Convex Difference Inclusions for Systems Analysis and Design

Inclusiones Convexas para el Análisis y el Diseño

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Ever tried. Ever failed. No matter.
Try again. Fail again. Fail better.

S. Beckett.

To my family.

To Eduardo and Teo.

Contents

Notation

$\mathbb{N}$	Set of natural number.
$\mathbb{Z}$	Set of integer number.
$\mathbb{R}$	Set of real number.
$\mathbb{R}_+$	Set of non-negative real numbers.
$\mathbb{R}_-$	Set of non-positive real numbers.
$\mathbb{N}_{[a,b]}$	Set of integers included between two integers $a,b \in \mathbb{N}$: $\mathbb{N}_{[a,b]} = \{x \in \mathbb{N} : a \leq x \leq b\}$.
$\mathbb{N}_a$	Set of positive integers smaller or equal than $a \in \mathbb{N}$, with $a \geq 1$: $\mathbb{N}_{[1,a]}$.
A_i	i-th row of matrix $A \in \mathbb{R}^{n\times m}$, with $i \in \mathbb{N}_n$ (i-th element of vector A if $m = 1$).
e^j	Vector in $\mathbb{R}^n$ whose entries are zero, but the j-th which is 1: $e^j = [0\ \dots\ 0\ 1\ 0\ \dots\ 0]^T$.
I_m	Identity matrix of dimension $m \in \mathbb{N}$.
$0_{m,n}$	Matrix in $\mathbb{R}^{m\times n}$ whose entries are 0.
$[A,\ B]$	Matrix concatenation of matrices A and B.
$\mathscr{S}(D)$	Set of subsets of D.
$\mathscr{K}(D)$	Set of convex, compact subsets of D.
$\mathscr{K}^0(D)$	Set of convex, compact subsets of D containing the origin in their interior, i.e., $0 \in \text{int}\,(D)$.
$\mathbf{B}_p^n$	p-norm unitary ball in $\mathbb{R}^n$ centered in the origin: $\mathbf{B}_p^n = \{x \in \mathbb{R}^n : \ \|x\|_p \leq 1\}$.
αD	Set $\alpha D = \{\alpha x : \ x \in D\}$, where $\alpha \geq 0$ and $D \subseteq \mathbb{R}^n$.
$\text{co}(D)$	Convex hull of set $D \subseteq \mathbb{R}^n$, i.e., the set of points that can be expressed as a convex combination of a subset of elements of D.
$\text{int}\,(D)$	Interior of a set $D \subseteq \mathbb{R}^n$.
$\text{ri}\,(D)$	Relative interior of a set $D \subseteq \mathbb{R}^n$.
∂D	Boundary of set D.

$\nabla_x f(x_0)$	Gradient of function $f : X \to Y$ with respect to $x \in X$ at $x_0 \in X$.
$f(D)$	Image of set $D \subseteq \mathbb{R}^n$ through function $f : \mathbb{R}^n \to \mathbb{R}^m$: $f(D) = \{f(x) : x \in D\}$.
$\mathrm{sgn}\ (\cdot)$	Sign operator.
$\phi_D(\eta)$	Support function of $D \subseteq \mathbb{R}^n$ evaluated at $\eta \in \mathbb{R}^n$, see Appendix C.
$\oplus$	Minkowski set addition. Given two sets $U,\ V \subseteq \mathbb{R}^n$, we have $U \oplus V = \{x = u + v :\ u \in U,\ v \in V\}$.
$\ominus$	Pontryagin (or Minkowski) set difference. Given two sets $U,\ V \subseteq \mathbb{R}^n$, we have $U \ominus V = \{x \in \mathbb{R}^n :\ x + v \in U,\ \forall v \in V\} = \{x \in \mathbb{R}^n :\ x \oplus V \subseteq U\}$.
$\bigoplus_{i=0}^{N} D^i$	Minkowski sum of sets $D^i \subseteq \mathbb{R}^n$, with $i \in \mathbb{N}_N$.
$\mathscr{E}(P)$	Ellipsoid determined by the positive definite matrix $P \in \mathbb{R}^{n \times n}$ with $P = P^T > 0$: $\mathscr{E}(P) = \{x \in \mathbb{R}^n :\ x^T P x \leq 1\}$.

Acronyms

CDI	Convex Difference Inclusions.
CCDI	Concave-Convex Difference Inclusions.
DC	Difference of convex.
LDI	Linear Difference Inclusions.
LPV	Linear Parameter Varying.
MPC	Model Predictive Control.
LMI	Linear Matrix Inequalities.

Chapter 1

Introducción

El objetivo principal de esta tesis es contribuir al desarrollo de métodos basados en la teoría de conjuntos para el análisis y el diseño de sistemas no lineales e inciertos. Particular atención será dedicada a los conjuntos invariantes y λ-contractivos, muy importantes en el contexto del diseño del control y del análisis de sistemas no lineales e inciertos.

Este capítulo describe la motivación y los objetivos de la tesis, presenta el problema tratado e introduce la estructura de la tesis y el trabajo desarrollado. Primero se introducirán algunos conceptos relacionados con la teoría de conjuntos en el ámbito del control y del análisis de sistemas dinámicos, haciendo particular hincapié en el concepto de invariancia. Se darán las definiciones básicas y se describirán algunas propiedades de los conjuntos invariantes. Se ilustrarán aspectos que hacen evidente la importancia de la invariancia y de los métodos basados en la teoría de conjuntos para el control.

Se proporcionará un resumen de las principales contribuciones, presentes en la literatura, que han llevado a los más recientes resultados en el campo de investigación tratado, seguido por una introducción al estado del arte sobre la invariancia y los métodos basados en la teoría de conjuntos.

Luego se proporcionará una introducción al concepto de convexidad, tanto para conjuntos como para funciones. La convexidad merece especial interés, siendo un ingrediente clave para muchos de los resultados presentados en la tesis. Las propiedades de los conjuntos convexos y de las funciones convexas serán extensivamente usadas a lo largo de la tesis, debido al hecho de que, dado un conjunto, su convexidad permite formular propiedades generales basadas en condiciones que implican solamente un subconjunto, posiblemente finito, de dicho conjunto.

Finalmente la motivación, los objetivos y las contribuciones del trabajo de investigación serán ilustrados y la estructura de la tesis presentada.

1.1 Teoría de conjuntos en el ámbito del control

Generalmente, con métodos basados en la teoría de conjuntos nos referimos a aquellas técnicas que conciernen propiedades compartidas por todos los elementos de unos conjuntos del espacio de estados. Dos importantes ejemplos, en el campo del análisis de sistemas dinámicos y del diseño de control, que implican la utilización de métodos basados en la teoría de conjuntos, son la invariancia y el enfoque "worst-case" (del caso peor) para el análisis y el diseño.

Además, los metodos basados en la teoría de conjuntos resultan muy útiles para el análisis y el diseño de control para sistemas inciertos, con incertidumbre desconocida pero acotada.

1.1.1 Enfoque worst-case para el análisis y el diseño

El enfoque clásico para tratar los problemas de análisis y control para sistemas inciertos estaba basado, hasta el final de los años sesenta, en hipótesis estocásticas sobre la naturaleza de la incertidumbre. El objetivo del control óptimo en este contesto es, en general, la determinación de la ley de control que minimiza el valor de una función de coste, bajo la suposición de una incertidumbre caracterizada por una distribución de probabilidad dada. Análogamente, asumiendo por ejemplo que el sistema es lineal y que la condición inicial, las medidas y los ruidos que afectan el sistema son modelados por procesos blancos Gaussianos, el problema de estimación es solucionado por el filtro de Kalman, que proporciona la solución óptima que minimiza el valor esperado del error de estimación.

Una manera paralela, y en cierto sentido dual, de tratar los problemas de análisis y control para sistemas inciertos es a través del enfoque worst-case (o garantista). Este enfoque está basado en hipótesis diferentes sobre la incertidumbre que afecta al sistema. En este escenario la incertidumbre es asumida desconocida, pero acotada en un conjunto. Tal enfoque está basado en las siguientes consideraciones:

- la asunción del conocimiento exacto de la función de distribución de probabilidad de los ruidos y de las perturbaciones puede ser demasiado restrictiva, mientras la suposición de presencia de cotas conocidas para la incertidumbre puede ser más realista en muchos casos. De hecho, el enfoque worst-case es justificado a menudo por el hecho de que ninguna suposición probabilística sobre las incertidumbres puede ser hecha, mientras que las cotas sobre los errores de modelos pueden ser establecidas en muchos casos. Se consideren sistemas con dinámica dependiente de parámetros, cuyos valores posibles están restringidos por cotas debidas a limitaciones físicas conocidas. En este caso el enfoque worst-case resulta más realista que el estocástico.

- Cuando el sistema presenta restricciones duras, el enfoque worst-case presenta algunas ventajas. Si consideramos el enfoque estocástico, ninguna garantía de cumplimiento de las restricciones puede ser asegurada, mientras los resultados proporcionados por el enfoque worst-case pueden asegurar la satisfacción de las restricciones, siempre que las hipótesis sobre la incertidumbre se cumplan.

- La asunción de presencia de incertidumbre en el sistema puede ser usada para tratar no linealidades. Por ejemplo, supóngase que las dinámicas no lineales del sistema sean conocidas. En este caso, un sistema lineal con incertidumbre acotada puede ser empleado, asumiendo que la incertidumbre modela la discrepancia con el sistema no lineal. Este procedimiento de aproximación, si bien introduce algún conservadurismo, permite aplicar resultados basados en la linealidad a sistemas no lineales. Los sistemas de inclusión de diferencias lineales (LDI) y los sistemas lineales con incertidumbre aditiva son modelos clásicos en este contexto, véase (Gurvits, 1995; Boyd, El Ghaoui, Feron and Balakrishnan, 1994).

Los métodos basados en la teoría de conjuntos se presentan como posibles alternativas a los estocásticos, para los problemas del análisis de estabilidad, de diseño de control robusto y de estimación de estado para sistemas afectados por incertidumbres. La suposición de modelar la incertidumbre como desconocida pero acotada, en vez de como proceso estocástico, fue presentada en los trabajos pioneros de Witsenhausen (1968*b*), Schweppe (1968), Bertsekas y Rhodes (1971).

Vale la pena notar que el objetivo del enfoque worst-case es obtener conjuntos de elementos que satisfacen las especificaciones requeridas, más que el particular elemento óptimo con respecto a un criterio de evaluación. Por ejemplo, las regiones del espacio de estado cuyos puntos aseguran la satisfacción de restricciones para el control son el análogo del control LQG estocástico, mientras la estimación garantizada es el análogo worst-case del filtro de Kalman.

1.1.2 Invariancia

El concepto de invariancia ha llegado a ser fundamental para el análisis de sistemas y el diseño de control. Aunque muchos esfuerzos de investigación hayan sido dirigidos a este tema a lo largo de la segunda mitad del siglo pasado, el campo se hizo particularmente activo en los ultimos años. La importancia de los conjuntos invariantes en control es debida a las propiedades de estabilidad implícitas en estas regiones del espacio de estados.

Un conjunto invariante para un sistema dinámico es una región del espacio de estados tal que la trayectoria generada por el sistema queda contenida en el conjunto, si la condición inicial está dentro de él, (Blanchini and Miani, 2008). Definiciones más formales de invariancia se proporcionan en el Apéndice A, una caracterización conceptual de invariancia es suficiente aquí para mostrar como la invariancia puede ser usada en el ámbito del control y sus principales propiedades.

Particularmente relevante es la propiedad de invariancia robusta (de control) para un conjunto, ya que ésta puede ser usada en los contextos del análisis de estabilidad y de la satisfacción de restricciones para sistemas dinámicos en presencia de incertidumbres desconocidas pero acotadas. También el problema de la convergencia de estrategias de control está fuertemente relacionado con el concepto de invariancia robusta de control. El esquema en la Figura 1.1 representa las relaciones entre invariancia, λ-contractividad y algunos de los más importantes conceptos implicados en la teoría del control.

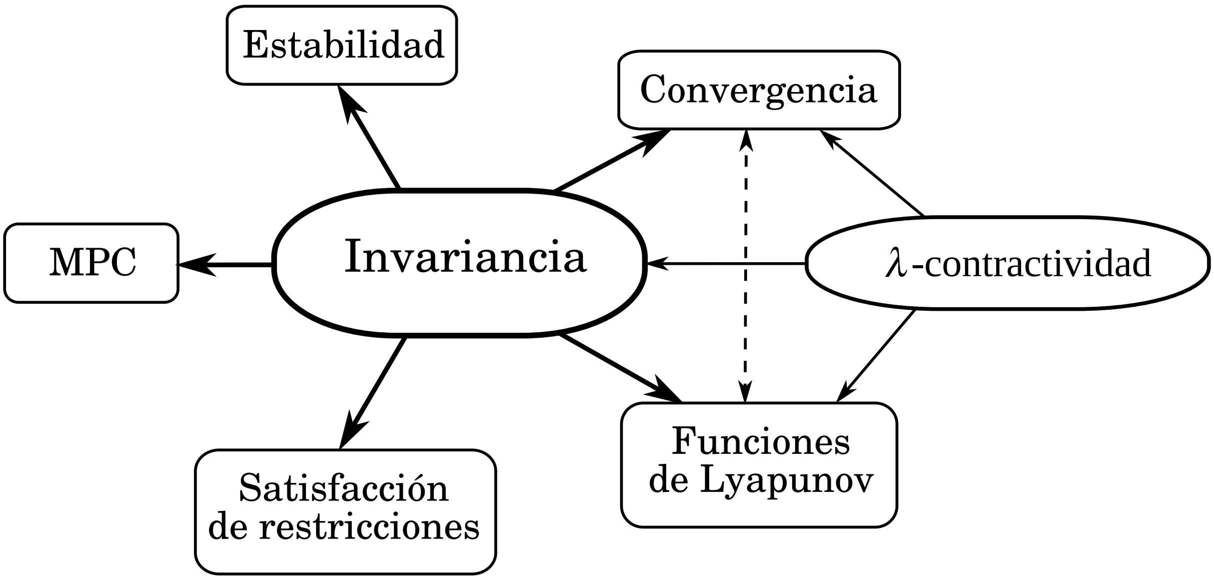

Figure 1.1: Invariancia en control.

1.1.2.1 Invariancia y satisfacción de restricciones duras

Se considera la definición estándar de invariancia para sistemas determinísticos autónomos tiempo discreto, véase (Blanchini and Miani, 2008). Más definiciones y propiedades relacionadas con la invariancia (por ejemplo, para sistemas inciertos, para sistemas no autónomos, conjuntos λ-contractivos, etc.) se presentan en el Apéndice A.

Considérese el sistema autónomo tiempo discreto

$$x^+ = f(x), \tag{1.1}$$

donde $x \in \mathbb{R}^n$ es el estado, $x^+ \in \mathbb{R}^n$ es el estado sucesor y $f : D \to \mathbb{R}^n$ es una función definida en el conjunto $D \subseteq \mathbb{R}^n$.

Un subconjunto del espacio de estados, $\Omega \subseteq D$, es un conjunto invariante positivo si cada trayectoria dada por x_k, con $k \in \mathbb{N}$, generada por (1.1) y con $x_0 \in \Omega$, es tal que $x_k \in \Omega$ para todo $k \in \mathbb{N}$. En la práctica, Ω es un conjunto invariante positivo si cada trayectoria generada por el sistema dinámico con condición inicial x_0 en Ω, permanece en el conjunto Ω.

Aunque la invariancia de un conjunto sea una propiedad que concierne a todas las trayectorias generadas por el sistema dinámico con condición inicial en Ω, puede ser enunciada a través de una definición alternativa, que no implica explícitamente las trayectorias. De hecho, un conjunto $\Omega \subseteq D$ es un invariante positivo para el sistema autónomo tiempo discreto (1.1) si $f(x) \in \Omega$, para todo $x \in \Omega$.

Se puede demostrar que cualquier elemento de un conjunto invariante Ω es mapeado por la función dinámica dentro de Ω si y sólo si la trayectoria entera generada por el sistema, con el estado inicial en Ω, permanece contenida en el conjunto invariante. En la práctica, si $x_0 \in \Omega$ entonces, por definición de invariancia, tenemos que $x_1 = f(x_0) \in \Omega$, que implica $x_2 = f(x_1) \in \Omega$ etcétera. Entonces $x_k \in \Omega$, para todo $k \in \mathbb{N}$.

Nótese que se ha empleado el término invariante *positivo*, para distinguirlo del concepto de invariancia simple. Históricamente, el término invariante denota un conjunto de condiciones iniciales cuyas trayectorias hacia atrás y adelante en el tiempo son contenidas en el mismo conjunto, mientras que para un invariante positivo sólo la parte futura de las trayectorias tiene que pertenecer al conjunto. Cómo en esta tesis estamos interesados exclusivamente en los conjuntos invariantes positivos, nos referiremos a ellos simplemente como conjuntos invariantes.

La invariancia también puede ser expresada en términos de la imagen de Ω a través de la función $f(\cdot)$. De hecho, un conjunto $\Omega \subseteq D$ es un conjunto invariante si

$$f(\Omega) \subseteq \Omega.$$

Se pueden dar definiciones análogas para sistemas no autónomos, es decir en presencia de una entrada de control. Un conjunto invariante de control es una región Ω del espacio de estados tal que, para cualquiera de sus elementos $x \in \Omega$, existe una entrada de control $u(x)$ que mantiene el estado sucesor dentro de Ω. Esto conlleva que, considerando un conjunto invariante de control, existe al menos una ley de control $u(x)$ definida en Ω tal que el conjunto es invariante para el sistema en bucle cerrado con $u(x)$.

Es evidente la relación entre satisfacción de restricciones duras e invariancia, para un sistema genérico. Supongamos que se requiere que el estado del sistema sea mantenido dentro de una región del espacio de estados, es decir, en el conjunto $X \subseteq \mathbb{R}^n$. La existencia de un conjunto invariante Ω contenido en X asegura que, si el estado actual del sistema está contenido en Ω, entonces ninguna violación de restricciones ocurrirá nunca, para todo $k \in \mathbb{N}$.

De hecho, cualquier conjunto invariante $\Omega \subseteq X$, por definición de invariancia, satisface

$$f(\Omega) \subseteq \Omega \subseteq X,$$

que significa que $x_k \in X$ para todo $k \in \mathbb{N}$, donde $x_{k+1} = f(x_k)$, con condición inicial $x_0 \in \Omega$. Esto implica que cualquier elemento de la trayectoria no deja el conjunto Ω, de ahí que ninguna violación de restricción ocurrirá en la futura evolución del sistema. Una vez más, vale la pena notar que, aunque la invariancia es una condición que concierne al comportamiento del sistema en cualquier instante de tiempo futuro, desde el presente hasta el infinito, puede ser caracterizada por una simple condición geométrica.

1.1.2.2 Máximo conjunto invariante y operador a un paso

La fuerte relación entre satisfacción de restricciones duras e invariancia justifica el interés por el máximo conjunto invariante contenido en una región del espacio de estado, ver referencias (Gutman and Cwikel, 1986; Gutman and Cwikel, 1987; Gilbert and Tan, 1991; Blanchini, 1999) y (Kolmanovsky and Gilbert, 1998).

Considerando una región X del espacio de estados, muchos conjuntos invariantes pueden estar contenidos en ella. Por ejemplo, es evidente que cualquier punto de equilibrio contenido en X es un conjunto invariante. El máximo conjunto invariante es un conjunto que es invariante para el sistema y contiene cualquier otro conjunto invariante. Es fácil demostrar que el máximo conjunto invariante, cuando existe y es no vacío, está formado por todos los elementos de X tales que sus evoluciones nunca abandonarán X. Esto significa que un punto x pertenece al máximo conjunto invariante si y sólo si la trayectoria generada por el sistema con condición inicial $x_0 = x$ nunca viola la restricción, es decir $x_k \in X$ para todo $k \in \mathbb{N}$. Por otra parte, si un punto no pertenece al máximo conjunto invariante, entonces seguramente habrá un instante del tiempo futuro en el cual una violación de restricción ocurrirá. En la práctica, el máximo conjunto invariante contenido en X puede ser considerado como el conjunto de puntos ”seguros” en X, en el sentido de que ninguna violación de restricción ocurrirá en el futuro.

Aunque se hayan propuesto muchos procedimientos algorítmicos para calcular el máximo conjunto invariante, hay una idea básica común a todos ellos. Los procedimientos iterativos están basados en el empleo del operador a un paso $Q(\cdot)$. Considerando un conjunto $\Omega \subseteq X$ en el espacio de estados y un sistema dinámico, el conjunto a un paso $Q(\Omega)$ viene dado por el conjunto de puntos en X cuya evolución a través de la función dinámica está contenida en Ω. Es decir, considerando $\Omega \subseteq X$, un punto x pertenece al conjunto a un paso $Q(\Omega)$ si $x \in X$ y $f(x) \in \Omega$. De ahí, $X_1 = Q(X)$ es el conjunto de puntos de X que permanecen en X al menos en el primer instante. Está claro que el uso iterativo genera una secuencia de conjuntos $X_{k+1} = Q(X_k) \cap X_k$ tales que un punto x pertenece a X_k si y sólo si la trayectoria generada con condición inicial $x_0 = x$ queda en X al menos durante los primeros k pasos, para todo $k \in \mathbb{N}$. Debería ser también evidente que el máximo conjunto invariante puede ser obtenido iterando el procedimiento para un número infinito de pasos.

El resultado no sería muy útil en la práctica si el máximo conjunto invariante no pudiera ser obtenido después de un número finito de iteraciones. En este caso el máximo conjunto invariante se dice finitamente determinado y el número finito de pasos que lo genera se denota índice de determinación. Existen importantes contribuciones en la literatura, principalmente para sistemas lineales, que permiten establecer condiciones para que el máximo conjunto invariante sea finitamente determinado.

Otra importante propiedad del operador a un paso es el hecho de que la aplicación del operador a un conjunto que es invariante genera otro conjunto invariante que contiene el anterior. Así, el uso iterativo del operador a un paso, con un conjunto invariante dado como elemento inicial, produce una secuencia creciente de conjuntos invariantes. Nótese que el mismo proceso iterativo con $X_0 = X$ como elemento inicial, genera una secuencia de conjuntos no necesariamente invariantes, lo que implica que la invariancia del conjunto a un paso $k \in \mathbb{N}$ no está garantizada, hasta que el índice de determinación sea alcanzado (si finito).

En algunos casos, se puede demostrar que las iteraciones inicializadas con un conjunto invariante convergen al dominio de atracción de un punto de equilibrio, es decir al conjunto de los estados cuyos elementos convergen al equilibrio. Claramente, se requieren asunciones sobre la estabilidad del sistema en este caso.

1.1.2.3 Conjuntos alcanzables y mínimo conjunto invariante

En esta sección se introducen dos importantes conceptos de la teoría de conjuntos como los conjuntos alcanzables y el mínimo conjunto invariante para sistemas dinámicos afectados por incertidumbre aditiva. Los dos conceptos están fuertemente relacionados, ya que el mínimo conjunto invariante puede ser visto como el conjunto límite de la secuencia de conjuntos alcanzables.

Considérese un sistema lineal asintóticamente estable afectado por incertidumbre aditiva, es decir

$$x^+ = Ax + w,$$

donde w es la incertidumbre y $w \in W$, con W subconjunto acotado del espacio de estados con $0 \in W$. Nótese que, debido a la presencia de incertidumbre aditiva, el sucesor de un estado depende de la realización de la incertidumbre y todos los posibles sucesores de un estado forman un conjunto. Es decir, dado un estado x, el conjunto de sucesores está dado por $(Ax \oplus W) \subseteq \mathbb{R}^n$.

En este contexto, es útil introducir el concepto de conjuntos alcanzables. El conjunto alcanzable en $k \in \mathbb{N}$ es el conjunto de los estados que pueden pertenecer en el instante k a una trayectoria, para condiciones iniciales dadas y para una realización admisible de la incertidumbre. Entonces, dado un conjunto inicial $R_0 \subseteq \mathbb{R}^n$, el conjunto alcanzable, para cada instante $k \in \mathbb{N}$, puede ser obtenido recursivamente como

$$R_{k+1} = AR_k \oplus W = A^{k+1}R_0 \oplus \bigoplus_{i=0}^{k} A^i W.$$

El conjunto R_k se denomina conjunto alcanzable en el instante $k \in \mathbb{N}$, la secuencia de R_k es denominada tubo alcanzable, alcanzable desde R_0. La secuencia de conjuntos alcanzables es interesante ya que contiene la información sobre todas las trayectorias posibles generadas por un sistema incierto con condición inicial contenida en R_0. Los conjuntos alcanzables para un sistema lineal incierto pueden ser usados para calcular acotaciones de la evolución real de un sistema no lineal, mientras que la discrepancia entre los dos modelos esté acotada por W. Además, el tubo alcanzable puede ser visto como el resultado de la estimación de estado en ausencia de medidas. La computación de conjuntos alcanzables es usada también en el enfoque worst-case para la estimación de estado, integrándolo con la información proporcionada por una medida.

Dignos de particular interés, son los conjuntos (y el tubo) alcanzables para los sistemas lineales inciertos con el origen como condición inicial, dados por

$$R_{k+1} = AR_k \oplus W = \bigoplus_{i=0}^{k} A^i W,$$

con $R_0 = \{0\}$.

En presencia de restricciones duras para sistemas lineales, los conjuntos alcanzables pueden ser usados para plantear una condición suficiente para la exclusión de cualquier violación de restricciones a lo largo de todas las trayectorias posibles. De hecho, si los conjuntos alcanzables están contenidos en la región admisible del espacio de estados, entonces ninguna violación de restricciones es posible.

Esta idea puede ser utilizada para diseñar leyes de control que garantizan la satisfacción de restricciones duras, ver (Chisci, Rossiter and Zappa, 2001). Aquellas leyes de control robustas basadas en la información proporcionada por los conjuntos alcanzables se denominan estrategias de control basadas en tubos. Los enfoques basados en tubos, presentados primero en (Witsenhausen, 1968*a*; Bertsekas and Rhodes, 1971*a*; Glover and Schweppe, 1971), proporcionan una solución para el problema del control robusto en presencia de incertidumbre desconocida pero acotada.

Más recientemente, este enfoque ha sido extendido a las estrategias de control predictivo basado en modelo (model predictive control, MPC), véase (Mayne, Rawlings, Rao and Scokaert, 2000; Chisci et al., 2001; Camacho and Bordóns, 2004; Limón, Álamo and Camacho, 2005; Bravo, Álamo and Camacho, 2006), y (Langson, Chryssochoos, Raković and Mayne, 2004; Magni, De Nicolao, Magnani and Scattolini, 2001), estrategias de control robusto muy apropiadas en presencia de restricciones duras. Es evidente cuanto útil puede resultar, de hecho, el concepto de los conjuntos alcanzables en el contexto del control predictivo basado en modelo para sistemas afectados por incertidumbre aditiva, donde se requiere una predicción del estado.

Cómo ya se ha mencionado, un conjunto invariante particularmente interesante es el mínimo. El mínimo conjunto invariante para un sistema es el conjunto invariante contenido en cada otro conjunto invariante. Se puede demostrar que el mínimo conjunto invariante es el conjunto de puntos en el espacio de estados que puede ser alcanzado desde el origen. Conceptualmente, es el conjunto de todos los estados que pueden pertenecer a todas las trayectorias posibles generadas por el sistema, con el origen como condición inicial. También puede ser demostrado que el mínimo conjunto invariante para un sistema incierto lineal es dado por

$$R_\infty = \bigoplus_{i=0}^{\infty} A^i W,$$

que es el conjunto alcanzable R_k, desde el origen, cuando k tiende a infinito. Es evidente por definición que el mínimo conjunto invariante exacto no puede ser obtenido, en general. Los métodos para calcular aproximaciones del mínimo conjunto invariante son el objetivo de recientes trabajos de investigación, véase (Raković, Kerrigan, Kouramas and Mayne, 2005; Ong and Gilbert, 2006).

