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AZAR GAME IN THE BOOK OF THE DICE OF ALFONSO X THE LEARNED. ITS RELATION WITH THE HAZARD GAMES OF MONTMORT, COTTON, HOYLE, DE MOIVRE AND JACOB BERNOULLI

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RÉSUMÉ – Le jeu appelé « Azar » dans le livre des dés d'Alphonse X le savant. Ses rapports avec les jeux de hasard de Montmort, Cotton, Hoyle, De Moivre et Jacob Bernoulli

Un jeu dénommé "Azar" est présenté dans la deuxième partie du livre de jeux « Le livre des échecs, dés et tables *» d'Alphonse X Le Sage (1221-1284). Les règles de ce jeu de Hasard dépendent de deux événements appelés « re-hasard » et « hasard ». La probabilité de gagner dans ce jeu Hasard implique un événement dont la probabilité dépend d'un nombre indéterminé de lancements de trois dés. Nous démontrons ici que cette probabilité est proche de 50 %.*

Nous avons associé ce jeu de Hasard avec d'autres jeux, tels le jeu de Hasard de De Montmort, Cotton, Hoyle et De Moivre, et aussi avec le jeu de Cinq et Neuf de J. Bernoulli. Enfin, nous montrons que le premier jeu de Hasard de De Moivre est le jeu populaire du Craps.

MOTS-CLÉS – Jeu de "Azar", Alphonse X Le Sage, Jeu de Hasard de De Montmort, Jeu de Craps de De Moivre

SUMMARY – *A game called "Azar" is presented in the second part of the Book* "The Book of Chess, Dice and Tables" *by Alfonso X El Sabio (The Learned) (1221-1284). The rules of this Azar game depend on two events called "chance" and "azar". The winning probability in this Azar game implies an event whose probability depends on an infinite number of three-dice rolls. We intend to demonstrate here that this probability is of around 50 %. We associated this game with other games, like the Hazard game by Montmort, Cotton, Hoyle and De Moivre, and also with the Cinq et Neuf game of J. Bernoulli. Finally, we see that the first Hazard game of De Moivre is the famous Craps game.*

KEYWORDS – Game of "Azar", Alfonso X The Learned, Hazard Game of Montmort, Craps Game of De Moivre

1. INTRODUCTION

Alfonso X, el Sabio or the Learned, was born in Toledo on November 23, 1221. He was the king of Castile, Leon and Andalusia, as he liked to be named at the right moments. In the Royal Palace of Seville, where Fernando III, father of Alfonso, died thirty two years ago, Alfonso X died on April 4, 1284 [González, 2004, Introduction, p. 1].

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Alfonso X used the centenary tradition of the prestigious Translation School of Toledo (1130-1187) commissioned many books for its translation to the vernacular Castilian language, for example, the "Libros del saber de astrología" [F. Gómez Redondo, 1998, section 4.4].

Alfonso X established two foundations, one in Seville in 1254, about "Estudios Generales de latín e de arábico" to learn medicine and science, and other in Murcia in 1269, which was managed by the mathematician al-Ricotí.

The Book of Chess, Dice and Tables dated 1283 (Seville) was his final work.

The Book has three parts, the first: folios 1 to 64 that contain the Introduction and the Book of Chess. Its number of folios reflects the sixty-four squares on a chessboard. The second: folios 65 to 71 that contain the Book of Dice and the third: folios 72 to 80 that contain the Book of Tables.

The Book of Dice contains the twelve following games: Mayores o de Menores; Tanto en uno como en dos; Triga; Otra manera de Triga; Azar; Marlota; Riffa; Par con As; Panquist; Medio Azar; Azar Pujado and Guirguiesca, and each one contains a miniature that illustrates the negative effect of the games of chance on the player moral.

This article is about the game of Azar, which begins on folio 67r of the Book of Chess, Dice and Tables.

The rest of the article is organized as follows. Section 2 describes the game of Azar. In section 3 we study the relationship between the game of Azar and the game called Hazard, this last game is described in the books of Montmort [1723], Cotton [1674] and Hoyle [1800]. In section 4 we study the Problem XLVI of De Moivre [1718], where De Moivre describes the games of Craps, Hazard and several generalizations of these games. Section 5 describes the Cinq et Neuf game of James Bernoulli and its relations with the games of Hazard and Azar. Section 6 concludes the article with a discussion of the results. The Appendix-I presents the miniature of Azar game; in the Appendix- II we calculate the probability of winning the Azar game and, finally, in the Appendix-III we proof a corollary related to the game of Azar.

2. THE GAME CALLED AZAR

In this game called Azar (az zahr, in arabic; hasard, in french; azzardo, in italian and hazard, in English) play two players with three dice. In order to make our exposition much easier, we name the players as G1 and G2. In most of the games gathered in the Book of the Dice it is decided which player will throw the dice in the beginning. In other games that appear in the Book of the Dices it is said that "Los que quisieren iogar an de lançar primeramientre batalla"; which means that one die must be thrown by each player and the one that obtain the highest point would be the first to roll three dice once. In this game nothing is said about whom must throw the dice.

The following text that we have gathered has been taken initially from the book of Vicent García Editores [1987] and from the manuscript "Libro de Ajedrez, dados y tablas". A copy of the manuscript can be consulted in the General Library of the University of Seville. For the reading of Hazard game, we have helped ourselves, also, with the Dictionary of the "Prosa Castellana" (Castilian Prose) of the King Alfonso X

(Kasten, and Nitti, 2002]. In this last book we have found, an illustration, whole sentences from the manuscript. We are presenting the Azar game.

