Progress In Electromagnetics Research C, Vol. 4, 85–102, 2008

GEOMETRICAL ANALYSIS OF WAVE PROPAGATION IN LEFT-HANDED METAMATERIALS. PART II

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Abstract—Geometrical analysis of basic equations of electromagnetics waves propagation in anisotropic dielectric materials with magnetic isotropy are presented in two complementary papers (Part I and Part II). In this paper, analysis arises from local properties of dielectric permittivity tensor and Möhr's plane graphical construction compatible with circumferences that represent general plane wave equation. This theoretical study allows to know how rays propagate in left-handed metamaterials (LHMs) exhibiting dielectric anisotropy and magnetic isotropy.

Geometrical analysis yields plane graphical procedures for ray tracing, that are extremely easy. In particular, *indefinite dielectric media*, where dielectric permittivities are not all the same sign, has been investigated and a joint study of materials having the same eigenvalues of $\tilde{\varepsilon}$, but with opposite values of μ_r is performed. The opposite sense of propagation of rays in "opposite media" (media with opposite values of ε_i and μ_r) has been also shown. It must be pointed out that the presented method always allows graphical plane constructions, even when dealing with biaxial media.

1. INTRODUCTION

Since the theoretical study of Veselago [1] and the works of Pendry [2] and Smith et al. [3], the interest of left-handed metamaterials (LHMs) is growing [4–19].

To take into account anisotropic character of LHMs, geometrical analysis of basic equations of wave propagation in anisotropic dielectric materials with magnetic isotropy are presented in two complementary papers. The methodology is valid also for negative values of dielectric permittivities and magnetic permeability and allows the study of propagation of locally plane electromagnetic waves in LHMs.

In this paper, geometrical analysis arises from local properties of dielectric permittivity tensor ($\tilde{\varepsilon}$) and Möhr's plane graphical construction compatible with circumferences that represent general plane wave equation (relation of dispersion) (See Section 2). It must be noted that the approach presents many advantages, because it facilitates both quantitative and qualitative analysis inherent to a plane geometry. Some examples on ray tracing in RHMs and LHMs with dielectric anisotropy and magnetic isotropy are shown in Section 3, where, the so called *indefinite* media are considered [4–6]. This term refers to a material for which the eigenvalues of the permittivity and permeability tensor are not all the same sign.

In the complementary part (Part I) of this paper [22], another kind of geometrical analysis of ray propagation in RH and LH media, involving the compatibility between quadrics associated with relative dielectric tensor ($\tilde{\varepsilon}$) and conical surfaces representing the general plane wave equation (relation of dispersion) has been carried out.

2. GEOMETRICAL PLANE CONSTRUCTION IN LHMS WITH DIELECTRIC ANISOTROPY

2.1. General

We are only concerned with linear media, with dielectric anisotropy and magnetic isotropy ($\mu = \mu_0 \, \mu_r$), free of currents and charges ($\mathbf{j} = \mathbf{0}$; $\varrho = 0$).

An orthonormal cartesian frame ($\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$) along the main directions of relative dielectric permittivity tensor $\tilde{\varepsilon}$ is used. The eigenvalues of $\tilde{\varepsilon}$, are named $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ and they can be positive or negative.

Our starting point is the corresponding *relation of dispersion* or *general plane wave equation* for these media, which can also be written as [20]:

$$\boldsymbol{\nu} - (\mathbf{n} \cdot \boldsymbol{\nu}) \, \mathbf{n} = a \, \boldsymbol{\varepsilon}_{\boldsymbol{\nu}} \tag{1}$$

where unit vector $\boldsymbol{\nu}$ is defined as $\boldsymbol{\nu} = \mathbf{E}/|\mathbf{E}|$, bound vector $\boldsymbol{\varepsilon}_{\boldsymbol{\nu}} = \tilde{\varepsilon} \, \boldsymbol{\nu} = \mathbf{D}/(\varepsilon_{\mathbf{0}} \, |\mathbf{E}|)$. If $\mathbf{n}$ is an unit vector along the direction of propagation, wave vector $\mathbf{k}$ is expressed as: $\mathbf{k} = k \, \mathbf{n} = (\omega/v_p) \, \mathbf{n}$, where ω and v_p, are the angular frequency and the phase velocity, respectively. Parameter a is equal to $\mu_r \, v_p^2/c^2$.

In an anisotropic medium, there are two normal eigen modes of locally plane waves for each direction of propagation. Our study only deals with the eigen mode that does not behave isotropically. This means that the directions of Poynting vector and wave vector $\mathbf{k}$ are non collinear. Thus, in uniaxial media, we are only interested in the *extraordinary ray.*

In this paper, intrinsic components (τ_ν, σ_ν) of bound vectors $\boldsymbol{\varepsilon_\nu} = \tilde{\varepsilon}\,\boldsymbol{\nu}$ play an important role. They are defined as [20]:

$$\sigma_\nu = \boldsymbol{\varepsilon_\nu} \cdot \boldsymbol{\nu}; \qquad \tau_\nu^2 = |\boldsymbol{\varepsilon_\nu}|^2 - \sigma_\nu^2$$