A diferencia del máximo conjunto invariante, interesante tanto para sistemas determinístico como inciertos (lineal o no lineal), el mínimo conjunto invariante es significativo sólo en presencia de incertidumbre aditiva. Además, hay que notar que el mínimo conjunto invariante ha sido estudiado principalmente para sistemas lineales.

El interés en la computación del mínimo conjunto invariante y sus propiedades ha surgido más recientemente. Los motivos que hacen al mínimo conjunto invariante interesante en el ámbito del control son menos intuitivos que aquellos del máximo conjunto invariante. El mínimo conjunto invariante es útil en los siguientes contextos:

- Importantes condiciones para la existencia y la determinación finita del máximo conjunto invariante para un sistema lineal incierto están basadas en el mínimo conjunto invariante. Nótese que, si el mínimo conjunto invariante, calculado en ausencia de restricciones, no está contenido en la región admisible X, entonces ningún conjunto invariante robusto puede ser obtenido. Esto quiere decir que si el mínimo conjunto invariante no está contenido en el conjunto X, entonces existe una secuencia de realizaciones de la incertidumbre que conduce al estado a violar las restricciones, para cualquier condición inicial en X.

- El concepto clásico de estabilidad asintótica de un punto de equilibrio para un sistema no es aplicable en caso de presencia de incertidumbre aditiva. Recordamos aquí, sólo conceptualmente, que un sistema es asintóticamente estable si las trayectorias son acotadas (por lo menos aquellas que empiezan en una vecindad del equilibrio) y la distancia entre el estado y el equilibrio converge a cero. Está claro que, a no ser que se asuma que la incertidumbre desaparece cuando el sistema se acerca al equilibrio (considérese por ejemplo el caso de incertidumbre modelada como una función del estado), el sistema no puede ser mantenido en el origen. De hecho, ningún equilibrio es admitido. Un concepto análogo a la estabilidad asintótica puede ser formulado para el caso de presencia de incertidumbre aditiva, substituyendo el punto de equilibrio con un conjunto del espacio de estados y la distancia desde el equilibrio con la distancia desde dicho conjunto. Este concepto se denomina acotación terminal (ultimate boundedness). Se puede demostrar que el conjunto al que el sistema converge es el mínimo conjunto invariante. Entonces, el mínimo conjunto invariante puede ser visto como el análogo para los sistemas inciertos del concepto de punto de equilibrio para sistemas determinísticos.

- Recientemente, un nuevo enfoque basado en tubos está ganando cada vez más popularidad en el campo del control robusto de sistemas lineales en presencia de incertidumbre aditiva, ver (Raković and Mayne, 2005; Limón, Alvarado, Álamo and Camacho, 2008). En la práctica, tales técnicas de control basadas en tubos proponen dividir la acción de control en una parte local y una parte nominal. El control local es diseñado para mantener el verdadero estado en una vecindad del estado nominal, mientras que la evolución nominal se hace converger al equilibrio. La evolución nominal es obtenida por la dinámica del sistema en ausencia de incertidumbre.

  A condición de que la vecindad del estado nominal sea un conjunto invariante para el sistema incierto en bucle cerrado con la ley de control local, se puede demostrar que el tubo compuesto por el conjunto invariante ”centrado” en los estados de la trayectoria nominal contiene la trayectoria real, para cualquier realización de la incertidumbre. Entonces el objetivo se reduce a controlar la trayectoria nominal de manera que el tubo se mantenga dentro de la región admisible X. Está claro que, en general, cuanto más pequeño es el conjunto invariante que determina la forma del tubo, más grande es el tubo admisible en el cual la trayectoria nominal tiene que ser mantenida. Una vez que la ley de control local es determinada, el uso del mínimo conjunto invariante como vecindad del estado nominal proporciona un control menos conservador.

1.1.2.4 Conjuntos λ-contractivos y funciones de Lyapunov inducidas

Ha sido mostrado que la invariancia de una región del espacio de estados es una propiedad que implícitamente caracteriza todas las posibles trayectorias generadas por sus elementos, concerniendo tanto el comportamiento transitorio del sistema como el estado en régimen permanente, es decir su comportamiento límite. Esto hace que los conjuntos invariantes sean un instrumento muy útil para ambos objetivos: garantizar la satisfacción de restricciones duras y la estabilidad. También la convergencia a un punto de equilibrio (o a un conjunto) puede ser relacionada con regiones del espacio de estados introduciendo el concepto de λ-contractividad.

Conceptualmente, un conjunto convexo, compacto Ω y que contiene el origen en su interior es un conjunto λ-contractivo para un sistema dinámico si cada estado inicial en Ω se mapea en el conjunto escalado, $\lambda\Omega$, con un factor de escala λ positivo y menor que uno. Esto conlleva que la imagen de Ω a través de la función dinámica que caracteriza el sistema está contenida en el interior de Ω. Claramente, si $\lambda = 1$, entonces la definición de invariancia es recuperada. Además, es evidente que λ-contractividad implica invariancia.

Consideraciones análogas son válidas también para los conjuntos invariantes de control. Es decir, también en presencia de una entrada de control, puede ser interesante determinar una región Ω del espacio de estado tal que existe una ley de control, definida en Ω, que permita mapear Ω en $\lambda\Omega$. Nótese que, si se elimina la condición de convexidad de Ω, entonces λ-contractividad no implica invariancia, ya que $\lambda\Omega$ no necesariamente está contenido en Ω en este caso.

El concepto de λ-contractividad de un conjunto para un sistema dinámico dado puede inducir una función de Lyapunov, y entonces estabilidad asintótica o acotación terminal. La relación entre conjuntos λ-contractivos y funciones de Lyapunov puede ser ilustrada mediante el concepto de función de Minkowski. Considerado un conjunto compacto y convexo Ω (con el origen contenido en su interior), su función de Minkowski es una función del estado $x \in \mathbb{R}^n$ definida como el mínimo α tal que x está contenido en $\alpha\Omega$ y se denota con $\Psi_\Omega(x)$.

En caso de sistemas lineales afectados tanto por la incertidumbre paramétrica como por la aditiva, considerando un conjunto λ-contractivo Ω, cualquier conjunto $\mu\Omega$, con $\mu \geq 1$, es λ-contractivo, véase la propiedad P1 en (Blanchini, 1994). También se puede demostrar que si no hay ningún término aditivo de la incertidumbre, entonces $\mu\Omega$ es λ-contractivo para todo μ positivo. En ausencia de incertidumbres aditivas y asumiendo que Ω es convexo, compacto λ-contractivo y contiene el origen en su interior, su función de Minkowski es una función de Lyapunov. De hecho, si la función de Minkowski en un punto x es $\Psi_\Omega(x) = \alpha$, su valor en su sucesor x^+ es menor o igual a $\alpha\lambda$, es decir $\Psi_\Omega(x^+) \leq \alpha\lambda$. Ésto conlleva que la función de Minkowski decrece a lo largo de las trayectorias del sistema, si $\lambda < 1$ y el estado no es el origen. Ésto, y el hecho que la función de Minkowski es una función del estado definida positiva, aseguran que es una función de Lyapunov.

Es importante notar que el hecho de que Ω es λ-contractivo implique que también $\alpha\Omega$ es λ-contractivo (con α positivo), no se cumple para sistemas no lineales. Entonces funciones de Lyapunov inducidas no pueden ser determinadas en general. Una contribución importante de esta tesis concierne a este aspecto. De hecho, se proponen modelos dinámicos que permiten asegurar la estabilidad asintótica (exponencial) para una amplia clase de sistemas no lineales, determinando funciones de Lyapunov inducidas para aproximaciones de los sistemas no lineales. Una vez más, esa importante propiedad se basa en la convexidad.

Estas consideraciones permiten tener en cuenta funciones de Lyapunov cuyos conjuntos de nivel no son elipsoidales, cómo los que caracterizan las clásicas funciones de Lyapunov cuadráticas. Esto quiere decir que la caracterización de conjuntos genéricos λ-contractivos implica un análisis implícito de propiedades de estabilidad para una más amplia clase de potenciales funciones de Lyapunov. El empleo de funciones de Lyapunov poliédricas, inducidas por conjuntos politópicos λ-contractivos, adquirió particular interés en las décadas pasadas, ver (Blanchini, 1994; Blanchini, 1995; Blanchini and Miani, 2008). Los polítopos son, de hecho, muy versátiles y permiten aproximar cada conjunto convexo.

1.1.2.5 Control predictivo basado en modelo y conjuntos invariantes

Los conjuntos invariantes son extensamente empleados para el diseño de reguladores estabilizantes y, en particular, para la aplicación de estrategias de control con horizonte deslizante. De hecho, muchas formulaciones del control predictivo basado en modelo (MPC) necesitan una región terminal dentro de la cual la convergencia (asintótica) puede ser asegurada implícitamente por una simple, a menudo lineal, ley de control, ver (Mayne et al., 2000; Bemporad, Morari, Dua and Pistikopoulos, 2002; Camacho and Bordóns, 2004).

Se resumen brevemente las importantes características del MPC, para mostrar la importancia de la invariancia para esta técnica de control muy popular. Aunque han sido formuladas muchas variaciones de reguladores predictivos, proporcionamos aquí los ingredientes que caracterizan al MPC estándar:

- Predicción basada en modelo. El control está basado en la predicción de la evolución del sistema. Un modelo dinámico, lineal o no lineal, del sistema real es supuesto conocido. Debido a que las computaciones se ejecutan en linea para cada paso, en general el modelo considerado se asume en tiempo-discreto e invariante en el tiempo. En cada instante, el estado real es medido y una predicción de la evolución del sistema es obtenida como función de las entradas, dentro de un intervalo de tiempo N_p llamado horizonte de predicción. El número de los elementos de la secuencia de futuras entradas N_c, denominado horizonte de control, puede ser diferente del horizonte de predicción. El modelo permite prevenir la violación de restricciones que puede ocurrir en el futuro, dentro del horizonte de predicción.

- Restricciones. La razón principal de la creciente popularidad del MPC es su capacidad de manejar restricciones duras. Debido a la presencia de una predicción basada en modelo, el control desecha implícitamente aquellas secuencias que llevan el sistema a la violación de las restricciones. Restricciones en el estado y en la entrada pueden ser consideradas en el problema de optimización resuelto en línea. El resultado es que sólo un subconjunto de todas las posibles secuencias de entrada se asume factible. Tal conjunto, la región de factibilidad del problema de optimización, es el subconjunto del espacio de secuencias de entrada (obtenido como producto cartesiano del espacio de entrada) formado por sólo aquellas secuencias que evitan la violación de restricciones. Así, a cualquier elemento de la región de factibilidad se le asocia una trayectoria admisible potencial.

- Función de coste. El problema de optimización es resuelto para obtener la trayectoria, entre todas las admisibles, que minimiza una función de coste. La función de coste por lo general está formada por una parte que penaliza una medida de la distancia entre la trayectoria predicha y la deseada y otra parte que penaliza el esfuerzo de control. Intuitivamente, el objetivo es calcular la secuencia de control y la trayectoria asociada que consigue un alto rendimiento con un bajo esfuerzo de control. Diferentes funciones de coste pueden ser consideradas. Un rasgo importante de la función de coste es que debería ser una función definida positiva de los estados predichos y de la secuencia de entradas de control. Esto puede ser usado para demostrar que tal función de coste decrece a lo largo de la trayectoria real del sistema controlado por MPC, siendo entonces una función de Lyapunov y garantizando estabilidad asintótica.

- Horizonte deslizante. El problema de optimización es solucionado en línea en cada paso. Una vez que la secuencia óptima de entradas de control ha sido calculada, sólo la primera acción de control se aplica. De este modo, las discrepancias entre el comportamiento del sistema real y la trayectoria predicha por el modelo pueden ser compensadas.

Las propiedades de convergencia del MPC son aseguradas a menudo a través de la definición de una región terminal y una ley de control local que garantiza la estabilidad y, posiblemente, la convergencia asintótica, ver (Mayne et al., 2000; Camacho and Bordóns, 2004; Limón et al., 2005; Álamo, Ramírez, Muñoz de la Peña and Camacho, 2007). Es aquí dónde la invariancia es fundamental para el control MPC.

De hecho, un modo muy común para garantizar la estabilidad asintótica del sistema controlado por MPC es imponiendo que el estado final de la secuencia predicha esté contenido en un conjunto invariante en el que una ley de control local y una función de Lyapunov son definidas.

Intuitivamente, se asuma que existe una ley de control en realimentación y una región invariante para el sistema en el bucle cerrado. Una vez que el sistema alcanza tal conjunto invariante, que contiene al origen, se puede asumir en la predicción que el MPC es ”apagado” y la ley de control local es aplicada. Esto garantiza que ninguna violación de restricción ocurriría en el futuro. Si también una función de Lyapunov es definida, entonces la convergencia asintótica puede ser asegurada. De ahí que, la concatenación de las primeras N_u acciones de control, solución del problema de optimización, con el resto de la secuencia de control obtenida mediante la ley en realimentación, determina una secuencia infinita de acciones de control y una trayectoria admisible en cualquier instante futuro y convergente al origen.

Resumiendo, la introducción de una restricción terminal que impone que el último estado de la secuencia predicha pertenece a un conjunto invariante, proporciona un instrumento útil para asegurar propiedades fundamentales, como la estabilidad asintótica y la satisfacción de restricción duras para la trayectoria completa.

Aunque la definición de una función de Lyapunov dentro de la región terminal de MPC no sea necesaria para asegurar la estabilidad (véase, por ejemplo, (Bravo et al., 2006)), muchos resultados sobre MPC para sistemas no lineales e inciertos están basados en esto. Por otro lado, no hay muchos resultados sobre como obtener ese importante ingrediente para el MPC que son los conjuntos invariantes para sistemas no lineales. Particular atención es dedicada en esta tesis a este problema central, de hecho una contribución clave de nuestra investigación consiste en la propuesta de métodos para obtener conjuntos convexos invariantes para sistemas no lineales.

También los conjuntos invariantes de control pueden ser usados para el diseño de leyes de control en presencia de restricciones duras, como el MPC. Asúmase que un conjunto invariante de control para el sistema es conocido. Una restricción adicional, con la cual se impone que el estado pertenezca al conjunto invariante de control en el instante sucesivo, garantiza, por definición de invariancia, la existencia de una acción de control apropiada que asegura que ninguna violación de restricción ocurrirá. Nótese que esta única restricción puede sustituir a todas las restricciones sobre el estado. Si, además, el conjunto invariante es λ-contractivo, entonces la convergencia puede ser asegurada en algunos casos. De hecho, supóngase que se conoce un conjunto invariante de control Ω que garantiza λ-contractividad de $\alpha\Omega$ para el sistema, para todo $\alpha \in [0, 1]$ y para una ley de control apropiada. A menudo hay que imponer la ausencia de incertidumbre aditiva para que $\alpha\Omega$ sea λ-contractivo para cualquier α positivo. Entonces, intuitivamente, considerando el estado actual x y su función de Minkowski $\Psi_\Omega(x)$, cualquier acción de control tal que la función de Minkowski en x^+ es menor que $\lambda\Psi_\Omega(x)$ hace que el conjunto sea λ-contractivo en bucle cerrado. Como, por construcción, existe al menos una acción de control que satisface tal condición, el problema de calcular un $u(x)$ tal que $\Psi_\Omega(x^+) \leq \lambda\Psi_\Omega(x)$ y sea óptima con respecto a alguna medida de las prestaciones, es siempre factible y asegura la convergencia exponencial al origen. La computación de conjuntos λ-contractivos para sistemas no lineales sujetos a una ley de control apropiada, que conlleva la síntesis de un control que asegure la convergencia asintótica en bucle cerrado, no es una tarea simple. Una solución para tal problema de diseño, para particulares sistemas no lineales, representa otra importante contribución de esta tesis.

También para el caso del MPC robusto, la definición de un conjunto invariante robusto (de control) como región terminal es generalmente requerida para que el sistema incierto controlado cumpla la acotación terminal. El dominio de atracción de las estrategias MPC es por lo general fuertemente dependiente del tamaño de tal región terminal.

1.2 Estado del arte sobre los métodos basados en la teoría de conjuntos

En los años pasados han sido obtenidos muchos resultados para aplicar el enfoque worst-case y para caracterizar los conjuntos invariantes. En esta sección se proporcionan algunas importantes contribuciones y resultados presentados en literatura que tratan estos temas, tanto para los sistemas lineales como para los no lineales.

1.2.1 Trasfondo histórico

Trabajos pioneros aparecieron al final de los años sesenta, ver (Schweppe, 1968; Witsenhausen, 1968*b*), y al principio de los años setenta, (Bertsekas and Rhodes, 1971*b*). La estimación de estado garantista para sistemas afectados por incertidumbres aditivas trata el problema de determinar una secuencia de conjuntos tales que el estado del sistema dinámico en el instante $k \in \mathbb{N}$ está seguramente contenido en el k-esimo elemento de la secuencia. Esto se consigue integrando la información de la medida con la actualización dinámica, obtenida esta ultima a través del conjunto alcanzable a un paso. Nótese que, en ausencia de medida, el concepto de tubo alcanzable es recuperado.

Motivado por el problema del seguimiento de un objetivo evasivo, en (Schweppe, 1968) el autor trata el problema de estimar en cada instante el conjunto del espacio de estados que contiene el estado real de un sistema lineal afectado por perturbaciones sobre el estado y sobre la salida. Las condiciones iniciales y las perturbaciones son desconocidas pero acotadas por elipsoides.

También Witsenhausen trata en (Witsenhausen, 1968*b*) el problema del cómputo de conjuntos en el espacio de estados que sean compatibles con las observaciones y con las condiciones iniciales. El sistema se supone afectado por perturbaciones sobre el estado y sobre la salida, asumidas acotadas por conjuntos compactos y convexos. Se proponen aproximaciones poliédricas, que llevan a problemas de programación lineal.

El trabajo (Bertsekas and Rhodes, 1971*b*) trata el mismo problema de estimación considerado por Schweppe pero es ampliado al análisis de los casos de “smoothing“ y predicción.

Además, la adaptación a sistemas en tiempo-discreto es explícitamente expuesta.

La primera contribución sobre invariancia en el campo de sistemas dinámicos ha sido probablemente (Bertsekas, 1972). Este trabajo trata el problema del cómputo y la caracterización del máximo conjunto invariante robusto. En este trabajo fundamental, el autor considera sistemas tiempo discreto no autónomos y no lineales, afectados por incertidumbre, es decir sistemas de la forma

$$x_{k+1} = f(x_k, u_k, w_k),$$

con restricciones sobre el estado $x_k \in X$ y sobre la entrada $u_k \in U(x_k)$, que pueden depender del estado. Las restricciones sobre la incertidumbre también pueden ser dependientes del estado y de la entrada, es decir $w_k \in W(x_k, u_k)$. Primero, una condición necesaria y suficiente para la invariancia de control de un conjunto es presentada, luego se da una caracterización del máximo invariante de control. Considerando el conjunto X, el operador a un paso ha sido empleado para definir la secuencia de conjuntos, denotados $S_k(X)$, cuyos elementos pueden ser mantenidos k veces en X, para $k \in \mathbb{N}$, mediante una apropiada secuencia de acciones de control. Ha sido demostrado un resultado muy interesante y no intuitivo, es decir, el hecho de que la intersección de tal secuencia de conjuntos no es igual, en general, al máximo conjunto invariante de control. Una condición para que esta igualdad se dé está basada en que los conjuntos implicados deben ser compactos. Un caso particularmente interesante para el que tal condición se satisface, es que el sistema sea afín en w_k, es decir

$$x_{k+1} = f(x_k, u_k) + w_k,$$

con U y W no dependientes de x y (x,u), respectivamente, que los conjuntos X y W sean compactos y que $f(\cdot)$ sea continua. En este caso la intersección de los conjuntos $S_k(X)$, para todo $k \in \mathbb{N}$, es igual al máximo conjunto invariante de control.

Los aspectos computacionales para el máximo conjunto invariante han sido considerados en dos trabajos publicados en 1991, que son (Gilbert and Tan, 1991) y (Blanchini, 1994) (aunque la segunda referencia sea relativa a un artículo publicado en 1994, una primera versión del trabajo fue presentada en 1991, en (Blanchini, 1991)).

En (Gilbert and Tan, 1991), el problema de la caracterización y de la computación del máximo conjunto invariante con salida admisible para un sistema lineal determinístico es considerado, es decir para $x^+ = Ax$ con $y = Cx$. En particular, las restricciones son definidas en el espacio de la salida, es decir, en la forma $y \in Y$. Esto no conlleva grandes diferencias con el caso de restricciones sobre el espacio de estados. El resultado principal se refiere a la condición de determinación finita del máximo conjunto invariante. Se demuestra que, si el sistema es asintóticamente estable, la pareja C, A es observable, el conjunto de salidas admisible Y es acotado y el origen está contenido en su interior, entonces el máximo conjunto invariante con salida admisible es finitamente determinado.

En (Blanchini, 1994), se consideran sistemas no autónomos lineales con incertidumbre paramétrica y aditiva. Se presenta el concepto de conjunto λ-contractivo y se proporciona un procedimiento iterativo para calcular el máximo conjunto λ-contractivo, para un dado $\lambda \in [0,1]$. Un resultado importante demostrado en el articulo es el hecho de que el máximo conjunto λ-contractivo es dado por la intersección de la secuencia de conjuntos calculados mediante una especie de operador a un paso, siempre que el conjunto inicial sea convexo, compacto y contenga el origen en su interior. El resultado es análogo al presentado por Bertsekas, pero para conjuntos λ-contractivos. Funciones de Lyapunov inducidas y el diseño del control también son analizados.

Un primer importante articulo survey sobre invariancia es (Blanchini, 1999), que resume los principales resultados sobre el tema. Se consideran sistemas en tiempo-continuo y en tiempo-discreto, se presentan las condiciones de invariancia para sistemas lineales y no lineales. Además, funciones de Lyapunov inducidas, así como problemas de diseño de control basados en invariancia, son analizados. También los aspectos computacionales son considerados, en particular para conjuntos invariantes politópicos y elipsoidales.

Otro trabajo muy importante y básico sobre el tema es (Kolmanovsky and Gilbert, 1998). El artículo trata el problema de la caracterización del máximo conjunto invariante con salida admisible para sistemas en tiempo-discreto lineales afectados por incertidumbre aditiva. Los resultados están fuertemente basados en instrumentos matemáticos, como las funciones suporte y la diferencia de Pontryagin (o de Minkowski), que son extensamente empleados en esta tesis. Se presentan condiciones necesarias y suficientes para la invariancia, se caracterizan conjuntos invariantes mínimos y máximos y se propone un procedimiento iterativo para calcular el máximo conjunto invariante. Resultados muy importantes presentados en (Kolmanovsky and Gilbert, 1998) son las condiciones necesarias y suficientes para la existencia del máximo conjunto invariante y para su determinación finita. El máximo conjunto invariante es no vacío si y sólo si el mínimo conjunto invariante está contenido en la región admisible, además es finitamente determinado si está contenido en el interior de dicha región.

Contribuciones más recientes que tratan el problema de la caracterización del mínimo conjunto invariante para sistemas lineales afectados por incertidumbre aditiva son (Raković et al., 2005; Ong and Gilbert, 2006). Ya que el mínimo conjunto invariante es la suma de Minkowski de infinitos términos, no puede ser calculado en general, y la atención de los autores se dirige al cómputo de aproximaciones del mínimo conjunto invariante.

1.2.2 Estado del arte para sistemas no lineales

Aquí se proporciona una breve visión general de los resultados presentados en los años pasados sobre cuestiones relacionadas con los métodos basados en la teoría de conjuntos y sobre invariancia para sistemas no lineales.

Uno de los principales problemas inherentes al uso de métodos basados en la teoría de conjuntos y al cómputo de conjuntos invariantes es el hecho de que, en general, la no linealidad del sistema o del controlador conlleva conjuntos no convexos y no poliédricos. Esto comporta una gran, a menudo no manejable, complejidad computacional. Entonces, en general, se emplean aproximaciones y se debe alcanzar una compensación entre el conservadurismo inducido y la complejidad computacional.

En primer lugar, se presenta una descripción de metodos de aproximación de los conjuntos alcanzables y de estimación garantista para sistemas no lineales. El problema planteado es el cómputo de la secuencia de conjuntos en el espacio de estados que proporciona la garantía de contener el estado del sistema. Nótese que sistemas no lineales y sistemas lineales inciertos están relacionados, ya que a menudo los métodos para calcular conjuntos alcanzables y de estimación para sistemas no lineales están basados en aproximaciones lineales.

El problema del cómputo de conjuntos alcanzables para sistemas no lineales es considerado en (Kühn, 1999) usando una técnica basada en el teorema del valor medio para acotar la evolución real del sistema no lineal. Es decir, considerado un sistema no lineal y un conjunto, una aproximación del conjunto alcanzable a un paso puede ser obtenida acotando la función dinámica no lineal con una función lineal con incertidumbre aditiva. La secuencia de conjuntos alcanzables, entonces, se obtiene a través de un mapeo lineal y de una suma de Minkowski, en cada paso. Ésto conlleva, en general, un excesivo aumento de la complejidad de los conjuntos. El problema de la complejidad es solucionado empleando zonotopes, que permiten controlar la complejidad computacional y de representación de los conjuntos, al precio de algún conservadurismo. En (Girard, LeGuernic and Maler, 2006) se han propuesto más desarrollos en esta dirección, usando zonotopes y cajas para acotar la evolución admisible del sistema.

Ha de recordarse que un modo para tratar el problema del cómputo de los conjuntos alcanzables para un sistema no lineal es aproximandolo con un sistema lineal incierto. Un nuevo enfoque que garantiza la convergencia de la secuencia de conjuntos alcanzables aproximados es presentado en (Raković and Fiacchini, 2008), en el que las propiedades de invariancia son empleadas para determinar una forma básica para acotar los conjuntos alcanzables exactos. Un método basado en la homotecia conduce a la determinación de un procedimiento computacional que combina el bajo esfuerzo computacional con la convergencia a cero del error de aproximación.

Análogamente, el problema de la estimación de estado garantista para sistemas no lineales ha sido tratada mediante enfoques basados en la teoría de conjuntos. El trabajo (Álamo, Bravo and Camacho, 2005) presenta un nuevo enfoque para la estimación garantista para sistemas en tiempo-discreto no lineales con perturbaciones acotadas afectando al estado y a la salida. Se proporciona un algoritmo para calcular un conjunto que contiene los estados compatibles con la salida medida y con el modelo del sistema. Este conjunto es representado por un zonotope. El tamaño del zonotope es minimizado en cada paso a través de una expresión analítica o solucionando un problema de optimización convexo. La aritmética intervalar se usa para calcular una secuencia garantista de conjuntos en el espacio de estados.

En (Álamo, Bravo, Redondo and Camacho, 2007) se presenta un método para la estimación de estado garantista para sistemas en tiempo discreto no lineales con perturbaciones acotadas. Los conjuntos de estados que son compatibles con la evolución del sistema, las salidas medidas y las perturbaciones acotadas son representados por zonotopes. La principal novedad es el uso de funciones DC para calcular la secuencia de conjuntos de aproximación. Las funciones DC resultan muy útiles para calcular acotaciones de las soluciones óptimas de problemas de programación no convexa, y también son usadas en esta tesis.

No es trivial adaptar al caso no lineal aquellos instrumentos matemáticos estándar, como el operador a un paso y los conjuntos alcanzables, extensamente empleados para el análisis de invariancia para sistemas lineales. Considérese por ejemplo el hecho que el conjunto a un paso $Q(\Omega)$ no es necesariamente convexo para sistemas no lineales, tampoco si Ω es convexo. Así, su uso puede conducir a la generación de secuencias de conjuntos sumamente complejos.