> *Otra manera ay de iuego de dados que llaman azar que se iuega en esta guisa.*
>
> *El qui primero ouiere de lançar los dados. si lançare XV. puntos o dizeseys. o dizesiete, o dizeocho. o las soçobras destas suertes. que son seys o cinco o quatro o tres; ganan. E qual quiere destas suertes en qual quier manera que uenga segundo los otros juegos que desuso dixiemos es llamado azar.*
>
> *E si por auentura no lança ninguno destos azares primeramientre. et da all otro por suerte una daquellas que son de seys puntos a arriba o de quinze ayuso; en qual quiere manera que pueda uenir. segundo en los otros iuegos dixiemos que uinien.*
>
> *E despues destas lançare alguna de las suertes que aqui dixiemos que son azar; esta suerte sera llamada; reazar. et perdera aquel que primero lançare. E otrossi si por auentura no lançare esta suerte. que se torna en reazar. et tomare pora sí una de las otras suertes que son de seys puntos a arriba o de quinze ayuso en qual quiere manera que uenga. Conuerna que lance tantas uegadas fasta que uenga una destas suertes o la suya por que gana. o la dell otro por que pierde. saluo ende si tomare aquella misma suerte que dio all otro; que serie llamada encuentro. E conuernie que tornassen a alançar como de cabo. E como quier que uiniesse alguna delas suertes que son llamadas azar o reazar, et entre tanto que uinie una daquellas que amos auian tomado pora ssi; non ganarie ninguno dellos por ella nin perderie fasta que se partiesse por las suertes; assí como desuso; diz.* [5fol. 67r° .The game called azar].

In the text of the game of Azar, the "soçobra" of a point is another one that is the complement of value 21. This means that, if on having thrown three dice the sum is 15 points, then the soçobra of 15 will be equal to 21 - 15 = 6 points. Where the value 21 = 3 (6 + 1) is, in this game of three dice, equal to 1 + 2 + 3 + 4 + 5 + 6, that is the sum of the points of the six faces of the die. Besides this, a point and its soçobra have the same probability (see appendix II). In case of throwing two dice, the point 4 is the "soçobra" of point 10, because 4 + 10 = 14 which at the same time is equal to 2 (6 + 1), that are, two dices, each one with six faces.

Supposing that player G1 throws three dice, we are going to describe the events that the Azar game considers.

(1) The set B = {15, 16, 17, 18, 6, 5, 4, 3} is called azar event, that is formed by the union of the points {15, 16, 17, 18} and {6, 5, 4, 3}. We also see that for every point of the set B, its corresponding "soçobra" is also in the set B. The name of the set B is motivated because it includes the less probable points.

(2) The set C = {7, 8, 9, 10, 11, 12, 13, 14} is called chance event and is the complementary of the set B.

Let's see now the rules of this game of Azar.

In the first throwing of player G1, he can extract an azar event or a chance event. If he extracts an azar event, wins the game, but if he extracts a chance event, for example, 9 points, then the point 9 is the chance that could make player G2 win the game.

If player G1 extracts a chance event in the first throwing, then he must throw three dice again (second throwing).

In this second throwing, player G1 can extract a point of the azar event, which will make player G1 lose the game, or he can extract a point of the chance event, for example 11 points. In this case, this point 11 is the chance that could make player G1 win the game.

If player G1 extracts his chance, for example 11 points, that is different from G2 chance, for example 9 points, then the player G1 must throw successively three dice until either he gets 11 points, his chance, before 9 appears, in which case G1 wins, or else G2 gets his chance, 9 points, before 11 appears, in which case G2 wins.

We observe that the set chance is the same to the players G1 and G2, which can make that the chance of G1 coincide with the chance of G2.

When the chances of G1 and G2 coincide, called "encuentro" in the Azar game language, the rule of the game says: "E conuerna que tornassen a alançar como de cabo", that means that there is not a winner and the game must begin again.

To continue describing this game we are using the following Table 1:

In the Table 1, the row headed by FC gathers the favorable cases associated with every point obtained by having thrown three dice.

We see in the Table 1 that player G1 wins the game if in the first throwing he extracts the azar event but if he extracts a point of chance set, that point will be the chance of player G2.

We also gather a second throwing in Table 1. Considering the chance belongs to player G2, if in the second throwing he extracts an azar event, then player G2 wins the game. If in this second throwing he extracts a chance point, this point will be the chance of player G1 and there will be two possibilities:

1) If the chances coincide (of G1 and G2), the game will start again (we gather this possibility with letter "C" in Table 1).

2) If the chances are different, that will lead the players to throw successively the three dice until player G1 get his chance before G2 get his chance and consequently player G1 will win the game. On else, if G2 gets his chance before than G1, G2 will win the game. These successively throwing are represented with the letter "R" in the Table 1.

In Appendix-II we have calculated that the probability of winning the player G1 is 0.5187. If both players bet the same quantity $\frac{A}{2}$, then the expected profit of G1 is $A \, 0.5187 - \frac{A}{2} = A \, 0.0187$, which is a positive small value, so player G1 has an advantage of 1.87 % over the whole bet.