Since $\mathbf{D}$ is parallel to $\boldsymbol{\varepsilon_\nu}$ (note that $\mathbf{D} = \varepsilon_0\,|\mathbf{E}|\,\boldsymbol{\varepsilon_\nu}$), and $\mathbf{D} \cdot \mathbf{n} = 0$, the dot product of Eq. (1) and vector $\boldsymbol{\varepsilon_\nu}$ gives:

$$\sigma_\nu = a\,|\boldsymbol{\varepsilon_\nu}|^2 \tag{2}$$

that can be written in the form:

$$\tau_\nu^2 + \sigma_\nu^2 - \frac{\sigma_\nu}{a} = 0 \tag{3}$$

Eq. (3) describes the *locus* of the end of the bound vector $\boldsymbol{\varepsilon_\nu}$ in the plane cartesian coordinate system (σ, τ). The locus is a circumference centered at point $(1/(2\,a), 0)$ of radius $R = 1/(2|a|)$. Since parameter a might be negative, the radius has always a positive value and its center defines, as a function of a, the location of this circumference. The method applies for both positive and negative values of a. We call this circumference, *vector* $\mathbf{D}$ *circumference*, since $\mathbf{D}$ is parallel to $\boldsymbol{\varepsilon_\nu}$. It follows immediately that the angle θ between $\mathbf{E}$ and $\mathbf{D}$ is given by:

$$\cos\,\theta = a\,|\boldsymbol{\varepsilon_\nu}| \tag{4}$$

An alternative expression of Fresnel's equation of wave normals in terms of intrinsic components of bound vector $\boldsymbol{\varepsilon_n}$ was derived in [20]:

$$\tau_n^2 + \sigma_n^2 - \sigma_n \left(I - \frac{1}{a}\right) + a\,\Delta = 0 \tag{5}$$

where (τ_n, σ_n) are the intrinsic componentes of $\boldsymbol{\varepsilon_n}$, I and Δ, trace and determinant of tensor $\tilde{\varepsilon}$, respectively, and $a = \mu_r\,v_p^2/c^2$.

Eq. (5) describes the locus of the end of $\boldsymbol{\varepsilon_n}$ in the plane cartesian coordinate system (σ, τ). This locus is a circumference centered at point $((I - 1/a)/2, 0)$ of radius R given by $R = \sqrt{((I - 1/a)/2)^2 - a\,\Delta}$. In what follows, this circumference is named *Fresnel's circumference.*

If $\mathbf{t}$ is an unit vector along the extraordinary ray direction, defined as $\mathbf{t} = (\mathbf{E} \times \mathbf{H})/|\mathbf{E} \times \mathbf{H}|$, it is shown [20] that the angle between $\mathbf{t}$ and

n is the same as the angle between **E** and **D** (given by Eq. (4)), and then $\mathbf{t} \cdot \mathbf{n} = a\,|\boldsymbol{\varepsilon}_{\boldsymbol{\nu}}|$.

Since $a = \mu_r\, v_p^2/c^2$, only media with $\mu_r = -1$ give values of θ greater than $\pi/2$. This is the case of LHMs.

2.2. Graphical Procedures of Ray Tracing

General graphical procedures of ray tracing based on Möhr's construction, can be found in [20, 21]. In short, Möhr's plane construction [23], allows to obtain the intrinsic components of bound vector $\boldsymbol{\varepsilon}_{\boldsymbol{\nu}}$ from the knowledge of $\boldsymbol{\nu}$. Reciprocally, if vector $\boldsymbol{\varepsilon}_{\boldsymbol{\nu}}$ is given, the direction $\boldsymbol{\nu}$ to which is associated, can be easily found. To perform this graphical construction only the knowledge of the eigenvalues of $\tilde{\varepsilon}$ is required.

Möhr's construction for uniaxial media is extremely easy and uses an unique circumference of diameter $\varepsilon_1 - \varepsilon_3$ drawn in plane (σ, τ), whose center lies on a point of σ-axis of coordinates $(\frac{\varepsilon_1+\varepsilon_2}{2}, 0)$. The equation of this circumference is:

$$\tau^2 + \sigma^2 - \sigma(\varepsilon_1 + \varepsilon_3) + \varepsilon_1\,\varepsilon_3 = 0 \tag{6}$$

If an unit vector **n** forms an angle ϕ with $\mathbf{u}_1$ axis, this angle is marked off on the vertical through ε_1. Let A be the intersection point between this sloped line and Möhr's circle. This point A is the end of bound vector $\boldsymbol{\varepsilon}_{\boldsymbol{n}}$ traced from the origin (see Fig. 1).

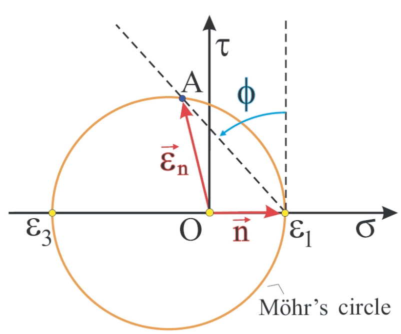


Figure 1. Möhr's construction for an uniaxial indefinite medium ($\varepsilon_1 > 0$ and $\varepsilon_3 < 0$) from the knowledge of angle ϕ between **n** (unit wave vector) and $\mathbf{u}_1$ axis. The end of bound vector $\boldsymbol{\varepsilon}_{\boldsymbol{n}}$ lies on a Möhr's circumference passing through ε_1 and ε_3.

In [21], a more simplified construction is also outlined, where there is no need to draw the Möhr's circle. The construction is shown in

Fig. 2. As an example, principal permittivities are not all the same sign ($\varepsilon_1 = \varepsilon_2 > 0$ and $\varepsilon_3 < 0$). These are the steps of the procedure:

(i) Values of ε_1 and ε_3 are marked on the straight line along the direction of propagation $\mathbf{n}$, that forms an angle ϕ with $\mathbf{u}_1$ axis.