Uno de los problemas principales, pasando de los sistemas lineales a los no lineales, es que algunas propiedades útiles relacionadas con la linealidad pierden validez. Un ejemplo muy interesante es ilustrado en las consideraciones siguientes, sobre la condición de invariancia, véase (Blanchini and Miani, 2008). En este trabajo, primero se considera el caso tiempo-continuo. Un resultado fundamental sobre invariancia es representado por el teorema de Nagumo, que proporciona una condición necesaria y suficiente para la invariancia para un subconjunto cerrado del espacio de estados, para sistemas en tiempo-continuo. Conceptualmente, tal teorema afirma que un conjunto es un invariante positivo si y sólo si el vector velocidad es dirigido hacia el interior (o tangente a la frontera) del conjunto en cualquier punto de la frontera. Intuitivamente, si el vector velocidad se dirige hacia el interior del conjunto, esto implica que las trayectorias en la frontera entran en el conjunto, entonces no puede haber trayectorias que empiezan dentro del conjunto y lo abandonan. Esto implica claramente invariancia.

Entonces, la atención en (Blanchini and Miani, 2008) pasa al caso tiempo-discreto

$$x^+ = f(x),$$

para el cual la invariancia de un conjunto S no puede ser asegurada a través de una condición de frontera.

Los autores expresamente declaran que:

"... Como se puede entender fácilmente, no hay ninguna extensión evidente de la condición de Nagumo "de tipo frontera" para sistemas en tiempo-discreto. Intuitivamente, el homólogo

natural de la condición de Nagumo... sería

$$f(x) \in S, \qquad \forall x \in \partial S,$$

que quiere decir, aproximadamente, el estado en la frontera ”salta dentro”. Sin embargo, esta condición no es suficiente para asegurar $f(x) \in S$ para todo $x \in S$. De hecho, es fácil proporcionar ejemplos tiempo discreto para los cuales la susodicha condición de frontera puede estar satisfecha, pero el conjunto no es un invariante positivo. Por lo tanto la única razonable ”extensión tiempo-discreto” del teorema de Nagumo es la tautología: S es positivamente invariante si y sólo si

$$f(S) \subseteq S.$$

Afortunadamente, la situación es completamente diferente si restringimos nuestra atención a la clase de sistemas homogéneos (incluyendo los lineales) ... ”

Esto quiere decir que, considerando genericos sistemas no lineales el análisis de invariancia tiene que implicar al conjunto entero, mientras que para sistemas lineales (y homogéneos) una condición de invariancia de tipo frontera puede ser formulada. Una de las principales contribuciones conceptuales de esta tesis es mostrar que una condición de invariancia de tipo frontera puede ser enunciada también para sistemas no lineales. El ingrediente que permite deducir propiedades que afectan al conjunto entero a través del análisis en la frontera es la convexidad.

1.2.2.1 Contribuciones sobre la computación de conjuntos invariantes para sistemas no lineales

En primer lugar, merece la pena mencionar el trabajo (Kerrigan and Maciejowski, 2000) que proporciona una revisión sobre invariancia para sistemas no lineales hasta el momento. El artículo se enfoca principalmente en la caracterización teórica de la invariancia y su empleo en el control, más que en las cuestiones computacionales.

El problema del diseño de un MPC para sistemas no lineales se trata en (Cannon, Deshmukh and Kouvaritakis, 2003). Con ese propósito, se considera el problema de cómo calcular un conjunto invariante para ser usado como conjunto terminal. En particular, un conjunto invariante politópico es calculado para un sistema LDI, válido dentro de una región. El problema es expresado como problema de programación lineal, cuyo objetivo es maximizar el volumen del polítopo, cuya complejidad geométrica está acotada. En particular, se considera la transformación lineal de una bola en norma infinito, es decir, un parallelotope, cuyos vértices son las variables de optimización.

El problema de sistemas lineales con particulares realimentaciones no lineales estáticas, como las afines a trozos y la saturación, ha sido considerado en el trabajo (Hu and Lin, 2004), donde se proporcionan condiciones de invariancia para un elipsoide.

El trabajo (Bravo, Limón, Álamo and Camacho, 2005) trata el problema del cómputo de conjuntos invariantes de control para sistemas no lineales con restricciones. El enfoque propuesto está basado en la computación de una aproximación interior del operador a un paso. Basado en este procedimiento, conjuntos invariantes de control pueden ser calculados por recursión. En este trabajo, la aritmética intervalar se emplea para calcular el conjunto a un paso.

Particular atención ha sido dedicada a una no linealidad muy común en los sistemas dinámicos reales, la saturación. El articulo (da Silva and Tarbouriech, 1999) trata el problema del análisis y la computación de conjuntos invariantes y λ-contractivos para sistemas en tiempo-discreto en presencia de saturación. En particular, los autores proponen particionar el espacio en las regiones donde los valores superiores o inferiores de la saturación son alcanzados y donde ninguna saturación ocurre. Así, el sistema es asumido lineal y perturbado por una perturbación constante dentro de cada región. La condición necesaria y suficiente para la contractividad de conjuntos poliédricos es enunciada.

En (Álamo, Cepeda, Limón and Camacho, 2006*b*), un método para estimar el dominio de atracción para sistemas saturados en tiempo-discreto es presentado. Se introduce una nueva noción de invariancia, denominada invariancia-SNS. Se proporciona un algoritmo para generar una secuencia de conjuntos invariantes anidados y se demuestra que la secuencia converge al conjunto invariante-SNS más grande para esta clase de sistemas. También se demuestra que los conjuntos invariantes-SNS generados por este algoritmo iterativo son conjuntos poliédricos convexos y que constituyen una estimación del dominio de atracción del sistema no lineal. Los autores han abordado el mismo problema también en (Álamo, Cepeda, Limón and Camacho, 2006*a*).

1.3 Convexidad e invariancia

Una de las claves de la tesis, el concepto de convexidad de conjuntos y funciones, es ilustrada brevemente en esta sección. Muchos esfuerzos han sido dirigidos al análisis de la convexidad, véase por ejemplo (Boyd and Vandenberghe, 2004; Rockafellar, 1970; Schneider, 1993; Ben-Tal and Nemirovski, 2001).

Hay muchas razones por las que se considera importante la convexidad para temas relacionados con la invariancia y la teoría de conjuntos en control. La primera razón es la alta complejidad de representación y computacional inducida por los conjuntos no convexos. Por ejemplo, desde un punto de vista práctico, los procedimientos algorítmicos estándar, por lo general, generan secuencias de conjuntos cuya complejidad explota después de pocos pasos, cuando se tratan con conjuntos no convexos. De hecho, las familias de conjuntos considerados en la literatura para problemas prácticos relacionadas con la invariancia comparten la propiedad de convexidad, por ejemplo elipsoides, polítopos, zonotopes y cajas.

De otra parte, la convexidad de las funciones permite deducir propiedades satisfechas por cualquier elemento de un conjunto mediante condiciones que implican sólo un subconjunto finito de puntos. Esta consideración conduce también a problemas de optimización convexos, que son computacionalmente manejables, y a algoritmos caracterizados por una complejidad asequible. Así que vale la pena recordar aquí algunas definiciones básicas y propiedades relacionadas con la convexidad de conjuntos y funciones.

Un conjunto $S \subseteq \mathbb{R}^n$ es convexo si, para cada pareja de elementos de S, es decir para cada $x, y \in S$, el segmento entero entre los dos puntos está contenido en S.

Una propiedad importante de los conjuntos convexos, véase (Rockafellar, 1970), es que un conjunto $S \subseteq \mathbb{R}^n$ es convexo si y sólo si contiene todas las combinaciones convexas de sus elementos. Esto quiere decir que, si S es convexo, cualquier punto que puede ser expresado como una combinación convexa de los elementos de S pertenece a S. Además, si cualquier punto que se puede expresar como combinación convexa de los elementos de S pertenece a S, entonces el conjunto es convexo.

Hay varios modos diferentes para definir la convexidad de una función. Una manera está basada en el concepto de conjunto convexo y proporciona un significado geométrico de las funciones convexas.

Considerando una función $f : \mathbb{R}^n \to \mathbb{R}$, se define su grafo como $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \operatorname{dom} f\}$ y su epígrafe como el conjunto de puntos en $\mathbb{R}^{n+1}$ que están por encima del grafo. Una función es convexa si su epígrafe es un subconjunto convexo de $\mathbb{R}^{n+1}$. Es evidente lo profundamente relacionados que están los conceptos de conjuntos convexos y funciones convexas.

Una característica importante de la convexidad, ya mencionada, es el hecho de que una propiedad que implica sólo algunos elementos de un conjunto puede ser extendida a un conjunto posiblemente incontable de puntos, cuando se usan conjuntos convexos y funciones convexas. Se verá que el hecho de que propiedades generales pueden ser deducidas a través de condiciones que implican sólo algunos elementos de un conjunto, permitirá formular procedimientos algorítmicos para el cómputo de conjuntos invariantes, algoritmos caracterizados por una complejidad asequible.

Como ejemplo, introducimos brevemente la programación convexa, que juega un papel clave tratando problemas de programación matemática. Los problemas de programación convexa son caracterizadas por la minimización de una función de coste convexa sobre un conjunto factible convexo, o problemas equivalentes, ver (Boyd and Vandenberghe, 2004).

La importancia de problemas de programación convexa es debida al hecho de que estos son caracterizados por resolubilidad eficiente, es decir, pueden ser solucionados en tiempo polinomial. Los problemas generales de programación no lineal suelen ser mucho más exigentes computacionalmente, ver (Ben-Tal and Nemirovski, 2001).

Esto puede ser entendido intuitivamente recordando que un mínimo local para un problema convexo es también un mínimo global, a diferencia del caso de problemas no convexos. En el contexto de problemas convexos, algoritmos iterativos, basados en el gradiente de la función de coste por ejemplo, pueden ser diseñados, ver (Bazaraa and Shetty, 1979). Así, a través de las iteraciones del algoritmo, la disminución del valor de la función de coste en cada paso asegura la convergencia al óptimo. Si, al contrario, óptimos locales no son también globales, como es para problemas no convexos, los algoritmos que explotan el gradiente de la función de coste no garantizan, en general, la convergencia a un mínimo global. Así que, en el contexto de la programación matemática, la disponibilidad de una representación en forma de programación convexa para un problema es crucial.

1.4 Motivación y objetivos de la tesis

Como se ha ilustrado, la invariancia y los métodos basados en la teoría de conjuntos son unas técnicas muy importantes para el análisis de sistemas no lineales e inciertos. Además, es evidente lo útil que es el uso de la invariancia y de las estructuras relacionadas con la teoría de conjuntos para la determinación de estrategias de control robustas, el diseño de leyes de control en presencia de restricciones duras, el control basado en predicciones etc., es decir, para la síntesis de control en general, para sistemas no lineales e inciertos.

A pesar de tal fundamental posición ganada por la invariancia y la teoría de conjuntos para el análisis de sistemas y el control, sobre todo en las ultimas décadas, sólo muy pocos resultados han sido proporcionados para sistemas no lineales. Los resultados presentados en la literatura han sido enunciados para casos particulares de sistemas no lineales (como sistemas saturados, sistemas bilineales, etc.) y a menudo no pueden ser aplicados a genéricos sistemas no lineales. Es decir, a pesar de la importancia de tales estructuras, su uso es restringido a casos particulares.

Considérese la importancia mencionada de los conjuntos invariantes para el control predictivo basado en modelo para sistemas no lineales. En este contexto, la disponibilidad de un conjunto invariante es usada a menudo para demostrar propiedades deseadas para el sistema controlado, como la estabilidad, la convergencia y la satisfacción de restricciones. De otra parte, no es trivial, en la práctica, obtener un conjunto invariante para un sistema no lineal, generalmente requerido para ser usado como la región terminal en el MPC. Es decir, en muchas contribuciones sobre el control predictivo no lineal la disponibilidad de un conjunto invariante, que conlleva resultados generales, es asumida, pero no se aborda el problema computacional de como obtener tal importante ingrediente. Este hecho podría contribuir a una cierta pérdida de generalidad para tales resultados en la práctica.

Existe claramente un vacío entre la importancia de la invariancia y de la teoría de conjuntos para objetivos teóricos y la aplicabilidad práctica de los resultados obtenidos, principalmente para el caso de sistemas no lineales. Hay que notar que este vacío es menor para el caso de sistemas lineales inciertos, para los cuales muchos métodos computacionales bien establecidos están presentes en la literatura. El objetivo de esta tesis es contribuir en rellenar este vacío. De hecho, se dedicará una atención particular a los sistemas no lineales.

El principal problema conceptual relativo a los métodos analíticos y computacionales para sistemas lineales, es el hecho de que muchas propiedades y características que son intrínsecas a la linealidad no pueden ser exportadas a los sistemas no lineales. El teorema de Nagumo es un claro ejemplo de condición que no es aplicable en caso de no linealidad.

La idea subyacente, común a muchos de los resultados presentados en la tesis, es la de adaptar aquellos métodos y propiedades que caracterizan el análisis y el diseño de control para sistemas lineales, a los no lineales. Se mostrará que en muchos casos la convexidad es el ingrediente ”que falta” para la formulación de métodos e instrumentos análogos, que garanticen las propiedades deseadas en presencia de no linealidades.

Primero introduciremos el principal marco de modelado, los llamados sistemas de inclusión de diferencias convexas (CDI). Esta clase de sistemas dinámicos es profundamente caracterizada por la convexidad, siendo su dinámica definida mediante un conjunto de funciones convexas. Ésto implica que la evolución dinámica de tales sistemas se caracteriza por restricciones convexas, y el uso de muchos instrumentos propios del análisis de sistemas lineales conduce a problemas de programación convexa, polinomialmente complejos. Es decir, intuitivamente, substituyendo la linealidad por la convexidad, algunos resultados para sistemas lineales son conservados para sistemas CDI, al precio de un leve aumento de la complejidad computacional.

Muchos resultados importantes, análogos a aquellos bien establecidos para sistemas lineales, son expuestos y demostrados. Por ejemplo, el cómputo del operador exacto a un paso, las condiciones necesarias y suficientes para invariancia y λ-contractividad, los algoritmos para generar secuencias de conjuntos que convergen al dominio de atracción, rigurosamente desarrollados para sistemas lineales, se proporcionan para sistemas CDI.

La importancia de ese marco de modelado es evidente considerando que los elementos de una muy amplia clase de sistemas no lineales pueden ser aproximados por sistemas CDI. Es decir, considerando un sistema no lineal, si por un lado el análisis de la aproximación CDI del sistema induce cierto conservadurismo, por el otro, algunas propiedades beneficiosas, válidas para sistemas lineales, son conservadas para sistemas CDI. Ésto lleva a resultados generales y fuertes para el sistema no lineal aproximado.

Otro aspecto del enfoque CDI es que muchas propiedades satisfechas por un conjunto para el sistema CDI (la invariancia por ejemplo), se cumplen también para cualquier sistema no lineal aproximado por el sistema CDI. Esto implica que considerar un sistema CDI es la base del análisis de una familia entera de sistemas no lineales, es decir todos los sistemas cuya función dinámica es acotada por la función que caracteriza el sistema CDI.

A pesar de que los resultados proporcionados para sistemas CDI son fuertes, las suposiciones que caracterizan tal marco de modelado no son demasiado restrictivas. Se mostrará que muchas clases comunes de sistemas son particulares sistemas CDI o, por lo menos, admiten aproximaciones CDI. Entonces, un importante problema práctico, relacionado con la teoría desarrollada, es cómo generar el sistema CDI que aproxima a un sistema no lineal dado. En este contexto introduciremos algunos aspectos computacionales sobre cómo obtener una representación CDI o una aproximación CDI para los elementos de algunas clases comunes de sistemas no lineales.

Los sistemas de inclusión de diferencias convexas y cóncavas (CCDI) son la primera clase de sistemas incluidos en el marco CDI. Tales sistemas son particulares sistemas CDI, caracterizados por un número finito de funciones que determinan su dinámica. Muchos sistemas no lineales pueden ser aproximados por un sistema CCDI, de manera que sólo un número finito de funciones de acotación tienen que ser calculadas.

La segunda clase de sistemas no lineales para los cuales una representación CDI es un instrumento muy útil, son los sistemas Lur'e. Estos sistemas estan formados por un sistema lineal en bucle cerrado con particulares leyes de realimentación con ganancia estática y son conocidos en el contexto de la teoría del control, principalmente en tiempo continuo. En la tesis se considerarán sistemas Lur'e en tiempo-discreto.

La clase más importante de sistemas para los cuales una aproximación CDI se obtiene facilmente son los sistemas llamados DC. Tales sistemas son caracterizados por funciones dinámicas que pueden ser expresadas como la diferencia de funciones convexas (DC). La importancia de las funciones DC se debe al hecho de que es fácil determinar funciones cota superior e inferior convexas para cualquiera de ellas. Esto llevará a la determinación implícita de un sistema CDI que aproxima al original DC no lineal. Además, una muy amplia clase de funciones no lineales puede ser representada por una DC.

Otra subclase de sistemas CDI son los sistemas lineales con incertidumbre paramétrica. Este marco de modelado, para el cual algunos resultados están disponibles en la literatura, permite aplicar técnicas propias de análisis y síntesis de sistemas lineales a la aproximación de un sistema no lineal. De hecho, por ejemplo, un sistema no lineal definido en una región acotada puede ser aproximado por un sistema lineal con incertidumbre paramétrica acotando el gradiente de la función no lineal en tal región.

La presencia de incertidumbre aditiva puede ser considerada para cualquiera de los marcos de modelado mencionados. La asunción de presencia de incertidumbre aditiva desconocida pero acotada hace el modelo más realista en muchos casos, pudiendo ser la suposición de conocimiento perfecto de la dinámica del sistema demasiado restrictiva.

Finalmente, presentamos la estructura de la tesis, junto con las contribuciones sobre los diferentes aspectos de invariancia y métodos basados en la teoría de conjuntos.

- El segundo capítulo trata el problema del modelado. Se recordarán definiciones y caracterizaciones generales de sistemas dinámicos no lineales, introduciendo los conceptos de incertidumbre y de de mapas con conjuntos cómo valor, extensamente empleadas en la tesis. Luego los nuevos modelos propuestos, como el marco de modelado CDI, serán presentados.

- Aspectos computacionales que relacionan los sistemas CDI con las comunes clases de sistemas no lineales e inciertos son desarrollados en el capítulo tres. Se presentan sistemas CCDI y sistemas Lur'e cómo subclases de sistemas CDI orientados a la práctica. Sus doble relación, con los sistemas CDI por un lado y con comunes sistemas no lineales por el otro, se enfatiza para demostrar que muchos sistemas reales estan incluidos en estas clases de modelos. Los sistemas DC son ilustrados posteriormente. Se proporcionan definiciones, propiedades y ejemplos para enfatizar las principales características de esto modelos, particularmente ricos y expresivos. Se proporciona una breve descripción de las funciones DC para aclarar los motivos que nos conducen a considerar esta clase particular de funciones no lineales. Finalmente, sistemas lineales con incertidumbre paramétrica son definidos. Dos subclases de sistemas lineales con incertidumbre paramétrica, como los lineales dependientes de parámetro variante (LPV) y los sistemas de inclusiones de diferencias lineales (LDI), también son ilustradas.

- En el capítulo cuatro se considerará la invariancia y temas relacionados para sistemas CDI. Importantes resultados, establecidos para sistemas lineales, son enunciados para esta clase de sistemas. Se proporcionarán condiciones necesarias y suficientes para que un conjunto convexo en el espacio de estados sea invariante y λ-contractivo, también en presencia de incertidumbre aditiva. Se demostrará que, en caso de ausencia de incertidumbre aditiva, la relación entre conjuntos convexos λ-contractivos para sistemas CDI y funciones de Lyapunov, propia de los sistemas lineales, es conservada para sistemas CDI. El operador a un paso es determinado y caracterizado, y un algoritmo para generar secuencias de conjuntos que convergen al dominio de atracción es propuesto. Finalmente, problemas computacionales sobre cómo obtener conjuntos invariantes convexos y λ-contractivos para sistemas CDI son abordados.

- El quinto capítulo trata el problema del cálculo de conjuntos invariantes convexos y conjuntos λ-contractivos para particulares sistemas no lineales autónomos. En particular, se considerarán clases de sistemas no lineales orientados a la práctica, ilustrados precedentemente, como los sistemas DC y Lur'e. Se darán condiciones suficientes para la invariancia y la λ-contractividad para sistemas DC. También se tratará el caso de sistemas DC en presencia de incertidumbre aditiva. Se abordará el problema de la computación práctica de un conjunto invariante convexo, que llevará a la definición de un procedimiento algorítmico para obtener un conjunto no vacío, convexo e invariante en ausencia de incertidumbre. Se propone un método ad-hoc para obtener una secuencia de conjuntos invariantes anidados para sistemas Lur'e. También se mostrará que tal secuencia de conjuntos converge a una aproximación convexa del dominio de atracción.

- El capítulo seis presenta resultados relacionados con el problema de la síntesis de control. La computación de leyes de control y de conjuntos invariantes de control para sistemas CDI no autónomos es el tema principal del capítulo. La primera parte se dedica a ilustrar las propiedades de los conjuntos invariantes de control convexos y λ-contractivos para sistemas DC. Se proporcionará una condición suficiente para la invariancia de control y la λ-contractividad de un conjunto convexo. En particular, en el caso de conjuntos politópicos, se demuestra que el cálculo de una acción de control en los vértices del polítopo que satisfaga una condición convexa, permite la determinación de una acción de control, definida sobre todo el conjunto y tal que la estabilidad asintótica (exponencial) es garantizada para el sistema no lineal. El operador a un paso, útil para obtener una secuencia de conjuntos invariantes de control anidados y una aproximación del máximo conjunto estabilizable, es analizado para sistemas DC. También cuestiones computacionales son consideradas, definiendo algoritmos para determinar la ley de control estabilizante.

- En el capítulo final se resumen las contribuciones y los resultados ilustrados en la tesis y las direcciones para la investigación futura.

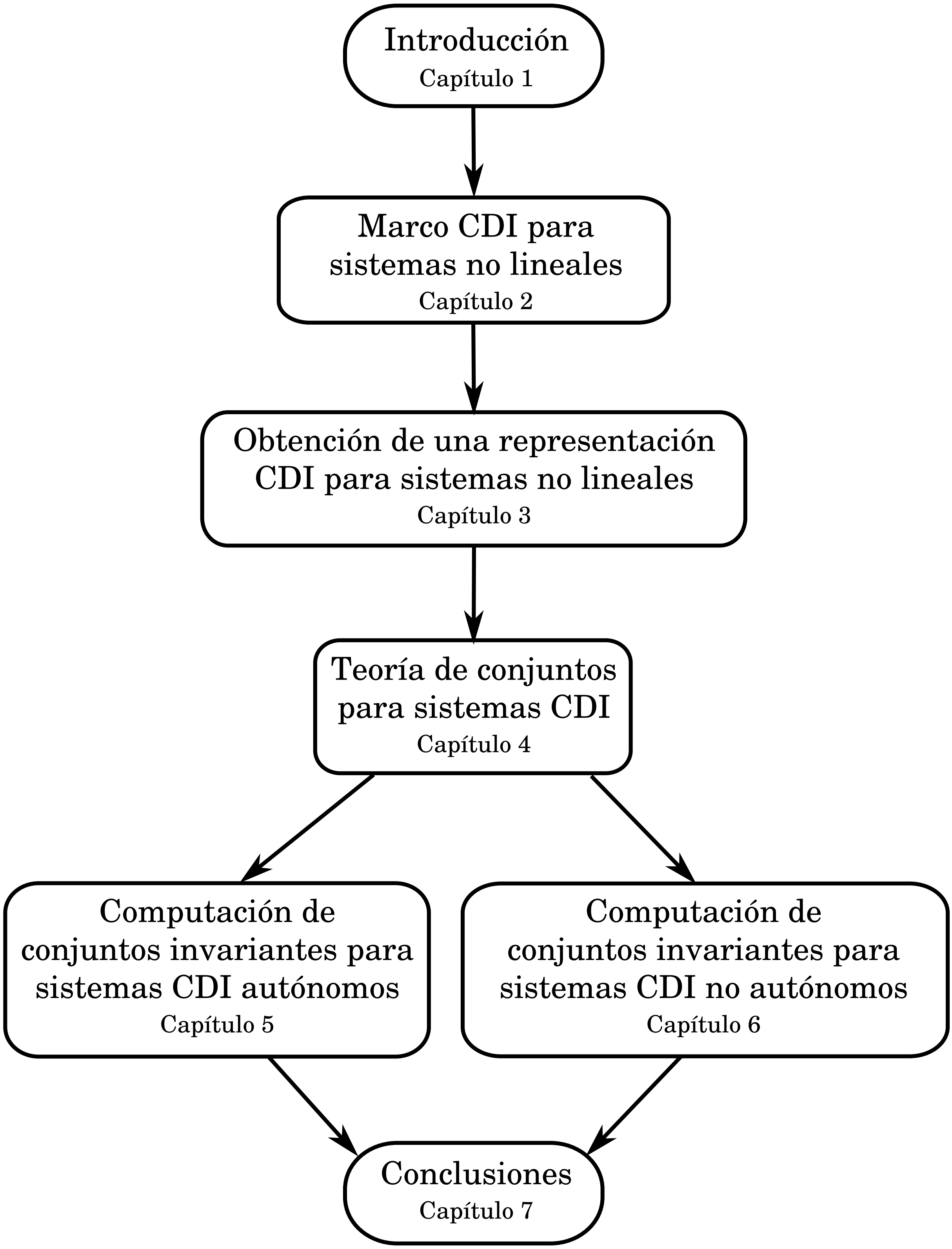

Figure 1.2: Estructura de la tesis.

Chapter 1

Introduction

The main objective of this thesis is to contribute to the development of set-theoretic methods for the analysis and design of nonlinear and uncertain systems. Particular attention will be devoted to invariance and λ-contractiveness, very important concepts in the context of modern control design and analysis for nonlinear and uncertain systems.

This chapter describes the motivation and objectives of this thesis, presents the problem we are dealing with and introduces the structure of the thesis and the work developed. We first introduce concepts related to set theory in control and dynamic systems analysis, focusing in particular on invariance. Basic definitions and generic description of properties of invariant sets will be given. Some aspects which make evident the importance of invariance and set-theoretic methods in modern control theory will be illustrated.

First an overview of the historical background which paved the way to main results in the field is presented and then an introduction to the state of the art on invariance and set theoretic methods follows.

An introduction to the concept of convexity, for both sets and functions, is then presented. Convexity deserves special interest being a keystone for many results presented in this thesis. Indeed, properties of convex sets and convex functions are widely exploited, since, given a set, convexity allows the formulation of general properties based on conditions involving a subset, possibly finite, of the set.

Finally the motivation, the objectives and the contributions of our research are illustrated and the thesis structure presented.

1.1 Set-theoretic methods in control

Generically, with set-theoretic methods we refer to those techniques concerning properties shared by all the elements of sets of the state space. Two important examples in the field of dynamic systems and control design involving set-theoretic methods are represented by invariance and the worst-case approach to the problems of analysis and design.

Set-theoretic approach is useful in the framework of analysis and control design for uncertain systems in presence of unknown but bounded uncertainty.

1.1.1 Worst-case approach to analysis and design

The classical approach to deal with the standard analysis and control problems for uncertain systems, up to the end of the sixties, was based on stochastic assumptions on the nature of the uncertainty. The objective of optimal control in this framework is usually the determination of the input action minimizing the expectation of a cost function, under the assumption of an uncertainty characterized by a given probability distribution. Analogously, assuming for instance that the system is linear, the initial conditions and the measurements and system noises are modelled by white Gaussian processes, the estimation problem is solved by the use of a Kalman filter, which provides the optimal solution minimizing the expected value of the estimation error.

A parallel, and in a certain sense dual, way of proceeding is through the so-called worst-case (or guaranteed) approach. This approach is based on different assumptions on the uncertainty affecting the system. Indeed, in this scenario the uncertainty is assumed to be unknown but bounded inside a set. Such approach is based on the following considerations:

- The assumption of full knowledge of the probability distribution of noises and disturbances can be too restrictive, while the assumption of presence of bounds on uncertainty can be more realistic in many cases. Indeed, worst-case approach is often justified by the fact that no probabilistic assumption on the uncertainties can be made, while bounds on the model errors can be established in many of the cases. Consider for example systems with dynamics depending on parameters whose value presents bounds due to known physical limitations. In this case the worst-case approach can be more realistic than the stochastic one.