Table 1. Game of Azar of Alfonso X

Points	3	4	5	6	7	8	9	10
FC	1	3	6	10	15	21	25	27
Points	18	17	16	15	14	13	12	11

AZAR SET

CHANCE SET

FIRST THROWING OF PLAYER G1

3	4	5	6	7	8	9	10
G1	G1	G1	G1	chance G2	chance G2	chance G2	chance G2
18	17	16	15	14	13	12	11

SECOND THROWING OF PLAYER G1

	7	8	9	10	11	12	13	14	CHANCES OF G2
3	G2	G2	G2	G2	G2	G2	G2	G2	
4	G2	G2	G2	G2	G2	G2	G2	G2	
5	G2	G2	G2	G2	G2	G2	G2	G2	
6	G2	G2	G2	G2	G2	G2	G2	G2	
15	G2	G2	G2	G2	G2	G2	G2	G2	
16	G2	G2	G2	G2	G2	G2	G2	G2	
17	G2	G2	G2	G2	G2	G2	G2	G2	
18	G2	G2	G2	G2	G2	G2	G2	G2	
7	C	R	R	R	R	R	R	R	
8	R	C	R	R	R	R	R	R	
9	R	R	C	R	R	R	R	R	
10	R	R	R	C	R	R	R	R	
11	R	R	R	R	C	R	R	R	
12	R	R	R	R	R	C	R	R	
13	R	R	R	R	R	R	C	R	
14	R	R	R	R	R	R	R	C	
CHANCES OF G1									

The Book of Dice does not specify if the Azar game is a fair game, nor the bets that were done. The fact of not deciding who can start the game leads us to think that the Azar game was a fair game for the authors. Our calculations corroborate that supposition, because the advantage of player G1 (the one that throws the dice) is only 1.87 % of the whole bet.

Also we want to highlight that the authors knew the behaviour of the points generated with the three dice, as they made differences between more probable points (the azar event), and the less probable one (the chance event).

Equally, the fact that a point and its “soçobra” were in chance set, make us think that the authors knew that a point and its “soçobra” must have the same behaviour. Let’s remember that if the faces of the die can appear with the same chance (due to the symmetry of the die), it would not be difficult for a mathematician to show that a point and its “soçobra” must appear with the same chance. Appendix III gathers a proof of this result.

The chances of each point are not calculated in this game of Azar, but they are calculated in other games of the Book of Dice: “E qual quiere destas suertes en qual quier manera que uenga segundo los otros juegos que desuso dixiemos es llamado azar”. If we go to a game named Triga (on folio 66r of the Book of Dice) we can read the following:

> *…los dizeochos, puntos, senas alterz; los dizesiete, senas cinco; los dizeséis, senas quatro e quinas seis;…; seis, quatro amas as o tres dos e as o dos dos alterz; cinco, tría e amas o dos dos as; quatro, dos e amas as; tres, amas as alterz.*

In the following Table 2 we gather the chances of the Triga game.

Table 2. Chances of Triga

Points	Chances
18	6-6-6
17	6-6-5
16	6-6-4; 5-5-6
15	6-6-3; 6-5-4; 5-5-5
6	4-1-1; 3-2-1; 2-2-2
5	3-1-1; 2-2-1
4	2-1-1
3	1-1-1

For example, the chance of point 17 is 6-6-5, where the order of the dice do not mind; if we consider the order of the dice, then the point 17 is generated by the following permutations (5-6-6), (6-5-6) y (6-6-5). Our doubt is if the sequence 6-6-5 represents the permutations (5-6-6), (6-5-6) y (6-6-5). It is clear that the notation is confusing and doesn’t let make difference between the combinations and permutations with repetition.

In the following paragraphs we are going to study the Hazard games that inherit many of the properties from the Azar game gathered in the Book of Dice.

3. RELATION AMONG THE GAMES OF HAZARD OF MONTMORT, COTTON AND HOYLE.

We have found a game called “Hazard” that it’s close to the Azar game.

The Hazard’s game is gathered (in p. 177-179) of the second edition of the book of Pierre Rémond de Montmort (1678-1719) *Essay d’Analyse sur les Jeux de Hazard* published in 1713.

The description of the Hazard's game begins (in p. 177):

PROBLÊME
SUR LE JEU DU HAZARD.
EXPLICATION DE CE JEU.

139. On y joue avec deux dés comme au Quinquenove. Nommons encore Pierre celui qui tient le dé, & ſuppoſons que Paul repréſente les autres Joueurs. Pierre pouſſera le dé juſqu'à ce qu'il ait amené ou 5, ou 6, ou 7, ou 8, ou 9; celui de ces nombres qui ſe préſentera le premier ſervira de chanſe à Paul, enſuite Pierre recommencera à pouſſer le dé pour ſe donner ſa chanſe. Or les chanſes de Pierre ſont ou 4, ou 5, ou 6, ou 7, ou 8, ou 9, ou 10; en ſorte qu'il en a deux plus que Paul, ſçavoir 4 & 10. Il faut encore ſçavoir ce qui ſuit :

1°. Si Pierre après avoir donné à Paul une chanſe qui ſoit ou 6 ou 8, amene au ſecond coup ou la même chanſe, ou douze, il gagne; mais s'il amene ou bezet, ou deux & as, ou onze, il perd. Z

2°. S'il a donné à Paul la chanſe de 5 ou de 9, & qu'il amene au coup ſuivant la même chanſe, il gagne; mais s'il amene ou bezet, ou deux & as, ou onze, ou douze, il perd.