(ii) A normal to axis $\mathbf{u}_1$ passing through ε_1 and a normal to axis $\mathbf{u}_3$ passing through ε_3 are traced. The intersection of these lines is point P, that becomes the end of $\boldsymbol{\varepsilon_n}$.

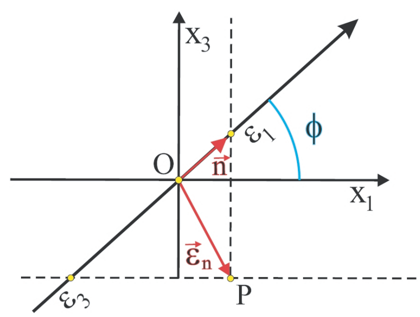


Figure 2. A more simplified Möhr's construction in the ($\mathbf{u}_1$, $\mathbf{u}_3$) plane for an uniaxial indefinite medium ($\varepsilon_1 > 0$ and $\varepsilon_3 < 0$) from the knowledge of angle ϕ between $\mathbf{n}$ (unit wave vector) and $\mathbf{u}_1$ axis. Tracing of two straight lines is only needed.

Figure 1 and Fig. 2 describe two simple graphical constructions to find $\boldsymbol{\varepsilon_n}$ if vector $\mathbf{n}$ is known. Similarly, one can easily find $\mathbf{n}$ from $\boldsymbol{\varepsilon_n}$, as it was also outlined in [21].

Nevertheless, Möhr's construction only states the tensor relation between $\boldsymbol{\nu}$ (direction of $\mathbf{E}$) and $\boldsymbol{\varepsilon_\nu}$ (parallel to $\mathbf{D}$) in the (σ, τ) plane. In order to take into account the relation of dispersion given by (1), Equations (5) or (3) are needed. In fact, (3) and (5) are the loci of the end of vectors $\boldsymbol{\varepsilon_\nu}$ and $\boldsymbol{\varepsilon_n}$, respectively. Moreover, in both equations parameter $a = \mu_r v_p^2/c^2$ is involved. Möhr's construction does not impose retrictions and can be applied to both positive and negative eigenvalues. Since Eq. (1) derives directly from Maxwell equations for plane waves, occurrence of negative values of μ is also regarded.

Figure 3 shows the plots of Möhr's circumference, $\mathbf{D}$ vector and Fresnel's circumferences for a LH metamaterial with $\varepsilon_1 = 2$; $\varepsilon_2 = \varepsilon_3 = -1$, and for $a = -1/2$.

Property 1. Angle θ between $\mathbf{E}$ and $\mathbf{D}$ and angle θ_n between $\mathbf{n}$ and $\boldsymbol{\varepsilon_n}$ verify the following relation in uniaxial *indefinite* media:

$$\cos^2\theta = \cos^2\theta_n$$

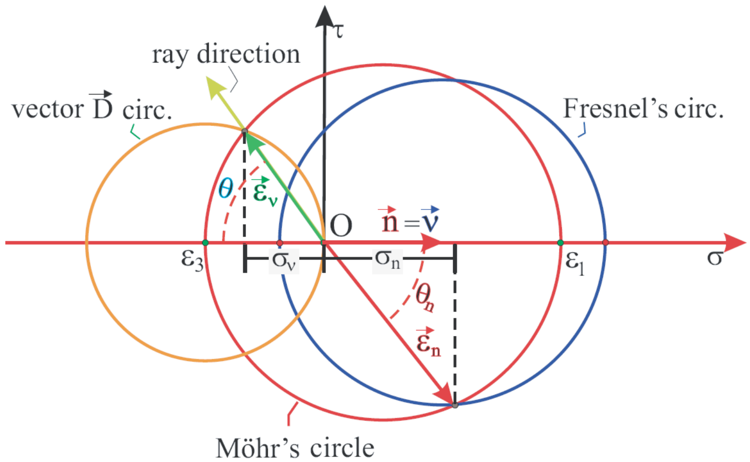


Figure 3. Möhr's circumference for a left-handed uniaxial indefinite medium ($\varepsilon_1 > 0$, $\varepsilon_3 < 0$, $\mu_r = -1$). **D** vector circumference (locus of the ends of bound vector $\boldsymbol{\varepsilon_\nu}$) and Fresnel's circumference (locus of the ends of bound vector $\boldsymbol{\varepsilon_n}$) are drawn for $v_p^2 = c^2/2$.

Proof: The abscissa of the point of intersection between Möhr's circle given by Eq. (6) and Eq. (3) becomes:

$$\sigma_\nu = \frac{\varepsilon_1\,\varepsilon_3}{\varepsilon_1 + \varepsilon_3 - \frac{1}{a}}$$

The angle θ between **E** and **D** can be obtained from the expression:

$$\cos\,\theta = \frac{\sigma_\nu}{|\boldsymbol{\varepsilon_\nu}|} \tag{7}$$

where $|\boldsymbol{\varepsilon_\nu}|^2 = \sigma_\nu/a$.