- When the system presents hard constraints, the worst-case approach has some advantages. If we consider the stochastic approach, no guarantee of constraints satisfaction can be assured, while the results obtained using the worst-case approach ensure constraints satisfaction, provided that the assumptions on the uncertainty hold.

- Nonlinearities can be handled by assuming uncertainties. If the dynamic system is known to be nonlinear, a linear system with bounded uncertainty can be used, supposing that the uncertainty models the mismatch with the nonlinear system. This approximation procedure, although introducing some conservativeness, permits to apply linear based results to nonlinear systems. Linear difference inclusion (LDI) systems and linear systems with additive bounded uncertainty are classical modelling frameworks raised in this context, see (Gurvits, 1995; Boyd et al., 1994).

The set-theoretic methods appear as sort of counterpart of the stochastic methods for problems of stability analysis, robust control design and state estimation for systems affected by unknown but bounded uncertainties. The assumption of modelling the uncertainty as unknown but bounded, rather than a stochastic process, was first introduced in the pioneering works by Witsenhausen (1968*b*), Schweppe (1968), Bertsekas and Rhodes (1971).

It is worth noticing that the objective of worst-case approach techniques is to obtain sets of elements satisfying the desired features, rather than the particular element optimal with respect to an evaluation criterion. For instance, regions of the state space whose points ensure constraints satisfaction for control is the counterpart of stochastic LQG, while the guaranteed set-membership estimation is the worst-case counterpart of Kalman filter.

1.1.2 Invariance

The concept of invariance has become fundamental for the analysis and design of control systems. Although many research efforts have been directed to related themes throughout the whole second half of the last century, the field became particularly active in the last years. The importance of invariant sets in control is due to the implicit stability properties of these regions of the state space.

An invariant set for a given dynamic system is a region of the state space such that the trajectory generated by the system remains confined in the set if the initial condition lies within it, (Blanchini and Miani, 2008). More formal definitions for invariance are provided in Appendix A, a conceptual characterization of invariance is sufficient here for showing how invariance can be used in control and its main properties.

Particularly relevant is the property of robust (control) invariance of a set, since it can be used in the context of stability and constraints satisfaction for dynamic systems in presence of unknown but bounded uncertainties. Also the issue of convergence of model predictive control strategies is strongly related to the concept of robust control invariance. The diagram in Figure 1.1 represents the relations between invariance, λ-contractiveness and the main topics involved in control theory.

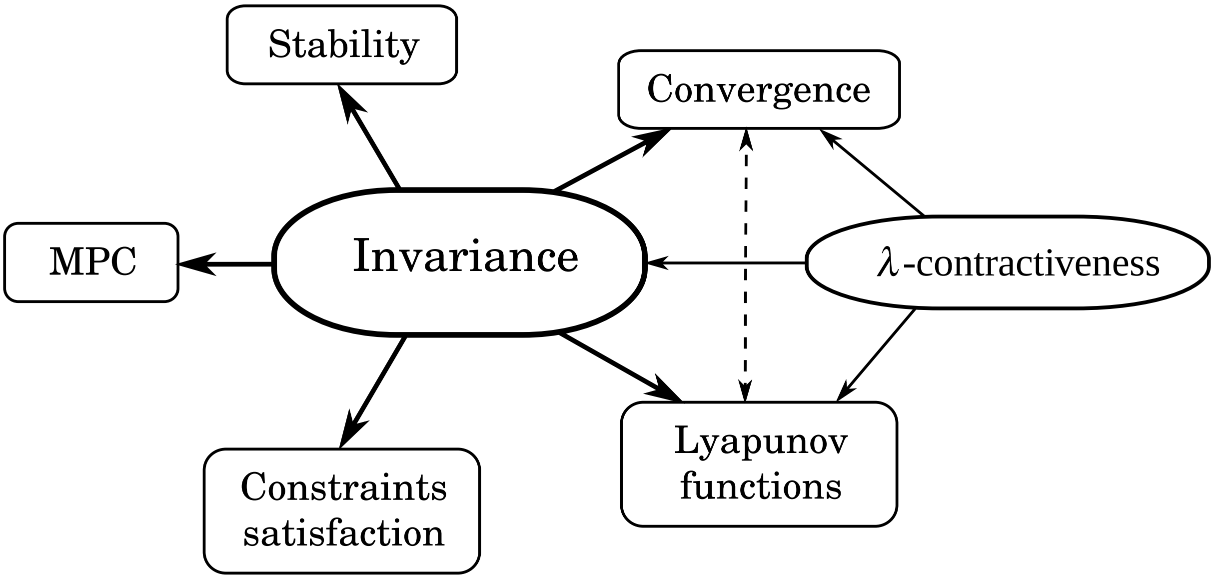

Figure 1.1: Invariance in control.

1.1.2.1 Invariance and hard constraints satisfaction

We consider the standard definition of invariance for discrete-time deterministic autonomous systems, see (Blanchini and Miani, 2008). More definitions and properties related to invariance (for instance, for uncertain systems, for non-autonomous systems, λ-contractive sets, etc.) are presented in the Appendix A.

Consider the autonomous discrete-time system

$$x^+ = f(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state, $x^+ \in \mathbb{R}^n$ is the successor state and $f : D \to \mathbb{R}^n$ is a function defined on the set $D \subseteq \mathbb{R}^n$.

A subset of the state space, $\Omega \subseteq D$, is a positive invariant set if every trajectory given by x_k, with $k \in \mathbb{N}$, generated by (1.1) and with $x_0 \in \Omega$, is such that $x_k \in \Omega$ for all $k \in \mathbb{N}$. Roughly speaking, Ω is a positive invariant set if every trajectory generated by the dynamic system with initial condition x_0 in Ω, remains confined in the set Ω.

Although invariance of a set is a property which concerns all the trajectories generated by the dynamic system with initial condition in Ω, it can be stated through an alternative definition, which does not explicitly involve the trajectories. In fact, a set $\Omega \subseteq D$ is a positive invariant set for the discrete-time autonomous system (1.1) if $f(x) \in \Omega$, for all $x \in \Omega$.

It can be proved that any element of an invariant set Ω is mapped through the dynamic function inside Ω if and only if the whole trajectory generated by the system, with initial state in Ω, remains contained in the invariant set. Indeed, if $x_0 \in \Omega$ then, by definition of invariance, we have that $x_1 = f(x_0) \in \Omega$, which implies $x_2 = f(x_1) \in \Omega$ and so on. Then $x_k \in \Omega$, for all $k \in \mathbb{N}$.

Notice that we employed the term *positive* invariant set, to distinguish the concept from simple invariance. Historically, the term invariant set denotes a set of initial conditions whose trajectory backward and forward in time is confined in the set, while for a positive invariant set only the future part of trajectories are required to belong to the set. Since in this thesis we are interested exclusively in positive invariant sets, we will refer to them simply as invariant sets.

Positive invariance can also be expressed in terms of image of Ω through function $f(\cdot)$. In fact a set $\Omega \subseteq D$ is a positive invariant set if

$$f(\Omega) \subseteq \Omega.$$

Analogous definitions can be given for non-autonomous systems, that is, in presence of control input. A control invariant set is a region Ω of the state space such that, for any of its elements $x \in \Omega$, there exists a control input $u(x)$ that maintains the successor state inside Ω. It follows that, given a control invariant set, there exists at least a control law $u(x)$ defined on Ω such that the set is an invariant set for the system in closed-loop with $u(x)$.

The relation between hard constraints satisfaction for a generic system and invariance is evident. Suppose that the state of the system is required to be maintained inside a region of the state space, say the set $X \subseteq \mathbb{R}^n$. The existence of an invariant set Ω contained in X ensures that, if the current state of the system is contained in Ω, then no constraints violation will occur, at any time step $k \in \mathbb{N}$.

In fact, for any invariant set $\Omega \subseteq X$ we have that, by definition of invariance,

$$f(\Omega) \subseteq \Omega \subseteq X,$$

which means $x_k \in X$ for all $k \in \mathbb{N}$, where $x_{k+1} = f(x_k)$, with initial condition $x_0 \in \Omega$. This entails that any element of the trajectory does not leave the set Ω, hence no constraint violation will occur in the whole future evolution of the system. Once more, it is worth pointing out that, although invariance is a condition which involves the behavior of the system at any time instant, from present to infinite, it can be given by a simple geometric set condition.

1.1.2.2 Maximal invariant set and one-step operator

The strong relation between hard constraints satisfaction and invariance justifies the interest in the maximal invariant set contained in a region of the state space, see references (Gutman and Cwikel, 1986; Gutman and Cwikel, 1987; Gilbert and Tan, 1991; Blanchini, 1999) and (Kolmanovsky and Gilbert, 1998).

Given a region X of the state space, many invariant sets can be contained in it. For instance, it is evident that any equilibrium point contained in X is an invariant set. The maximal invariant set is a set which is invariant for the system and contains any other invariant set. It is easy to prove that the maximal invariant set, when exists and is non-empty, is given by all the elements of X such that their evolutions will never abandon X. That means that a point x belongs to the maximal invariant set if and only if the trajectory generated by the system with initial condition $x_0 = x$ never violates the constraint, i.e. $x_k \in X$ for all $k \in \mathbb{R}^n$. On the other hand, if a point does not belong to the maximal invariant set, then there will certainly be a time step at which a constraint violation will occur. Roughly speaking, the maximal invariant set contained in X can be seen as the set of "safe" points in X, in the sense that no constraints violation will occur in the future.

Although many algorithmic procedures for computing the maximal invariant set have been proposed, there is a basic idea common to all of them. The iterative procedures are based on the use of the one-step operator $Q(\cdot)$. Given a set $\Omega \subseteq X$ in the state space and a dynamic system, the one-step set $Q(\Omega)$ relates Ω to the set of points in X whose evolution through the dynamic function is contained in Ω. That is, given $\Omega \subseteq X$, a point x belongs to the one-step set $Q(\Omega)$ if $x \in X$ and $f(x) \in \Omega$. Hence, $X_1 = Q(X)$ is the set of points in X which remain in X at least at the first instant. It is clear that iterative application generates a sequence of sets $X_{k+1} = Q(X_k) \cap X_k$, such that a point x belongs to X_k if and only if the trajectory generated with initial condition $x_0 = x$ remains in X at least during the first k steps, for all $k \in \mathbb{N}$. It should be also evident that the maximal invariant set can be obtained iterating the procedure for an infinite number of steps.

The result would be not very useful unless the maximal invariant set can be obtained after a finite number of iterations. In this case the invariant set is said to be finitely determined and the number of iterations is denoted as determination index. Important contributions have been provided in literature, mainly for linear systems, which permit to establish conditions under which the maximal invariant set is finitely determined.

Another important property of the one-step operator is the fact that applying the operator to a set which is invariant, generates another invariant set which contains the previous one. Thus, the iterative application of the one-step operator, with a given invariant set as initial element, produces a growing sequences of invariant sets. Notice that the same iterative process with $X_0 = X$ as initial element, generates a sequence of sets not necessarily invariant, which entails that invariance of the current set is not guaranteed until the determination index is reached (if finite).

In some cases, it can be proved that iterations initialized with an invariant set converge to the domain of attraction of an equilibrium point, that is, to the set of points which converge to the equilibrium. Clearly, assumptions on stability of the system are required in this case.

1.1.2.3 Reachable sets and minimal invariant set

In this section we introduce two important set-theoretic concepts such as the reachable sets and the minimal invariant set for dynamic systems affected by additive uncertainty. Those two concepts are strongly related, since the minimal invariant set can be viewed as the limit set of the sequence of reachable sets, as illustrated below.

Consider a linear asymptotically stable system in presence of additive uncertainty, that is

$$x^+ = Ax + w,$$

where w is the uncertainty and $w \in W$, with W bounded subset of the state space with $0 \in W$. Note that, due to the presence of additive uncertainty, the successor of a state depends on the realization of the uncertainty and all the possible successors of a state form a set. That is, given a state x the successor set is given by the set $(Ax \oplus W) \subseteq \mathbb{R}^n$.

In this context, it is useful to introduce the concept of reachable sets. The reachable set at $k \in \mathbb{N}$ is the set of states that can belong at time k to a trajectory for given initial conditions, for a proper admissible uncertainty realization. We have that, given an initial set $R_0 \subseteq \mathbb{R}^n$, the reachable set, at any time instant $k \in \mathbb{N}$, can be obtained recursively as

$$R_{k+1} = AR_k \oplus W = A^{k+1} R_0 \oplus \bigoplus_{i=0}^{k} A^i W.$$

The set R_k is called reachable set, at time $k \in \mathbb{N}$, the sequence of R_k is the reachable tube, reachable from R_0. The sequence of reachable sets is interesting since it holds the information about all the possible trajectories generated by an uncertain system with initial condition contained in R_0. The reachable sets for a linear uncertain system can be used to compute bounds on the real evolution of a nonlinear system, provided that the mismatch between the two models is bounded in W. Moreover, the reachable tube can be viewed as the result of state estimation in absence of measurement. Reachable sets computation is used also in the worst-case approach to the problem of state estimation, when integrated with the information given by a measurement.

Particularly interesting are the reachable sets (and tube) for a linear uncertain system with the origin as initial condition, given by

$$R_{k+1} = AR_k \oplus W = \bigoplus_{i=0}^{k} A^i W,$$

with $R_0 = \{0\}$.

In presence of hard constraints for linear systems, reachable sets can be used to pose a sufficient condition for excluding any constraints violation along all the possible trajectories. In fact, if the reachable sets are contained in the admissible region of the state space then no constraint violation is possible.

This idea can be exploited to design control laws guaranteeing hard constraints satisfaction, see (Chisci et al., 2001). Those robust control laws based on the information provided by the reachable sets are referred to as tube based control strategies. The reachable tube approaches presented first in (Witsenhausen, 1968*a*; Bertsekas and Rhodes, 1971*a*; Glover and Schweppe, 1971) provides a solution for robust control in presence of unknown but bounded uncertainty.

More recently, the approach has been extended to model predictive control based strategies, see (Mayne et al., 2000; Chisci et al., 2001; Camacho and Bordóns, 2004; Limón et al., 2005; Bravo et al., 2006), and (Langson et al., 2004; Magni et al., 2001), which has been revealed to be one of the most appropriate robust control strategies in presence of hard constraints. It is evident how useful can result, in fact, the concept of reachable sets in the context of model predictive control of systems affected by additive uncertainty, where a prediction of the state is required.

As claimed above, a particularly interesting invariant set is the minimal one. The minimal invariant set for a system is the invariant set contained in every other invariant set. It can be proved that the minimal invariant set is the set of points in the state space that can be reached from the origin. Conceptually, it is the set of all possible states that can belong to all the possible trajectories generated by the system, with the origin as initial condition. It can also be proved that the minimial invariant set for the uncertain linear system is given by

$$R_\infty = \bigoplus_{i=0}^{\infty} A^i W,$$

which is the reachable set R_k, from the origin, when k tends to infinity. It is evident by definition that, in general, the exact minimal invariant set can not be obtained. Methods to compute approximations of the minimal invariant set are the objective of recent research efforts, see (Raković et al., 2005; Ong and Gilbert, 2006).

Unlike the maximal invariant set which is interesting to be computed and analyzed for both deterministic and uncertain systems (linear or nonlinear), minimal invariant set is meaningful only in presence of additive uncertainty. Moreover, it has to be pointed out that minimal invariant set has been studied mainly for linear systems.

The interest in the computation of the minimal invariant set and its properties is more recent. The reasons that make the minimal invariant set interesting to the control community are less intuitive than those of the maximal invariant set. The minimal invariant set is useful in the following contexts:

- Important conditions on the existence and the finite determination of the maximal invariant set for linear uncertain systems are based on the minimal invariant set. Notice that, if the minimal invariant set computed in absence of constraints is not contained in the admissible region X, then no robust invariant set can be obtained. This means that if the minimal invariant set is not contained in the set X then there exists a sequence of uncertainty realizations which leads the state to violate the constraints, for any initial condition in X.

- The classical concept of asymptotic stability to an equilibrium point for a system is not applicable in case of presence of additive uncertainty. We recall here, only conceptually, that a system is asymptotically stable if the trajectories stay bounded (at least those starting in a neighborhood of the equilibrium) and the distance from the state and the equilibrium converges to zero. It is clear that, unless the uncertainty is assumed to vanish as the system approaches the equilibrium (consider for instance the case of uncertainty modelled as function of the state), the system can not be maintained at the origin. In fact, no equilibrium is admitted. A concept analogous to asymptotic stability can be formulated, for the case of presence of additive uncertainty, by replacing the equilibrium point with a set in the state space and the distance from the equilibrium with the distance of a state from a set. This concept is referred to as ultimate boundedness. The set to which the system can be proved to converge is the minimal invariant set. Then, the minimal invariant set can be viewed as the analogous for uncertain systems of the equilibrium point for deterministic systems.

- Recently, a new approach based on tubes gained more and more popularity in the field of robust control for linear systems in presence of additive uncertainty, see (Raković and Mayne, 2005; Limón et al., 2008). Roughly speaking, such tube-based control techniques propose to split the control action in a local and a nominal part. First, the local control is designed to maintain the real state in a neighborhood of a nominal state, then the nominal evolution can be steered to the equilibrium. The nominal evolution is obtained by the system dynamics in absence of uncertainty.

  Provided that the neighborhood of the nominal state is an invariant set for the uncertain system in closed-loop with the local control law, it can be proved that the tube composed by the invariant set "centered" at nominal states contains the real trajectory, regardless on the uncertainty realization. The objective is then reduced to control the nominal trajectory maintaining the tube inside the admissible region X. It is clear that, in general, the smaller is the invariant set determining the tube shape, the greater is the feasibility tube in which the nominal trajectory has to be confined. Once the local control law is determined, using the minimal invariant set provides the less conservative reachable tube ensuring to contain the real evolution of the system.

1.1.2.4 λ-contractive sets and induced Lyapunov functions

It has been shown that invariance of a region of the state space is a property which implicitly characterizes all the possible trajectories generated by its elements, involving the transient behavior of the system as well as the steady state, that is, its behavior at the limit. This makes invariant sets a very useful tool for both purposes: guaranteeing hard constraints satisfaction and ensuring stability. Convergence to an equilibrium (or to a set) can also be related to regions of the state space introducing the concept of λ-contractiveness.

Conceptually, a convex, compact set Ω containing the origin in the interior is a λ-contractive set for a dynamic system if every initial state in Ω evolves into the scaled set, $\lambda\Omega$, with a positive scaling factor λ smaller than one. It follows that the image of Ω through the dynamic function characterizing the system is contained in the interior of Ω. Clearly, if $\lambda = 1$, then the definition of invariance is recovered. Furthermore, it is evident that λ-contractiveness implies invariance.

Analogous considerations are also valid for control invariant sets. That is, also in presence of a control action it can be of interest to determine a region Ω of the state space such that there exists a control law mapping Ω into $\lambda\Omega$. Notice that, if the condition of convexity of Ω drops, then the λ-contractiveness does not imply invariance, since $\lambda\Omega$ is not necessarily contained in Ω in this case.

The concept of λ-contractiveness of a set for a given dynamic system, can induce a Lyapunov function, and then asymptotic stability or ultimate boundedness. The relation between λ-contractive sets and Lyapunov functions can be illustrated by means of the concept of Minkowski function. Given a compact, convex set Ω (containing the origin in the interior), its Minkowski function is a function of the state $x \in \mathbb{R}^n$ defined as the minimal α such that x is contained in $\alpha\Omega$ and it is denoted as $\Psi_\Omega(x)$.

In case of linear systems affected by both parametric and additive uncertainty, given a λ-contractive set Ω, any set $\mu\Omega$, with $\mu \geq 1$, is λ-contractive, see property P1 in (Blanchini, 1994). It can also be easily proved that if there is no additive term of the uncertainty, then $\mu\Omega$ is λ-contractive for all positive μ. In the absence of additive uncertainties and assuming that Ω is a convex, compact λ-contractive set containing the origin in its interior, its Minkowski function is a Lyapunov function. In fact, if the Minkowski function at a point x is $\Psi_\Omega(x) = \alpha$, its value at its successor x^+ is smaller than or equal to $\alpha\lambda$, i.e. $\Psi_\Omega(x^+) \leq \alpha\lambda$. It follows that the Minkowski function decreases along the system trajectories, if $\lambda < 1$ and the state is not the origin. This, and the fact that the Minkowski function is a definite positive function of the state, ensures that it is a Lyapunov function.

It is important to point out that the fact that Ω is λ-contractive implies that also $\alpha\Omega$ is λ-contractive (for positive α), is not valid for nonlinear systems. Then induced Lyapunov functions cannot be determined in general. An important contribution of this thesis concerns this aspect. In fact, we propose modelling frameworks which permit to ensure asymptotic (exponential) stability for a wide class of nonlinear systems, determining induced Lyapunov functions for systems bounding the nonlinear ones. Once more, such important property relies on convexity.

These considerations permit to take into account Lyapunov functions whose level sets are not the ellipsoidal sets obtained with classical quadratic Lyapunov functions. This means that the characterization of generic λ-contractive sets entails an implicit analysis of stability properties through a wider class of potential Lyapunov functions. The use of polyhedral Lyapunov functions, induced by polytopic λ-contractive sets, gained particular interest in the last decades, see (Blanchini, 1994; Blanchini, 1995; Blanchini and Miani, 2008). Polytopes are in fact very versatile and permit to approximate every convex set.

1.1.2.5 Model predictive control and invariant sets

Invariant sets are widely employed to design stabilizing controllers and, in particular, for applying receding horizon control strategies. In fact, many formulations of the model predictive control (MPC) need a terminal region within which (asymptotic) convergence can be implicitly assured by a simple, often linear, control law, see (Mayne et al., 2000; Bemporad et al., 2002; Camacho and Bordóns, 2004).

We shortly recall the key features of MPC, to show the importance of invariance for this very popular control technique. Although many variations of predictive controllers have been formulated, we provide here the ingredients characterizing standard MPC:

- Model based prediction. The control is based on the prediction of the evolution of the system. A dynamic model, linear or nonlinear, of the real system is assumed to be known. Since on-line computations are required to be performed at every time step, usually the model considered is discrete-time and assumed time-invariant. At any instant, the real state is measured and a prediction of the system evolution is obtained as a function of the input, within a time range called prediction horizon N_p. The number N_c of elements of the sequence of future inputs, called control horizon, can be different from the prediction horizon. The model permits to prevent constraints violation that can occur in the future, within the prediction horizon.
- Constraints. The main reason for the increasing popularity of MPC is its capability to cope with hard constraints. Due to the presence of a model based prediction, the control input sequences leading the system to constraints violation are implicitly discarded from the set of all the possible ones. State and input constraints can be posed in the optimization problem solved on-line. The result is that only a subset of all possible input sequences are assumed feasible. Such set, the feasibility region for the

optimization problem, is the subset of the space of input sequences (obtained through the Cartesian product of the input space) composed by only those sequences that avoid constraints violation. Thus, to any element of the feasibility region is associated a potential admissible trajectory.

- Cost function. The optimization problem is solved to obtain the trajectory, among all the admissible ones, minimizing a cost function. The cost function usually encompasses a part penalizing a measure of the distance between the predicted trajectory and the desired one and another part penalizing the control effort. Intuitively, the objective is to compute the control sequence and the associated trajectory providing high performance with low control effort. Different cost functions can be considered. An important feature of the cost function is that it should be a positive definite function of the predicted states and the control input in the sequence. This can be used to prove that such cost function decreases along the real trajectory of the system controlled through the MPC strategy, resulting then in a Lyapunov function and guaranteeing asymptotic stability.

- Receding horizon. The optimization problem is solved on-line at each time step. Once the optimal control input sequence has been computed, only the first control action is applied. In that way, mismatches between the real behavior of the system and the trajectory predicted through the model can be compensated.

Convergence properties of MPC are often ensured through the definition of a terminal region and a local control law which guarantees stability and, possibly, asymptotic convergence, see (Mayne et al., 2000; Camacho and Bordóns, 2004; Limón et al., 2005; Álamo, Ramírez, Muñoz de la Peña and Camacho, 2007). It is here where invariance is fundamental for MPC control.

In fact, a very common way to ensure asymptotic stability of the system controlled by MPC is imposing that the final state of the predicted sequence is contained in an invariant set, where a local control law and a Lyapunov function are defined.

Intuitively, assume that there is a feedback control law and a region invariant for the system in closed-loop. Then, once the system reaches such invariant set, it can be assumed in the prediction that the MPC is "switched off" and the local control law is applied. This guarantees that no constraints violation would occur in future. If also a Lyapunov function is defined, then asymptotic convergence can be assured. Hence, the concatenation of the first N_u control actions, solution of the optimization problem, with the rest of the control sequence obtained by means of the feedback control law, determines an infinite sequence of control actions and a trajectory admissible at any time instant and converging to the origin.

Summarizing, the introduction of a terminal constraint to impose that the last state of the predicted sequence belongs to an invariant set, provides a useful tool to ensure fundamental properties, such as asymptotic stability and hard constraints satisfaction for the whole trajectory.

Although the definition of a Lyapunov function within the terminal region of MPC is not necessary for assuring stability (see, for example, (Bravo et al., 2006)), many results on MPC for nonlinear and uncertain systems are based on this. On the other hand, only few results on how to obtain such important ingredient for MPC like the invariant set for nonlinear systems have been proposed. Particular attention is devoted in this thesis to this central problem, in fact a key contribution of our research consists in the proposed methods for obtaining convex invariant sets for nonlinear systems.

Also control invariant sets can be used in the design of control laws in presence of hard constraints, such as MPC. Assume that a control invariant set for the system is available. An additional constraint, with which the state is imposed to belong to the control invariant set at next time step, guarantees, by definition of invariance, the existence of a proper control action ensuring no constraints violation. Notice that this unique constraint can replace all the constraints on the state. If, moreover, the invariant set is λ-contractive, then convergence can be ensured, in some cases. Indeed, suppose that it is available a control invariant set Ω ensuring λ-contractiveness of $\alpha\Omega$ for the system, for $\alpha \in [0, 1]$ and for a proper control law. Absence of additive uncertainty is often required to have contractiveness of $\alpha\Omega$ for any positive α. Then, intuitively, given the current state x and its Minkowski function $\Psi_\Omega(x)$, any control action such that the Minkowski function at x^+ is smaller than $\lambda\Psi_\Omega(x)$ makes the set λ-contractive in closed-loop. Since, by construction, there exists at least a control action satisfying such condition, the problem of computing a $u(x)$ such that $\Psi_\Omega(x^+) \leq \lambda\Psi_\Omega(x)$ and it is optimal with respect to some performance measure is always feasible and ensures exponential convergence to the origin. The computation of λ-contractive sets for nonlinear systems under a proper control law, leading to the synthesis of a control ensuring asymptotic convergence in closed-loop, is not a simple task. A solution for such design problem, for particular nonlinear systems, represents another important contribution of this thesis.

Also in case of robust MPC, the definition of a robust (control) invariant set as terminal region is usually required to ensure ultimate boundedness of the controlled uncertain system. The domain of attraction of the MPC strategies is usually strongly dependent on the size of such terminal region.

1.2 State of the art on set-theoretic methods

Many results have been obtained in the last years for dealing with the worst-case approach and for characterizing invariant sets. In this section we provide some important contributions and results present in literature dealing with these themes, for both linear and nonlinear systems.

1.2.1 Historical background

The pioneering works on this field appeared at the end of the sixties, see (Schweppe, 1968; Witsenhausen, 1968*b*), and the beginning of the seventies, (Bertsekas and Rhodes, 1971*b*). The problem of set-membership estimation for systems affected by additive uncertainties concerns the issue of determining a sequence of sets such that the state of the dynamic system at time $k \in \mathbb{N}$ is ensured to be contained in the k-th element of the sequence. This is achieved by integrating the measurement information with the dynamic update, given in practice by the computation of the reachable set for one time step ahead. Notice that, in absence of measurement, the reachable tube concept is recovered.