3°. S'il a donné à Paul la chanſe de ſept, & qu'il amene le coup ſuivant ou la même chanſe, ou onze, il gagne; mais s'il amene ou bezet, ou deux & as, ou douze, il perd.

4°. Pierre s'étant donné une chanſe differente de celle de Paul, il gagnera s'il amene ſa chanſe avant que d'amener celle de Paul, & il perdra s'il amene la chanſe de Paul avant que d'amener la ſienne.

5°. Quand Pierre & Paul ont perdu, on recommence le jeu, en donnant de nouvelles chanſes; mais Pierre ne quitte le dé pour le donner à celui qui le ſuit, que lorſqu'il a perdu.

6°. S'il y a pluſieurs Joueurs, ils ont tous la même chanſe.

In this game, where two dice are thrown, play two players, which Montmort names Pierre and Paul. If there are more than two players, these bet together with Paul. Pierre does the function of the bank. Montmort supposes that Pierre begins the game throwing two dice until he extract a point of the event {5, 6, 7, 8, 9}, for example, 6 points. These six points will be the chance for player Paul.

Following this, Pierre throws the two dice again. If the dice show a value of the set {4, 5, 6, 7, 8, 9, 10}, for example, 5 points, these point will be the chance of Pierre.

To continue describing this game we are using the following Table 3:

Table 3. Game of Hazard of Montmort

Points	2	3	4	5	6	7	Paul
FC	1	2	3	4	5	6	
Points	12	11	10	9	8		
			4	5	6	7	Pierre
			3	4	5	6	
			10	9	8		
	Second throwing						
	Paul						
Pierre	5	6	7	8	9		
2	Paul	Paul	Paul	Paul	Paul		
3	Paul	Paul	Paul	Paul	Paul		
4	R	R	R	R	R		
5	Pierre	R	R	R	R		
6	R	Pierre	R	R	R		
7	R	R	Pierre	R	R		
8	R	R	R	Pierre	P		
9	R	R	R	R	Pierre		
10	R	R	R	R	R		
11	Paul	Paul	Pierre	Paul	Paul		
12	Paul	Pierre	Paul	Pierre	Paul		

We can see, in Table 3, that the chance's set of Paul is {5, 6, 7, 8, 9}, whereas the chance's set of Pierre is {4, 5, 6, 7, 8, 9, 10}. In the second throwing, if the dice show 2 or 3 points, Paul wins. If the chances of Paul and Pierre coincide, then Paul wins the game. Equally, if the chance of Paul is 5 or 9 points and the chance of Pierre is 11 or 12 points, Paul wins. Also, when the chance of Paul is 6 or 8 points, if the dice show 11 points, Paul wins, but if the dice show 12 points, Pierre wins. Finally, when the chance of Paul is 7 points, if the dice show 11 points, Pierre wins, but if the dice show 12 points, Paul wins.

When the chances don't coincide, event gathered with letter "R" in Table 3, Pierre must throw the two dice until either Paul gets his chance before the chance of Pierre appears, in which case Paul wins, or else Pierre gets his chance before the chance of Paul appears, in which cases Pierre wins.

Comparing this Hazard game with the game of Azar, we can observe that the set of chances of Paul corresponds to the chance set in the Azar game. Also, the set of chances of Pierre, which corresponds to the chance set in the game of Azar, is different from the set corresponding to Paul. In the game of Hazard there are different chance sets, which is useful to reduce the coincidences between the chances of both players.

We also see that, in the first throwing of the Azar game, if the event azar appears, which makes player G1 win the game, then the game ends. In the Hazard game, Pierre must throw the dice until the event {4, 5, 6, 7, 8, 9, 10} happen, which gives the chance to Paul. As we can observe, this last game doesn't finish when the event azar happens.

Another comparison between the games, Azar and Hazard, could be the different behaviour of the second throwing. In the Azar game, player G1 loses the game if the event azar happens; in the Hazard game, Pierre wins the game if happens a certain chance of Paul and a certain point of the azar set, as we can observe in Table 3.

Finally, Table 3 shows that the Hazard's game has eliminated the problem of beginning a new game, in the Azar game when there are coincidences between the players' chances. The Hazard's game proposes that Pierre win the game if these coincidences happen.

Montmort calculates (in p. 178 of his book), that the probability of winning at Hazard is $\frac{1979}{4032}$, which is approximately 0.4908. If each player bet the same quantity $\frac{A}{2}$, then the expected gain of player Pierre is $\frac{A.1979}{4032} - \frac{A}{2} = \frac{A.37}{4032}$, a negative value but small, which means that the disadvantage of Pierre is 0,917 % of the total bet.

We can see that if the Azar game is a favorable game to the player that throws the dice, G1, the game of Hazard is a favorable game to the player that doesn't throw the dice, Paul.

This game of Hazard described by Montmort (1678-1719) is gathered by Cotton (1641-1700), in chapter 34 (p. 168-173), of his book *The Compleat Gamester*, edition of 1674, where we can find a game named Hazzard.

Although the game of Hazzard of Cotton is the same game than the one of Montmort, Cotton name Caster to player Pierre and Setter to player Paul. Also he introduces new words, as "Main" to describe the chances of Setter (Paul) and "Chance" to describe the chances of Caster (Pierre). Cotton also use the word "Nick" to describe the event of coincidences between the chances of Caster and Setter or when the chances happen in cells [12, 6],[11, 7] or [12, 8], in Table 3 of Montmort, where Pierre wins the game, as we have seen before.

