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FIXED POINTS AND BOUNDARY BEHAVIOUR OF THE KOENIGS FUNCTION

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Abstract. We analyze the relationship between the fixed points of different iterates of an analytic self-map of the unit disk. We show that, in general, a boundary fixed point of such a function is not a fixed point of its iterates. However, in the context of fractional iteration, all the iterates have the same fixed points. We also present results, in terms of the Koenigs function, of self-maps whose behaviour are not so extreme as above.

1. Introduction and statement of the results

1.1. Let φ be a holomorphic map in the unit disc $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$. A point $a \in \partial\mathbf{D}$ is called a *boundary contact point* of φ, if the non-tangential or angular limit $L := \angle\lim_{z\to a}\varphi(z)$ lies in $\partial\mathbf{D}$. Moreover, boundary contact points have multipliers. That is, and also in the non-tangential sense, φ always has a derivative $\varphi'(a) \in \mathbf{C}_\infty\backslash\{0\}$ at that point a. If $L = a$, the point a is called a *boundary fixed point* of φ and, in this case, $\varphi'(a) \in (0,+\infty)\cup\{\infty\}$.

In what follows, a fixed point of φ will mean a fixed point in the classical sense ($\varphi(a) = a$ with $a \in \mathbf{D}$) or a boundary fixed point. The famous Denjoy–Wolff point (DW-point) of every non-trivial φ will be denoted by τ_φ or, when there is no confusion, simply by τ. The study of fixed points and boundary contact points is one of the central topics in iteration theory in the unit disk (see [8], [10], [11]) as well as in those related mathematical branches like composition operators. It is worth mentioning that boundary contact points are playing a more and more important role in many situations (see [2], [3] and the references therein).

This paper is about the collection of the boundary fixed points and the boundary contact points of the *different iterates* φ_n of the function φ. Of course, many

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things are known, so we are trying to determine clearly what our contributions are. Almost all of our results assume that φ has an inner DW-point ($\tau \in \mathbf{D}$ and $|\varphi'(\tau)| \neq 1$). We consider the case of a boundary DW-point ($\tau \in \partial\mathbf{D}$) only in Theorem 5. To be precise with the terminology, we recall that φ is usually said to have an elliptic DW-point, whenever φ is not trivial, $\tau \in \mathbf{D}$ and $|\varphi'(\tau)| = 1$.

So, suppose that φ has an inner DW-point. If $a \in \partial\mathbf{D}$ is a boundary fixed point of φ with $\varphi'(a) \neq \infty$, by [13, p. 80], the curve $r \in [0,1) \mapsto \varphi(ra) \in \mathbf{D}$ tends non-tangentially to a and, according to the Lehto–Virtanen theorem [13, Chapter 4], we get that a is also a boundary fixed point of φ_n, for all $n \in \mathbf{N}$. On the other hand, if a is a boundary fixed point of φ_n with finite derivative, we can only claim that a is a boundary contact point of φ. In other words, and under certain regularity assumptions, boundary fixed points "grow" in the forward direction. In spite of several ambiguous comments elsewhere, we have to say that the hypothesis $\varphi'(a) \neq \infty$ is crucial there. In fact, in Section 3, we provide an example (Example 1) of a *univalent* holomorphic map $\varphi\colon \mathbf{D} \to \mathbf{D}$ having a boundary fixed point $a \in \partial\mathbf{D}$ such that a is not even a boundary contact point of φ_2. Necessarily, $\varphi'(a) = \infty$.

1.2. As one may expect, we can go further in the framework of fractional iteration. Some words are in order. We recall that a semigroup of analytic functions (φ_t) is a family (indexed by the non-negative real numbers) of holomorphic self-maps of the unit disk, satisfying the following three conditions:

(a) φ_0 is the identity in $\mathbf{D}$,
(b) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
(c) for every $z \in \mathbf{D}$, $\lim_{t\to 0} \varphi_t(z) = z$.

The fractional iterates φ_t are always univalent. Moreover, if one of the iterates φ_t ($t > 0$) has an inner (resp. boundary) DW-point, all the φ_t ($t > 0$) have an inner (resp. boundary) DW-point and, indeed, all of the DW-points are the same [15]. In that case, we say that (φ_t) is a semigroup with inner (resp. boundary) DW-point. There are other two types of semigroups (trivial and elliptic) but we remit to the literature for more information.

Theorem 1. *Let (φ_t) be a semigroup of analytic functions with inner DW-point and $a \in \partial\mathbf{D}$. Then, the point a is a boundary fixed point of φ_t for some $t > 0$ if and only if it is a boundary fixed point for all the iterates φ_t.*

In particular, if a holomorphic mapping $\varphi\colon \mathbf{D} \to \mathbf{D}$ with inner DW-point can be embedded into a semigroup of analytic functions, then all the iterates of φ have the same collection of boundary fixed points. This theorem was also enunciated by Cowen in the relevant paper [7]. Unfortunately, Cowen's proof uses that a boundary fixed point of a holomorphic map $\varphi\colon \mathbf{D} \to \mathbf{D}$ is also a boundary fixed point of φ_2, which we know to be incorrect in general. Anyway, our approach is different and, perhaps, clearer.

Boundary fixed points can be characterized by some properties of the classical Koenigs function, also in the context of fractional iteration. The famous result of Koenigs [14, Section 6.1] asserts the existence of a unique holomorphic map $\sigma\colon \mathbf{D}\to\mathbf{C}$ such that

$$\sigma\circ\varphi = \varphi'(\tau)\sigma$$

with $\sigma'(\tau)=1$, for every holomorphic self-map φ of $\mathbf{D}$ with inner DW-point and $\varphi'(\tau)\neq 0$. This map σ is called the Koenigs function associated to φ and σ is univalent, whenever φ is. It is possible to prove that if (φ_t) is a semigroup with inner DW-point, all the Koenigs functions associated to φ_t $(t>0)$ coincide, so we can talk about *the* Koenigs function of the semigroup (φ_t).

Theorem 2. *Let (φ_t) be a semigroup of analytic functions with inner DW-point and let σ be the corresponding Koenigs function. Assume that $a\in\partial\mathbf{D}$. Then the following are equivalent*:

(1) *The point a is a boundary fixed point of φ_t for some (resp. all) $t>0$.*
(2) *The radial limit $\lim_{r\to 1^-}\sigma(ra)$ exists and is ∞.*
(3) *The unrestricted limit $\lim_{z\to a}\sigma(z)$ exists and is ∞. That is, the function σ admits a continuous extension from $\mathbf{D}\cup\{a\}$ into $\mathbf{C}_\infty$, where we assign $\sigma(a):=\infty$.*

In terms of prime end theory and denoting the corresponding Carathéodory map by $\hat\sigma$, we notice that the above statement two says that $\hat\sigma(a)$ is an accessible prime end and the statement three that the impression of $\hat\sigma(a)$ is singleton. This theorem is false outside of the fractional iteration world. In fact, we can show, see Example 2, a *univalent* function verifying statement two but failing the first one and another *univalent* function, see Example 3, satisfying the second but not the third condition.