Motivated by the problem of the tracking of an evasive target, in (Schweppe, 1968) the author addresses the problem of estimating the regions of the state space set containing the true state of a linear system affected by disturbances on the state and on the output. The system is assumed time-invariant and continuous-time, while the observations are made at discrete instants. Initial conditions and output disturbance are unknown but bounded by two ellipsoids, while two kinds of bounds for the state disturbance are considered: an ellipsoidal set and an energy type bound (i.e. a bound on the integral). Ellipsoidal approximations are proposed.

Also Witsenhausen deals in (Witsenhausen, 1968*b*) with the problem of computation of sets in the state space which are compatible with the observations and the initial conditions. In particular, a linear discrete-time time-varying system is considered. The system is supposed to be affected by disturbances on the state and on the output, assumed bounded by compact and convex sets. The issues of complexity and approximation are mentioned and the ellipsoidal framework provided by Schweppe is referred to. The alternative proposal is to use polyhedra, which yield to linear programming problems.

The work (Bertsekas and Rhodes, 1971*b*) is concerned with the same estimation problem considered by Schweppe but it is extended to the analysis of the cases of smoothing and prediction. Moreover, the adaptation to discrete-time systems is explicitly exposed. The main advantages of the methods proposed here with respect to the Schweppe's approach are the pre-computability of the matrix defining the prediction ellipsoid and the convergence, for time-invariant systems, to a steady-state solution as time tends to infinity.

More recently, in (Maksarov and Norton, 1996), the issue of estimation of the feasible sets (the set of state consistent with the model, the bounds and the measurements) for a discrete-time time-varying linear system with state and observation noises, with ellipsoidal bounds, is considered. The noises are assumed to be unknown but bounded by ellipsoids, as well as the initial condition. The estimation process is given by alternating the time-update step with the observation update. Each step requires an ellipsoidal approximation to be performed. The minimal-volume ellipsoidal approximations of the sum and of the intersection of two ellipsoids are employed. Three algorithms, based on different approximation procedures, are provided and compared.

In (Chernousko, 2002), the problem of estimation of the state feasible set for a continuous-time time-varying linear (affine) system is considered. The aim is to determine an optimal ellipsoidal outer approximation of the attainable set. The initial set and disturbance bounds are two ellipsoids and different criteria of optimality are considered, among them the volume and the sum of squared axes.

The first contribution on invariance in the field of dynamic systems has been probably (Bertsekas, 1972). This work dealt with the problem of computation and characterization of maximal robust control invariant set. In this seminal work, the author considers non-autonomous discrete-time nonlinear systems affected by uncertainty, that is, systems of the form

$$x_{k+1} = f(x_k, u_k, w_k),$$

with bounds on the state $x_k \in X$ and on the input $u_k \in U(x_k)$ which may depend on the state. Bounds on the uncertainty also can be dependent on the state and the input, i.e. $w_k \in W(x_k, u_k)$. First a necessary and sufficient condition for control invariance is presented, then a characterization of the maximal control invariant is given. Given the set X, the one-step operator has been employed to define the sequence of sets, denote them $S_k(X)$, whose elements can be maintained k times in X, for $k \in \mathbb{N}$, by means of a proper sequence of control actions. It has been proved a very interesting non-intuitive result, that is the fact that the intersection of such sequence of sets is not equal, in general, to the maximal control invariant set. A condition for this equality to hold is based on compactness of the involved sets. A particularly interesting case for which such condition is satisfied, is given by systems affine with respect to w_k, that is

$$x_{k+1} = f(x_k, u_k) + w_k,$$

with U and W not dependent on x and (x, u), respectively, sets X and W compact and $f(\cdot)$ continuous. In this case the intersection of sets $S_k(X)$, for $k \in \mathbb{N}$, converges to the maximal control invariant set.

The computational aspects for the maximal invariant set have been addressed in two works published in 1991, that is in (Gilbert and Tan, 1991) and (Blanchini, 1994) (although the second reference is relative to a paper published in 1994, a preliminary version of the work was presented in 1991, in (Blanchini, 1991)).

In (Gilbert and Tan, 1991), the problem of characterization and computation of the maximal output admissible invariant set for a linear deterministic system is studied, that is, for $x^+ = Ax$ with $y = Cx$. In particular, the constraints are assumed to be defined in the space of the output, i.e. in the form $y \in Y$. This does not entail major differences with the case of bounds on the state space. The main result concerns the condition of finite determination of the maximal invariant set. It is proved that, if the system is asymptotically stable, the pair C, A is observable, the admissible output set Y is bounded and the origin is contained in its interior, then the maximal output admissible set is finitely determined.

In (Blanchini, 1994), the non-autonomous systems taken into account are linear with parametric and additive uncertainty. The concept of λ-contractive set is introduced and an iterative procedure to compute the maximal λ-contractive set for a given $\lambda \in [0,1]$ is provided. An important result proved in the paper is the fact that the maximal λ-contractive set is the intersection of the sequence of sets computed by means of a sort of one-step operator, provided that the initial set is convex, compact and contains the origin in its interior. The result is analogous to that presented by Bertsekas, but for λ-contractive sets. Lyapunov induced functions and control design are also analyzed.

A first important survey paper on invariance is (Blanchini, 1999) summarizing the main results on the field. Continuous-time and discrete-time systems are considered, condition for invariance for linear and nonlinear systems are given. Moreover, induced Lyapunov functions as well as control design problems based on invariance are analyzed. Also the computational aspects are considered, in particular for polytopic and ellipsoidal invariant sets.

Another very important and basic work on the field is (Kolmanovsky and Gilbert, 1998). The paper deals with the problem of characterization of the maximal output admissible invariant set for discrete-time linear systems affected by additive uncertainty. The results are strongly based on mathematical tools, such as support functions and Pontryagin (or Minkowski) difference, which are widely employed in this thesis. Necessary and sufficient conditions for invariance are given, minimal and maximal invariant sets are characterized and an iterative procedure to compute the maximal invariant set is proposed. Very important results presented in (Kolmanovsky and Gilbert, 1998) are the necessary and sufficient conditions for the existence of the maximal invariant set and for its finite determination. The maximal invariant set is non-empty if and only if the minimal invariant set is contained in the admissible region, it is finitely determined if it is contained in the interior of the admissible region.

More recent papers dealing with the problem of characterization of the minimal invariant set for linear systems affected by additive uncertainty are, see (Ong and Gilbert, 2006; Raković et al., 2005). Since the minimal invariant set is the Minkowski summation of infinite terms, it cannot be computed in general, and the attention of the authors is focused on the computation of approximations of the minimal invariant set.

1.2.2 State of the art for nonlinear systems

Here we provide a short review on the results presented in the last years on issues related to set-theoretic methods and on invariance for nonlinear systems.

One of the main problems inherent to set-theoretic methods application and invariant set computation is the fact that, in general, nonlinearity of the system or the controller leads to non-convex and non-polyhedral sets. This yields to a great, often unaffordable, computational complexity. Then approximations are usually employed and a trade-off between the induced conservativeness and the computational complexity has to be reached.

First, an overview on reachability approximations and set-membership identification for nonlinear systems is recalled. The problem is the computation of the sequence of sets in the state space providing the guarantee of containing the state of the system. Notice that nonlinear systems and linear uncertain systems are related, since often the methods for computing reachable sets and estimation sets for nonlinear systems are based on linear approximations.

The problem of reachable sets computation for nonlinear systems is addressed in (Kühn, 1999) using mean value theorem related techniques to approximate the nonlinear evolution. That is, given a nonlinear system and a set, its reachable set can be obtained by approximating the nonlinear dynamic function with a linear function with additive uncertainty. The sequence of reachable sets is, then, obtained through a linear mapping and a Minkowski summation, at every step. This yields, in general, to an unaffordable growth of the complexity of the sets. The problem of complexity is solved by employing zonotopes, which allows to control the representation and computational complexity, at the price of some conservativeness. Further developments on this direction have been proposed by (Girard et al., 2006) which uses zonotopes and boxes to bound the system admissible evolution.

We recall that a way to deal with the problem of reachable sets computation for a nonlinear system is by approximating it with a linear uncertain system. A novel approach guaranteeing convergence of the sequence of approximated reachable sets is presented in (Raković and Fiacchini, 2008), where properties of invariance are employed to determine a basic shape for bounding the exact reachable sets. A procedure based on homothety leads to the determination of a computational procedure which combines low computational effort and convergence of the approximation error to zero.

Analogously, the problem of set-membership state estimation for nonlinear systems has been addressed by means of set-theoretic approaches. The work (Álamo et al., 2005) presents a new approach to guaranteed state estimation for nonlinear discrete-time systems with bounded disturbances on the state and on the output. An algorithm to compute a set that contains the states consistent with the measured output and the system model is provided. This set is represented by a zonotope. The size of the zonotope is minimized at each time step by an analytic expression or by solving a convex optimization problem. Interval arithmetic is used to calculate a guaranteed sequence of sets in the state space.

In (Álamo, Bravo, Redondo and Camacho, 2007) a method for guaranteed state estimation for nonlinear discrete-time systems with bounded disturbances is presented. The sets of states that are consistent with the evolution of the system, the measured outputs and bounded disturbances are represented by zonotopes. The main novelty is the usage of DC functions to compute the approximating sequence of sets. DC functions result very useful in order to compute bounds on the optimal solutions of non-convex programming problems, and are also used in this thesis.

It is not trivial to adapt to the nonlinear case those standard mathematical tools, such as the one-step operator and reachable sets, widely employed for analysis of invariance for linear systems. Consider for instance the fact that the one-step set $Q(\Omega)$ is not necessarily convex for nonlinear systems, neither for Ω convex. Then its application can lead to the generation of sequences of highly complex sets.

One of the major problems when moving from linear to nonlinear systems comes from the fact that some useful properties related to linearity are lost. A very interesting and clear example is illustrated in the following considerations on the condition for invariance, cited from (Blanchini and Miani, 2008). In this work, first the continuous-time case is considered. A fundamental result on invariance is represented by the Nagumo theorem, which provides a necessary and sufficient condition for invariance for a closed subset of the state space, for continuous-time systems. [1] Such theorem claims that a set is positively invariant if and only if the velocity vector is directed towards the interior (or tangent to the boundary) of the set at any point of the boundary. Intuitively, if the velocity vector heads inside the set, it implies that the trajectories on the boundary enter the set, then there cannot be any trajectory starting inside and leaving the set. This entails clearly invariance.

Then the attention, in (Blanchini and Miani, 2008), moves to the discrete-time case

$$x^+ = f(x),$$

for which invariance of a set S cannot be ensured by a boundary condition.

The authors expressly state that:

"... As it can be easily understood, there is no evident extension of Nagumo's "boundary-type" condition for discrete-time systems. Intuitively, the natural counterpart of the Nagumo's condition,... would be

$$f(x) \in S, \qquad \forall x \in \partial S,$$

which means, roughly, the state on the boundary "jumps inside". However, this condition is not sufficient to assure $f(x) \in S$ for all $x \in S$. Indeed, it is easy to provide discrete-time examples in which the above boundary condition can be satisfied, yet the set is not positively invariant. Therefore the only reasonable "discrete-time extension" of Nagumo's theorem is the tautology: S is positively invariant if and only if

$$f(S) \subseteq S.$$

Luckily enough, the situation is completely different if we restrict our attention to the class of homogeneous systems (including the linear ones) ..."

[1] The Nagumo theorem is not given formally here, only its geometrical meaning is described.

This means that focussing on general nonlinear systems the analysis of invariance has to involve the whole set, while for linear (and homogeneous) systems a boundary-type condition for invariance can be given. One of the main conceptual contribution of this thesis is to show that boundary-type condition can be stated also for nonlinear systems. The ingredient which permits to infer properties of the whole set from boundary-based analysis is convexity.

1.2.2.1 Contributions on invariant sets computation for nonlinear systems

First it is worth mentioning the work (Kerrigan and Maciejowski, 2000) which provides a survey on invariance for nonlinear systems up to the moment. The paper focuses mainly on theoretical characterization of invariance and its use in control, rather than on the computational issues.

The problem of designing an MPC control for nonlinear systems is addressed in (Cannon et al., 2003). For that purpose, the issue of computing an invariant set to be used as target set is considered. In particular, a polytopic invariant set is computed for an LDI system valid within a region. The problem is posed as a linear programming problem whose objective is to maximize the volume of the polytope, with bounded geometric complexity. In particular, the linear image of an infinity-norm ball is considered, i.e., a parallelotope, whose vertices are the optimization variables.

The problem of linear system with particular static nonlinear feedbacks, such as piecewise affine and saturation, has been addressed in the work (Hu and Lin, 2004), where conditions for invariance for an ellipsoid are provided.

The work (Bravo et al., 2005) deals with the problem of computation of control invariant sets for constrained nonlinear systems. The proposed approach is based on the computation of an inner approximation of the one-step operator, that is, the set of states that can be steered to a given target set by an admissible control action. Based on this procedure, control invariant sets can be computed by recursion. In this work, interval arithmetic is employed to compute the one-step set.

Particular attention has been devoted to a common nonlinearity present in real dynamic system, the saturation. The paper (da Silva and Tarbouriech, 1999) addresses the problem of analysis and computation of invariant and λ-contractive sets for discrete-time systems in presence of saturation. In particular, the authors divided the space in regions where the upper or lower value of saturation is attained or where no saturation occurs. Then, the system is assumed linear and perturbed by constant perturbations within any region. Necessary and sufficient condition for contractiveness of polyhedral sets is given.

In (Álamo, Cepeda, Limón and Camacho, 2006*b*) a method to estimate the domain of attraction for discrete-time saturated systems is presented. A new notion of invariance, denoted SNS-invariance, is introduced. An algorithm to generate a sequence of nested invariant sets is provided and it is proved that the sequence converges to the largest SNS-invariant set for this class of systems. It is also proved that the SNS-invariant sets generated by this iterative algorithm are polyhedral convex sets and constitute an estimation of the domain of attraction of the non-linear system. The same problem has been tackled by the authors also in (Álamo, Cepeda, Limón and Camacho, 2006*a*).

1.3 Convexity and invariance

One of the keystones of the thesis, the concept of convexity of sets and functions, is briefly illustrated in this section. Many efforts have been directed to the analysis of convexity, see for instance (Boyd and Vandenberghe, 2004; Rockafellar, 1970; Schneider, 1993; Ben-Tal and Nemirovski, 2001).

The importance of convexity for topics related to invariance and set-theory in control is manifold. The first reason is the high complexity of representation and computation induced by non-convex sets. For example, from the practical point of view, the standard algorithmic procedures usually generate sequence of sets whose complexity explodes after few steps when dealing with non-convex sets. As a matter of fact, the families of sets considered in literature for practical issues related to invariance share the property of convexity, for instance ellipsoids, polytopes, zonotopes and boxes.

On the other hand, convexity of functions permits to infer properties which are satisfied by any element of a set by means of conditions involving only a finite subset of points. This consideration leads also to convex optimization problems, which are computationally tractable, and then to algorithms characterized by affordable complexity. Hence, it is worth recalling here some basic definitions and properties related to convexity of sets and functions.

A set $S \subseteq \mathbb{R}^n$ is said to be convex if, for every pair of elements of S, that is for every $x, y \in S$, we have that the whole segment between the two points is contained in S.

An important property of convex sets, see (Rockafellar, 1970), is the fact that a set $S \subseteq \mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements. This means that, if S is convex, any point that can be expressed as a convex combination of elements of S belongs to S. Conversely, if any point expressable as a convex combination of elements of S belongs to S, then convexity holds.

There are several different ways to define convexity of a function. One way is based on the concept of convex set and provides a geometrical meaning of convex functions.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ define its graph as $\{(x, f(x)) \in \mathbb{R}^{n+1} : \ x \in \mathrm{dom}\ f\}$ and its epigraph the set of points in $\mathbb{R}^{n+1}$ lying above the graph. A function is convex if its epigraph is a convex subset of $\mathbb{R}^{n+1}$. It is evident how deeply related are the concepts of convex sets and convex functions, as the latter can be defined in terms of the former.

An important feature of convexity is the fact that a property involving only some elements of a set can be extended to a possibly uncountable set of points, when dealing with convex sets and convex functions. We will see that the fact that general properties can be inferred from conditions involving few elements of a set will permit to formulate algorithmic procedures for invariant set computation, algorithms characterized by affordable complexity.

As an example, we shortly introduce convex programming, which plays a key role when dealing with mathematical programming problems. Convex programming problems are characterized by the minimization of a convex cost function over a convex feasible set, or equivalent problems, see (Boyd and Vandenberghe, 2004).

The importance of convex programming problems is due to the fact that they are characterized by efficient resolvability, that is, they can be solved in polynomial time. The general nonlinear programming problems are often much more computationally demanding, see (Ben-Tal and Nemirovski, 2001).

This can be intuitively understood recalling that a local minimum for a convex problem is also a global minimum, unlike the case of non-convex problem. In the context of convex problems, iterative algorithms, based on the gradient of the cost function for instance, can be designed, see (Bazaraa and Shetty, 1979). Thus, through the algorithm iterations, the cost function value decreases at every step and then convergence to the optimum is ensured. If, on the contrary, local optima are not also global, as for non-convex problems, the algorithms exploiting the gradient of the cost function does not guarantee, in general, convergence to a global minimum. Hence, in the context of mathematical programming, the availability of a convex programming representation for a problem is crucial.

1.4 Motivation and objectives of the thesis

As illustrated above, invariance and set-theoretic methods are very important for the analysis of nonlinear and uncertain systems. Moreover, it is evident how useful is the application of invariance and set-theory related structures for the determination of robust control strategies, the design of control laws in presence of hard constraints, prediction based control etc, hence for control synthesis in general, for nonlinear and uncertain systems.

Despite such fundamental position gained by invariance and set-theory in systems analysis and control, mainly in the last decades, only very few results have been provided for nonlinear systems. Results presented in literature have been derived for particular cases of nonlinear systems (such as saturated systems, bilinear systems, etc) and they often cannot be applied to general nonlinear systems. That is, despite of the importance of such structures, their application are restricted to particular cases.

Consider the mentioned importance of invariant sets in model predictive control for nonlinear systems. In this context, the availability of an invariant set is often used to prove the desired properties for the controlled system, such as stability, convergence and constraints satisfaction. On the other hand, it is not trivial in practice to obtain an invariant set for a nonlinear system, usually required to be used as terminal region in MPC. Roughly speaking, in many contributions on nonlinear predictive control the availability of an invariant set leading to general results is assumed, but the computational problem of how to obtain such important ingredient is not tackled. This fact might contribute to lose some of the generality of such results, in practice.

There is a clear gap between the importance of invariance and set-theory for theoretical purposes and the practical applicability of the obtained results, especially for nonlinear systems case. It has to be pointed out that this gap is less wide for the case of linear uncertain systems, for which many well established computational methods are given in literature. The objective of this thesis is to contribute in filling this gap. Particular attention is in fact devoted to nonlinear systems.

The main conceptual problem when moving from analytical and computational methods for linear systems, is the fact that many properties and features which are intrinsic to linearity cannot be exported to nonlinear systems. The Nagumo theorem is a clear example of a condition based on local analysis that is not applicable in case of nonlinearity.

The underlying idea, common to many of the results presented in the thesis, is to adapt those methods and properties characterizing analysis and control design for linear systems to nonlinear ones, exploiting convexity. It will be showed, that in many cases convexity is the ”missing” ingredient which allows the formulation of analogous methods and tools, preserving the desired properties in presence of nonlinearity.

First we will introduce the main systems modelling framework, called convex difference inclusion (CDI) systems. This class of dynamic systems are deeply characterized by convexity, as their dynamics are defined by means of a set of convex functions. This implies that the dynamic evolution of such systems is characterized by convex constraints, and the application of many tools proper to the analysis of linear systems leads to convex programming problems, polynomially complex. That is, intuitively, replacing linearity by convexity, some results for linear systems are preserved for CDI systems, at the price of a slight increase of the computational complexity.

Many important results, analogous to those well established for linear systems, are exposed and proved. For instance, computation of the exact one-step operator, necessary and sufficient conditions for invariance and λ-contractiveness, algorithms to generate sequences of sets converging to the domain of attraction, rigorously developed for linear systems, are provided for CDI systems.

The importance of such framework is evident considering that any element of a very wide class of nonlinear systems can be approximated by a CDI system. That is, given a nonlinear system, if on one hand the analysis of the approximating CDI system leads to some conservativeness, on the other, some beneficial properties valid for linear systems are preserved for CDI systems. This yields to general and strong results for the approximated nonlinear system.

Another aspect of the CDI approach is that many properties satisfied by a set for the CDI system (invariance for instance), is fulfilled for any nonlinear system approximated by the CDI system. This implies that considering a CDI system underlies the analysis of a whole family of nonlinear systems, that is, all the systems whose dynamic function is bounded by the function characterizing the CDI system.

Although the results provided for CDI systems are strong, the assumptions characterizing such framework are not too restrictive. It will be shown that many common classes of systems are particular CDI systems or, at least, admit tight CDI approximations. Then, an important practical issue related to the theory developed is how to generate the approximating CDI system, given a nonlinear one. In this context we will introduce some computational aspects on how to obtain a CDI representation or a CDI approximation for the elements of some common classes of nonlinear systems.

The first class of systems enclosed in the CDI framework is given by the convex-concave difference inclusion (CCDI) systems. Such systems are particular CDI systems, characterized by a finite number of functions that determine their dynamics. Many nonlinear systems can be approximated by a CCDI system, since only a finite number of bounding functions are required to be computed.

The second class of nonlinear systems, for which a CDI representation is a powerful tool, are the Lur'e systems. Those systems are formed by a linear system in closed-loop with particular static gain feedback laws and are well known in the context of control theory, mainly in the continuous-time. Discrete-time Lur'e systems will be considered in the thesis.

An important class of systems for which a CDI approximation is easily obtained is given by the called DC systems. Such systems are characterized by dynamic functions that can be expressed as the difference of convex (DC) functions. The importance of DC functions is due to the fact that it is straightforward to determine convex lower and upper bounding functions for any of them. This will lead to the implicit determination of a CDI system approximating the original nonlinear DC one. Moreover, a very wide class of nonlinear functions can be represented by a DC one.

Another subclass of CDI systems is given by linear parametric uncertain systems. This modelling framework, for which some results are available in literature, permits to apply techniques proper to linear systems analysis and synthesis to the approximation of a nonlinear system. In fact, for instance, a nonlinear system defined on a bounded region can be approximated by a linear parametric uncertain system determined by bounding the gradient of the nonlinear function over such region.

The presence of additive uncertainty can be considered for any of the mentioned modelling frameworks. Assuming the presence of additive unknown but bounded uncertainty makes the model more realistic in some cases, since supposing the perfect knowledge of the system dynamics can be too restrictive.

Finally, we provide the thesis structure, along with the contributions on the different aspects of invariance and set-theoretic methods.

- The second chapter deals with the modelling problem. We recall general definitions and characterizations of nonlinear dynamic systems, introducing the concepts of uncertainty and the concept of set valued map, widely employed in the thesis. Then the proposed novel models, the CDI framework, will be presented.

- Computational aspects relating CDI systems to classes of common nonlinear and uncertain systems are developed in chapter three. CCDI systems and Lur'e systems are introduced as practice-oriented subclasses of CDI systems, and their twofold relation, with CDI systems on one hand and with common nonlinear systems on the other, is stressed to point out that many real systems are enclosed in this classes of models. DC systems are then illustrated. Definitions, properties and examples are provided to stress the main features of this particularly rich and expressive models. A short overview on DC functions is provided to make clear some reasons which lead us to consider this particular class of nonlinear functions. Finally, linear parametric uncertain systems are defined. Two subclasses of linear parametric uncertain systems, as the linear parameter varying (LPV) systems and the linear difference inclusions (LDI) systems, are also illustrated.

- In chapter four invariance and related topics for CDI systems are considered. Important results, well known for linear systems, are stated for this class of systems. Necessary and sufficient conditions for a convex set in the state space to be invariant and λ-contractive, also in presence of additive uncertainty, are provided. It is proved that, in case of absence of additive uncertainty, the relation between convex λ-contractive sets for CDI systems and Lyapunov functions, characterizing linear systems, are preserved for CDI systems. The one-step operator is determined and characterized, and a sketch of the algorithm for generating sequences of sets converging to the domain of attraction is given. Finally, computational issues on how to obtain convex invariant and λ-contractive sets for CDI systems is illustrated.

- The fifth chapter deals with the problem of computing convex invariant sets and λ-contractive sets for particular classes of autonomous nonlinear systems. In particular, practice-oriented classes of nonlinear systems previously illustrated, such as DC and Lur'e ones, are considered. Sufficient conditions for invariance and λ-contractiveness for DC systems are given. Also the case of DC systems in presence of additive uncertainty is treated. Practical issues on computing a convex invariant set are tackled, yielding to the algorithmic procedure ensuring to provide a non-empty convex invariant set in the absence of uncertainty. An ad-hoc method to obtain a sequence of nested invariant sets is provided for Lur'e systems. It is also shown that such sequence of sets converges to a convex approximation of the domain of attraction.

- Chapter six presents results concerning the problem of control synthesis. Computation of control laws and control invariant sets for non-autonomous CDI systems is the main topic of the chapter. The first part is devoted to illustrate properties of convex control invariant and λ-contractive sets for DC systems. A sufficient condition for control invariance and λ-contractiveness of a convex set is provided. In particular, in case of polytopic sets, it is proved that the computation of a control action at the vertices of the polytope satisfying a local (convex) condition, allows the determination of a control action defined over the set such that asymptotic (exponential) stability is guaranteed for the nonlinear system. The one-step operator, useful to obtain a sequence of nested control invariant sets and an approximation of the maximal stabilizable set, is analyzed for DC systems. Also computational issues are considered, defining algorithms to determine the stabilizing control law.

- In the final chapter we summarize the contributions and results illustrated in thesis and the directions for future research.

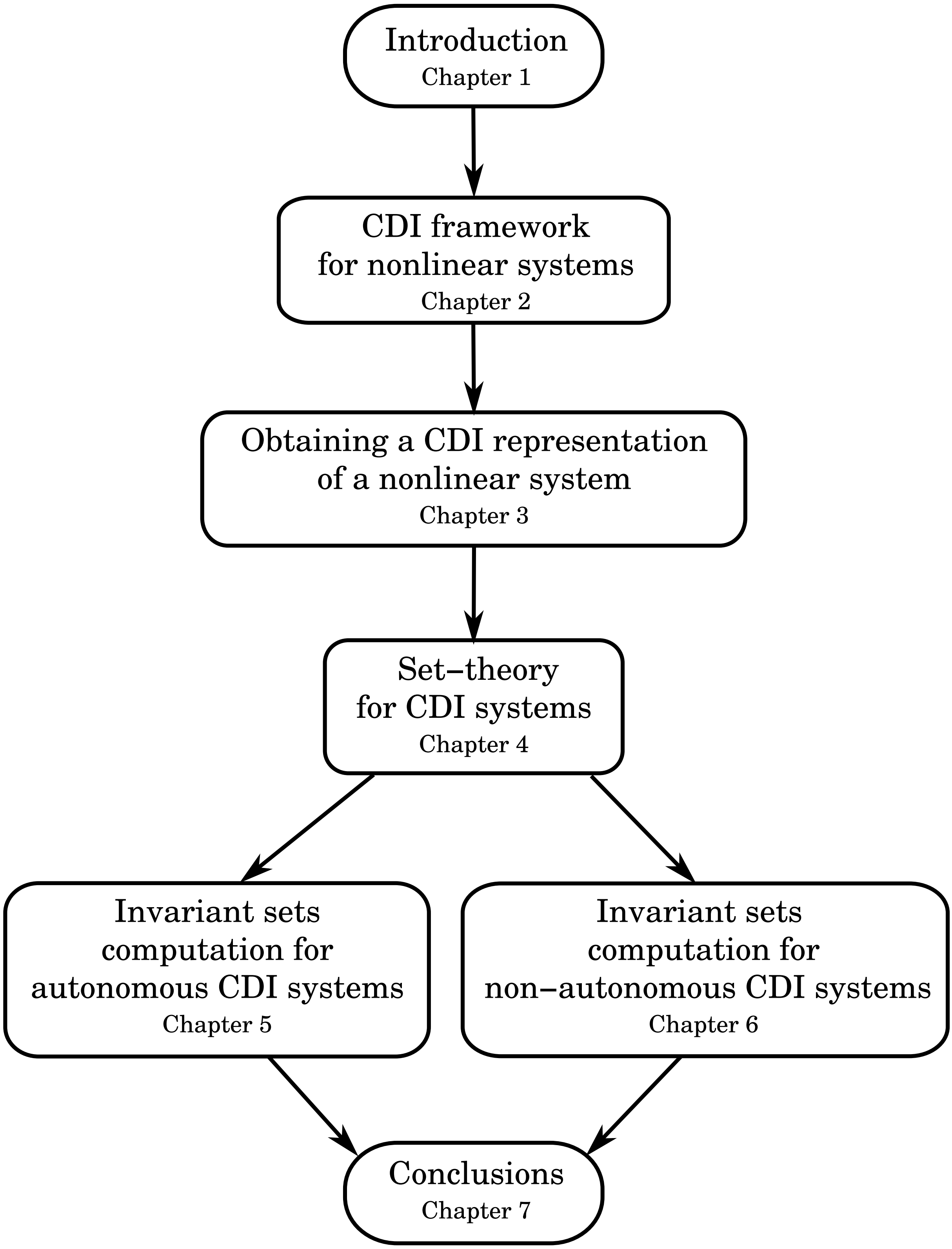

Figure 1.2: Thesis structure.