Cotton doesn't give a solution to this game. He only calculates the advantages of some points. Let's see an example. Cotton says that 5 points have "two chances": 3 - 2 and 4 - 1, while 7 points have "three chances": 5 - 2, 6 - 1 y 4 - 3. So, he concludes that 7 points have more advantage than 5 points. But, if we have into account 6 points, which have "three chances" this doesn't mean that 7 points have the same advantage than 6 points. This calculation is the same that the one found in the game of "Triga" of the Book of the Dice. In this game of Triga we can observe that there is not distinction between the combinations and the permutations, which leaves Cotton to calculate some chances with errors

About the advantages or disadvantages of this game of Hazzard, Cotton says (in p. 171-172) of his book that he had been seen an old man playing with a younger man, where the chance of the old man was 7 points and the chance of the other was 6. Cotton says that the bets of the old man were "ten pound" by day, and that he had lost in six days all his bests. Cotton trusted in a perfect squared dice. He had also realized some experiments with the dice that demonstrated that the old man had not much more advantage than the younger one. But he adds that the younger man had more probability of winning this game when he threw the dice.

It's very interesting how Cotton is trying to solve this game: the chance of Setter (the old man) is 7 points, and Caster's (the younger man) 6 points, and Cotton doubts about the advantages of both players although he says in the end that player Setter will have less advantage comparing to player Caster.

Following Montmort, in this game that Cotton is trying to solve, it's easy to proof that Setter has the probability of $\frac{6}{11}$ to win the game against the probability $\frac{5}{11}$ that player Caster has to win the game. When both players bet the same quantity the advantage of Setter is $\frac{A6}{11} - \frac{A}{2}$, which is approximately equal to $A0.04545$. So the advantage of Setter is 4.5 % over the whole bet, which confirms us what Cotton said about the advantage of Caster against Setter.

If we go to century XVIII, Bellhouse [1993, p. 416] says to us: "a revolution in the literature of gambling occurred in the 1740s with the publication of several boobs by Edmond Hoyle". Hoyle's first book was in 1743: "A Short Treatise on the Game of Whist". Hoyle's books were so popular that they went through several editions very quickly- fifteen editions by 1770. We are going to comment a reprint and a correction of the book of 1770 that was published in 1800, "Hoyle's games improved consisting of practical treatises on whist, quadrille, piquet, chess, back-gammon, draughts, cricket …".

In p. 231 to 235 is gathered a game called Hazard, which is the same game than the Hazard game of Montmort. Hoyle's book uses: the set Main; the chances of Setter; the set of Chances; the chances of Caster and the event called Nicks, which had been described by Cotton. The event {2, 3} is called "Crabs" by Hoyle, that as we have seen before in the Hazard game of Montmort, this event makes that Paul win the game.

Hoyle doesn't give a solution to this game of Hazard. He only gathers solutions based on supposing that the Main set of Setter and the Chance set of Caster are known. For example, if the Main Setter is 7 points and the Chance of Caster is 4 points, then the number of favorable chances to Setter is two and there is only one favorable chance to Caster. We know, thanks to Montmort, that this is a correct calculation.

4. RELATION WITH THE HAZARD GAME AND THE GAME OF CRAPS OF DE MOIVRE

De Moivre (1756) gathers in his book *The Doctrine of Chances: Or, A Method of Calculating the Probabilities of Events in Play* (p. 160), the following Problem XLVI of Hazard: "To find at Hazard the Advantage of the Setter upon all Suppositions of Main and Chance".

In this game there are two players, as in Cotton's Setter and Caster. The Main of Setter is 7 and the chances of Caster are {4, 5, 6, 8, 9, 10}.

Caster is the player that rolls a pair of dice once.

The rules of this game of Hazard of De Moivre are:

(a) If in the first throw, the dice shows a point of the chances of Caster {4, 5, 6, 8, 9, 10}, for example, if the dice shows 4 points, then Caster will throw successively two dice until either Caster get 4 points before 7 appears, in which case Caster wins, or else Setter get his chance 7 before the chance 4 of Caster appears, which cases Setter wins. If the chance of Caster is 5 point, we repeat the anterior process and so on until the chance 10 of Caster.

(b) If in the first throw, the dice shows 2, 3, or 12, Setter will win the game.

(c) If in the first throw, the dice shows 7 or 11, Caster will win the game.

160 *The* DOCTRINE *of* CHANCES.

PROBLEM XLVI.

Of HAZARD.

To find at Hazard the Advantage of the Setter *upon all Suppoſitions of Main and Chance.*

SOLUTION.

Let the whole Money played for be conſidered as a common Stake, upon which both the Caſter and the Setter have their ſeveral Expectations; then let thoſe Expectations be determined in the following manner.

Firſt, Let it be ſuppoſed that the Main is VII: then if the Chance of the Caſter be VI or VIII, it is plain that the Setter having 6 Chances to win, and 5 to loſe, his Expectation will be $\frac{6}{11}$ of the Stake: but there being 10 Chances out of 36 for the Chance to be VI, or VIII, it follows, that the Expectation of the Setter reſulting from the Probability of the Chance being VI or VIII, will be $\frac{10}{36}$ multiplied by $\frac{6}{11}$ or $\frac{60}{11}$ divided by 36.

Secondly, If the Main being VII, the Chance ſhould happen to be V or IX, the Expectation of the Setter would be $\frac{24}{5}$ divided by 36.