Corollary 3. *Let (φ_t) be a semigroup of analytic functions with inner DW-point and $a\in\partial\mathbf{D}$ a boundary fixed point of φ_t for some (resp. all) $t>0$. Then, each fractional iterate φ_t admits a continuous extension from $\mathbf{D}\cup\{a\}$ into $\mathbf{D}\cup\{a\}$, where $\varphi_t(a):=a$.*

Apparently, if we want a more general (univalent) version of the last theorem we have to weaken the hypothesis of being a boundary fixed point. As we will see, the concept of boundary contact point will be really useful for this task.

We point out that, as a consequence of the Julia–Carathéodory theorem, if $a\in\partial\mathbf{D}$ is a boundary contact point of some φ_n with $\varphi_n'(a)\neq\infty$, then a is also a boundary contact point of each iterate φ_k, $k<n$. Thus, once more under certain regularity assumptions, boundary contact points "grow" in the backward direction. For fractional iteration, there is also a strong relation between boundary fixed points and boundary contact points. Here, the dynamical concept of ω-limit appears in a very natural way. We want to point out that a point $\xi\in\mathbf{C}_\infty$ is

called an ω-limit point of a curve $\gamma\colon [0,1) \to \mathbf{C}$ if there exists a strictly increasing sequence $(r_n) \subset [0,1)$ convergent to 1 such that $\gamma(r_n) \to \xi$. The set of all ω-limit points of γ is called its ω-limit and it is denoted by $\omega(\gamma)$.

Theorem 4. *Let (φ_t) be a semigroup of analytic functions with inner DW-point and $a \in \partial\mathbf{D}$. Then the following are equivalent:*

(1) *The point a is a boundary fixed point of φ_t for some (resp. all) $t > 0$.*
(2) *The point a is a boundary contact point of φ_t for all $t > 0$.*
(3) *There is $t > 0$ such that the point a is a boundary contact point of φ_{nt}, for every $n \in \mathbf{N}$.*
(4) *For all $t > 0$, the ω-limit of the curve*
$$r \in [0,1) \longrightarrow \varphi_t(ra) \in \mathbf{D}$$
is completely included in $\partial\mathbf{D}$.
(5) *For some $t > 0$ and every $n \in \mathbf{N}$, the ω-limit of the curves*
$$r \in [0,1) \longrightarrow \varphi_{nt}(ra) \in \mathbf{D}$$
are completely included in $\partial\mathbf{D}$.

In general, the above analysis cannot be quickly translated to the DW-boundary context. Clearly, new phenomena appear and it deserves a proper and further study. Anyway, we want to point out that our initial theorem also holds in this situation.

Theorem 5. *Let $\Phi = (\varphi_t)$ be a semigroup of analytic functions with boundary DW-point and $a \in \partial\mathbf{D}$. Then, the point a is a boundary fixed point of φ_t for some $t > 0$ if and only if it is a boundary fixed point for all φ_t.*

Obviously, we can drop "boundary" above since fixed points of fractional iterates of semigroups with boundary DW-point are all of them contained in $\partial\mathbf{D}$.

This theorem was also stated by Cowen in [7]. However, the same remarks given for the inner DW-point case are still valid here.

1.3. Now we leave the fractional iteration and give the univalent version of Theorems 2 and 3. Due to the undoubtedly dynamical aspect of the equivalences, we have decided to give an initial version with "general" curves and, after that, to show the corresponding corollary with "radial" curves.

Theorem 6. *Let $\varphi\colon \mathbf{D} \to \mathbf{D}$ be a* univalent *function with inner DW-point and let σ be the corresponding Koenigs function. Likewise, let $\gamma\colon [0,1) \to \mathbf{D}$ be a curve. Then, the following statements are equivalent.*

(1) *For all $n \in \mathbf{N}$, the limit $\lim_{r\to 1} \varphi_n\bigl(\gamma(r)\bigr)$ exists and lies in $\partial\mathbf{D}$.*
(2) *For all $n \in \mathbf{N}$, the ω-limit of the curve $\varphi_n \circ \gamma\colon [0,1) \to \mathbf{D}$ is completely contained in $\partial\mathbf{D}$.*
(3) *The ω-limit of the curve $\sigma\circ\gamma\colon [0,1) \to \mathbf{C}$ is ∞. That is, $\lim_{r\to 1} \sigma\bigl(\gamma(r)\bigr) = \infty$.*

Corollary 7. *Let* $\varphi\colon \mathbf{D}\to\mathbf{D}$ *be a* univalent *function with inner DW-point and let* σ *be the corresponding Koenigs function. Assume that* $a\in\partial\mathbf{D}$. *Then, the following statements are equivalent.*

(1) *The point* a *is a boundary contact point of* φ_n, *for every* $n\in\mathbf{N}$.
(2) *For all* $n\in\mathbf{N}$, *the* ω*-limit of the curve* $r\in[0,1)\to\varphi_n(ra)\in\mathbf{D}$ *is completely contained in* $\partial\mathbf{D}$.
(3) *The radial limit* $\lim_{r\to 1^-}\sigma(ra)$ *exists and is* ∞.

1.4. It is quite natural to ask about what can happen in the non-univalent situation. Different examples unequivocally tell us that, in this context, everything is much more complicated and, roughly speaking, they suggest replacing the verb "to contain" by "to touch" in the corresponding assertions.

Anyway, it is worth mentioning that it is not possible to replace in the above corollary "$\omega(\varphi_n\circ\gamma)\subset\partial\mathbf{D}$, for all n" by "$\omega(\varphi_n\circ\gamma)\bigcap\partial\mathbf{D}\neq\emptyset$, for all n" (see Example 4).

Theorem 8. *Let* $\varphi\colon \mathbf{D}\to\mathbf{D}$ *be a function with inner DW-point* τ *and* $\varphi'(\tau)\neq 0$, *and let* σ *be the corresponding Koenigs function. Assume that* $a\in\partial\mathbf{D}$ *and let* $\gamma\colon[0,1)\to\mathbf{D}$ *be a curve such that* $\omega(\gamma)=\{a\}$. *If* a *is a boundary contact point for every iterate of* φ, *then the following two equivalent statements hold*:

(1) *For every* $n\in\mathbf{N}$, *the intersection* $\omega(\varphi_n\circ\gamma)\bigcap\partial\mathbf{D}$ *is not empty.*
(2) *The* ω*-limit of the curve* $\sigma\circ\gamma\colon[0,1)\to\mathbf{C}$ *contains* ∞.

Even for radial curves, the behaviour of the just mentioned ω-limit of the curve $\sigma\circ\gamma$ can be very "pathological" in a certain sense. In fact, in Section 3, we present a non-univalent holomorphic function $\varphi\colon\mathbf{D}\to\mathbf{D}$ having the point 1 as a boundary fixed point and 0 as the DW-point with $0<|\varphi'(0)|<1$, such that the ω-limit of the curve $r\in[0,1)\to\sigma(r)\in\mathbf{C}$ is a compact connected subset of $\mathbf{C}_\infty$ including ∞ and the DW-point 0 (see Example 5).

Thinking about the meaning, in this non-univalent framework, of having a non-common boundary contact point for the iterates of φ, we have arrived at an extreme dichotomy for the behaviour of the ω-limits of $\varphi_n\circ\gamma$, where γ is curve in $\mathbf{D}$ tending to the corresponding boundary fixed point of φ. In a certain sense, this complements the former theorem.