1.5 List of publications

1.5.0.2 Book chapters:

1. T. Alamo, M. Fiacchini, A. Cepeda, D. Limon, E. F. Camacho, J. M. Bravo. (2007). On the Computation of Robust Control Invariant Sets for Piecewise Affine Systems. *Assessment and Future Directions of Nonlinear Model Predictive Control (Lncis)*. New York. Springer-Verlag. 131 - 140.

1.5.0.3 Journal papers:

1. R. Gonzalez, M. Fiacchini, T. Alamo, J. L. Guzman, F. Rodriguez. Adaptive Control for a Mobile Robot under Slip Conditions using an LMI-based Approach. Accepted for publication in *European Journal of Control.*

2. T. Alamo, A. Cepeda, M. Fiacchini, E.F. Camacho. (2009) Convex Invariant Sets for Discrete-time Lur'e Systems. *Automatica* **45**. 1066 - 1071.

3. M. Fiacchini, T. Alamo, I. Alvarado, E. F. Camacho. (2008) Safety Verification and Adaptive Model Predictive Control of the Hybrid Dynamics of a Fuel Cell System. *International Journal of Adaptive Control and Signal Processing* **22**. 142 - 160.

4. L. A. Viguria, A. Prieto, M. Fiacchini, R. Cano, F. Rodriguez, J. Aracil, C. Canudas de Wit. (2006) Desarrollo y Exprementación de un Vehículo Basado en Péndulo Invertido (Ppcar). *Riai: Revista Iberoamericana de Automática e Informática Industrial* **3**. 54 - 63.

1.5.0.4 Submitted journal papers:

1. S. Tarbouriech, I. Queinnec, T. Alamo, M. Fiacchini, E. F. Camacho. Stability Analysis and Stabilization of Linear Systems Interconnected with Nonlinear Actuators: a Nonlinear Differential Inclusion Approach. Submitted to *Automatica*.

2. M. Fiacchini, T. Alamo, E. F. Camacho. On the Computation of Convex Robust Control Invariant Sets for Nonlinear Systems. Submitted to *Automatica*.

3. R. Gonzalez, M. Fiacchini, J. L. Guzman, T. Alamo, F. Rodriguez. Robust Tube-based Predictive Control for Mobile Robots under Slip Conditions. *IEEE Transactions on Robotics*.

1.5.0.5 Conference paper:

1. R. Gonzalez, M. Fiacchini, T. Alamo, J. L. Guzman, F. Rodriguez. (2009) Adaptive Control for a Mobile Robot under Slip Conditions using an LMI-based Approach. In *Proceedings of the European Control Conference 2009*. Budapest, Hungary.

2. R. Gonzalez, M. Fiacchini, J. L. Guzman, T. Alamo. (2009) Robust Tube-based MPC for Constrained Mobile Robots under Slip Condition. In *Proceedings of the 48th IEEE Conference on Decision and Control 2009*. Shangai, China.

3. S. V. Rakovic, M. Fiacchini. (2008) Approximate Reachability Analysis for Linear Discrete Time Systems Using Homothety and Invariance. In *Proceedings of the 17th IFAC World Congress 2008*. Seoul. Korea.

4. S. V. Rakovic, M. Fiacchini. (2008) Invariant Approximations of the Maximal Invariant Set or "Encircling the Square". In *Proceedings of the 17th IFAC World Congress 2008*. Seoul. Korea.

5. J. M. Bravo, T. Alamo, M. Fiacchini, E. F. Camacho. (2007) A Convex Approximation of the Feasible Solution Set for Nonlinear Bounded-Error Identification Problems. In *Proceedings of the 46th IEEE Conference on Decision and Control 2007*. New Orleans. USA.

6. M. Fiacchini, T. Alamo, C. Albea, E. F. Camacho. (2006) Adaptive Model Predictive Control of the Hybrid Dynamics of a Fuel Cell System. In *Proceedings of the 16th IEEE International Conference on Control Applications*. Singapore.

7. M. Fiacchini, T. Alamo, E. F. Camacho. (2007) On the Computation of Local Invariant Sets for Nonlinear Systems. In *Proceedings of the 46th IEEE Conference on Decision and Control 2007*. New Orleans. USA.

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Chapter 2

CDI framework for nonlinear systems

The thesis deals with nonlinear and uncertain systems. In this chapter, the main modelling framework of dynamic systems employed will be illustrated.

First, a basic classification of dynamic systems will be given, starting with generic characterizations of nonlinear systems and following with the description of uncertainty. The concept of set valued map will be introduced: intuitively, it is a function relating a set to any point of the space. Many systems considered in the thesis are characterized by set valued maps as dynamic functions, for instance parametric and additive uncertain systems.

Then, the main modelling framework used in the thesis, i.e., Convex Difference Inclusions (CDI) systems, will be described and their properties will be given. CDI systems are characterized by particular set valued maps. Convexity properties are assumed for the set valued maps, such that beneficial invariance related features characterizing linear systems are valid also for CDI systems. These properties will enable us to state necessary and sufficient conditions for invariance and λ-contractiveness for CDI systems, in particular boundary conditions.

The fact that CDI systems represent a tool to approximate a very wide class of nonlinear and uncertain systems, provides generality to this framework in order to characterize invariance and design computational procedures for nonlinear systems.

CDI systems provide the more general modelling framework employed in this thesis. Other models, more practice-oriented, presented in the following chapter can be approximated by CDI systems or they are particular cases of CDI systems.

2.1 Nonlinear systems

In this thesis we consider and analyze discrete-time systems in state space representation, that is, systems determined by difference equations, rather than differential equations. Hence, we implicitly assume in what follows that time variable k is an element of natural numbers, $k \in \mathbb{N}$, and the state vector $x \in \mathbb{R}^n$, at time k, is a function of the state and, possibly, of the control input and uncertainty, at the previous instant.

To define and characterize nonlinear systems, it can be helpful to recall the definition of linear system and the main property of these systems, the superposition principle.

Consider a discrete-time autonomous system

$$x^+ = f(x),$$

where $x \in \mathbb{R}^n$ is the current state and $x^+ \in \mathbb{R}^n$ is the successor, or a non-autonomous system

$$x^+ = f(x,u),$$

with $u \in \mathbb{R}^m$ control input.

The system is linear if function $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, that is, if it is such that

- $f(x+y) = f(x) + f(y), \qquad \forall x, y \in \mathbb{R}^n,$
- $f(\alpha x) = \alpha f(x), \qquad \forall x \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}.$

Hence, it is easy to see that a linear system has the form $x^+ = Ax$, for the autonomous case, and $x^+ = Ax + Bu$, for the non-autonomous one.

Intuitively the superposition principle says that, if the "cause" a leads, through the dynamic system, to the "effect" b, then $2a$ leads to $2b$. Moreover, if c leads to d, then $a+c$ leads to $b+d$. This entails that the analysis of particular pairs cause-effect permits to completely characterize the whole relation cause-effect represented by the system, or, equivalently, that the complete behavior of the systems can be inferred independently from the particular contingency.

In fact, linear systems are completely characterized by the square state-transition matrix $A \in \mathbb{R}^{n \times n}$ and, possibly, by $B \in \mathbb{R}^{n \times m}$, and their analysis can be treated with linear algebra.

Nonlinear systems are systems for which the superposition principle does not hold. This is the key reason for which it is far more complex to deal with nonlinearity. The relation between any "cause" and its "effect" should be taken into account independently, no general characteristic can be inferred by the analysis of a particular occurrence. In the nonlinear framework, local results are often pursued, in contrast with the globality and generality of properties usually ensured for linear systems.

Another aspect to be underlined is the fact that frequency based methods for analysis and design, widely exploited in the context of linear systems, are not applicable for general nonlinear systems. This is because the output of a nonlinear system excited by a sinusoidal input is not another sinusoid with same frequency, property fulfilled by stable linear systems (after the transient). Hence, many classical methods for analysis and control design for linear systems are not valid when dealing with nonlinear systems.

On the other hand, nonlinear systems permit to model a much wider class of dynamic systems, see (Vidyasagar, 1993; Khalil, 2002). Nonlinear systems provide a far richer framework, it could be claimed that no real system is linear, actually. Many interesting phenomena of dynamic systems are due to nonlinearity, for instance, see (Khalil, 2002), multiple isolated equilibria, limit cycles, chaos, etc.

Then, if on one hand nonlinear systems provide a very powerful tool to model the reality, on the other, the complexity involved can be often an insurmountable obstacle to generality of properties and results.

2.1.1 Uncertain nonlinear systems

The concept of uncertainty in systems analysis and control design is fundamental. Many research efforts have been directed to the problem of robustness. In practice, the assumption of full and complete knowledge of a dynamic system is not realistic. This lack of knowledge of the dynamics of a system is modelled as uncertainties on the dynamic functions. Considering uncertain models is natural, in fact, since in general either an exact model of reality is not available, because we are unable to recover the whole dynamic richness of a system, or because a too complex model would not be suitable for the analysis and control objective.

Then, it is reasonable to assume that the model used to represent a reality, a dynamic system, is not perfectly known, that there is a certain mismatch between the ideal behavior and the real evolution of the system. The mismatches between the real system and the model, denoted in general as uncertainties, can have several origins and different representations. A first discrimination between uncertain models can be due to assumptions on the nature of uncertainty. In the stochastic scenario, uncertainty is supposed to be characterized by a probability distribution.

In this thesis we consider unknown but bounded uncertainties. That is, the effects of uncertainty are assumed to be dependent on a signal or a parameter, usually varying with time, unknown but bounded in a region of the space in which it lies. It is worth noticing that also state and input dependent uncertainties are taken into account in this framework, provided that bounds on their effects are assumed. This assumption leads to the so-called worst-case approach, whose objective is usually to ensure that properties are preserved no matter the realization of the uncertainty, provided the uncertainty bounds are not violated. Then, the analysis is directed to guarantee properties and/or performance assuming that the uncertainty is the worst possible, supposing that the uncertainty plays an active role contrasting the control aims.

It is clear, then, why an important role on the robust analysis and robust control design is played by game theory, see (Basar and Olsden, 1999). Game theory is applied to problems, among all, in which two players act pursuing opposite objectives, which is what we have when considering the uncertainty as an active agent whose aim is to prevent the achievement of the control objective, see (Bertsekas and Rhodes, 1971*a*; Glover and Schweppe, 1971).

2.1.1.1 Set valued maps

We introduce here the concept of set valued maps, which will allow to define a class of dynamic systems that encloses many of the models considered in the thesis. Suppose that the system dynamics is not given by a function with values on the state space, but by a set valued map, that is, by a function defined on the state space (possibly on the Cartesian product of the state space and the input space) with subsets of the state space as values. We first define some particular sets of subsets of $D \subseteq \mathbb{R}^n$, useful in the following.

Definition 2.1 *Given any $D \subseteq \mathbb{R}^n$, we denote with $\mathscr{S}(D)$ the set of subsets of D, with $\mathscr{K}(D)$ the set of convex, compact subsets of D and with $\mathscr{K}^0(D)$ the set of convex, compact subsets of D containing the origin in their interior.*

By definition, we have the following relation: $\mathscr{K}^0(D) \subseteq \mathscr{K}(D) \subseteq \mathscr{S}(D)$ for all $D \subseteq \mathbb{R}^n$.

A set valued map $F(\cdot)$ is a function defined on $\mathbb{R}^n$ and whose values are elements of $\mathscr{S}(\mathbb{R}^n)$, that is $F : \mathbb{R}^n \to \mathscr{S}(\mathbb{R}^n)$. Therefore, for any $x \in \mathbb{R}^n$, we have that $F(x)$ is a subset of $\mathbb{R}^n$. Particular interest will be devoted in this thesis to set valued maps whose values are elements of $\mathscr{K}(\mathbb{R}^n)$.

We consider the autonomous discrete-time systems

$$x^+ \in F(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, x^+ is the successor state, and function $F(\cdot)$ is a set valued map, i.e., $F : \mathbb{R}^n \to \mathscr{S}(\mathbb{R}^n)$.

For this kind of dynamic systems, the trajectories generated are sequences of subsets in the state space rather than sequences of points. We define the map $\mathscr{M}_F(\cdot)$ associated to the dynamic function of system (2.1) and the set valued map $F(\cdot)$, as

$$\mathscr{M}_F(D) = \bigcup_{x\in D} F(x), \tag{2.2}$$

where $D \subseteq \mathbb{R}^n$.

Property 2.2 *For any set valued map $F(\cdot)$, map $\mathscr{M}_F : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ defined in (2.2) is monotone, that is, for every $D, C \in \mathscr{S}(\mathbb{R}^n)$ with $D \subseteq C$ it follows that*

$$\mathscr{M}_F(D) \subseteq \mathscr{M}_F(C).$$

Given an initial set of points in the state space $X_0 \subseteq \mathbb{R}^n$, the sequence of sets ensuring to contain the state generated in time by the dynamic system (2.1) with initial condition $x_0 \in X_0$, are obtained through the following iteration

$$X_{k+1} = \mathscr{M}_F(X_k),$$

with initial set X_0 and where the map $\mathscr{M}_F(\cdot)$ is defined in (2.2). The set X_k is the reachable set at time k, for $k \in \mathbb{N}$, mentioned in the introduction.

It can be proved that the set of points reachable at time $k \in \mathbb{N}$ from X_0 by the dynamic system (2.1) is given by

$$\begin{aligned} X_k(X_0) = \{x \in \mathbb{R}^n : \ \text{there exist } x^0, \ldots, x^k \in \mathbb{R}^n \text{ such that:} \\ x^i \in F(x^{i-1}), \forall i \in \mathbb{N}_k, \ \ x^0 \in X_0, \ \ x = x^k\}. \end{aligned} \tag{2.3}$$

In the thesis, both autonomous and non-autonomous systems are taken into account. A discrete-time non-autonomous system characterized by a set valued map is given by

$$x^+ \in F(x,u), \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. Function $F(\cdot,\cdot)$ is a set valued map, i.e., $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{S}(\mathbb{R}^n)$ and then $F(x,u) \subseteq \mathbb{R}^n$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Analogously to the case of autonomous system, given a sequence of inputs $u(k) \in \mathbb{R}^m$ and an initial set $X_0 \subseteq \mathbb{R}^n$, the trajectory of the uncertain system is a sequence of sets in the state space.

2.1.1.2 Parametric uncertainty

A particular case of systems whose dynamic function is a set valued map are the parametric uncertain ones, that is, systems whose dynamic function depends on a parameter. Given a set $R \subseteq \mathbb{R}^{n_r}$, consider the autonomous discrete-time system

$$x^+ \in F(x) = \{p(x,r) : \; r \in R\},$$

where $x \in \mathbb{R}^n$ is the state, x^+ is the successor state, vector $r \in \mathbb{R}^{n_r}$ is the parameter and $p : \mathbb{R}^n \times \mathbb{R}^{n_r} \to \mathbb{R}^n$ is a function defined for every $r \in R$.

A discrete-time non-autonomous system affected by parametric uncertainty is given by

$$x^+ \in F(x,u) = \{p(x,u,r) : \; r \in R\},$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, vector $r \in \mathbb{R}^{n_r}$ is the parameter and $p : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_r} \to \mathbb{R}^n$ is a function defined for every $r \in R$.

Different assumptions on the knowledge of the parameter r lead to different frameworks. Supposing that only the set R is known, all the possible successor states for $x \in \mathbb{R}^n$, i.e., the whole set $F(x) \subseteq \mathbb{R}^n$, have to be taken into account for state estimation, control design, etc.

In case that set valued function determining the dynamics of the uncertain system is implicitly defined through linear functions, we have the called Linear Difference Inclusion (LDI) system, see (Boyd et al., 1994; Gurvits, 1995) or the Linear Parameter Varying (LPV) system, see (Shamma and Athans, 1991; Shamma and Xiong, 1999), depending on the assumed knowledge of the dynamics, as illustrated in Section 3.5. Autonomous linear parametric uncertain systems are given by (2.1) with the set valued map given by

$$F(x) = \{Ax : \; A \in \mathscr{A}\}, \tag{2.5}$$

where $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$, while non-autonomous linear parametric uncertain systems have the form (2.4) with

$$F(x,u) = \{Ax + Bu : \; [A,B] \in \mathscr{M}\}, \tag{2.6}$$

where $\mathscr{M} \subseteq \mathbb{R}^{n\times(n+m)}$ and $[A,B]$ denotes here the matrix obtained concatenating matrices A and B.

A very effective tool to deal with parametric uncertainty related problems for linear systems are the LMI-based (Linear Matrix Inequality) optimization solvers, which permit to efficiently solve robust analysis and control design problems as convex programming problems, see (Boyd et al., 1994; Ben-Tal and Nemirovski, 2001; Álamo, Normey-Rico, Arahal, Limón and Camacho, 2006; Kothare, Balakrishnan and Morari, 1996).

2.1.1.3 Additive uncertainty

Another possibility is considering the uncertainty as a signal which is added to the dynamic function. The common assumption in the worst case approach is that it is unknown but bounded inside a set. Recall that also state (and input) dependent uncertainty are considered with this approach, through properly determined bounds. It is worth noticing that also additive uncertain systems are characterized by a set valued map.

Additive uncertainty can be considered affecting linear and nonlinear systems, as well as systems affected also by parametric uncertainty. For instance, consider the linear discrete-time system

$$x^+ = Ax + w,$$

where now the state-transition matrix $A \in \mathbb{R}^n$ is assumed to be constant and known while the signal $w \in \mathbb{R}^n$ is supposed to be an element of the set $W \subseteq \mathbb{R}^n$, subset of the state space in this case, that is $w \in W$. Usually the bounding set W is supposed to be compact. In fact we pose the following assumption, used throughout the thesis when dealing with systems affected by additive uncertainty.

Assumption 2.3 *We assume that $W \subseteq \mathbb{R}^n$ is a compact set in the state space with $0 \in$ int $(co\ (W))$.*

Notice that the uncertainty term w could be a function of the state x (and, possibly, of the input u) and other terms representing noises and exogenous disturbances, as assumed in some works in the literature related to robust control.

As an example of problems related to dynamic systems affected by additive uncertainty, consider the problem of robust fulfillment of hard constraints. If it is ensured that the system evolution is maintained inside the admissible set for any sequence of $w(k)$, $k \in \mathbb{N}$, then the system real trajectory does not violate the constraints, no matter the realization of the uncertainty. Particular importance is devoted, in problems of hard constraints satisfaction, to the sequence of uncertainty which opposes more effectively to such aim. Ensuring that the system satisfies the constraints under the worst possible uncertainty realization, excludes any constraint violation. For this reason, this way of dealing with uncertainty is often denoted as worst-case approach.

Additive uncertainty is commonly employed with linear systems, as the uncertainty can be read as the effect of the nonlinearity. Roughly speaking, the behavior of a nonlinear system can be approximated by a linear one and the effects of the mismatch between the two systems can be modelled by the additive uncertainty component.

Remark 2.4 *The effect of uncertainty is often considered affecting the linear systems additively or parametrically, it is not usual to take into account both uncertainty structures.*

A reason is that, in many cases, the selection of the uncertainty framework stems from assumptions made on the system and from the nature of the system itself. For instance, when uncertainty reflects the dependence of the real system dynamics on endogenous or exogenous signals or physical parameters, the parametric uncertainty framework might fit better, while when uncertainty models the effect of noises and disturbances, additive unknown but bounded uncertainty is more appropriate.

Nevertheless, we can assume that the system is affected by the two uncertainty contributions, for the analysis and control design process.

Finally, it is worth noticing that it is not common, in the field of dynamic systems analysis and control design, to deal with both nonlinearity and uncertainty. This is due to the fact that the uncertainty is often employed to model the effects of nonlinearity, allowing to apply analysis and robust control techniques proper to linear systems. That is, deterministic nonlinear systems are often treated as linear uncertain system, while in this thesis we direct our attention also to nonlinear uncertain systems.

2.2 Convex difference inclusions: CDI systems

We introduce here the modelling framework used in the thesis to determine the behavior of families of nonlinear and uncertain systems. The systems taken into account are named Convex Difference Inclusions (CDI) systems and are characterized by a particular class of set valued maps as dynamic functions. Recall that for this kind of systems, a sequence of sets can be generated, the reachable sets. In particular, we will see that the set valued map determining a CDI system is such that, given a point in the state space, its image through the map is a convex and compact set. That is, denoting with $\mathscr{F}(\cdot)$ the set valued map determining the dynamics of the CDI system, then $\mathscr{F}:\mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$, which means that $\mathscr{F}(x) \in \mathscr{K}(\mathbb{R}^n)$ for every state $x \in \mathbb{R}^n$. This assumption will assure that the reachable sets generated are convex and compact. Furthermore, we will see that the one-step operator for CDI systems, applied to convex, compact sets provides convex, closed sets.

Moreover, we require a further convexity condition to be satisfied by the set valued map. Given a state x, the support function of its successor, i.e., of the set $\mathscr{F}(x)$, with respect to a given direction $\eta \in \mathbb{R}^n$ is the maximum of $\eta^T z$ for all $z \in \mathscr{F}(x)$, by definition, see Appendix C. Now, if we fix the direction $\eta \in \mathbb{R}^n$, the value of the support function of the set $\mathscr{F}(x)$ depends, clearly, on x. The required condition is that such dependence satisfies a convexity property.

Such structure will allow us to determine necessary and sufficient conditions for invariance and λ-contractiveness of sets, for the CDI systems. Moreover, such conditions are boundary conditions, that is, they involve only the elements on the boundary, just a finite number of points (the vertices) in case of polytopic sets. Recalling here that such very desirable properties are lost when moving from linear to nonlinear systems, we see that convexity conditions on the dynamic set valued functions are the missing ingredient for generic nonlinear systems.

Similar considerations will lead us to the definition of design methods for determining control laws which ensure exponential convergence for CDI systems, in absence of additive uncertainty.

Finally, it is important to stress that many nonlinear systems admit CDI representations or can be approximated by CDI systems. This means that the results valid for CDI systems can be used to obtain invariant sets, λ-contractive sets, control invariant sets, approximated reachable sets, for a very wide class of nonlinear systems.

As a matter of fact, the analysis of a CDI system can be considered as the analysis of families of systems, since any nonlinear system bounded by a CDI one (we will clarify below the meaning of bounding systems) share important invariance related properties with the CDI system.

Formal definition of a CDI systems follows. Let the system be

$$x^+ \in \mathscr{F}(x), \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the state, x^+ is the successor and $\mathscr{F}(\cdot)$ is a set valued map on $\mathbb{R}^n$, that is a function which relates a set to every point $x \in \mathbb{R}^n$.

Particular importance in the following is devoted to set valued dynamic functions, such that $\mathscr{F}(x) \in \mathscr{K}(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$, and the graph of $\mathscr{F}(\cdot)$ is determined by a set of functions convex with respect to x, as stated below.

Assumption 2.5 *Assume that the set valued map $\mathscr{F} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics (2.7) is such that, for every $\eta \in \mathbb{R}^n$, function $\check{f}_\eta : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$\check{f}_\eta(x) = \sup_{z \in \mathscr{F}(x)} \eta^T z, \tag{2.8}$$

is convex on $\mathbb{R}^n$, and $\check{f}_\eta(0) = 0$.

In what follows, we will refer to dynamic systems (2.7) for which Assumption 2.5 holds, as Convex Difference Inclusions (CDI) systems and to functions $\check{f}_\eta(\cdot)$ as convex bounding functions.

Notice that, under Assumption 2.5 and for any $x \in \mathbb{R}^n$, functions $\check{f}_\eta(\cdot)$ can be considered as support function at $\eta \in \mathbb{R}^n$, determining the set $\mathscr{F}(x)$ (see Appendix C for the definition and properties of support function). That is, given $x \in \mathbb{R}^n$ and by convexity and compactness of $\mathscr{F}(x)$, we have that

$$\mathscr{F}(x) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{f}_\eta(x), \forall \eta \in \mathbb{R}^n\}, \tag{2.9}$$

with

$$\check{f}_\eta(x) = \sup_{z \in \mathscr{F}(x)} \eta^T z = \phi_{\mathscr{F}(x)}(\eta), \tag{2.10}$$

for every $\eta \in \mathbb{R}^n$, where $\phi_{\mathscr{F}(x)}(\eta)$ is the support function of set $\mathscr{F}(x)$ evaluated at $\eta \in \mathbb{R}^n$, see Appendix C. The only, important, requirement of this set of support functions (depending on x) is that $\check{f}_\eta(\cdot)$ has to be convex with respect to x in $\mathbb{R}^n$.

Remark 2.6 *It is also important to note that for every $\eta \in \mathbb{R}^n$ and every $x \in \mathbb{R}^n$, there exists a point $z(x,\eta) \in \mathscr{F}(x)$ such that*

$$\eta^T z(x,\eta) = \check{f}_\eta(x),$$

which means that the plane $\{z \in \mathbb{R}^n : \ \eta^T z = \check{f}_\eta(x)\}$ is a support hyperplane of $\mathscr{F}(x)$.

Remark 2.7 *In the following, with a slight abuse of notation, we say that a set valued map $F : \mathbb{R}^n \to \mathscr{S}(\mathbb{R}^n)$ is overbounded by the set valued map $\mathscr{F} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ if*

$$F(x) \subseteq \mathscr{F}(x), \quad \forall x \in \mathbb{R}^n, \tag{2.11}$$

and we denote such condition as $F \subseteq \mathscr{F}$.We recall here that, from (2.9), we have that

$$\mathscr{F}(x) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{f}_\eta(x), \ \forall \eta \in \mathbb{R}^n\},$$

and then $F \subseteq \mathscr{F}$ means

$$\eta^T z \leq \check{f}_\eta(x), \quad \forall \eta \in \mathbb{R}^n, \quad \forall z \in F(x),$$

for all $x \in \mathbb{R}^n$.

Clearly, for functions with values on $\mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that $f(\cdot)$ is overbounded by $\mathscr{F}(\cdot)$ if, defining

$$S_{\mathscr{F}} = \{f : \ f(x) \in \mathscr{F}(x), \ \forall x \in \mathbb{R}^n\}, \tag{2.12}$$

we have that $f \in S_{\mathscr{F}}$, which means

$$\eta^T f(x) \leq \check{f}_\eta(x), \quad \forall \eta \in \mathbb{R}^n,$$

for all $x \in \mathbb{R}^n$.