When dealing with the so called left-handed and right-handed *indefinite* media, values of a can be either positive or negative, so it is advisable to use the square of Eq. (7):

$$\cos^2\,\theta = \frac{\sigma_\nu^2}{|\boldsymbol{\varepsilon_\nu}|^2} = a\,\sigma_\nu = \frac{a\,\varepsilon_1\,\varepsilon_3}{\varepsilon_1 + \varepsilon_3 - \frac{1}{a}} \tag{8}$$

Similarly, the abscissa of the point of intersection between Möhr's circumference and Fresnel's circumference, given by (5), is:

$$\sigma_n = a\,\varepsilon_1\,\varepsilon_3 = \frac{a\,\Delta}{\varepsilon_2} \tag{9}$$

Taking into account that $\sigma_n^2 + \tau_n^2 = |\boldsymbol{\varepsilon_n}|^2$, direct substitution of Eq. (9) into Eq. (5) leads to:

$$|\boldsymbol{\varepsilon_n}| = \sqrt{a\,\varepsilon_1\,\varepsilon_3\left(\varepsilon_1 + \varepsilon_3 - \frac{1}{a}\right)}$$

The angle θ_n between $\mathbf{n}$ and $\boldsymbol{\varepsilon_n}$ can be obtained from:

$$\cos\,\theta_n = \frac{\sigma_n}{|\boldsymbol{\varepsilon_n}|} \tag{10}$$

and, then

$$\cos^2\,\theta_n = \frac{\sigma_n^2}{|\boldsymbol{\varepsilon_n}|^2} = \frac{\sigma_n}{\left(\varepsilon_1 + \varepsilon_3 - \frac{1}{a}\right)} = \frac{a\,\varepsilon_1\,\varepsilon_3}{\varepsilon_1 + \varepsilon_3 - \frac{1}{a}} \tag{11}$$

Then, for *indefinite media*, we can write:

$$\cos^2\,\theta = \cos^2\,\theta_n \quad \square \tag{12}$$

Property 2. The locus of the ends of vectors $\boldsymbol{\varepsilon_n}$ in plane $(\mathbf{u}_1,\, \mathbf{u}_3)$ is an ellipse of semiaxes $|\varepsilon_1|$ and $|\varepsilon_3|$.

Proof: From Fig. 4, one can writes:

$$\boldsymbol{\varepsilon_n} = \varepsilon_1\,\cos\,\phi\,\mathbf{u}_1 - |\varepsilon_3|\,\sin\,\phi\,\mathbf{u}_3 = x_1\,\mathbf{u}_1 + x_3\,\mathbf{u}_3$$

and, consequently:

$$\left(\frac{x_1}{\varepsilon_1}\right)^2 + \left(\frac{x_3}{\varepsilon_3}\right)^2 = 1 \qquad \square \tag{13}$$

The locus of the ends of vectors $\boldsymbol{\varepsilon_n}$ is the ellipse with semiaxes $a = |\varepsilon_1|$ and $b = |\varepsilon_3|$.

To relate vector $\boldsymbol{\varepsilon_n}$ with unit vector $\mathbf{t}$, we recall that in uniaxial media with positive eigenvalues of $\tilde{\varepsilon}$, vector $\boldsymbol{\varepsilon_n}$ is collinear with the extraordinary ray direction $\mathbf{t}$, as it was shown in [20].

In uniaxial indefinite media, $\boldsymbol{\varepsilon_n}$ also lies in the same plane that $\boldsymbol{\nu}$, $\boldsymbol{\varepsilon_\nu}$ and $\mathbf{n}$. Since the angle between $\mathbf{E}$ and $\mathbf{D}$ is the same as the angle between $\mathbf{t}$ and $\mathbf{n}$ and Cauchy theorem [20] states that $\boldsymbol{\varepsilon_n} \cdot \boldsymbol{\nu}(= \mathbf{n} \cdot \boldsymbol{\varepsilon_\nu})$ also vanishes, vector $\boldsymbol{\varepsilon_n}$ is collinear with $\mathbf{t}$, but may be antiparallel.

Nevertheless, from a graphical point of view, there is no uncertainty in tracing the direction of extraordinary ray. If $a > 0$, the angle between $\mathbf{t}$ (the ray) and $\mathbf{n}$ is acute (less than $\pi/2$), and if $a < 0$, the corresponding angle is greater than $\pi/2$, as it is depicted in Fig. 3.

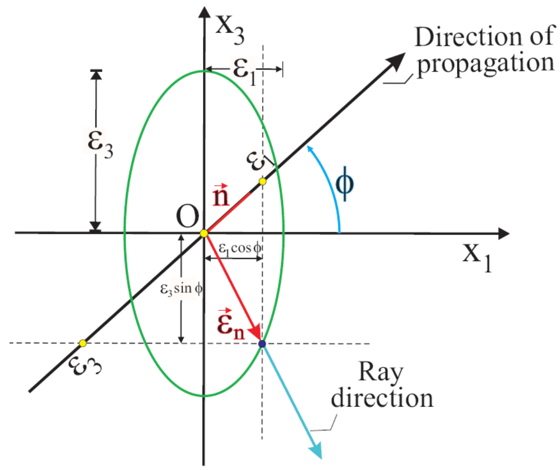


Figure 4. Locus of the ends of bound vector $\boldsymbol{\varepsilon_n}$ in an uniaxial indefinite medium ($\varepsilon_1 > 0$ and $\varepsilon_3 < 0$). The locus is an ellipse of semiaxes ε_1 and ε_3.

In order to state the correspondence between the proposed methods and those used by other authors, uniaxial media are considered. In [24–27], a relation between the slopes of the direction of propagation and the extraordinary ray direction is obtained. The alternative method leads to the same relation.