Theorem 9. *Let* $\varphi\colon \mathbf{D}\to\mathbf{D}$ *be a function with inner DW-point* τ *and* $\varphi'(\tau)\neq 0$, *and let* σ *be the corresponding Koenigs function. Let* $\gamma\colon[0,1)\to\mathbf{D}$ *be a curve. Then, only one of the two following conditions is satisfied*:

(i) *The* ω*-limit of the curve* $\sigma\circ\gamma\colon[0,1)\to\mathbf{C}$ *does not contain* ∞ *and*

$$\sup\bigl\{|\varphi_n\circ\gamma(t)-\tau| : t\in[0,1)\bigr\}\overset{n\to\infty}{\longrightarrow}0.$$

(ii) *The* ω*-limit of the curve* $\sigma\circ\gamma\colon[0,1)\to\mathbf{C}$ *contains* ∞ *and* $\omega(\varphi_n\circ\gamma)\bigcap\partial\mathbf{D}\neq\emptyset$, *for every* $n\in\mathbf{N}$.

It is worth mentioning that there are functions φ such that the associated Koenigs function touches the point ∞ for every radial limit (see Example 6).

2. Proofs of the results

In order not to repeat arguments, we will give the proofs in a different order.

Proof of Theorem 6. Let Ω be equal to $\sigma(\mathbf{D})$. Denote $\lambda = \varphi'(\tau)$. Since σ is univalent we have that $\lambda \neq 0$. If the limit $\lim_{r\to 1} \varphi_n(\gamma(r)) = \eta$ exists, then we have that $\omega(\varphi_n \circ \gamma)$ is only the single point η. So, it is clear that (1) implies (2).

Let us see that (2) implies (3). We know that $\omega(\sigma \circ \gamma) \subset \mathbf{C}_\infty$. Suppose that there is a point $w \in \omega(\sigma \circ \gamma) \cap \mathbf{C}$. Since $w \in \mathbf{C}$, $0 \in \Omega$ (which is a open set), and $\lambda \in \mathbf{D}$ there is a natural number n such that $\lambda^n w \in \Omega$. Moreover, $w \in \omega(\sigma \circ \gamma)$. That is, there is a sequence (r_m) in the interval $(0,1)$ converging to 1 such that $\sigma\big(\gamma(r_m)\big) \to w$. Therefore, bearing in mind that σ is univalent, we have

$$\varphi_n\big(\gamma(r_m)\big) = \sigma^{-1}\big(\lambda^n \sigma\big(\gamma(r_m)\big)\big) \stackrel{m\to\infty}{\longrightarrow} \sigma^{-1}(\lambda^n w) \in \mathbf{D}.$$

That is, $\omega(\varphi_n \circ \gamma) \cap \mathbf{D} \neq \emptyset$. A contradiction.

Finally, we see that (3) implies (1). Fix a natural number n and consider the curve $r \in [0,1) \mapsto \lambda^n \sigma\big(\gamma(r)\big)$. By hypothesis, $\lim_{r\to 1} \lambda^n \sigma\big(\gamma(r)\big) = \infty$. Since σ is univalent it follows from [14, p. 162] that

$$\varphi_n\big(\gamma(r)\big) = \sigma^{-1}\big(\lambda^n \sigma\big(\gamma(r)\big)\big)$$

tends to a limit as $r \to 1$, and this limit lies in $\partial\mathbf{D}$ because σ is finite in $\mathbf{D}$. □

Proof of Theorems 1, 2 *and* 4. Let Ω be equal to $\sigma(\mathbf{D})$. To fix the notation, we have that there is c with $\operatorname{Re} c > 0$ such that $\varphi_t(z) = \sigma^{-1}\big(e^{-ct}\sigma(z)\big)$ for all $t \geq 0$ and $z \in \mathbf{D}$.

To prove these three theorems, we have to get that the following eight assertions are equivalent:

(i) The point a is a boundary fixed point of φ_t for some $t > 0$.
(ii) The point a is a boundary fixed point of φ_t for all $t > 0$.
(iii) The unrestricted limit $\lim_{z\to a} \sigma(z)$ exists and is ∞.
(iv) The radial limit $\lim_{r\to 1^-} \sigma(ra)$ exists and is ∞.
(v) The point a is a boundary contact point of φ_t for all $t > 0$.
(vi) There is $t > 0$ such that the point a is a boundary contact point of φ_{nt}, for every $n \in \mathbf{N}$.
(vii) For all $t > 0$, the ω-limit of the curve

$$r \in [0,1) \longrightarrow \varphi_t(ra) \in \mathbf{D}$$

is completely included in $\partial\mathbf{D}$.

(viii) For some $t>0$ and every $n\in\mathbf{N}$, the ω-limits of the curves

$$r\in[0,1)\longrightarrow\varphi_{nt}(ra)\in\mathbf{D}$$

are completely included in $\partial\mathbf{D}$.

First of all, bearing in mind that all functions of the semigroup have the same Koenigs function, by Theorem 6, we have that (iv), (v), (vi), (vii), and (viii) are equivalent. Moreover, it is obvious that (ii) implies (v), (ii) implies (i), and that (iii) implies (iv). To close the cycle we are going to prove that

(iv) implies (iii), (iv) implies (ii), and (i) implies (vi).

(iv) implies (iii). To simplify this implication we introduce some notation. Given $c\in\mathbf{C}$ with $\operatorname{Re}c>0$ and $w\in\mathbf{C}$, $w\neq 0$, we define the spiral $\operatorname{spir}_c[w]=\{e^{-cs}w:s\in\mathbf{R}\}\bigcup\{0\}\bigcup\{\infty\}$; given real numbers $s<t$, we define the spiral segment $\operatorname{spir}_c[e^{-s}w,e^{-t}w]$ as the subarc of $\operatorname{spir}_c[w]$ that goes from $e^{-s}w$ to $e^{-t}w$; finally, $\operatorname{spir}_c[\infty,w]=\{e^{-cs}w:s\leq 0\}\bigcup\{\infty\}$.

Take a zero-chain (C_m) in $\mathbf{D}$ converging to a such that $\big(\sigma(C_m)\big)$ is a zero-chain in Ω such that $\hat\sigma(a)=\big[\sigma(C_m)\big]$. To get (iii) we have to prove that the impression of the prime end $\hat\sigma(a)$ is the single point ∞. By [5, Lemma 3.3] the corresponding impression $I(\hat\sigma(a))\subset\partial_\infty\Omega$ must be of one of the following types:

(a) $I\big(\hat\sigma(a)\big)$ is a single point.
(b) There is $\lambda\in\partial\mathbf{D}$ and $\eta_1<\eta_2$ such that $I\big(\hat\sigma(a)\big)=\operatorname{spir}_c[e^{-\eta_1 c}\lambda,e^{-\eta_2 c}\lambda]$.
(c) There is $\lambda\in\partial\mathbf{D}$ and $\eta\in\mathbf{R}$ such that $I\big(\hat\sigma(a)\big)=\operatorname{spir}_c[\infty,e^{-\eta c}\lambda]$.