Analogously, we say that a dynamic system is overbounded by a CDI system if the dynamic function of the former is overbounded by the set valued map of the latter.

We present here an example of how useful the CDI framework can result to deal with nonlinear systems. In fact, we will show that for any nonlinear system whose dynamic function is twice differentiable, its Taylor series expansion determines a CDI bounding system valid in a given region.

Example 2.8 *Consider a generic discrete-time nonlinear system*

$$x^+ = f(x),$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function assumed twice differentiable in a set $D \subseteq \mathbb{R}^n$. This implies that for every $f_j(\cdot)$, with $j \in \mathbb{N}_n$, the gradient $\nabla f_j(\cdot)$ and the Hessian, denoted $H(f_j)(\cdot) = H^j(\cdot)$, exist at every $x \in D$.

Then, exploiting the Taylor series expansion and, in particular the Lagrange form of Remainders, which is based on the Mean value theorem, we have that given a $x_0 \in D$, for every $x \in D$ there exists a $\tilde{x}(x) = \tilde{x} \in D$ such that the following equality holds

$$f_j(x) = f_j(x_0) + (x - x_0)^T \nabla f_j(x_0) + \frac{1}{2!}(x - x_0)^T H^j(\tilde{x})(x - x_0),$$

for every $j \in \mathbb{N}_n$. Define the convex bounding functions as in the following

$$\check{f}_\eta(x) = \sum_{j=1}^{n} \left\{ \eta_j (f_j(x_0) + (x - x_0)^T \nabla f_j(x_0)) + \rho_j |\eta_j| (x - x_0)^T (x - x_0) \right\},$$

for every $\eta \in \mathbb{R}^n$, with $\rho \in \mathbb{R}^n$ such that

$$\left| \frac{1}{2!}(x - x_0)^T H^j(\tilde{x})(x - x_0) \right| \leq \rho_j (x - x_0)^T (x - x_0),$$

for all $x \in D$ and $\tilde{x} \in D$, with $j \in \mathbb{N}_n$. A possible choice for ρ_j is the maximal absolute value of the eigenvalues of $0.5H^j(\tilde{x})$, that is its spectral norm, for all $\tilde{x} \in D$.

The related CDI system overbounds the nonlinear one, in fact, for all $x \in D$, we have that

$$\begin{aligned}
\eta^T f(x) = \sum_{j=1}^{n} \eta_j (f_j(x_0) + (x - x_0)^T \nabla f_j(x_0) + \tfrac{1}{2!}(x - x_0)^T H^j(\tilde{x}^j)(x - x_0)) \leq \\
\leq \sum_{j=1}^{n} \eta_j (f_j(x_0) + (x - x_0)^T \nabla f_j(x_0)) + |\eta_j| \left| \tfrac{1}{2!}(x - x_0)^T H^j(\tilde{x}^j)(x - x_0) \right| \leq \\
\leq \sum_{j=1}^{n} \eta_j (f_j(x_0) + (x - x_0)^T \nabla f_j(x_0)) + \rho_j |\eta_j| (x - x_0)^T (x - x_0)) = \check{f}_\eta(x),
\end{aligned}$$

for every $\eta \in \mathbb{R}^n$, which means that $f \in S_{\mathscr{F}}$, see Remark 2.7, where $\mathscr{F}(\cdot)$ is the set valued map defined by functions $\check{f}_\eta(\cdot)$.

We give now a one-dimensional and a two-dimensional examples of CDI systems. The main interest of these particular examples lies in the geometrical interpretations of set valued maps and convex bounding functions, as we provide graphical representations of these structures.

Example 2.9 *Consider the system (2.7) with set valued function $\mathscr{F} : \mathbb{R} \to \mathscr{K}(\mathbb{R})$ given by*

$$\mathscr{F}(x) = \{y \in \mathbb{R} : \ -|x| \le y \le |x|\}. \tag{2.13}$$

This means that, for instance, we have

$$\mathscr{F}(2) = \mathscr{F}(-2) = \{y \in \mathbb{R} : -2 \le y \le 2\}.$$

The graph of function $\mathscr{F}(\cdot)$ is depicted in Figure 2.1.

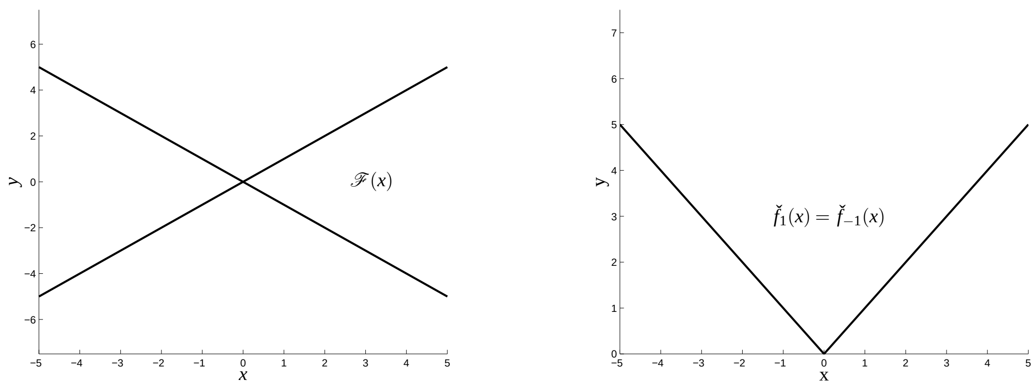

Figure 2.1: *Left*: function $\mathscr{F}(\cdot)$. *Right*: functions $\check{f}_1(\cdot)$ and $\check{f}_{-1}(\cdot)$.

It is also straightforward to determine convex bounding functions for $\eta = 1$ and $\eta = -1$, in fact we have that

$$\check{f}_1(x) = |x|, \qquad \check{f}_{-1}(x) = |x|,$$

also represented in Figure 2.1, are convex functions of $x \in \mathbb{R}$ satisfying Assumption 2.5.

Example 2.10 *We consider here a two-dimensional CDI system, that is, the state is $x \in \mathbb{R}^2$. The dynamic system is given by (2.7) with set valued map $\mathscr{F} : \mathbb{R}^2 \to \mathscr{K}(\mathbb{R}^2)$ defined as*

$$\mathscr{F}(x) = \{z \in \mathbb{R}^2 : \ z^T P z \le x^T x\} = \{z \in \mathbb{R}^2 : \ z^T P z \le \|x\|_2^2\}. \tag{2.14}$$

where $P \in \mathbb{R}^{2\times 2}$ is a positive definite matrix: $P = P^T > 0$. For instance, take

$$P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix},$$

whose eigenvalues are 0.7929 *and* 2.2071.

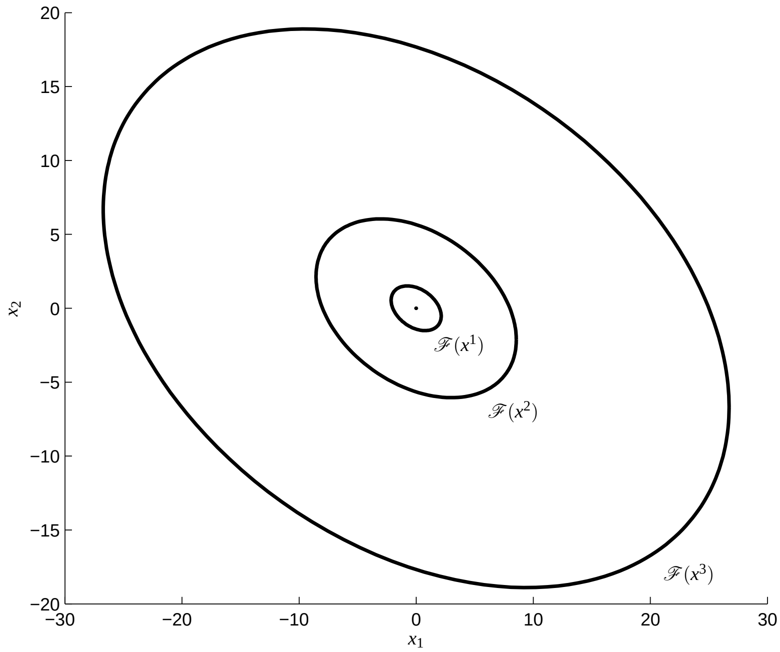

Figure 2.2: Image sets $\mathscr{F}(x^1)$, $\mathscr{F}(x^2)$ and $\mathscr{F}(x^3)$, with $\mathscr{F}(\cdot)$ defined 2.14.

We list below the expressions of the images of three values of the state vector, $x^1 = [1,\ 1]^T$, $x^2 = [-2,\ 2]^T$ and $x^3 = [-3,\ -4]^T$, through the set valued map $\mathscr{F}(\cdot)$, that is the sets related to the three elements of the state space are given by:

$$\begin{aligned} \mathscr{F}(x^1) &= \{z \in \mathbb{R}^2 : z^T P z \leq 2\}, \\ \mathscr{F}(x^2) &= \{z \in \mathbb{R}^2 : z^T P z \leq 8\}, \\ \mathscr{F}(x^3) &= \{z \in \mathbb{R}^2 : z^T P z \leq 25\}. \end{aligned}$$

The sets are depicted in Figure 2.2.

Clearly $\mathscr{F}(x^1)$ is the image through $\mathscr{F}(\cdot)$ of any point x in the state space such that $x^T x = 2$, which is a circle in the state space. Analogously the sets of the state space for which $\mathscr{F}(x) = \mathscr{F}(x^2)$ and $\mathscr{F}(x) = \mathscr{F}(x^3)$ are circles with radius $\sqrt{8}$ and 5, respectively.

It is not easy to represent properly the graph of the set valued map $\mathscr{F}(\cdot)$ for the two dimensional CDI system, since it lies in the four dimensional space $\mathbb{R}^4$. If we fix the value of x_2, assume for instance $x_2 = 0$, the graph of the set valued map as a function of x_1 can be depicted, just for sake of geometrical insight, see Figure 2.3.

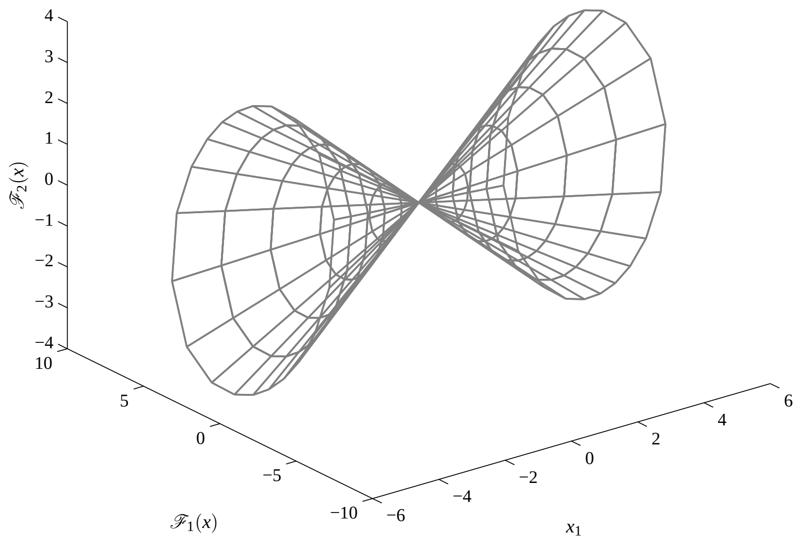

Figure 2.3: Graph of set valued map $\mathscr{F}(\cdot)$, defined in the (2.14), projected on $x_2 = 0$.

Also the convex bounding functions can be computed. Given a $\eta \in \mathbb{R}^2$ and a $x \in \mathbb{R}^2$, we have that

$$\begin{aligned} \check{f}_\eta(x) &= \sup_{z\in\mathbb{R}^2}\{\eta^T z:\ z^T P z \le x^T x\} = \sup_{z\in\mathbb{R}^2}\{\eta^T z:\ z^T\left(\frac{1}{\|x\|_2^2}P\right)z \le 1\} = \\ &= \sqrt{\eta^T P^{-1}\eta\,\|x\|_2^2} = \sqrt{\eta^T P^{-1}\eta}\ \|x\|_2 \end{aligned} \tag{2.15}$$

convex with respect to $x \in \mathbb{R}^2$ for every $\eta \in \mathbb{R}^2$, since the triangular inequality $\|a+b\|_2 \le \|a\|_2 + \|b\|_2$ is satisfied for all $a, b \in \mathbb{R}^n$. Function $\check{f}_\eta(\cdot)$ for particular value of $\eta = [-2,\ 0.7]^T$ is depicted in Figure 2.4.

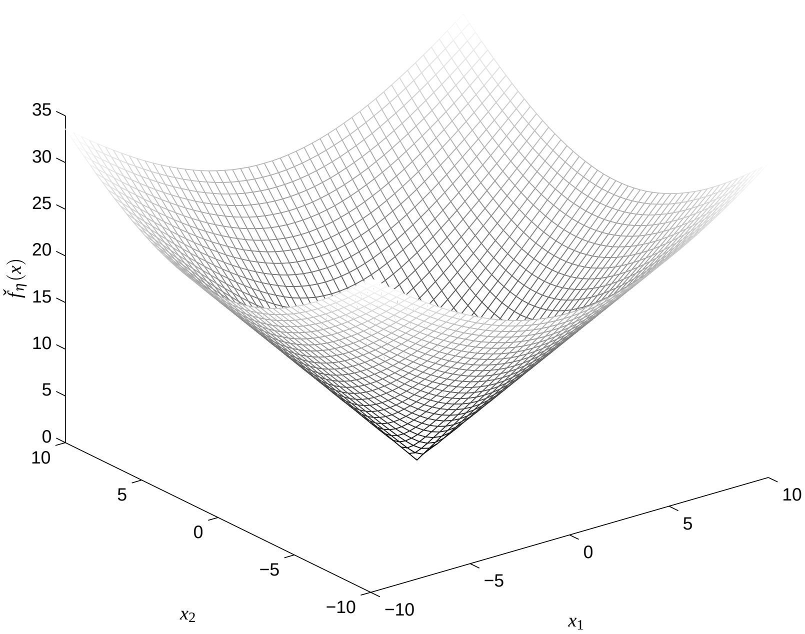

Figure 2.4: Convex bounding function $\check{f}_\eta(\cdot)$ for $\eta = [-2,\ 0.7]^T$ and $\mathscr{F}(\cdot)$, defined in (2.15).

Since the triangular inequality is satisfied by every norm (by definition of norm, in fact), any set valued map defined as

$$\mathscr{F}(x) = \{z \in \mathbb{R}^2 :\ z^T P z \leq \|x\|_p^2\}. \tag{2.16}$$

determines a CDI system, for every $p \in \mathbb{R}$ such that $p \geq 1$. In fact a derivation analogous to (2.14) leads to

$$\check{f}_\eta(x) = \sqrt{\eta^T P^{-1} \eta}\ \|x\|_p,$$

convex with respect to x.

In Figure 2.5 convex bounding function for $\eta = [-2,\ 0.7]$ and $p = 1$ and $p = 4$ are represented.

Remark 2.11 *CDI systems enclose a large class of nonlinear and uncertain systems and can be used to approximate many others, see Example 2.8, for instance. We will see, in the next chapter, other modelling frameworks, more practice-oriented, which are particular cases of CDI systems or can be easily approximated by them. Hence for more practical examples of CDI systems, for instance saturated and Lur'e systems, we refer to those presented in the following chapter.*

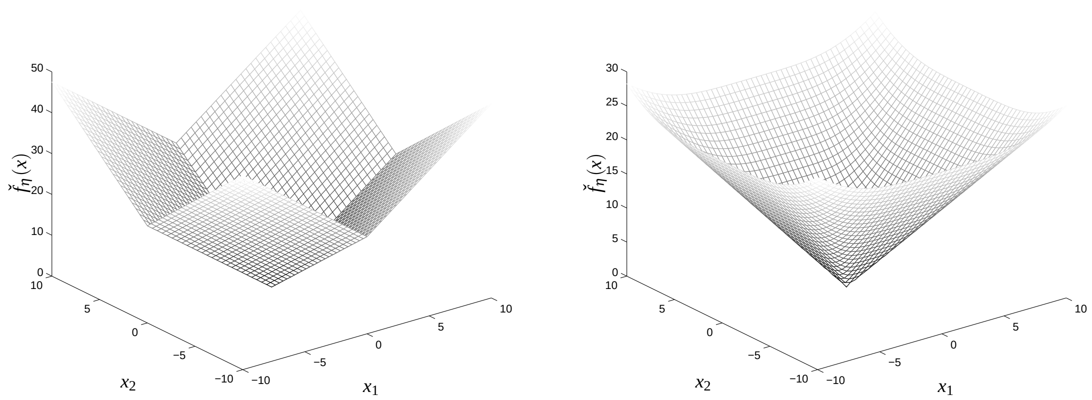

Figure 2.5: Convex bounding functions $\check{f}_\eta(\cdot)$ for $\eta = [-2,\ 0.7]^T$ and $\mathscr{F}(\cdot)$, defined in (2.16), for $p = 1$ (*left*) and $p = 4$ (*right*).

Assuming the existence of a convex bounding function $\check{f}_\eta(\cdot)$ for every direction $\eta \in \mathbb{R}^n$ can appear a quite restrictive condition, at least from the practical point of view. Looking at the CDI system as a bounding system of a, more common, nonlinear system, we have to be able to determine such uncountably many (one for any $\eta \in \mathbb{R}^n$) bounding functions to construct the CDI system, and this can result a hard task in general. Nevertheless, it can be sufficient to define $\check{f}_\eta(\cdot)$ only for a finite number of direction $\eta \in \mathbb{R}^n$, provided they are convex in $x \in \mathbb{R}^n$ and the sets bounded by such $\check{f}_\eta(\cdot)$ are compact (and convex) for every $x \in \mathbb{R}^n$. From those finite number of $\check{f}_\eta(\cdot)$, the convex bounding functions can be inferred for any other $\eta \in \mathbb{R}^n$, as shown in the following.

Remark 2.12 *First, notice that the set of $\eta \in \mathbb{R}^n$ under analysis can be restricted to the boundary of the unitary sphere, i.e., to $\eta \in \partial \mathbf{B}_2^n$ where*

$$\partial \mathbf{B}_2^n = \{\eta \in \mathbb{R}^n : \ \|\eta\|_2 = 1\}. \tag{2.17}$$

In fact, the support function is positively homogeneous of order one with respect to η, see Appendix C, that is, for any $\Omega \subseteq \mathbb{R}^n$

$$\phi_\Omega(\alpha\eta) = \sup_{z\in\Omega}(\alpha\eta)^T z = \alpha \sup_{z\in\Omega} \eta^T z = \alpha\phi_\Omega(\eta),$$

for any $\alpha > 0$. The case of $\eta = 0$, that is $\alpha = 0$, can be obtained fixing $\check{f}_0(x) = 0$. Then assuming $\check{f}_\eta(\cdot)$ defined for all elements of $\partial \mathbf{B}_2^n$, we have that functions

$$\check{f}_\eta(x) = \phi_{\mathscr{F}(x)}(\eta) = \|\eta\|_2 \phi_{\mathscr{F}(x)}\left(\frac{1}{\|\eta\|_2}\eta\right) = \|\eta\|_2 \check{f}_{\left(\frac{1}{\|\eta\|_2}\eta\right)}(x), \tag{2.18}$$

for all $x \in \mathbb{R}^n$, where, clearly $\left(\frac{1}{\|\eta\|_2}\eta\right) \in \partial \mathbf{B}_2^n$, are the convex bounding functions defined for all $\eta \in \mathbb{R}^n$.

The consequence is that, in many cases, the analysis can be restricted to the η such that $\|\eta\|_2 = 1$, i.e., to $\eta \in \partial \mathbf{B}_2^n$. On the other hand it has to be pointed out that elements of $\partial \mathbf{B}_2^n$ are still uncountable many.

In the following we show that a finite number of convex bounding functions can be sufficient to determine the CDI system. First we prove that convex bounding functions defined for $\eta \in E$ with $E \subseteq \partial \mathbf{B}_2^n$ are sufficient to determine a CDI system, provided that the related set valued map $\mathscr{F}(\cdot)$ has compact values. Clearly such subset of directions E can be finite.

Property 2.13 *Consider the set valued map defined by*

$$\mathscr{F}(x) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{f}_\eta(x), \forall \eta \in E\} \tag{2.19}$$

where $\check{f}_\eta : \mathbb{R}^n \to \mathbb{R}$*, for all* $\eta \in E \subseteq \partial \mathbf{B}_2^n$*, are functions convex on* $\mathbb{R}^n$*, with* $\check{f}_\eta(0) = 0$*, such that* $\mathscr{F}(x) \in \mathscr{K}(\mathbb{R}^n)$ *for every* $x \in \mathbb{R}^n$ *and condition (2.8) is satisfied for all* $\eta \in E$ *and all* $x \in \mathbb{R}^n$*.*

Then Assumption 2.5 holds for $\mathscr{F}(\cdot)$*.*

Proof: We have to prove that, given the convex bounding functions $\check{f}_\eta(\cdot)$ for $\eta \in \mathbb{R}^n$ in a subset $E \subseteq \partial \mathbf{B}_2^n$, bounding functions can be defined also for every $\eta \in \mathbb{R}^n$ such that $\mathscr{F}(\cdot)$ satisfies Assumption 2.5. We prove such condition for all $\eta \in \partial \mathbf{B}_2^n$, from Remark 2.12 this determines bounding functions on the whole space $\mathbb{R}^n$.

The proof is constructive: we provide a method to determine $\check{f}_\eta(\cdot)$ for any $\eta \in \partial \mathbf{B}_2^n$, from functions $\check{f}_\eta(\cdot)$ defined for $\eta \in E$. We proceed defining functions $\tilde{f}_{\tilde{\eta}}(\cdot)$ for any $\tilde{\eta} \in \partial \mathbf{B}_2^n$ and proving that such functions satisfy (2.8) and are convex with respect to x. This entails that proposed functions $\tilde{f}_{\tilde{\eta}}(\cdot)$ are bounding functions of Assumption 2.5.

Given $\tilde{\eta} \in \partial \mathbf{B}_2^n$ and a $x \in \mathbb{R}^n$, define

$$\tilde{f}_{\tilde{\eta}}(x) = \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z : \ \eta^T z \leq \check{f}_\eta(x), \ \forall \eta \in E\}, \tag{2.20}$$

that is, the maximal value of $\tilde{\eta}^T z$ taken by an element of $\mathscr{F}(x)$, determined by $\check{f}_\eta(x)$ for $\eta \in E$.

Condition (2.8) follows from definition of $\tilde{f}_{\tilde{\eta}}(\cdot)$ and (2.19), in fact

$$\tilde{f}_{\tilde{\eta}}(x) = \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z : \ \eta^T z \leq \check{f}_\eta(x), \ \forall \eta \in E\} = \sup_{z \in \mathscr{F}(x)} \tilde{\eta}^T z.$$

for every $\tilde{\eta} \in \partial \mathbf{B}_2^n$.

We have to prove that $\tilde{f}_{\tilde{\eta}}(\cdot)$ are convex in x for any $\tilde{\eta} \in \partial \mathbf{B}_2^n$. Given $x^1,\ x^2 \in \mathbb{R}^n$ and $\lambda \in [0,\ 1]$, sets $\lambda \mathscr{F}(x^1)$ and $(1-\lambda)\mathscr{F}(x^2)$ are given by

$$\begin{aligned}
\lambda \mathscr{F}(x^1) \ &= \{\lambda z \in \mathbb{R}^n :\ z \in \mathscr{F}(x^1)\} = \\
&= \{\lambda z \in \mathbb{R}^n :\ \eta^T z \leq \check{f}_\eta(x^1), \forall \eta \in E\} = \\
&= \{y = \lambda z \in \mathbb{R}^n :\ \eta^T \tfrac{1}{\lambda} y \leq \check{f}_\eta(x^1), \forall \eta \in E\} = \\
&= \{y \in \mathbb{R}^n :\ \eta^T y \leq \lambda\, \check{f}_\eta(x^1), \forall \eta \in E\} = \\
&= \{z \in \mathbb{R}^n :\ \eta^T z \leq \lambda\, \check{f}_\eta(x^1), \forall \eta \in E\}
\end{aligned}$$

and

$$\begin{aligned}
(1-\lambda)\mathscr{F}(x^2) \ &= \{(1-\lambda) z \in \mathbb{R}^n :\ z \in \mathscr{F}(x^2)\} = \\
&= \{z \in \mathbb{R}^n :\ \eta^T z \leq (1-\lambda)\, \check{f}_\eta(x^2), \forall \eta \in E\}.
\end{aligned}$$

The support functions at vector $\eta \in E$ are given by

$$\begin{aligned}
&\phi_{\lambda \mathscr{F}(x^1)}(\eta) = \lambda\, \check{f}_\eta(x^1), \\
&\phi_{(1-\lambda)\mathscr{F}(x^2)}(\eta) = (1-\lambda)\, \check{f}_\eta(x^2),
\end{aligned} \tag{2.21}$$

for all $\eta \in E$.

Now we can prove convexity of $\tilde{f}_{\tilde{\eta}}(\cdot)$. We have that, by convexity of $\check{f}_\eta(\cdot)$ for all $\eta \in E$,

$$\begin{aligned}
\tilde{f}_{\tilde{\eta}}(\lambda x^1 + (1-\lambda)x^2) &= \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z :\ \eta^T z \leq \check{f}_\eta(\lambda x^1 + (1-\lambda)x^2),\ \forall \eta \in E\} \leq \\
&\leq \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z :\ \eta^T z \leq \lambda\, \check{f}_\eta(x^1) + (1-\lambda)\, \check{f}_\eta(x^2),\ \forall \eta \in E\} = \\
&= \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z :\ \eta^T z \leq \phi_{\lambda \mathscr{F}(x^1)}(\eta) + \phi_{(1-\lambda)\mathscr{F}(x^2)}(\eta),\ \forall \eta \in E\},
\end{aligned} \tag{2.22}$$

where the last equality follows from (2.21). Since for all convex $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ we have that $\phi_{\Omega_1 \oplus \Omega_2}(\eta) = \phi_{\Omega_1}(\eta) + \phi_{\Omega_2}(\eta)$, see Appendix C, then

$$\phi_{\lambda \mathscr{F}(x^1) \oplus (1-\lambda)\mathscr{F}(x^2)}(\eta) = \phi_{\lambda \mathscr{F}(x^1)}(\eta) + \phi_{(1-\lambda)\mathscr{F}(x^2)}(\eta), \quad \forall \eta \in E.$$

From this and (2.22), it follows that

$$\tilde{f}_{\tilde{\eta}}(\lambda x^1 + (1-\lambda)x^2) \leq \sup_{z \in \mathbb{R}^n} \{\tilde{\eta}^T z :\ \eta^T z \leq \phi_{\lambda \mathscr{F}(x^1) \oplus (1-\lambda)\mathscr{F}(x^2)}(\eta),\ \forall \eta \in E\}. \tag{2.23}$$

Now, since set $\lambda \mathscr{F}(x^1) \oplus (1-\lambda)\mathscr{F}(x^2)$ is compact and convex, then $z \in \lambda \mathscr{F}(x^1) \oplus (1-\lambda)\mathscr{F}(x^2)$ if and only if $\eta^T z \leq \phi_{\lambda \mathscr{F}(x^1) \oplus (1-\lambda)\mathscr{F}(x^2)}(\eta)$, for all $\eta \in E$. This is equivalent to say that there exist $z^1 \in \lambda\mathscr{F}(x^1)$ and $z^2 \in (1-\lambda)\mathscr{F}(x^2)$ such that $z = z^1 + z^2$ if and only if $\eta^T z \leq \phi_{\lambda\mathscr{F}(x^1)\oplus(1-\lambda)\mathscr{F}(x^2)}(\eta)$, for all $\eta \in E$. Then from (2.23) we have

$$\begin{aligned}
\tilde{f}_{\tilde{\eta}}(\lambda x^1 + (1-\lambda)x^2) \leq & \sup_{z^1,z^2\in\mathbb{R}^n} \{\tilde{\eta}^T(z^1+z^2) :\ z^1 \in \lambda\mathscr{F}(x^1),\ z^2 \in (1-\lambda)\mathscr{F}(x^2)\} = \\
&= \sup_{z^1\in\mathbb{R}^n} \{\tilde{\eta}^T z^1 :\ z^1 \in \lambda\mathscr{F}(x^1)\} + \sup_{z^2\in\mathbb{R}^n} \{\tilde{\eta}^T z^2 :\ z^2 \in (1-\lambda)\mathscr{F}(x^2)\} = \\
&= \sup_{z^1\in\mathbb{R}^n} \{\tilde{\eta}^T z^1 :\ \eta^T z^1 \leq \lambda \check{f}_\eta(x^1),\ \forall\eta \in E\} + \\
&\quad + \sup_{z^2\in\mathbb{R}^n} \{\tilde{\eta}^T z^2 :\ \eta^T z^2 \leq (1-\lambda)\,\check{f}_\eta(x^2),\ \forall\eta \in E\} = \\
&= \lambda\,\tilde{f}_{\tilde{\eta}}(x^1) + (1-\lambda)\,\tilde{f}_{\tilde{\eta}}(x^2),
\end{aligned}$$

that is

$$\tilde{f}_{\tilde{\eta}}(\lambda x^1 + (1-\lambda)x^2) \leq \lambda\,\tilde{f}_{\tilde{\eta}}(x^1) + (1-\lambda)\,\tilde{f}_{\tilde{\eta}}(x^2), \tag{2.24}$$

which means convexity of $\tilde{f}_{\tilde{\eta}}(\cdot)$ on x, for all $\tilde{\eta} \in \partial\mathbf{B}_2^n$. Then $\mathscr{F}(\cdot)$ satisfies Assumption 2.5, with convex bounding function $\check{f}_\eta(x) = \tilde{f}_\eta(x)$ for all $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. ■

Since functions $\check{f}_\eta(\cdot)$ determine the hyperplanes defining any set $\mathscr{F}(x)$, for all $x \in \mathbb{R}^n$, if the set valued map is characterized by a finite number of convex bounding functions, then $\mathscr{F}(x)$ are polytopes.