Thirdly, If the Main being VII, the Chance ſhould happen to be IV or X, it follows that the Expectation of the Setter would be 4 divided by 36.

Fourthly, If the Main being VII, the Caſter ſhould happen to throw II, III, or XII, then the Setter would neceſſarily win, by the Law of the Game; but there being 4 Chances in 36 for throwing II, III, or XII, it follows that before the Chance of the Caſter is thrown, the Expectation of the Setter reſulting from the Probability of the Caſter's Chance being II, III, or XII, will be 4 divided by 36.

Laſtly, If the Main being VII, the Caſter ſhould happen to throw VII, or XI, the Setter loſes his Expectation.

To continue describing this game we are using the following Table 4:

Table 4. Game of Hazard of De Moivre

Points	2	3	4	5	6	7	Caster
FC	1	2	3	4	5	6	□
Points	12	11	10	9	8	□	□
□							□
□					Points	7	Setter
□					FC	6	□
FIRST	THROWING					□	□
□	2	SETTER					□
□	3	SETTER					□
□	4	R					□
□	5	R					□
□	6	R					□
□	7	CASTER					□
□	8	R					□
□	9	R					□
□	10	R					□
□	11	CASTER					□
□	12	SETTER	□	□	□	□	□

We can see in Table 4 that player Setter wins the game if Caster throws 2, 3, or 12 points. If Caster throws 7 or 11, then Caster will win the game. In other cases the successively throwing (rule (a)), is represented by letter "R" in Table 4.

If we compare this game with the Azar game, we see that the chance 7 of Setter corresponds with the chance event of player G2 in the Azar game. If the chance event of the Azar game is determined when G1 throws the tree dice, the chance 7 of Setter is now determined before Caster throw the two dice. In addition, the chances {4, 5, 6, 7, 8, 9, 10} of Caster corresponds with the chance event of player G1 of Azar game.

From the rules of the game of De Moivre, the probability of winning the player Setter is $\frac{251}{495}$. This value is approximately 0.5071. If each player bet a monetary unit, then the expected gain of Setter is equal to $\frac{7}{495}$ (see the p. 160-161 of De Moivre).

If we compare this game of De Moivre with the game of Craps gathered in R. Isaac [1995], we conclude that De Moivre game is the Craps game. This game of Craps corresponds to column 7 in Table 3 of Montmort, where Setter and Caster are the players Paul and Pierre, respectively.

Returning to Problem XLVI of the book De Moivre (p. 161), he proposes the following game: "By the same Method of Process, if will be found that the Main being VI or VIII, the Gain of the Setter will be $\frac{167}{7128}$". In this game, we see that the Main set of Setter is now {6, 8}, while Caster has the same chances, {4, 5, 6, 7, 8, 9, 10}, as in the Craps game. This game of De Moivre corresponds to column 6 and 8 in Table 3 of Montmort, where Setter and Caster are Paul and Pierre, respectively. Another game that

De Moivre proposes is: "It will be also that the Main being V or IX, the Gain of the Setter will be $\frac{43}{2835}$". This game of De Moivre corresponds to column 5 and 9 in Table 3 of Montmort, where Setter and Caster are Paul and Pierre, respectively.

From the last generalizations of the game of Craps, De Moivre proposes the following Corollary 1:

> *If each particular Gain made by the Setter, in the Caster of any Main, be respectively multiplied by the number of Chances which there are for that Main to come up, and the Sum of the Products be divided by the number of the all those Chances, the Quotient will express the Gain of the Setter* before a Main is thrown*: from whence it follows that the Gain of the Setter, if he be resolved to set upon the first Main that may happen to be thrown, is to be estimated by* $\frac{42}{495}+\frac{1670}{7128}+\frac{344}{2835}$*, the whole to be divided by 24, which being reduced will be* $\frac{37}{2016}$.

Other expression of the gain of Setter is:

$$\frac{6\frac{7}{495}+10\frac{167}{7128}+8\frac{43}{2835}}{(6+10+8)}=\frac{37}{2016},$$

where if we return to Table 3 of Montmort, we see that the favourable cases of 7 are 6, of 5 or 9 are 10 and of 6 or 8 are 8. In addition, the phrase "before a Main is thrown" means that Caster must throw the two dice until rolling 5, 6, 7, 8, or 9; as player Pierre does in the game of Hazard of Montmort.

From the last comparison with the game of Monmort, we conclude that Corollary 1 of De Moivre is the game of Hazard of Montmort.

This last coincidence was gathered by the book of Todhunter [1865]. In his p. 163, he says:

> *Probl. XLVI is the game of Hazard; there is no description of the game here, but there is one given by Montmort (on his p. 177); and from this description, De Moivre's solution can be understood; his results agree with Montmort.*

In the Corollary 2, De Moivre calculates the odds of the game of Hazard of Montmort.

In the following Corollary 3, De Moivre returns to game of Craps. This Corollary said:

> *If it be agreed between the Caster and Setter, that de Main shall always be VII, and it be father agreed, that the next Chance happening to be Ames-ace, the Caster shall lose but half his stake, then the Caster's loss is only* $\frac{1}{3960}$ *of his Stake.*

We see that in this game of Craps, De Moivre proposes that when Caster throws 2 points, he loses but half his stake. This game modified the original loss $\frac{1}{495}$ by $\frac{1}{3960}$.