Property 3. There holds the following relation between the slope of the direction of propagation ($\tan\phi$) and that of the extraordinary ray direction ($\tan\psi$), in the plane $(\mathbf{u}_1, \mathbf{u}_3)$:

$$\tan\psi = -\frac{\varepsilon_3}{\varepsilon_1}\tan\phi$$

Proof: Let us consider the simplified construction given in Fig. 5. From the scheme, one has that:

$$\tan\psi = \frac{\overline{AB}}{\overline{OA}} = \frac{|\varepsilon_3|\sin\phi}{|\varepsilon_1|\cos\phi} = -\frac{\varepsilon_3}{\varepsilon_1}\tan\phi \qquad \square$$

3. SOME EXAMPLES OF RAY TRACING IN RHM AND LHM UNIAXIAL INDEFINITE MEDIA

In this section, the method is applied to four media with dielectric anisotropy and magnetic isotropy [4] with the following material parameters:

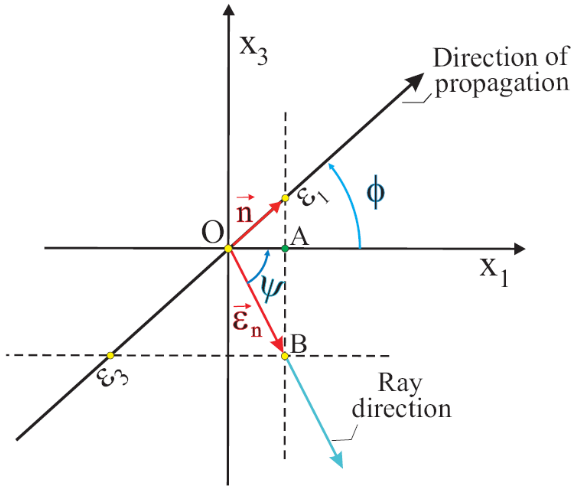


Figure 5. Graphical construction showing the equivalence between traditional and alternative methods in ray tracing, when propagation along an uniaxial indefinite medium is considered.

Medium	Class	ε_1	ε_2	ε_3	μ_r
"1"	RHM	1	1	-2	1
"2"	LHM	2	-1	-1	-1
"3"	RHM	2	-1	-1	1
"4"	LHM	1	1	-2	-1

It must be pointed out that solutions of plane and non-evanescent waves impose some restrictions on the allowed values of velocity of propagation, v_p and then on possible directions of propagation.

In these media allowed values of $1/a$ are either greater than ε_1 or less than ε_3. Thus, values of a in the interval $\varepsilon_3 < 1/a < \varepsilon_1$ are not possible. For instance, for right-handed medium "3" ($\mu_r = 1$ and $\theta \leq \pi/2$), Eq. (3) restricts possible values of velocity of propagation v_p, which must fulfill that: $v_p < c/\sqrt{2}$, because $\frac{1}{a} > 2$ to give non-evanescent waves. Consequently, possible values of directions of propagation are restricted.

Let us now consider certain cases that deserve attention in the study of light propagation of plane waves in these media. It may be noted that Eq. (6) determine the bound vectors, whereas Eqs. (3) and (5) give the phase velocity v_p and relative permeability μ_r in terms of $\boldsymbol{\varepsilon_\nu}$ and $\boldsymbol{\varepsilon_n}$, respectively.

3.1. Direction of Propagation Is Known: Joint Treatment for LHMs and RHMs

When direction of propagation **n** is known and, for instance, the angle that **n** forms with axis $\mathbf{u}_1$ is given, any of the procedures outlined in Section 2 (Fig. 1 and Fig. 2) can be followed.

To find extraordinary ray direction (vector **t**) we must only take into account that the sign of the dot product $\mathbf{t} \cdot \mathbf{n}$ coincides with the sign on μ_r.

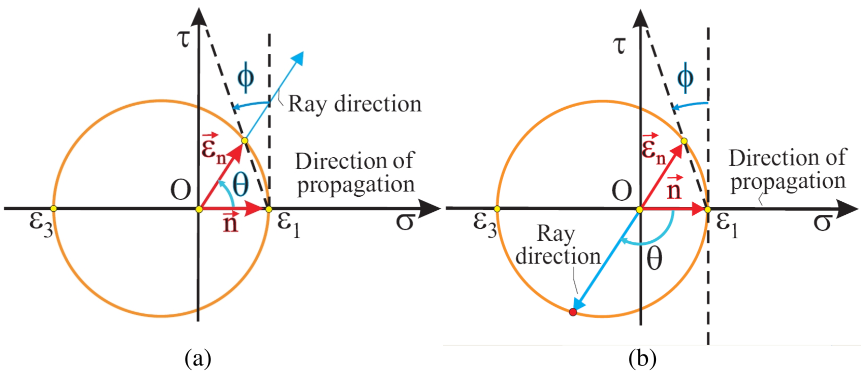


Figure 6. Graphical determination of the extraordinary ray direction from the knowledge of angle ϕ between **n** (unit wave vector) and $\mathbf{u}_1$ axis, for RH and LH uniaxial indefinite media. (a) Construction for a RH medium. Angle θ between direction of propagation and ray direction is less than $\pi/2$. (b) Construction for a LH medium. Angle θ between direction of propagation and ray direction is greater than $\pi/2$. Dielectric eigenvalues of both media are identical ($\varepsilon_1 > 0$ and $\varepsilon_3 < 0$). Procedure is based on Möhr's construction.

Thus, in Figs. 6 and 7 two constructions dealing with dielectric indefinite media are depicted. If the medium is right-handed, ($\mu_r = 1$) the θ angle between **n** and **t** is less than $\pi/2$. If the medium is left-handed, ($\mu_r = -1$), θ is greater than $\pi/2$. There is no uncertainty in tracing the extraordinary ray direction. This dual construction allows the joint study of RH and LH media.