Since $\lim_{r\to1}\sigma(ra)=\infty$, we have that $\infty\in I\big(\hat\sigma(a)\big)$. Therefore, the possibility (b) cannot occur. If (c) is satisfied, then $\lim_m w_m$ always exists and it is equal to $\lambda e^{-\eta c}$ whenever $w_m\in\sigma(C_m)$ for all $m\in\mathbf{N}$ (see the proof of [5, Theorem 1.2. (2) implies (3)]). Moreover, for each m, there is $r_m\in(0,1)$ such that $r_m a\in C_m$. So, $\lim_m\sigma(r_m a)=e^{-\eta c}$. A contradiction because the sequence (r_m) must tend to 1 and, by hypothesis, $\lim_m\sigma(r_m a)=\infty$. Therefore, $I\big(\hat\sigma(a)\big)$ is a single point and, again using that $\infty\in I\big(\hat\sigma(a)\big)$ we get that $I\big(\hat\sigma(a)\big)=\{\infty\}$.

(iv) implies (ii). Let us fix $t>0$. We have to show that

$$\lim_{r\to1}\varphi_t(ra)=\lim_{r\to1}\sigma^{-1}\big(e^{-ct}\sigma(ra)\big)=a.$$

First of all, notice that by [14, p. 162], the limit $\lim_{r\to1}\sigma^{-1}\big(e^{-ct}\sigma(ra)\big)$ does exist. So we only have to prove that the limit is equal to a.

Clearly, we have that $\lim_{r\to1}\varphi_0(ra)=a$. Take an increasing sequence (r_n) tending to 1 with $r_0=0$. For each n, define the following curves

$$\gamma_n:r\in\big[r_{n-1},r_n\big]\longmapsto\gamma_n(r):=\sigma(ra),$$
$$\eta_n:s\in\big[0,t\big]\longmapsto\eta_n(s):=e^{-cs}\sigma(r_n a).$$

Since Ω is c-spirallike, the curve η_n is in Ω. Now we build the curve

$$\Gamma = \bigoplus_{n=1}^{\infty} (\gamma_{2n-1} + \eta_{2n-1} + e^{-ct}\gamma_{2n} + \eta_{2n}^{-}).$$

Notice that Γ is a curve in Ω joining 0 to ∞. So, by [14, p. 162], we have that there is $\omega \in \partial\mathbf{D}$ such that $\lim_{w\in\Gamma, w\to\infty} \sigma^{-1}(w) = \omega$. On the one hand, the points $\sigma(r_{2n-1}a) \in \gamma_{2n-1} \subset \Gamma$ and the sequence $\big(\sigma(r_{2n-1}a)\big)$ tends to ∞. So, $\omega = \lim_{n\to\infty} \sigma^{-1}\big(\sigma(r_{2n-1}a)\big) = \lim_{n\to\infty} r_{2n-1}a = a$. On the other hand, the points $e^{-ct}\sigma(r_{2n-1}a) \in \eta_{2n-1} \subset \Gamma$ and the sequence $\big(e^{-ct}\sigma(r_{2n-1}a)\big)$ tends to ∞. So,

$$\omega = \lim_{n\to\infty} \sigma^{-1}\big(e^{-ct}\sigma(r_{2n-1}a)\big) = \lim_{n\to\infty} \varphi_t(r_{2n-1}a).$$

Therefore, $\lim_{n\to\infty} \varphi_t(r_{2n-1}a) = a$. Since the limit $\lim_{r\to 1} \varphi_t(ra)$ does exist, we have that it must be a.

(i) implies (vi). First of all, notice that, since Ω is c-spirallike, by [9, p. 431], the limit $\lim_{r\to 1} \sigma(r\xi)$ does exist and belong to $\mathbf{C} \bigcup \{\infty\}$ for all $\xi \in \partial\mathbf{D}$. In particular, we have that the limit $\lim_{r\to 1} e^{-cnt}\sigma(r\xi)$ exists for all n and all ξ. On the one hand, if this limit belongs to Ω, then the continuity of σ^{-1} in Ω implies that the limit $\lim_{r\to 1} \varphi_{nt}(r\xi) = \lim_{r\to 1} \sigma^{-1}\big(e^{-cnt}\sigma(r\xi)\big)$ exists. On the other hand, if this limit belongs to $\partial_\infty\Omega$, by [14, p. 162], the limit $\lim_{r\to 1} \varphi_{nt}(r\xi) = \lim_{r\to 1} \sigma^{-1}\big(e^{-cnt}\sigma(r\xi)\big)$ exists.

Summing up, we have that for all n and for all ξ, the limit $\lim_{r\to 1} \varphi_{nt}(r\xi)$ exists. It is worth pointing out that we want to prove (vi) and, at this moment, we only know that the limit $\lim_{r\to 1} \varphi_{nt}(ra)$ exists for all n (notice that we have not used the hypothesis (i) to get this preliminary fact) but we do not know if these limits belong to the boundary of the unit disc.

Now, we are going to get this last assertion. Namely, we show that

$$\lim_{r\to 1} \varphi_{nt}(ra) = a \quad \text{for all } n.$$

We argue by induction. By hypothesis, we have that $\lim_{r\to 1} \varphi_t(ra) = a$. Now suppose that $\lim_{r\to 1} \varphi_{nt}(ra) = a$. Then, by the Lehto–Virtanen theorem [13, Chapter 4], we have that

$$\lim_{r\to 1} \varphi_{(n+1)t}(ra) = \lim_{r\to 1} \varphi_t\big(\varphi_{nt}(ra)\big) = \lim_{z\to a,\, z\in\varphi_{nt}([0,1)a)} \varphi_t(z) = \lim_{r\to 1} \varphi_t(ra) = a. \ \square$$

Proof of Corollary 3. By Theorem 2, we have that the function σ has a continuous extension to the point a. What we are going to do is *to pass this continuous extension property* from σ to φ_t.

So, fix $t > 0$ and let

$$U_k = \big\{z \in \mathbf{D} : |z - a| < 1/k\big\}.$$

We have to show that $\operatorname{diam}\varphi_t(U_k) \to 0$ as $k \to \infty$. Suppose this is false. Then, for some subsequence, there are Jordan arcs C_{k_n} in $\varphi_t(U_{k_n})$ with $\operatorname{diam}C_{k_n} > b > 0$. We have

$$\sigma(C_{k_n}) \subset \sigma\bigl(\varphi_t(U_{k_n})\bigr) = e^{-ct}\sigma(U_{k_n}) \overset{n\to\infty}{\longrightarrow} \infty$$

by Theorem 2. Hence (C_{k_n}) is a sequence of Koebe arcs [12, p. 267] for the function σ. This is a contradiction because a univalent function has no Koebe arcs. □

Proof of Theorem 5. Denote $\tau \in \partial\mathbf{D}$ the DW-point of the semigroup. Then there is a *unique* univalent function $\sigma\colon \mathbf{D} \to \mathbf{C}$ with $\sigma(0) = 0$ verifying the property

$$(***) \qquad \Omega + t \subset \Omega, \quad \text{for each } t > 0, \text{ where } \Omega := \sigma(\mathbf{D}),$$

and such that

$$\varphi_t(z) = \tau\sigma^{-1}\bigl(\sigma(\overline{\tau}z) + t\bigr), \quad t \ge 0,\ z \in \mathbf{D}$$

(see [1] or [15]). Fix $t_0 > 0$ such that a is a fixed point for φ_{t_0}. We claim that the following properties hold:

(a) For all $\xi \in \partial\mathbf{D}$, the limit $\lim_{r\to1}\sigma(r\xi)$ exists and belongs to $\partial_\infty\Omega$;
(b) For all $\xi \in \partial\mathbf{D}$, the limit $\lim_{r\to1}\varphi_{t_0}(r\xi)$ exists;
(c) a is a fixed point of φ_{nt_0} for all n;
(d) $\lim_{r\to1}\sigma(ra) = \infty$.