In the following Corollary 4, De Moivre applies the Corollary 3 to two generalisations of Craps. This Corollary said:

The Main being VI or VIII, and the Caster has $\frac{3}{4}$ *of his money returned in case he throws Ames-ace (equal to 2 points), what is his Loss? And if the Main being V or IX, and he has* $\frac{1}{2}$*?.*

In answer to the first, the Gain of the Setter or Loss of the Caste is $\frac{1}{385\frac{11}{37}}$.

In answer to the second, the Loss of the Caste would be but $\frac{1}{782\frac{2}{29}}$

The phrase "Ames-ace" has been used also, in some games with two dice, in the Book of Dice.

Finally, De Moivre proposes the following Corollary 5:

If it be made a standing Rule, that whatever the Main may happen to be, if the Caster throws Ames-ace immediately after the Main, or in other words, if the Chance be Ames-ace, the Caster shall only lose $\frac{1}{3}$ *of his own Stake, then the Play will be brought so near an Equality, that it will hardly be distinguishable from it; the Gain of the Caster being upon the whole but* $\frac{1}{6048}$ *of his own Stake.*

In this last Corollary, De Moivre modifies the game of Hazard of Montmort. He proposes that when Caster throws 2 point, then Caster will lose but $\frac{2}{3}$ his stake. If in the original game Caster loses $\frac{37}{2016}$, in this game modified Caster loses $\frac{1}{6048}$.

5. RELATION WITH THE "CINQ ET NEUF" GAME OF J. BERNOULLI

In Part 3 of Ars Conjectandi (1713), James Bernoulli solves a variety of problems of games of chances. He formulates and solves 24 problems related to popular games. We consider the problem 16 called Cinq et neuf (see [Hald 1990, p. 236-237]).

To continue describing this game we are using the following table 5:

Table 5. Game of Cinq et Neuf of Bernoulli

Die 2 \ Die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Two players, G1 and G2, play with two dice that we have called red die and black die. When G1 throws the dice, the outcomes are, for example, (5, 4), that is, the red die show 5 points and the black die show 4 points, or , for example, he throws 5 points, that is, he throws the outcomes (1, 4), (2, 3), (3, 2) or (4, 1). In Table 5 the event {5, 9}contains the chances of player G2 and it is equivalent to the chance event of the game of Azar, while the event {5, 9} or a doublet ((1, 1),(2, 2),(3, 3),(,4, 4), (5.5) or (6.6)) is equivalent to the azar event of the Azar game.

The rules of Cinq et neuf game are:

1) Player G1 throws two dice, if the dice show 5 or 9 points, G2 wins.
2) If the dice show 3 or 11 points or a doublet, G2 wins.
3) For the remaining outcomes, if we suppose that G1 gets the outcome (1,3), that is, the red die show 1 point and the black die show 3 points, then the point is 4. That point 4 is the chance of G1.
4) G1 continues playing until either G2 wins by getting 5 or 9 (chances of G1) points, or G1 wins by getting the same number of points that make him to continue, for example, 4 points (the chance of G1).

When we compare this game of Bernoulli with the Azar game, we see that in the game of Bernoulli the chances of G2 are selected before starting the game and, in addition, in both games, the player G1 wins the game when he throws the azar event in the first throws.

The G1's expectation is equal to $\frac{4189}{9009}$, which is approximately 0.4649. If each player stakes $\frac{A}{2}$, then the G1's expectation loss is $\frac{A.4189}{9009} \square \frac{A}{2} = \square \frac{A.631}{18018}$, and this loss represents 3.502 % of the whole bet.

6. CONCLUSIONS

In this article we have proposed a possible interpretation of the Azar game, gathered in the Book of Dice of Alfonso X the Learned. Note that the games, of The Book of Dice, had great difficulty of knowing its rules and forms of gambling. These problems can be explained from the interest of Alfonso X on the regulation of the games. His book on "Ordenamiento de las tafurerías" is a legal code regarding the operation of gaming houses. For example, while in his Law XXV we note that the priest can't play games in the gaming house, so:

> *... que el clerigo saca tablaje, e seguiere los dados, viene contra aquello que defiende la santa iglesia e las ordenes, porque debe pasar e juzgarse por el juicio que pasan los otros tafures…*

in a miniature of the Book of Chess, we can see a novice monk learning from a professed monk the most noble game, the chess, during the time off of their gentle monastery.

We have seen that while the chances of players G1 and G2 are the same in the Azar game, these chances are differences in the Hazard game, thus reducing its coincidences in the Hazard game.

We have also seen that when in the Azar game the chances of G1 and G2 coincide, the game must begin again. This solution has been simplified in the Hazard game, where, if both players coincide in their chances, then the player that throws the dice wins the game.

In the Azar game, the player G1 loss the game if in the second thrown obtains the azar event; this solution depends on Paul's chance in the Hazard game.