The dielectric permittivities of the materials are $\varepsilon_1 = \varepsilon_2 = 1$; $\varepsilon_3 = -2$, that correspond to media "1" and "4". Vector **n** forms an angle of 19° with the $\mathbf{u}_1$ axis, and the procedure allows to determine the extraordinary ray direction from the direction of $\mathbf{n} = \mathbf{k}/k$.

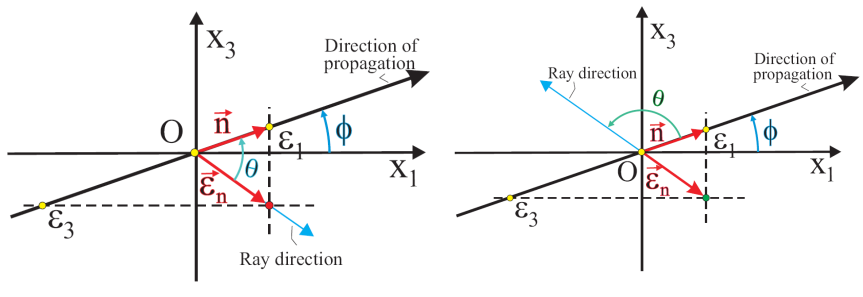


Figure 7. A more simplified procedure to obtain extraordinary ray direction from the knowledge of angle ϕ between $\mathbf{n}$ (unit wave vector) and $\mathbf{u}_1$ axis, for RH and LH uniaxial indefinite media. Dielectric eigenvalues of both media are identical ($\varepsilon_1 > 0$ and $\varepsilon_3 < 0$). (a) Construction for a RH medium. Angle θ between direction of propagation and ray direction is less than $\pi/2$. (b) Construction for a LH medium. Angle θ between direction of propagation and ray direction is greater than $\pi/2$.

3.2. Direction of the Extraordinary Ray Is Known

From Property 3 and taking into account a procedure outlined in [21], it is immediate to find the direction of propagation from the direction of the extraordinary ray.

Let us consider the plane ($\mathbf{u}_1$, $\mathbf{u}_3$). Let us assume that the extraordinary ray direction forms an angle ψ with axis $\mathbf{u}_1$.

(i) Values of ε_1 and ε_3 are marked on the straight line along the extraordinary ray.

(ii) A parallel to axis $\mathbf{u}_1$ passing through ε_1 and a parallel to axis $\mathbf{u}_3$ passing through ε_3 are traced. If P is the intersection point between these lines, the straight line OP is the direction of propagation given by $\mathbf{n}$ (see Fig. 8).

Figure 8 shows the graphical construction for medium "2".

3.3. Inverse Problem: Phase Velocity Is Known

In this case, parameter a is also known, since $a = \mu_r v_p^2/c^2$. There are two procedures to find the angle between $\mathbf{t}$ and $\mathbf{n}$.

<u>First procedure</u>

(i) Let us trace the corresponding Möhr's circumference, in plane σ, τ, and assume that vector $\boldsymbol{\nu}$ lies along σ axis.

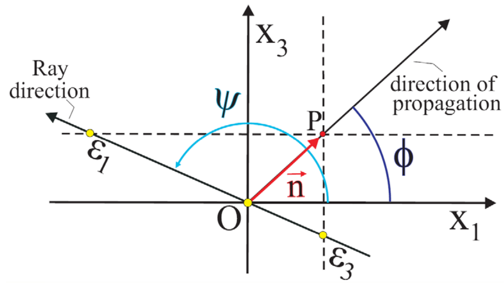


Figure 8. Graphical determination of the direction of propagation $\mathbf{n} = \mathbf{k}/k$ from the knowledge of extraordinary ray direction in an uniaxial indefinite medium (medium "2").

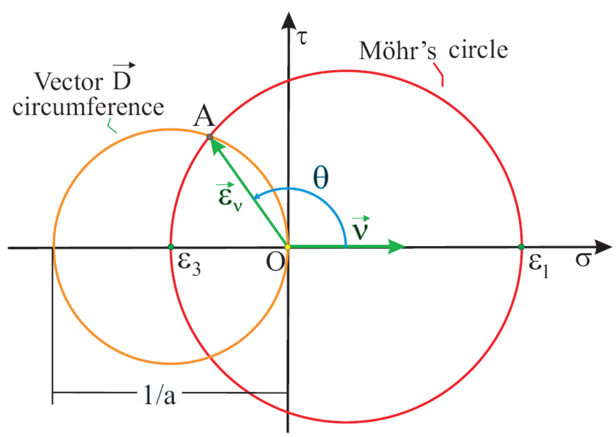


Figure 9. First graphical procedure to obtain angle θ between the extraordinary ray and direction of propagation $\mathbf{n} = \mathbf{k}/k$ when phase velocity v_p is known in a LH uniaxial indefinite medium. Point A is the intersection between Möhr's circumference and $\mathbf{D}$ vector circumference, for $v_p^2 = c^2/2$ in medium "2".

(ii) $\mathbf{D}$ vector circumference (Eq. (3)) for this value of a is drawn.

(iii) The intersection point between these circumferences, A, is the end point of vector $\boldsymbol{\varepsilon_\nu}$.

(iv) The angle θ between $\boldsymbol{\nu}$ and $\boldsymbol{\varepsilon_\nu}$ is the same angle between $\mathbf{n}$ and $\mathbf{t}$ and can be measured directly from the drawing.

Figure 9 depicts the procedure for medium "2" and $a = -1/2$.