Once we know that (d) is satisfied we conclude the proof arguing as in the proof of (iv) implies (ii) above.

Let us check (a). In [4], K. Ciozda obtained that, given a univalent function σ satisfying $(***)$, there is $\alpha \in \bigl[-\frac12\pi, \frac12\pi\bigr]$, such that

$$\operatorname{Re} e^{i\alpha}(1-z)^2\sigma'(z) \ge 0 \quad \text{for all } z \in \mathbf{D}.$$

In particular, denoting $g(z) = z/(1-z)$, we have that

$$\operatorname{Re}\frac{e^{i\alpha}\sigma'(z)}{g'(z)} \ge 0 \quad \text{for all } z \in \mathbf{D}.$$

Since g is convex, we see that the function $e^{i\alpha}\sigma$ is close-to-convex with associated function g. Notice that $g(\mathbf{D}) = \bigl\{w \in \mathbf{C} : \operatorname{Re} w > -\frac12\bigr\}$. So,

$$\partial g(\mathbf{D}) = \Bigl\{w \in \mathbf{C} : \operatorname{Re} w = -\frac12\Bigr\}$$

and we parametrize this curve by

$$w(t) = -\frac12 + \frac{\sin t}{2(1-\cos t)}i$$

for all $0 \le t \le 2\pi$. Take

$$\beta(t) = \begin{cases} \lim_{\tau\to t^+} \arg\big[w(\tau) - w(t)\big] & \text{if } w(t) \neq \infty, \\ \lim_{\tau\to t^+} \arg w(\tau) + \pi & \text{if } w(t) = \infty. \end{cases}$$

On the one hand, since the function $t \in (0, 2\pi) \mapsto \sin t/2(1-\cos t)$ is increasing, we get that $\beta(t) = \frac{3}{2}\pi$ for all $t \in (0, 2\pi)$. On the other hand,

$$\beta(0) = \lim_{\tau\to 0^+} \arg w(\tau) + \pi = \lim_{\tau\to 0^+} \arg\left[-\frac{1}{2} + \frac{\sin\tau}{2(1-\cos\tau)} i\right] + \pi = \frac{1}{2}\pi + \pi = \frac{3}{2}\pi.$$

Therefore, by [13, Theorem 3.21], the limit $\lim_{r\to 1} \sigma(r\xi)$ does exist and belongs to $\mathbf{C}\bigcup\{\infty\}$ for all $\xi \in \partial\mathbf{D}$. The univalence of σ implies that this limit belongs to $\partial_\infty\Omega$.

Now we obtain (b). On the one hand, if this limit belongs to Ω, then the continuity of σ^{-1} in Ω implies that the limit $\lim_{r\to 1} \varphi_{nt_0}(r\xi) = \lim_{r\to 1} \sigma^{-1}(\sigma(r\xi) + nt_0)$ exists. On the other hand, if this limit belongs to $\partial_\infty\Omega$, by [14, p. 162], the limit $\lim_{r\to 1} \varphi_{nt_0}(r\xi) = \lim_{r\to 1} \sigma^{-1}\big(\sigma(r\xi) + nt_0\big)$ exists. Summing up, we have that for all n the limit $\lim_{r\to 1} \varphi_{nt_0}(ra)$ exists.

To get (c), we argue by induction. We have to see that $\lim_{r\to 1} \varphi_{nt_0}(ra) = a$ for all n. By hypothesis, we have that $\lim_{r\to 1} \varphi_{t_0}(ra) = a$. Now suppose that $\lim_{r\to 1} \varphi_{nt_0}(ra) = a$. Then, by the Lehto–Virtanen theorem [13, Chapter 4], we have that

$$\begin{aligned} \lim_{r\to 1} \varphi_{(n+1)t_0}(ra) &= \lim_{r\to 1} \varphi_{t_0}\big(\varphi_{nt_0}(ra)\big) \\ &= \lim_{z\to a,\, z\in\varphi_{nt_0}([0,1)a)} \varphi_{t_0}(z) = \lim_{r\to 1} \varphi_{t_0}(ra) = a. \end{aligned}$$

Finally, we deduce (d). Denote $\xi = \lim_{r\to 1} \sigma(ra)$ (this limit does exist by (a)). Let us see that $\xi = \infty$. Suppose on the contrary that $\xi \neq \infty$. Then there are two possibilities: either there is n such that $\xi + nt_0 \in \Omega$ or $\xi + st_0 \in \partial\Omega$ for all $s > 0$. In the first case, $a = \lim_{r\to 1} \varphi_{nt_0}(ra) = \lim_{r\to 1} \sigma^{-1}\big(\sigma(ra) + nt_0\big) \in \Omega$. A contradiction. In the second case, we have that the half-line $\{\xi + s : s > 0\}$ is included in the boundary of Ω. Moreover, we have one of the following two situations:

$$\operatorname{Im}\sigma(rb) < \operatorname{Im}\xi \text{ for all } r \text{ or } \operatorname{Im}\sigma(rb) > \operatorname{Im}\xi \text{ for all } r.$$

Suppose we have that $\operatorname{Im}\sigma(rb) < \operatorname{Im}\xi$ for all r. Then, there is $c \in \mathbf{R}$, such that the half-strip

$$\big\{w \in \mathbf{C} : 0 < \operatorname{Im} w < \operatorname{Im}\xi,\ \operatorname{Re} w > c\big\}$$

is included in $\Omega^{\#}$.This implies that the points of the half-line $\{\xi + s : s > 0\}$ are simple boundary points and, by [6, Theorem 14.5.12(b)], the function σ^{-1} has a continuous one-to-one extension to $\Omega\bigcup\{\xi + s : s > 0\}$. But we know that $\sigma^{-1}(\xi + nt_0) = a$ for all n. A contradiction. So, ξ cannot be different from ∞. □

Proof of Theorem 9. Without loss of generality, we may assume that $\tau = 0$. Denote $\lambda = \varphi'(0)$. First, assume that $r := \sup_{z\in\gamma} |\varphi_m(z)| < 1$ for some natural number m. If $z \in \gamma$ and $n > m$, then

$$|\varphi_n(z)| = \big|\varphi_{n-m}\big(\varphi_m(z)\big)\big| \le \sup_{|\zeta|\le r} |\varphi_{n-m}(\zeta)|.$$

Since 0 is the Denjoy–Wolff point, we get that $\sup_{|\zeta|\le r} |\varphi_{n-m}(\zeta)| \underset{n\to\infty}{\longrightarrow} 0$. The Koenigs function satisfies $\sigma(z) = \sigma\big(\varphi_n(z)\big)/\lambda^n$ for all n and for all z. So, given $z \in \gamma$, we have

$$|\sigma(z)| = \frac{1}{|\lambda|^m}\big|\sigma\big(\varphi_m(z)\big)\big| \le \frac{1}{|\lambda|^m} \sup_{|\zeta|\le r} |\sigma(\zeta)| < \infty.$$

Therefore, if $\sup_{z\in\gamma} |\varphi_m(z)| < 1$ for some natural number m, we have that (1) is satisfied.

