential inequality, we deduce that

$$\begin{aligned}\|z_t - y_n\|^2 &= \|t(u-y_n) + (1-t)(T_\varepsilon z_t - y_n)\|^2 \\ &\le (1-t)^2\|T_\varepsilon z_t - y_n\|^2 + 2t\langle u - y_n, \mathcal{J}(z_t - y_n)\rangle \\ &\le (1-t)^2(\|T_\varepsilon z_t - T_\varepsilon y_n\| + \|T_\varepsilon y_n - y_n\|)^2 + 2t\|z_t - y_n\|^2 + 2t\langle u - z_t, \mathcal{J}(z_t - y_n)\rangle \\ &\le \left(1+t^2\right)\|z_t - y_n\|^2 + M\|T_\varepsilon y_n - y_n\| + 2t\langle u - z_t, \mathcal{J}(z_t - y_n)\rangle,\end{aligned} \tag{3.55}$$

where $M > 0$ is a constant such that

$$M > \max\left\{\|z_t - y_n\|^2, 2\|z_t - y_n\| + \|T_\varepsilon y_n - y_n\|\right\}, \quad t \in (0,1),\ n \in \mathbb{N}. \tag{3.56}$$

Then it follows from (3.55) that

$$\langle u - z_t, \mathcal{J}(y_n - z_t)\rangle \le \frac{M}{2}t + \frac{M}{2t}\|T_\varepsilon y_n - y_n\|. \tag{3.57}$$

Taking $\limsup_{n\to\infty}$ yields

$$\limsup_{n\to\infty}\langle u - z_t, \mathcal{J}(y_n - z_t)\rangle \le \frac{M}{2}t. \tag{3.58}$$

Then, letting $t \to 0$ and noting the fact that the duality map $\mathcal{J}$ is norm-to-norm uniformly continuous on bounded sets, we get (3.54) as desired. Due to (3.53), we can apply Lemma 2.5(ii) to (3.52) to conclude that $s_n \to 0$; that is, $y_n \to z$.

Case 2. Suppose that there exists $n_1 \in \mathbb{N}$ such that $s_{n_1} \le s_{n_1+1}$. Let us define

$$I_n := \{n_1 \le k \le n : s_k \le s_{k+1}\}, \quad n \ge n_1. \tag{3.59}$$

Obviously $I_n \ne \emptyset$ since $n_1 \in I_n$ for any $n \ge n_1$. Set

$$\tau(n) = \max I_n. \tag{3.60}$$

Note that the sequence $\{\tau(n)\}$ is nonincreasing and $\lim_{n\to\infty}\tau(n) = \infty$. Moreover, $\tau(n) \le n$ and

$$s_{\tau(n)} \le s_{\tau(n)+1}, \tag{3.61}$$

$$s_n \le s_{\tau(n)+1}, \tag{3.62}$$

for any $n \geq n_1$ (see Lemma 3.1 of Maingé [33] for more details). From inequality (3.47) we get

$$s_{\tau(n)+1} \leq \left(1 - \alpha_{\tau(n)}\right) s_{\tau(n)} - \delta\phi\left(\left\| y_{\tau(n)} - r_{\tau(n)} A y_{\tau(n)} - T_{\tau(n)} y_{\tau(n)} + r_{\tau(n)} A z \right\|\right) \\ - \delta\left\| A y_{\tau(n)} - Az \right\|^q + q\alpha_{\tau(n)} \left\langle u - z, \mathcal{J}_q\left(y_{\tau(n)+1} - z\right)\right\rangle. \tag{3.63}$$

It turns out that

$$\lim_{n\to\infty} \left\| A y_{\tau(n)} - Az \right\| = 0, \qquad \lim_{n\to\infty} \left\| y_{\tau(n)} - r_{\tau(n)} A y_{\tau(n)} - T_{\tau(n)} y_{\tau(n)} + r_{\tau(n)} A z \right\| = 0. \tag{3.64}$$

Consequently,

$$\lim_{n\to\infty} \left\| T_{\tau(n)} y_{\tau(n)} - y_{\tau(n)} \right\| = 0. \tag{3.65}$$

Now repeating the argument of the proof of (3.53) in Case 1, we can get

$$\limsup_{n\to\infty} \left\langle u - z, \mathcal{J}_q\left(y_{\tau(n)} - z\right)\right\rangle \leq 0. \tag{3.66}$$