Second procedure

(i) Möhr's circumference is traced, assuming that vector **n** lies along σ axis.

(ii) Fresnel's circumference (Eq. (5)) for this value of a is drawn.

(iii) The intersection point between these circumferences, B, is the end point of vector $\boldsymbol{\varepsilon_n}$.

(iv) The sign of a determines the sign of the dot product $\mathbf{t} \cdot \mathbf{n}$.

Figure 10 depicts the procedure for medium "2" and $a = -1/2$. In this case, $a < 0$, and then the **t** is along $-\boldsymbol{\varepsilon_n}$. Angle θ is shown in the drawing.

Obviously, the direction of **n** is immediate to find, following Möhr's construction, as one can see from Fig. 10: The angle that segment $\overline{B\varepsilon_1}$ forms with the vertical through ε_1 is the angle ϕ that **n** forms with $\mathbf{u}_1$ axis. The angle that **n** forms with $\mathbf{u}_3$ is $\pi/2 - \phi$.

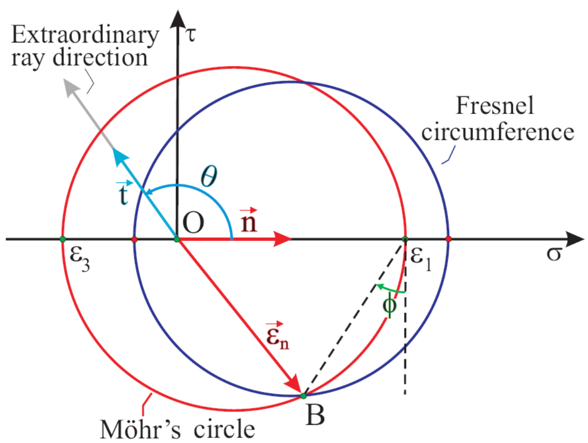


Figure 10. Second graphical procedure to obtain angle θ between the extraordinary ray and direction of propagation $\mathbf{n} = \mathbf{k}/k$ when phase velocity v_p is known in a LH uniaxial indefinite medium. Point B is the intersection between Möhr's circumference and Fresnel's circumference for $v_p^2 = c^2/2$ in medium "2".

Finally, Fig. 11 shows graphical construction for both procedures in the same picture: **D** vector circumference (Eq. (3)) and Fresnel's circumference (Eq. (5)) are shown jointly.

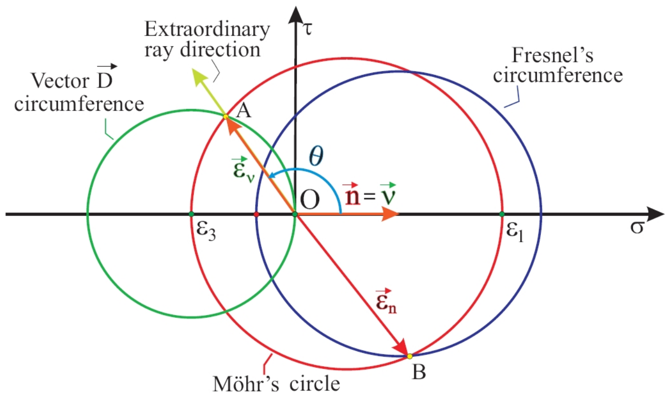


Figure 11. Joint representation of both procedures to obtain angle θ between the extraordinary ray and the direction of propagation $\mathbf{n} = \mathbf{k}/k$ when phase velocity v_p is known in a LH uniaxial indefinite medium (medium "2") for $v_p^2 = c^2/2$.

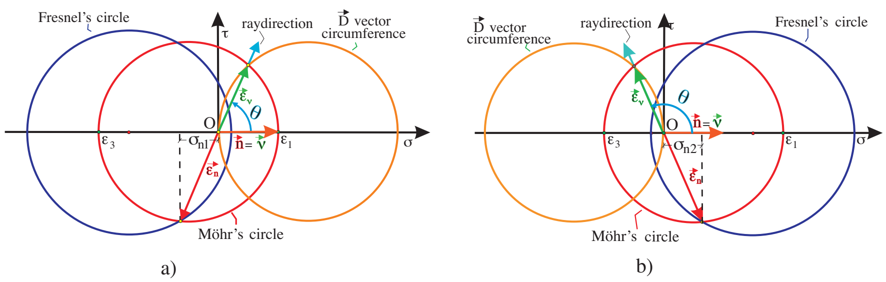


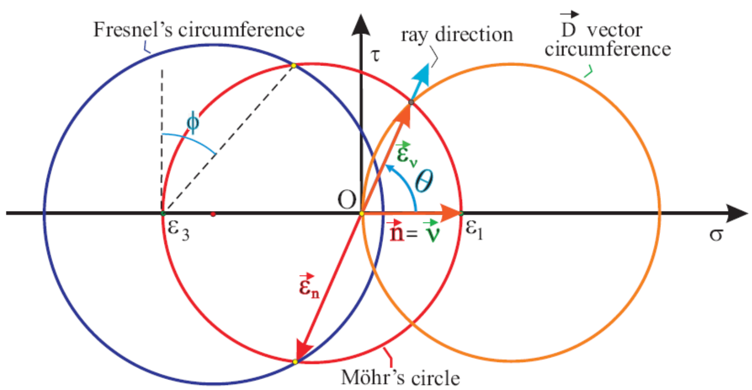
Figure 12. Ray tracing in *"opposite" media*. *"Opposite" media* are media with opposite values of their dielectric and magnetic parameters (eigenvalues ε_i of $\tilde{\varepsilon}$ and μ_r). For a given $\mathbf{n} = \mathbf{k}/k$, the angle between $\mathbf{n}$ and the extraordinary ray in medium "1" and the angle between $\mathbf{n}$ and the extraordinary ray in medium "2" become a pair of supplementary angles. (a) Construction for medium "1" ($\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -2$, $\mu_r = 1$). The medium is RH and $\theta < \pi/2$. (b) Construction for medium "2", opposite of medium "1" ($\varepsilon_1 = 2$, $\varepsilon_2 = \varepsilon_3 = -1$, $\mu_r = -1$). The medium is LH and $\theta > \pi/2$.