Now, assume that

$$(*) \qquad \sup_{z\in\gamma} |\varphi_n(z)| = 1 \quad \text{for all natural } n.$$

In this case, it is trivial that $\omega(\varphi_n \circ \gamma) \bigcap \partial\mathbf{D} \ne \emptyset$, for every $n \in \mathbf{N}$. Take $0 < \varrho < 1$ such that σ has no zeros on $|\zeta| = \varrho$. Then there is $c > 0$ such that

$$(**) \qquad |\sigma(\zeta)| \ge c \quad \text{for } |\zeta| = \varrho.$$

Since $\varphi_n(0) = 0$, the curve $\gamma_n = \{\varphi_n(z) : z \in \gamma\}$ has 0 as initial point and, by $(*)$, we have that $\overline{\gamma_n} \bigcap \partial\mathbf{D} \ne \emptyset$. So there is $z_n \in \gamma$ such that $|\varphi_n(z_n)| = \varrho$. Now, using again that $\sigma(z) = \sigma\big(\varphi_n(z)\big)/\lambda^n$ and $(**)$, we have that

$$|\sigma(z_n)| = \frac{1}{|\lambda|^n}\big|\sigma\big(\varphi_n(z)\big)\big| \ge \frac{c}{|\lambda|^n} \underset{n\to\infty}{\longrightarrow} \infty$$

and this implies that $|z_n| \to 1$. □

Proof of Theorem 8. Denote $a_n := \lim_{r\to 1} \varphi_n(ra) \in \partial\mathbf{D}$ for all n. Since $\varphi_n(\mathbf{D}) \subset \mathbf{D}$, the function

$$\psi_n(z) = \log\big(1 - \overline{a_n}\varphi_n(z)\big) \quad (z \in \mathbf{D})$$

satisfies that $|\operatorname{Im}\psi_n(z)| < \frac{1}{2}\pi$. Therefore, ψ_n is a Bloch function.

Suppose that $\sup_{z\in\gamma} |\varphi_n(z)| < 1$ for some natural number n. Then $|\psi_n|$ is bounded on γ. On the one hand, by the Anderson–Clunie–Pommerenke theorem [13, Proposition 4.4], we get that $|\psi_n|$ is bounded on the radial segment $[0, a)$. On the other hand, since $1 - \overline{a_n}\varphi_n(ra) \underset{r\to 1}{\longrightarrow} 0$, we have that $|\psi_n|$ is not bounded on the radial segment $[0, a)$. A contradiction.

So, $\sup_{z\in\gamma} |\varphi_n(z)| = 1$ for all n and, by Theorem 9, we obtain that

$$\limsup_{z\to a,\, z\in\gamma} |\sigma(z)| = +\infty. \ \square$$

3. The examples

Example 1. There is a univalent function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with $\varphi(0) = 0$ and boundary fixed point 1 such that φ_2 does not have a radial limit at 1.

Proof. (a) We define $R = [0, 1)$ and

$$A = \left\{ (1 - e^{-1/t}) e^{it} : 0 < t \le \pi \right\}, \tag{1}$$

furthermore, for $k = 1, 2, \ldots$,

$$r_k = 1 - e^{-k}, \quad B_k = \left\{ r_k e^{it} : \frac{1}{k} + \exp(-e^{2k}) \le t \le \pi \right\}. \tag{2}$$

We consider the simply connected domain

$$\Omega = \mathbf{D} \setminus \left(A \bigcup [-1, -1 + e^{-1}] \bigcup \left(\bigcup_{k=1}^{\infty} B_k \right) \right) \tag{3}$$

which contains 0. Let φ be the Riemann map of $\mathbf{D}$ onto Ω that satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$ in the sense that $\gamma = \varphi(R)$ lies between the arc A and the arcs B_k; see Figure 1. This function φ has a continuous extension to $\overline{\mathbf{D}} \setminus \{1\}$. There are two important aspects: the arc A is very tangential at 1 and the gaps between A and the B_k are exceedingly small.

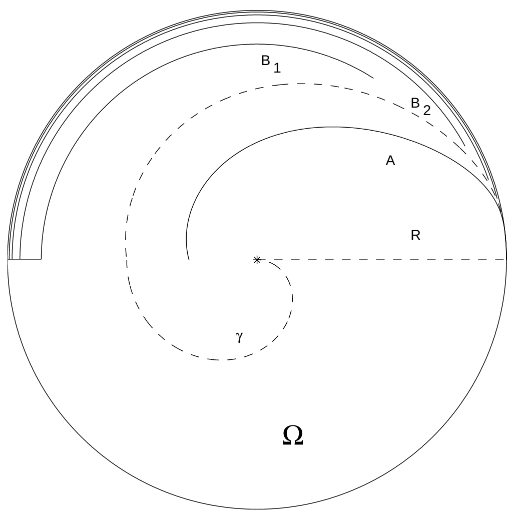

Figure 1 (not to scale).

Since $\gamma = \varphi(R)$ lies to the left of R, the image $\varphi \circ \varphi(R)$ lies to the left of $\varphi(R)$. Furthermore $\varphi \circ \varphi(R)$ goes from 0 to $\partial\mathbf{D}$. Hence there exists z_k with

$$z_k \in \varphi(R), \quad |\varphi(z_k)| = \tfrac{1}{2}(r_k + r_{k+1}), \quad \arg\varphi(z_k) < \pi. \tag{4}$$

We claim that $\arg\varphi(z_k) > \frac{1}{2}\pi$ for large k.

(b) Suppose that $\arg\varphi(z_k) \le \frac{1}{2}\pi$. Let $c_1, c_2, \ldots,$ denote suitable positive constants. Since $\varphi(R)$ goes from 0 to 1, there exists $b_k \in R$ with $|\varphi(b_k)| = \frac{1}{2}(r_k + r_{k+1})$; see Figure 2. The hyperbolic distance λ in $\mathbf{D}$ satisfies [13, p. 92]

$$\lambda(b_k, z_k) \le \inf_C \int_C \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)}, \tag{5}$$

where C runs through all curves in Ω from $\varphi(b_k)$ to $\varphi(z_k)$. It follows from the geometry of Ω (see (1) and (2)) that this infimum is essentially attained if C is the arc C_k of $\{|w| = \frac{1}{2}(r_k + r_{k+1})\}$ from $\varphi(b_k)$ to $\varphi(z_k)$. Now $\varphi^{-1}(A)$ lies on $\partial\mathbf{D}$ above R and the $\varphi^{-1}(B_k)$ lie on $\partial\mathbf{D}$ below R. Hence $\varphi(b_k)$ has about the same distance [12, p. 318] from A and $\bigcup B_k$.