By the asymptotic regularity of $\{y_{\tau(n)}\}$ and (3.65), we deduce that

$$\lim_{n\to\infty} \left\| y_{\tau(n)+1} - y_{\tau(n)} \right\| = 0. \tag{3.67}$$

This implies that

$$\limsup_{n\to\infty} \left\langle u - z, \mathcal{J}\left(y_{\tau(n)+1} - z\right)\right\rangle \leq 0. \tag{3.68}$$

On the other hand, it follows from (3.64) that

$$s_{\tau(n)+1} - s_{\tau(n)} + \alpha_{\tau(n)} s_{\tau(n)} \leq q\alpha_{\tau(n)} \left\langle u - z, \mathcal{J}_q\left(y_{\tau(n)+1} - z\right)\right\rangle. \tag{3.69}$$

Taking the $\limsup_{n\to\infty}$ in (3.69) and using condition (i) we deduce that $\limsup_{n\to\infty} s_{\tau(n)} \leq 0$; hence $\lim_{n\to\infty} s_{\tau(n)} = 0$. That is, $\|y_{\tau(n)} - z\| \to 0$. Using the triangle inequality,

$$\left\| y_{\tau(n)+1} - z \right\| \leq \left\| y_{\tau(n)+1} - y_{\tau(n)} \right\| + \left\| y_{\tau(n)} - z \right\|, \tag{3.70}$$

we also get that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$ which together with (3.42) guarantees that $\|y_n - z\| \to 0$.

4. Applications

The two forward-backward methods previously studied, (3.29) and (3.41), find applications in other related problems such as variational inequalities, the convex feasibility problem, fixed point problems, and optimization problems.

Throughout this section, let C be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$. Note that in this case the concept of monotonicity coincides with the concept of accretivity.

Regarding the problem we concern, of finding a zero of the sum of two accretive operators in a Hilbert space $\mathcal{H}$, as a direct consequence of Theorem 3.7, we first obtain the following result due to Combettes [34].

Corollary 4.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be monotone and $B : \mathcal{H} \to \mathcal{H}$ maximal monotone. Assume that κA is firmly nonexpansive for some $\kappa > 0$ and that*

(i) $\lim_{n\to\infty}\alpha_n > 0$,

(ii) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\kappa$,

(iii) $\sum_{n=1}^{\infty} \|a_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|b_n\| < \infty$,

(iv) $S := (A+B)^{-1}(0) \ne \emptyset$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_{\lambda_n}((x_n - \lambda_n(Ax_n + a_n)) + b_n) \tag{4.1}$$

converges weakly to a point in S.

Proof. It suffices to show that κA is firmly nonexpansive if and only if A is κ-inverse strongly monotone. This however follows from the following straightforward observation:

$$\langle \kappa Ax - \kappa Ay, x-y\rangle \ge \|\kappa Ax - \kappa Ay\|^2 \Longleftrightarrow \langle Ax - Ay, x-y\rangle \ge \kappa\|Ax - Ay\|^2, \tag{4.2}$$

for all $x, y \in \mathcal{H}$. □

4.1. Variational Inequality Problems

A monotone variational inequality problem (VIP) is formulated as the problem of finding a point $x \in C$ with the property:

$$\langle Ax, z - x\rangle \ge 0, \quad \forall z \in C, \tag{4.3}$$

where $A : C \to \mathcal{H}$ is a nonlinear monotone operator. We shall denote by S the solution set of (4.3) and assume $S \ne \emptyset$.

One method for solving VIP (4.3) is the projection algorithm which generates, starting with an arbitrary initial point $x_0 \in \mathcal{H}$, a sequence $\{x_n\}$ satisfying

$$x_{n+1} = P_C(x_n - rAx_n), \tag{4.4}$$

where r is properly chosen as a stepsize. If in addition A is κ-inverse strongly monotone (ism), then the iteration (4.4) with $0 < r < 2\kappa$ converges weakly to a point in S whenever such a point exists.

By [35, Theorem 3], VIP (4.3) is equivalent to finding a point x so that

$$0 \in (A + B)x, \tag{4.5}$$

where B is the normal cone operator of C. In other words, VIPs are a special case of the problem of finding zeros of the sum of two monotone operators. Note that the resolvent of the normal cone is nothing but the projection operator and that if A is κ-ism, then the set Ω is closed and convex [36]. As an application of the previous sections, we get the following results.