3.4. "Opposite" Media

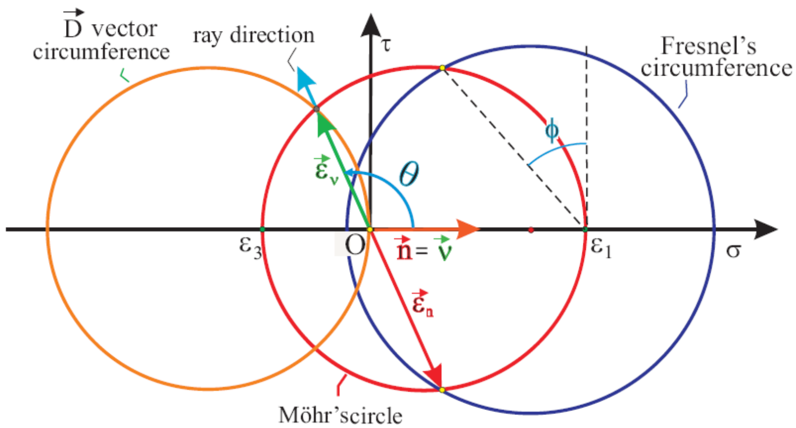
One can observe that parameters of media "2" and "3" are the opposite values of those corresponding to "1" and "4". Thus an interesting property raises:

Property 4: For a given direction of propagation **n**, the angle between the ray $\mathbf{t_1}$ in medium "1" and **n** and the angle between the ray $\mathbf{t_2}$ in medium "2" and **n**, become a pair of supplementary angles.

Proof: A given direction of propagation forms an angle ϕ with the optic axis ($\mathbf{u}_3$) in medium "1", and the same angle ϕ with the optic axis of medium "2" ($\mathbf{u}_1$). Then, the phase velocity is the same in both



(a)



(b)

Figure 13. Ray tracing in *"opposite" media*. The angle between **n** and $\mathbf{u}_3$ axis in medium "1" is equal to the angle between **n** and $\mathbf{u}_1$ axis in medium "2". (a) Construction for medium "1" ($\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -2$, $\mu_r = 1$). (b) Construction for medium "2", opposite of medium "1" ($\varepsilon_1 = 2$, $\varepsilon_2 = \varepsilon_3 = -1$, $\mu_r = -1$).

media, but a changes sign. From Eq. (9), one has:

$$\sigma_{n1} = -\sigma_{n2} \qquad \square$$

where σ_{n1} and σ_{n2} are the abscissas of the points of intersection between Möhr circumference and Fresnel's circumference in medium "1" and medium "2", respectively. Figs. 12 and 13 show the property for medium "1" and medium "2".

4. CONCLUSIONS

An extension of the geometrical interpretation of the basic equations of wave propagation in terms of the intrinsic components of bound vectors $\boldsymbol{\varepsilon_\nu} = \tilde{\varepsilon}\,\boldsymbol{\nu}$, eigenvalues ε_i of tensor $\tilde{\varepsilon}$, relative magnetic permeability μ_r and phase velocity v_p has been carried out in order to study left-handed metamaterials with dielectric anisotropy and magnetic isotropy.

The locus described by Fresnel's equation of wave normals in the plane of the intrinsic components of $\boldsymbol{\varepsilon_n}$ is a circumference. And a circumference is the locus of another main equation describing the projection $\overline{D}$ onto $\overline{E}$. The use of plane geometry is one of the advantages of the method, which facilitates both quantitative and qualitative analysis.

The correspondence with other analogous works has been also stated. Characteristic phenomena of LHMs, like *imperfect backward-wave* propagation have been easily explained.

A detailed discussion on wave propagation along *indefinite dielectric media*, where dielectric permittivities are not all the same sign, has been carried out and, as an example, four dielectric uniaxial media with given parameters have been investigated. Several properties of propagation of light along these media are demonstrated.

The discussion proposes extremely easy graphical constructions to obtain the extraordinary ray direction $\mathbf{t}$ from the knowledge of $\mathbf{n} = \mathbf{k}/k$ and, reciprocally, to obtain $\mathbf{n}$ if $\mathbf{t}$ is given: it suffices to trace two straight lines. Moreover, graphical procedures allow a joint study of materials having the same eigenvalues of $\tilde{\varepsilon}$, but with opposite values of μ_r.

If phase velocity v_p is known, two graphical and versatile constructions are also outlined. They involve intersections between Möhr's circumference and the other circumferences (loci of $\boldsymbol{\varepsilon_\nu}$ and $\boldsymbol{\varepsilon_n}$). The knowledge of v_p also allows the immediate obtention of $\mathbf{n}$ direction: it suffices to trace a straight line. Finally, the opposite sense of propagation of light in "opposite media" (media with opposite values of ε_i and μ_r) has been also discussed.

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