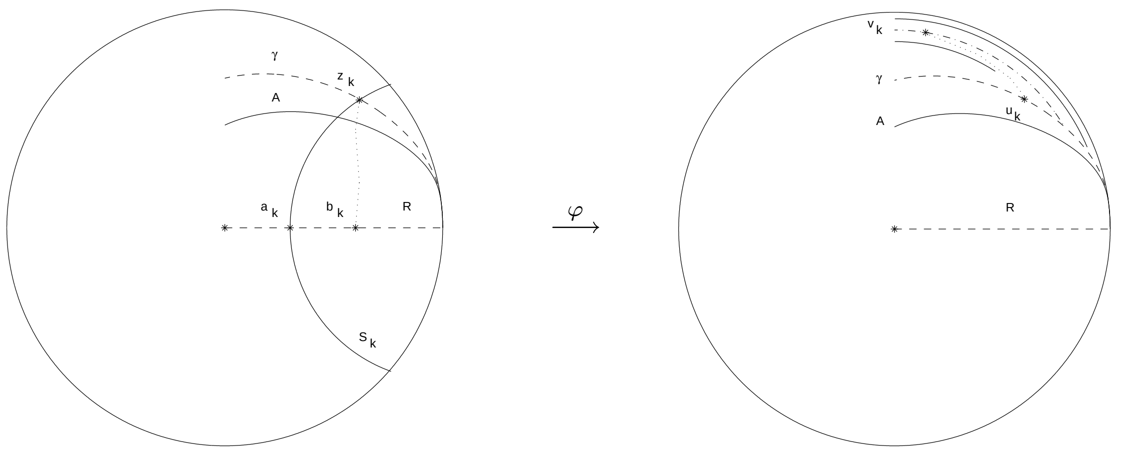

Figure 2 (not to scale). The dotted lines are $\varphi^{-1}(C_k)$ and C_k. Note that the arc appears in both parts of the figure. We have denoted $v_k = \varphi(z_k)$ and $u_k = \varphi(a_k)$.

Since $\arg\varphi(z_k) \le \frac{1}{2}\pi$ we therefore obtain that $\operatorname{dist}(w, \partial\Omega) \ge c_1 e^{-k}$ for $w \in C_k$. Hence we see from (5) that $\lambda(b_k, z_k) \le c_2 e^k$. If S_k is the hyperbolic segment from z_k to $a_k \in R$, that is orthogonal to R, then it follows that

$$\lambda(a_k, z_k) \le \lambda(b_k, z_k) \le c_2 e^k; \tag{6}$$

see Figure 2. The curve A is tangential to $\partial\mathbf{D}$ at 1 and $\varphi(R)$ lies between A and $\partial\mathbf{D}$. Hence, by (1), there exists t_k such that

$$(1 - e^{-1/t_k})e^{it_k} \in S_k, \quad |z_k| > 1 - e^{-1/t_k}. \tag{7}$$

Since the euclidean radius of S_k is $\sim 1 - a_k$ as $k \to \infty$, we have $t_k \sim 1 - a_k$. Hence, by (7),

$$\lambda(0, z_k) > c_3 \log \frac{1}{1 - |z_k|} > \frac{c_3}{t_k} \geq \frac{c_4}{1 - a_k} \geq \exp\bigl(c_5 \lambda(0, a_k)\bigr)$$

and we obtain from (6) that, for large k,

$$c_2 e^k > \exp\bigl(c_5 \lambda(0, a_k)\bigr) - \lambda(0, a_k) > \exp\bigl(c_6 \lambda(0, a_k)\bigr). \tag{8}$$

There are $x_k \in R$ such that $|\varphi(x_k)| = r_k$. We have [13, Corollary 1.5]

$$\lambda(\varsigma_0, \varsigma_1) \geq \frac{1}{4} \log \frac{|\varphi(\varsigma_0) - \varphi(\varsigma_1)|}{4 \operatorname{dist}(\varphi(\varsigma_0), \partial\Omega)} \quad \text{for } \varsigma_0, \varsigma_1 \in \mathbf{D}. \tag{9}$$

Now suppose that $a_k \leq x_k$. Then there exists $z'_k \in S_k$ with $|\varphi(z'_k)| = r_k$. Since

$$\operatorname{dist}\bigl(\varphi(z'_k), \partial\Omega\bigr) \leq \exp(-e^{2k}),$$

by (1) and (2), we obtain from (9) that

$$\lambda(a_k, z_k) \geq \lambda(z'_k, z_k) \geq c_7(e^{2k} - k) \quad \text{for large } k$$

which contradicts (6). Hence $x_k < a_k$ and it follows as above from (9) that

$$\lambda(0, a_k) \geq \lambda(x_{k-1}, x_k) \geq c_8 e^{2k}.$$

Thus we conclude from (8) that $c_2 e^k > \exp(c_9 e^{2k})$ for large k and we have arrived at a contradiction.

(c) Hence we have $\pi > \arg \varphi(z_k) > \frac{1}{2}\pi$ for large k. Since $\varphi(z_k) \in \varphi_2(R)$ by (4), we conclude the ω-limit of $\varphi_2(R)$ contains points different from 1.

On the other hand, since φ is univalent and $\varphi(1) = 1$, we obtain from a Theorem of Lindelöf [13, Theorem 2.16] that this ω-limit contains 1. Hence $\varphi_2(R)$ does not have a limit as $x \to 1$, with $x \in R$. □

Example 2. There are a univalent function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with an inner DW-point and associated Koenigs function σ and a point $a \in \partial\mathbf{D}$ such that $\lim_{r\to 1} \sigma(ra) = \infty$ and a is not a boundary fixed point of φ.

Proof. Take $\sigma(z) = z/(1-z^2)$ for all $z \in \mathbf{D}$. We have that $\sigma'(z) = (1+z^2)/(1-z^2)^2$, $\sigma(0) = 0$, and $\sigma'(0) = 1$. The function $p(z) = z\sigma'(z)/\sigma(z) = (1+z^2)/(1-z^2)$ satisfies that $p(0) = 1$ and $\operatorname{Re}\bigl(p(z)\bigr) > 0$ for all $z \in \mathbf{D}$. So, by [12, Theorem 2.5], σ is starlike. Moreover, $\sigma(-z) = -\sigma(z)$ for all $z \in \mathbf{D}$. Therefore, $-\frac{1}{2}\sigma(\mathbf{D}) \subset \sigma(\mathbf{D})$.

Consider the univalent function $\varphi(z) = \sigma^{-1}\bigl(-\frac{1}{2}\sigma(z)\bigr)$ for all $z \in \mathbf{D}$. It is clear that the Koenigs function of φ is the function σ. We have that $\lim_{r\to 1}\sigma(r) = \infty$ but

$$\begin{aligned}\lim_{r\to 1}\varphi(r) &= \lim_{r\to 1}\sigma^{-1}\bigl(-\tfrac{1}{2}\sigma(r)\bigr)\\ &= \lim_{s\to+\infty,\, s\in\mathbf{R}}\sigma^{-1}\bigl(-\tfrac{1}{2}s\bigr) = \lim_{s\to-\infty,\, s\in\mathbf{R}}\sigma^{-1}\bigl(\tfrac{1}{2}s\bigr) = -1.\end{aligned}$$

So 1 is not a fixed point of φ. □

Example 3. There is a univalent function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with an inner DW-point and associated Koenigs function σ and a point $a \in \partial\mathbf{D}$ such that

$$\lim_{r\to 1}\sigma(ra) = \infty$$

(so, a is a boundary contact point of φ_n for all n) and $\lim_{z\to 1}\sigma(z) \neq \infty$.