Corollary 4.2. *Let $A : C \to \mathscr{H}$ be κ-ism for some $\kappa > 0$, and let the following conditions be satisfied:*

(i) $\lim_{n\to\infty}\alpha_n > 0$,

(ii) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\kappa$.

Then the sequence $\{x_n\}$ generated by the relaxed projection algorithm

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n A x_n) \tag{4.6}$$

converges weakly to a point in S.

Corollary 4.3. *Let $A : C \to \mathscr{H}$ be κ-ism and let the following conditions be satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \lim_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\kappa$.

Then, for any given $u \in C$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n A x_n), \tag{4.7}$$

converges strongly to $P_S u$.

Remark 4.4. Corollary 4.3 improves Iiduka-Takahashi's result [37, Corollary 3.2], where apart from hypotheses (i)-(ii), the conditions $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$ and $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$ are required.

4.2. Fixed Points of Strict Pseudocontractions

An operator $T : C \to C$ is said to be a strict κ-pseudocontraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{4.8}$$

for all $x, y \in C$. It is known that if T is strictly κ-pseudocontractive, then $A = I - T$ is $(1-\kappa)/2$-ism (see [38]). To solve the problem of approximating fixed points for such operators, an iterative scheme is provided in the following result.

Corollary 4.5. *Let $T : C \to C$ be strictly κ-pseudocontractive with a nonempty fixed point set* $\mathrm{Fix}(T)$. *Suppose that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < \lim_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1-\kappa$.

Then, for any given $u \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n)((1-\lambda_n)x_n + \lambda_n T x_n) \tag{4.9}$$

converges strongly to the point $P_{\mathrm{Fix}(T)}u$.

Proof. Set $A = I - T$. Hence A is $(1-\kappa)/2$-ism. Moreover we rewrite the above iteration as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n). \tag{4.10}$$

Then, by setting B the operator constantly zero, Corollary 4.3 yields the result as desired. □

4.3. Convexly Constrained Minimization Problem

Consider the optimization problem

$$\min_{x\in C} f(x), \tag{4.11}$$

where $f : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function. Assume (4.11) is consistent, and let Ω denote its set of solutions.

The gradient projection algorithm (GPA) generates a sequence $\{x_n\}$ via the iterative procedure:

$$x_{n+1} = P_C(x_n - r\nabla f(x_n)), \tag{4.12}$$

where ∇f stands for the gradient of f. If in addition ∇f is $(1/\kappa)$-Lipschitz continuous; that is, for any $x, y \in \mathcal{H}$,

$$\|\nabla f(x) - \nabla f(y)\| \le \left(\frac{1}{\kappa}\right)\|x-y\|, \tag{4.13}$$

then the GPA with $0 < r < 2\kappa$ converges weakly to a minimizer of f in C (see, e.g, [39, Corollary 4.1]).

The minimization problem (4.11) is equivalent to VIP [40, Lemma 5.13]:

$$\langle \nabla f(x), z - x\rangle \ge 0, \quad z \in C. \tag{4.14}$$

It is also known [41, Corollary 10] that if ∇f is $(1/\kappa)$-Lipschitz continuous, then it is also κ-ism. Thus, we can apply the previous results to (4.11) by taking $A = \nabla f$.

Corollary 4.6. *Assume that $f : \mathcal{H} \to \mathbb{R}$ is convex and differentiable with $(1/\kappa)$-Lipschitz continuous gradient ∇f. Assume also that*

(i) $\lim_{n\to\infty}\alpha_n > 0$,

(ii) $0 \le \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n \le 2\kappa$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C\left(x_n - \lambda_n \nabla f(x_n)\right) \tag{4.15}$$

converges weakly to $x \in \Omega$.

Corollary 4.7. *Assume that $f : \mathcal{H} \to \mathbb{R}$ is convex and differentiable with $(1/\kappa)$-Lipschitz continuous gradient ∇f. Assume also that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \lim_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\kappa$.

Then for any given $u \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\left(x_n - \lambda_n \nabla f(x_n)\right) \tag{4.16}$$

converges strongly to $P_\Omega u$ whenever such point exists.