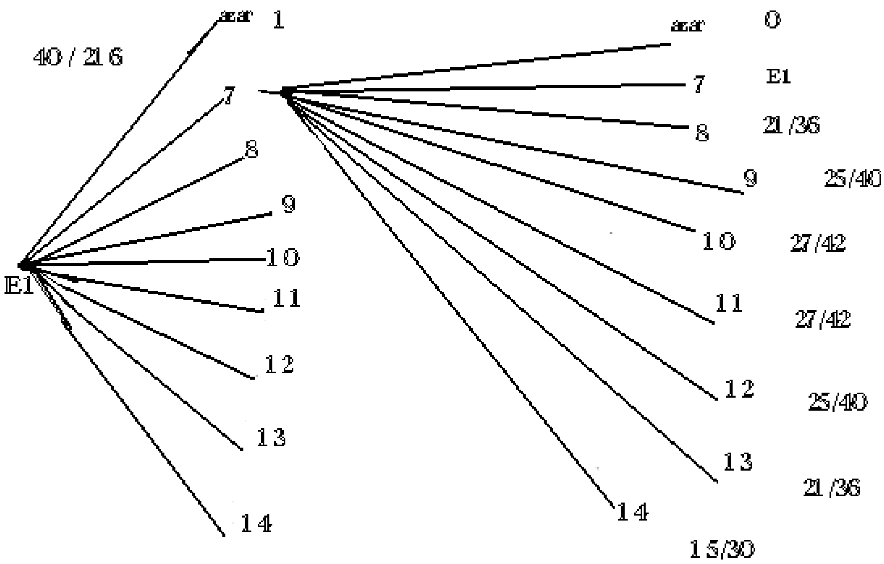
Finally, we have seen that the Problem XLVI of book *The Doctrine of Chances*, De Moivre solves the Craps game in his first problem and the Hazard game of Montmort in the corollary 1. The remainder problems are generalizations of Craps and Hazard games, where De Moivre wants to make fair games.

Bellhouse [2005] remind us that the Liber de Ludo Aleae (16th century) of Cardano could be viewed as a gambling manual. We want to point up that the Book of Dice (13th century) of Alfonso X is also a gambling games book.

APPENDIX-I

APPENDIX-II

The decision tree of the first two throw is:



We suppose the whole stake of both players is 1. The quantity E1 is the expectation or the probability of winning the player G1. The first branch of tree is equivalent to the player G1 throws the azar event, so the player G1 wins the game with probability (chance) equal to $\frac{40}{216}$; each one of the remaining branches is equivalent to the player G1 throw the points 7, 8, 9, 10, 11, 12, 13, or 14, respectively.

From the branch where the player G1 throws 7 points, if in the second throw the player G1 has thrown the azar event, then G1 loss the game and so his gain is zero; each one of the remaining branches is equivalent to G1 throw the point 7, 8, 9, 10, 11, 12, 13, or 14, respectively.

If in the second throw, G1 throws anew 7 points, then the G1's expectation is E1. The remaining branches, for example, the branch equivalent to the player G1 throws 8 points, G1 continues playing until either G2 wins by getting 7 points (chance of G1), or G1 wins by getting 8 points (chance of G1). In this last case, the probability of winning the player G1 is $\frac{21}{(21+15)} = \frac{21}{36}$. Other cases are similarly calculated.

Now, if in the first throw G1 throws 7 points, the G1's expectation is:

$$0\frac{40}{216} + E1\frac{15}{216} + A(7), \text{ where}$$

$$A(7) = \frac{21}{216}\left(\frac{21}{21+15}\right) + \frac{25}{216}\left(\frac{25}{25+15}\right) + \frac{27}{216}\left(\frac{27}{27+15}\right) +$$

$$+ \frac{27}{216}\left(\frac{27}{27+15}\right) + \frac{25}{216}\left(\frac{25}{25+15}\right) + \frac{21}{216}\left(\frac{21}{21+15}\right) + \frac{15}{216}\left(\frac{15}{15+15}\right)$$

In general, if in the first throw G1 throws i points, where I = 7, 8, 9, 10, 11, 12, 13, or 14, then the G1's expectation for i points is:

$$EP(i)A(i) + \quad ,$$

where the general expression of A(i) is:

$$A(i) = \sum_{k=7}^{14} \left(\frac{FC(k)}{216}\right)\left(\frac{FC(k)}{FC(k)+FC(i)}\right),$$

and P(i) is the ratio between the favorable cases(FC) of point i and 216. The favorable cases are calculated in the following table.

Table 6. Chances of Azar game

Points	3	4	5	6	7	8	9	10
Favorable Cases(FC)	1	3	6	10	15	21	25	27
Points	18	17	16	15	14	13	12	11

Now, the expectation E1, at the beginning of tree, is equal to expression:

$$E1 = \frac{40}{216} + E1 \sum_{i=7}^{14} P(i)^2 + \sum_{i=7}^{14} A(i)P(i). \qquad (1)$$

From expresión (1), we obtain that the player G1 wins the game with a probability equal to 0.5187.

APPENDIX-III

Let r dice be, where each die has k faces which are marked with numbers 1, 2 … k.

The F set contains the possible total points when we throw the r dice. Two points, x and y, of F, are complementary if $x + y = r(k+1)$.

COROLLARY. If x and y are two complementary points of F, then the total of permutations with repetition, m, generate from x is equal to the total of permutations with repetition, m, generate from y.

Proof. If $(a_1, a_2, \ldots, a_r)$ is a permutation with repetition generated from x, that is, $x = \sum_{i=1}^{r} a_i$, then the transformation $b_i = k + 1 - a_i$, i = 1, 2,…, r, generate the permutation $(b_1, b_2, \ldots, b_r)$ such that $\sum_{i=1}^{r} b_i = r(k+1) - x$, and thus $x + y = r(k+1)$. The transformation $b_i = k + 1 - a_i$ is one-to-one and thus the points x and y have the same chances.

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