Proof. For each natural number n take

$$H_n = \left\{z \in \mathbf{C} : \operatorname{Re} z = -1 + \frac{1}{2n},\ \operatorname{Im} z \geq -n\right\}$$

and consider the domain

$$\Omega = \{z \in \mathbf{C} : \operatorname{Re} z > -1\} \setminus \left(\bigcup_{n\geq 1} H_n\right).$$

Let σ be the normalized Riemann map from $\mathbf{D}$ onto Ω. Let C_n be the arc in Ω of the circumference centered in 0 that goes from $-1 + (1/2n) - ni$ to H_1. The sequence of Jordan arcs (C_n) generates a prime end p in Ω. Take the point $a \in \partial\mathbf{D}$ such that $\hat{\sigma}(a) = p$. Then the impression of the prime end is $I(p) = \{z \in \mathbf{C} : \operatorname{Re} z = -1\} \bigcup \{\infty\}$. So the limit $\lim_{z\to 1}\sigma(z)$ does not exist. Moreover, by [13, Theorem 2.16], $\lim_{r\to 1}\sigma(ra) = \infty$.

Notice that $\frac{1}{2}\Omega \subset \Omega$. Therefore, the univalent function $\varphi(z) = \sigma^{-1}\bigl(\frac{1}{2}\sigma(z)\bigr)$ for all $z \in \mathbf{D}$ is well-defined and it is clear that the Koenigs function of φ is the function σ. So, by Theorem 6, a is a boundary contact point of φ_n for all n. □

Example 4. There are a univalent function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with an inner DW-point and a point $a \in \partial\mathbf{D}$ such that, for all n, the ω-limit of the curve $r \in [0,1) \mapsto \varphi_n(ra)$ touches $\partial\mathbf{D}$ but it is not contained in $\partial\mathbf{D}$.

Proof. For each natural number n take

$$H_{2n} = \left\{ z \in \mathbf{C} : \operatorname{Re} z = -1 + \frac{1}{2n},\ \operatorname{Im} z \geq -2n \right\},$$
$$H_{2n+1} = \left\{ z \in \mathbf{C} : \operatorname{Re} z = -1 + \frac{1}{2n+1},\ \operatorname{Im} z \leq 2n+1 \right\}$$

and consider the domain

$$\Omega = \{ z \in \mathbf{C} : \operatorname{Re} z > -1 \} \setminus \left(\bigcup_{n \geq 1} H_n \right).$$

Let σ be the normalized Riemann map from $\mathbf{D}$ onto Ω. The sequence of Jordan arcs (C_n) given by

$$C_{2n} = \left[-1 + \frac{1}{2n+1} - i, -1 + \frac{1}{2n} - i \right],$$
$$C_{2n+1} = \left[-1 + \frac{1}{2n+2} + i, -1 + \frac{1}{2n+1} + i \right],$$

for all n, generates a prime end p in Ω. Take the point $a \in \partial\mathbf{D}$ such that $\hat{\sigma}(a) = p$. Then the set of principal points of the prime end p is $\{z \in \mathbf{C} : \operatorname{Re} z = -1\} \bigcup \{\infty\}$ and, by [13, Theorem 2.16], this set coincides with the ω-limit of the curve $r \in [0,1) \mapsto \sigma(ra)$. In particular, ∞ belongs to this ω-limit and $\lim_{r \to 1} \sigma(ra) \neq \infty$.

Notice that $\frac{1}{2}\Omega \subset \Omega$. Therefore, the univalent function $\varphi(z) = \sigma^{-1}\big(\frac{1}{2}\sigma(z)\big)$ for all $z \in \mathbf{D}$ is well-defined and it is clear that the Koenigs function of φ is the function σ. Since ∞ belongs to ω-limit of the curve $r \in [0,1) \mapsto \sigma(ra)$, by Theorem 9, the ω-limit of the curve $r \in [0,1) \mapsto \varphi_n(ra)$ touches $\partial\mathbf{D}$ for all n. Finally, since $\lim_{r \to 1} \sigma(ra) \neq \infty$, by Theorem 6, a is not a boundary contact point of φ_n for all n. □

Example 5. There are a function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with DW-point 0, $\varphi'(0) \neq 0$, and associated Koenigs function σ, and a boundary fixed point $a \in \partial\mathbf{D}$ of φ_n, for all n, such that, the ω-limit of the curve $r \in [0,1) \mapsto \sigma(ra)$ contains the DW-point 0 and ∞.

Proof. Take $0 < \lambda < 1$ and $\varphi(z) = z(z-\lambda)/(1-\lambda z)$. Then $\varphi'(0) = -\lambda$, $\varphi(1) = 1$, and $1 < \varphi'(1) < \infty$. Consider $x_1 = \lambda$ and $x_n = \varphi(x_{n+1})$ for all n. Then $\lambda < x_{n+1} < 1$, $x_n < x_{n+1}$ for all n and $x_n \to 1$. Moreover,

$$\sigma(x_n) = \sigma\big(\varphi(x_n)\big) = -\lambda\sigma(x_{n+1}).$$

Therefore, bearing in mind that $0 = \sigma(0) = \sigma\big(\varphi(\lambda)\big) = -\lambda\sigma(\lambda)$, we have that $\sigma(x_n) = 0$ for all n and 0 belongs to the ω-limit of the curve $r \in [0,1) \mapsto \sigma(ra)$. Since 1 is a fixed point of φ_n for all n, by Theorem 8, we have that ∞ belongs to this ω-limit, too. □

Example 6. There is a function $\varphi\colon \mathbf{D} \to \mathbf{D}$ with DW-point 0, $\varphi'(0) \neq 0$, and associated Koenigs function σ such that ∞ is in $\omega(\sigma \circ \Gamma)$ for all curves Γ in $\mathbf{D}$ with $\omega(\Gamma)$ in $\partial\mathbf{D}$.

Proof. Consider the finite Blaschke product

$$\varphi(z) = cz \prod_{k=1}^{m-1} \frac{z - z_k}{1 - \overline{z_k} z}, \quad |c| = 1,\ 0 < |z_k| < 1 \text{ for all } k.$$

Since φ is continuous on $\overline{\mathbf{D}}$ and $\varphi(\partial\mathbf{D}) = \partial\mathbf{D}$, the ω-limit of the curves $r \mapsto \varphi_n(\Gamma(r))$ are in $\partial\mathbf{D}$ for all curves Γ in $\mathbf{D}$ with $\omega(\Gamma)$ in $\partial\mathbf{D}$ and for all n. Hence we obtain from Theorem 9 that

$$\limsup_{r\to 1} \bigl|\sigma\bigl(\Gamma(r)\bigr)\bigr| = +\infty. \ \square$$

Valiron [16, p. 123] has given a more precise description of the behaviour of the Koenigs function for finite Blaschke products. His results show that $|\sigma(z)| \to +\infty$ as $|z| \to 1$ except in smaller and smaller neighbourhoods of the points $\varphi_n^{-1}(z_k)$ for all n and $k = 0, \ldots, m-1$ where $z_0 = 0$.

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