4.4. Split Feasibility Problem

The split feasibility problem (SFP) [42] consists of finding a point $\hat{x}$ satisfying the property:

$$\hat{x} \in C, \quad A\hat{x} \in Q, \tag{4.17}$$

where C and Q are, respectively, closed convex subsets of Hilbert spaces $\mathcal{H}$ and K and $A : \mathcal{H} \to K$ is a bounded linear operator. The SFP (4.17) has attracted much attention due to its applications in signal processing [42]. Various algorithms have, therefore, been derived to solve the SFP (4.17) (see [39, 43, 44] and reference therein). In particular, Byrne [43] introduced the so-called CQ algorithm:

$$x_{n+1} = P_C\left(x_n - \lambda A^*(I-P_Q)Ax_n\right), \tag{4.18}$$

where $0 < \lambda < 2\nu$ with $\nu = 1/\|A\|^2$.

To solve the SFP (4.17), it is very useful to investigate the following convexly constrained minimization problem (CCMP):

$$\min_{x\in C} f(x), \tag{4.19}$$

where

$$f(x) := \frac{1}{2}\left\|(I-P_Q)Ax\right\|^2. \tag{4.20}$$

Generally speaking, the SFP (4.17) and CCMP (4.19) are not fully equivalent: every solution to the SFP (4.17) is evidently a minimizer of the CCMP (4.19); however a solution to the CCMP (4.19) does not necessarily satisfy the SFP (4.17). Further, if the solution set of the SFP (4.17) is nonempty, then it follows from [45, Lemma 4.2] that

$$C \cap (\nabla f)^{-1}(0) \neq \emptyset, \tag{4.21}$$

where f is defined by (4.20). As shown by Xu [46], the CQ algorithm need not converge strongly in infinite-dimensional spaces. We now consider an iteration process with strong convergence for solving the SFP (4.17).

Corollary 4.8. *Assume that the SFP* (4.17) *is consistent, and let S be its nonempty solution set. Assume also that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*

(ii) $0 < \lim_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 2\nu$*.*

Then for any given $u \in C$, the sequence (x_n) generated by the algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\left[x_n - \lambda_n A^*(I-P_Q)Ax_n\right] \tag{4.22}$$

converges strongly to the solution $P_S u$ of the SFP (4.17)*.*

Proof. Let f be defined by (4.19). According to [39, page 113], we have

$$\nabla f = A^*(I-P_Q)A, \tag{4.23}$$

which is $(1/\nu)$-Lipschitz continuous with $\nu = 1/\|A\|^2$. Thus Corollary 4.7 applies, and the result follows immediately. □

Remark 4.9. Corollary 4.8 improves and recovers the result of [44, Corollary 3.7], which uses the additional condition $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, condition (i), and the special case of condition (ii) where $\lambda_n \equiv \lambda$ for all $n \in \mathbb{N}$.

4.5. Convexly Constrained Linear Inverse Problem

The constrained linear system

$$\begin{gathered} Ax = b \\ x \in C, \end{gathered} \tag{4.24}$$

where $A : \mathcal{H} \to K$ is a bounded linear operator and $b \in K$, is called convexly constrained linear inverse problem (cf. [47]). A classical way to deal with this problem is the well-known projected Landweber method (see [40]):

$$x_{n+1} = P_C\left[x_n - \lambda A^*(Ax_n - b)\right], \tag{4.25}$$

where $0 < \lambda < 2\nu$ with $\nu = 1/\|A\|^2$. A counterexample in [8, Remark 5.12] shows that the projected Landweber iteration converges weakly in infinite-dimensional spaces, in general. To get strong convergence, Eicke introduced the so-called damped projection method (see [47]). In what follows, we present another algorithm with strong convergence, for solving (4.24).

Corollary 4.10. *Assume that* (4.24) *is consistent. Assume also that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \lim_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\nu$.

Then, for any given $u \in \mathcal{H}$, *the sequence* $\{x_n\}$ *generated by the algorithm*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C[x_n - \lambda_n A^*(Ax_n - b)] \tag{4.26}$$

converges strongly to a solution to problem (4.24) *whenever it exists.*

Proof. This is an immediate consequence of Corollary 4.8 by taking $Q = \{b\}$. □

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