In the case of non-autonomous systems with set valued map as dynamic function, an analogous definition of CDI system can be given, recalling that a function $f(x,u)$ is said to be convex if its epigraph is convex, see Remark B.9. In the case of presence of control input, the discrete-time non-autonomous system is given by

$$x^+ \in \mathscr{F}(x,u), \tag{2.25}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $\mathscr{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{K}(\mathbb{R}^n)$ is a set valued map.

A set valued map, for which the following assumption holds, determine the CDI non-autonomous system.

Assumption 2.14 *Assume that the set valued map $\mathscr{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics (2.25) is such that, for every $\eta \in \mathbb{R}^n$, function $\check{f}_\eta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined as*

$$\check{f}_\eta(x,u) = \sup_{z\in\mathscr{F}(x,u)} \eta^T z, \tag{2.26}$$

is convex on $\mathbb{R}^n \times \mathbb{R}^m$ and $\check{f}_\eta(0,0) = 0$.

Considerations similar to those related to autonomous CDI systems could be stated, with proper adaptations, in the case of presence of input. A non-autonomous CDI systems is such that, for any $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and for every $\eta \in \mathbb{R}^n$ we have

$$\mathscr{F}(x,u) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{f}_\eta(x,u), \forall \eta \in \mathbb{R}^n\},$$

with

$$\check{f}_\eta(x,u) = \sup_{z \in \mathscr{F}(x,u)} \eta^T z = \phi_{\mathscr{F}(x,u)}(\eta),$$

convex, in $(x,u) \in \mathbb{R}^{n \times m}$. As for the autonomous CDI systems, convex bounding functions $\check{f}_\eta(\cdot,\cdot)$ can be viewed as support functions of sets $\mathscr{F}(x,u)$, depending on state and input. Furthermore, also in this case, a finite number of bounding functions can be employed to determine a non-autonomous CDI system, the set valued dynamic function of which has polytopic values.

The fact that Assumption 2.5 (Assumption 2.14 for non-autonomous systems) holds for the dynamic function of a system allows us to exploit features inherited by properties of convex functions and convex sets. Some useful properties are listed below.

- Set relations, such as set inclusion, involving the image of a state x through the set valued map, i.e., $\mathscr{F}(x)$, for any $x \in \mathbb{R}^n$, can often be posed as a set of convex constraints. The condition of set inclusion of the successor of the state is often required to be checked in order to ensure invariance and set-theory related properties. It will be made clearer in the following that, then, for systems as in (2.7) and under Assumption 2.5, condition of inclusion of the successor state can be imposed through a set of convex constraints, which can yield to convex problems, efficiently solvable, see (Ben-Tal and Nemirovski, 2001; Boyd and Vandenberghe, 2004).

- Convexity related properties of the dynamic set valued function, in particular convexity of the directional bounding functions $\check{f}_\eta(\cdot)$, for all $\eta \in \mathbb{R}^n$, permits to infer features shared by all the elements of a set by means of conditions involving only a subset of elements, possibly finite.

- Assuming that the effect of the parametric uncertainty is bounded by convex functions is not very restrictive, therefore the family of dynamic systems under analysis encloses a large class of functions. Many methods to approximate nonlinear systems lead to systems with a structure that can be reduced to CDI systems, as defined in (2.7), see Example 2.8 for instance.

  This means that, although a generic system defined by either a deterministic function $f(\cdot)$ or a set valued function $F(\cdot)$, is not in form of CDI systems, it is often possible to determine a CDI system with dynamic function $\mathscr{F}(\cdot)$ for which Assumption 2.5 holds and such that

  $$F(x) \subseteq \mathscr{F}(x), \qquad \text{or} \qquad f(x) \in \mathscr{F}(x),$$

for all $x \in \mathbb{R}^n$ or more shortly, such that $F(\cdot)$ or $f(\cdot)$ are overbounded by $\mathscr{F}(\cdot)$, as defined in Remark 2.7. Therefore, any invariant set for the approximating CDI system is also an invariant set for the original system. More precisely, once an invariant set $\Omega \subseteq \mathbb{R}^n$ is obtained for a CDI system, it is also an invariant for any discrete-time system whose dynamic function satisfies condition $F \subseteq \mathscr{F}$ or $f \in S_{\mathscr{F}}$, see Remark 2.7.

Intuitively, then, the analysis of CDI systems should be thought to as the study of characteristics and properties of a family of dynamic systems, nonlinear and/or affected by parametric uncertainty.

- In the case that the system presents a form of CDI systems as in (2.7), with Assumption 2.5, the results presented are quite strong: the maximal invariant set, for instance, can be well approximated. It is worth recalling that computation of the maximal (robust) invariant set can be an hard task also for linear systems, for nonlinear systems very few results have been provided in literature. Similar considerations are valid, evidently, for non-autonomous CDI systems (2.25) supposing that Assumption 2.14 holds.

2.2.1 Uncertain CDI systems

Definitions and assumptions in the context of CDI systems have been given for the deterministic case, or, if we regard the set valued dynamic map as a way of representing the effect of uncertainty, for the parametric uncertainty framework. No additive uncertainty have been considered so far. It is, anyway, direct to extend the considerations on CDI system to the case of additive uncertainty.

Consider the following discrete-time autonomous system affected by additive uncertainty

$$x^+ \in \mathscr{F}(x) \oplus W, \tag{2.27}$$

where $x \in \mathbb{R}^n$ is the state, x^+ is the successor, W is the additive uncertainty bounding set and $\mathscr{F}(\cdot)$ is a set valued map on $\mathbb{R}^n$.

If Assumption 2.5 holds for $\mathscr{F}(\cdot)$, then the system is denoted as an uncertain CDI system.

Remark 2.15 *Although the fact that function $\mathscr{F}(\cdot)$ is set valued rather than single valued can be viewed as the representation of parametric uncertainty, we will refer to CDI systems (2.27) as uncertain CDI systems, to distinguish them from CDI systems of the form (2.7).*

Analogously, the discrete-time non-autonomous system affected by additive uncertainty

$$x^+ \in \mathscr{F}(x,u) \oplus W, \tag{2.28}$$

where $u \in \mathbb{R}^m$ is the input, is denoted as uncertain non-autonomous CDI system if Assumption 2.14 holds for the set valued map $\mathscr{F}(\cdot,\cdot)$.

2.3 Conclusions

In this chapter the key modelling framework, i.e., CDI systems, used in the following has been presented. CDI systems are systems whose dynamic function is a set valued map rather than a function from the state space to the state space. The set valued map is, hence, defined on the state space, but its values are convex, compact subsets of the space and are determined by convex bounding functions. The assumption of convexity of bounding functions will be useful in next chapters to determine, for a given set, general properties based on boundary type conditions.

Chapter 3

Computation of CDI systems

As claimed in the previous chapter, CDI systems represent the basic framework which will permit to develop the main results of this work. We prove in this chapter that mild assumptions are required to be fulfilled by a dynamic system to be a CDI one and that in many common cases it is easy to obtain a CDI approximation of a nonlinear system.

In this chapter some methods to obtain a CDI representation, as well as to compute a CDI approximation, of a nonlinear and/or uncertain system are presented. The classes of nonlinear and uncertain systems treated in this chapter are listed below.

- Concave-Convex Difference Inclusions (CCDI) systems. This class of systems is composed by particular cases of CDI systems. CCDI systems are characterized by set valued maps determined by a finite number of bounding functions, in particular by n pairs of concave and convex functions.
- Lur'e systems. They are particular nonlinear systems, widely studied in classical literature, mainly in continuous-time. They are linear systems in closed-loop with static nonlinear feedbacks which satisfy a sector condition. Here, we consider a particular case of discrete-time Lur'e systems. We show that, for this class of nonlinear systems, a CDI approximation is available.
- Generalized saturated systems. They are systems composed by a linear system in closed-loop with particular nonlinear functions, the so-called generalized saturated functions. Generalized saturated functions are characterized by convex and concave bounding functions and it will be showed that a CDI system overbounding a general saturated one can be determined directly from such bounding functions.
- Difference-of-convex (DC) systems. They are characterized by dynamic functions which can be expressed as the difference of two convex functions. DC systems permit

to model a very wide class of nonlinear systems, since modest assumptions on the dynamic function guarantee the existence of a DC representation or at least an arbitrarily close approximation. The fact that DC functions are easily upper and lower bounded by concave and convex functions makes direct the computation of a CDI system overbounding the DC one.

- Linear parametric uncertain systems. They are characterized by a dynamic function which belongs, in some sense, to a family of linear (or affine) functions. Depending on the assumptions on the knowledge of the current realization of the dynamic function, different frameworks raise. It will be shown that such, very popular, modelling framework is composed by a subclass of CDI systems, in particular those systems whose convex bounding functions are linear or affine (hence convex).

3.1 Concave-Convex Difference Inclusions: CCDI systems

An interesting subclass of CDI dynamic systems is the family of systems of the form

$$x^+ \in \mathscr{F}(x), \tag{3.1}$$

and with map $\mathscr{F}(\cdot)$ elementwise bounded by functions which are concave and convex with respect to the state x. We consider in what follows systems characterized by set valued maps such that, for any $j \in \mathbb{N}_n$, the projection of set $\mathscr{F}(x)$ over the j-th axis is bounded by functions $\check{f}_j(\cdot)$ and $\hat{f}_j(\cdot)$, convex and concave respectively, and $\check{f}_j(0) = \hat{f}_j(0) = 0$.

Assumption 3.1 *Assume that the set valued map $\mathscr{F} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics (3.1) is given by*

$$\mathscr{F}(x) = \{z \in \mathbb{R}^n : \; \hat{f}_j(x) \leq z_j \leq \check{f}_j(x), \forall j \in \mathbb{N}_n\}, \tag{3.2}$$

where functions $\check{f}_j : \mathbb{R}^n \to \mathbb{R}$ and $\hat{f}_j : \mathbb{R}^n \to \mathbb{R}$ are convex and concave on $\mathbb{R}^n$, respectively, and $\hat{f}_j(0) = \check{f}_j(0) = 0$, for every $j \in \mathbb{N}_n$.

We will refer to Concave-Convex Difference Inclusions (CCDI) systems, when treating systems with dynamic function satisfying Assumption 3.1. A CCDI system is then characterized by convex and concave bounding functions, a pair for any dimension of the state space, then sets $\mathscr{F}(x) \subseteq \mathbb{R}^n$ are parallelograms.

In terms of CDI representation for CCDI systems, the bounding functions of $\mathscr{F}(\cdot)$ are defined only for a finite subset of $\eta \in \mathbb{R}^n$, that is for vectors e^j and $-e^j$, with $j \in \mathbb{N}_n$, where e^j is the vector with all zeros but a 1 as the j-th entry. A CDI representation can be obtained for any CCDI system, see Properties 3.3 and 3.5 below.

The set valued functions $\mathscr{F}(\cdot)$ with values in $\mathbb{R}^n$ satisfying Assumption 3.1 form a subset of those set valued functions fulfilling Assumption 2.5. Therefore, since this means that a CCDI system is also a CDI one, theory developed for CDI systems is also valid in this framework.

The following example shows that many generic convex and concave functions can be employed to define the bounding functions for a CCDI system, for instance exponential, absolute value, quadratic functions.

Example 3.2 *Consider the two dimensional CCDI system given by (3.1) with dynamic set valued map given by (3.2) with*

$$\begin{aligned} \check{f}_1(x) &= x_1 - 1 + e^{x_2}, \qquad & \hat{f}_1(x) &= x_1 + x_2; \\ \check{f}_2(x) &= 10(x_1^2 + |x_2|), \qquad & \hat{f}_2(x) &= 1 - e^{x_1^2 + x_2^2}. \end{aligned} \tag{3.3}$$

The convex and concave bounding functions for $j = 1$ and $j = 2$ are depicted in Figure 3.1.

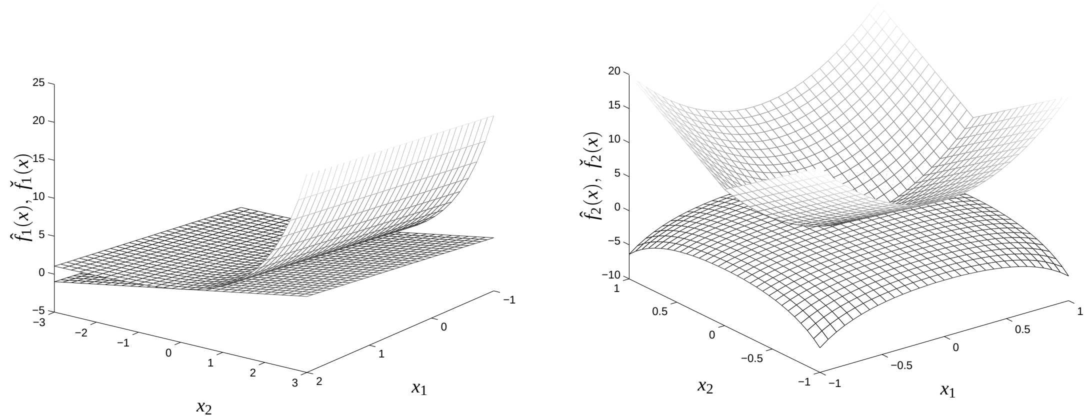

Figure 3.1: Convex and concave bounding functions $\check{f}_1(\cdot)$ and $\hat{f}_1(\cdot)$ (*left*) and $\check{f}_2(\cdot)$ and $\hat{f}_2(\cdot)$ (*right*), defined in (3.3).

For any $x \in \mathbb{R}^2$ the image through set valued map $\mathscr{F}(\cdot)$ is a box, whose maximal and minimal values with respect to both directions are determined by (3.3). A projection of the graph of $\mathscr{F}(\cdot)$ on $\mathbb{R}^3$ is represented in Figure 3.2 (recall that the graph of the set valued map lies on $\mathbb{R}^4$).

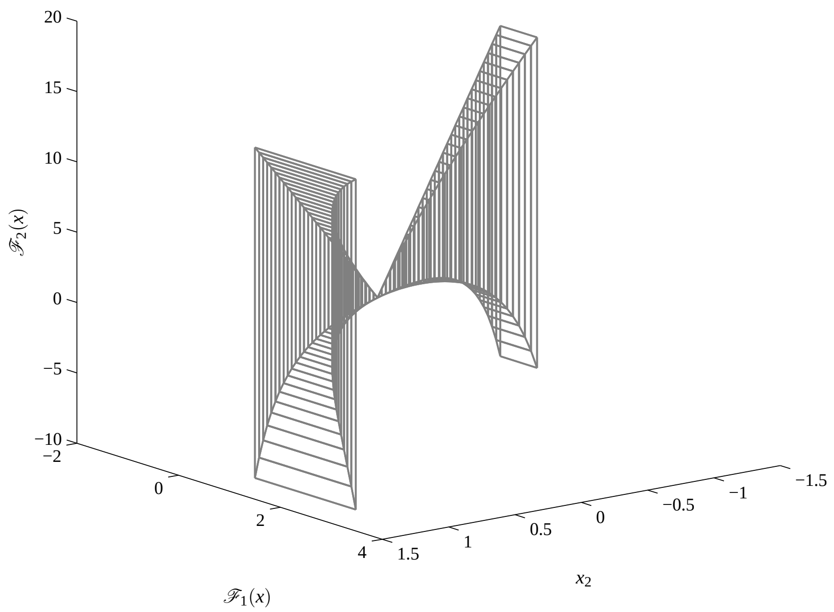

Figure 3.2: Graph of set valued map $\mathscr{F}(\cdot)$, defined by (3.3), projected on $x_1 = 0$.

It is proved below that CCDI systems are also CDI systems, that is, that it is possible to obtain convex bounding functions fulfilling Assumption 2.5 from a CCDI representation, by imposing a proper convex constraint for every $\eta \in \mathbb{R}^n$.

Property 3.3 *For any set valued map $\mathscr{F}(\cdot)$ defined on $\mathbb{R}^n$ for which Assumption 3.1 holds, also Assumption 2.5 holds with convex bounding functions given by*

$$\check{f}_\eta(x) = \sum_{j\in k_+} \eta_j \check{f}_j(x) + \sum_{j\in k_-} \eta_j \hat{f}_j(x), \tag{3.4}$$

for all $x \in \mathbb{R}^n$ and where $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n : \eta_j \geq 0\}$ and $k_- = k_-(\eta) = \{j \in \mathbb{N}_n : \eta_j < 0\}$.

Proof: Although the result is direct consequence of Property 2.13, a constructive proof is given below.

Consider the set valued map $\mathscr{F}(\cdot)$ for which Assumption 3.1 is valid. First, notice that the convex bounding functions $\check{f}_\eta(\cdot)$ for $\eta = e^j \in \mathbb{R}^n$, and $\eta = -e^j \in \mathbb{R}^n$, for all $j \in \mathbb{N}_n$, are given by

$$\begin{aligned}\check{f}_{e^j}(x) &= \check{f}_j(x),\\ \check{f}_{-e^j}(x) &= -\hat{f}_j(x),\end{aligned} \tag{3.5}$$

for all $x \in \mathbb{R}^n$. Clearly $\check{f}_{e^j}(\cdot)$ and $\check{f}_{-e^j}(\cdot)$ in (3.5) are convex, for every $j \in \mathbb{N}_n$. For these particular $\eta \in \mathbb{R}^n$, condition

$$\check{f}_\eta(x) = \sup_{z \in \mathscr{F}(x)} \eta^T z,$$

characterizing $\mathscr{F}(x)$, is satisfied, in fact $z \in \mathscr{F}(x)$ implies $z_j \leq \check{f}_j(x)$, with $j \in \mathbb{N}_n$, and for $\eta = e^j$ we have

$$\eta^T z = (e^j)^T z = z_j \leq \check{f}_j(x) = \check{f}_{e^j}(x).$$

Similarly, $z_j \geq \hat{f}_j(x)$, for every $j \in \mathbb{N}_n$, and then for $\eta = -e^j$ we have

$$\eta^T z = (-e^j)^T z = -z_j \leq -\hat{f}_j(x) = \check{f}_{-e^j}(x),$$

for all $j \in \mathbb{N}_n$.

Hence, so far we have proved that convex and concave functions assumed in Assumption 3.1 determine bounding function $\check{f}_\eta(\cdot)$ as in Assumption 2.5 for particular vectors η. We have to demonstrate that bounding functions $\check{f}_\eta(\cdot)$ can be defined for all $\eta \in \mathbb{R}^n$ in such a way that Assumption 2.5 holds.

For all $\eta \in \mathbb{R}^n$ we have that the bounding convex functions are obtained as

$$\check{f}_\eta(x) = \sum_{j=1}^n |\eta_j| \check{f}_{\{\text{sgn}\,(\eta_j) e^j\}}(x), \tag{3.6}$$

where sgn $(\cdot)$ is the sign operator. Convexity of $\check{f}_\eta(\cdot)$ stems from convexity of functions $\check{f}_{e^j}(\cdot)$ and $\check{f}_{-e^j}(\cdot)$ defined in (3.5), for $j \in \mathbb{N}_n$, and the fact that the sum of convex functions is still convex (notice that, trivially, $|\eta_j| \geq 0$ for all $j \in \mathbb{N}_n$). Inequality characterizing $\mathscr{F}(x)$ as in (2.8) is satisfied for every $\eta \in \mathbb{R}^n$, in fact

$$\begin{aligned}\eta^T z &= \sum_{j=1}^n \eta_j z_j = \sum_{j=1}^n |\eta_j| \text{sgn}\,(\eta_j) z_j = \\ &= \sum_{j=1}^n |\eta_j| (\text{sgn}\,(\eta_j)\, e^j)^T z \leq \sum_{j=1}^n |\eta_j| \check{f}_{\{\text{sgn}\,(\eta_j)\, e^j\}}(x) = \check{f}_\eta(x),\end{aligned}$$

for all $x \in \mathbb{R}^n$ and every $z \in \mathscr{F}(x)$.

Now, we prove the equivalence between definitions (3.4) and (3.6). In fact, recalling that $\check{f}_j(x) = \check{f}_{e^j}(x)$ and $\hat{f}_j(x) = -\check{f}_{-e^j}(x)$, for $j \in \mathbb{N}_n$, it follows that

$$|\eta_j| \check{f}_{\{\text{sgn}\,(\eta_j)\, e^j\}}(x) = \begin{cases} \eta_j \check{f}_{e^j}(x) = \eta_j \check{f}_j(x), & if\ \eta_j \geq 0, \\ -\eta_j \check{f}_{-e^j}(x) = \eta_j \hat{f}_j(x), & otherwise, \end{cases}$$

for every $j \in \mathbb{N}_n$ and every $\eta \in \mathbb{R}^n$, which means that the terms of the summatories defining $\check{f}_\eta(x)$ in (3.4) and (3.6) are the same.

Thus, we proved that for every $x \in \mathbb{R}^n$, if $z \in \mathscr{F}(x)$ then $\eta^T z \leq \check{f}_\eta(x)$, for every $\eta \in \mathbb{R}^n$, with $\check{f}_\eta(\cdot)$ defined in (3.4). This geometrically means that the set $\mathscr{F}(x)$ is contained in the set determined through convex bounding functions (3.4), for all $x \in \mathbb{R}^n$. It can be proved that such sets are the same, simply considering that the constraints relative to convex and concave functions for the CCDI systems are also present in the CDI representation, from (3.5).

It remains to be proved that the convex bounding functions $\check{f}_\eta(\cdot)$ are tight. In other words, we have to prove that, for every $\eta \in \mathbb{R}^n$ and every $x \in \mathbb{R}^n$, there exists a point $\tilde{z}(\eta, x) = \tilde{z} \in \mathscr{F}(x)$ such that $\eta^T \tilde{z} = \check{f}_\eta(x)$. Given $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$, the point defined by

$$\tilde{z}_j = \begin{cases} \check{f}_j(x) & if\ \eta_j \geq 0, \\ \hat{f}_j(x) & otherwise, \end{cases}$$

for $j \in \mathbb{N}_n$, is an element of $\mathscr{F}(x)$ and such that $\eta^T \tilde{z} = \check{f}_\eta(x)$, from (3.4). ■

Hence, the link between a CCDI system and the related CDI representation is direct. For any given CCDI system, the convex bounding functions determining the CDI representation are explicitly expressed in (3.4).

As for CDI systems, the definition of CCDI function can be extended to non-autonomous systems. In presence of an input control, in practice, the bounding functions depend on both the state $x \in \mathbb{R}^n$ and the input $u \in \mathbb{R}^m$ and are convex and concave in the Cartesian product of the spaces of x and u, as stated below.

Assumption 3.4 *Assume that the set valued map $\mathscr{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics*

$$x^+ \in \mathscr{F}(x, u), \tag{3.7}$$

is given by

$$\mathscr{F}(x, u) = \{z \in \mathbb{R}^n : \ \hat{f}_j(x, u) \leq z_j \leq \check{f}_j(x, u), \forall j \in \mathbb{N}_n\}, \tag{3.8}$$

where functions $\check{f}_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\hat{f}_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are convex and concave on $\mathbb{R}^n \times \mathbb{R}^m$, respectively, and $\hat{f}_j(0,0) = \check{f}_j(0,0) = 0$, for every $j \in \mathbb{N}_n$.

Also for the case of a non-autonomous CCDI system, the relation with its CDI representation can be expressed defining the convex bounding functions.

Property 3.5 *For any set valued map $\mathscr{F}(\cdot,\cdot)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ for which Assumption 3.4 holds, also Assumption 2.14 holds with convex bounding functions given by*

$$\check{f}_\eta(x, u) = \sum_{j \in k_+} \eta_j \check{f}_j(x, u) + \sum_{j \in k_-} \eta_j \hat{f}_j(x, u), \tag{3.9}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ *and where* $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n : \eta_j \geq 0\}$ *and* $k_- = k_-(\eta) = \{j \in \mathbb{N}_n : \eta_j < 0\}$.

Similarly to the case of CDI systems, also CCDI systems can admit the presence of additive uncertainty. Uncertain CCDI systems are given by dynamics (2.27) with the set valued maps $\mathscr{F}(\cdot)$ fulfilling Assumption 3.1, for the autonomous case, and by dynamics (2.28) with function $\mathscr{F}(\cdot,\cdot)$ satisfying Assumption 3.4 for the non-autonomous one.

3.1.1 CCDI subsystems

Now, suppose that the dynamic function defining a CCDI system is given by a linear part and the linear combination of a function, denote it $\varphi : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^m)$, with $m < n$, such that Assumption 3.1 is satisfied by $\varphi(\cdot)$. In some cases it can be beneficial to determine a proper linear mapping that permits to confine the nonlinearity effects in a subspace of lower dimension than the dimension of the whole state space. This allows one to consider, in the mapped space, the subspace relative to the linear dynamics and the subspace relative to the nonlinear dynamics. The procedure will be illustrated below with an example, and it can be applied also to CDI subsystems.

Suppose that the nonlinear system is a discrete-time system of the form

$$x^+ \in Ax \oplus B\varphi(x), \tag{3.10}$$

where $x \in \mathbb{R}^n$ is the state and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, with $m < n$. Assume also that the columns of B are linearly independent. Function $\varphi : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^m)$ is a set valued map such that Assumption 3.1 is satisfied by $\varphi(\cdot)$. Recall that, from Property 3.3, such set valued map satisfies also Assumption 2.5, then the system is a CDI system too.

By assumption, there exists $\check{\varphi} : \mathbb{R}^n \to \mathbb{R}^m$ convex on $\mathbb{R}^n$ and $\hat{\varphi} : \mathbb{R}^n \to \mathbb{R}^m$ concave on $\mathbb{R}^n$, with $\check{\varphi}(0) = \hat{\varphi}(0) = 0$ and such that, for every $x \in \mathbb{R}^n$ we have that

$$\hat{\varphi}_j(x) \leq z_j \leq \check{\varphi}_j(x), \quad \forall j \in \mathbb{N}_m, \tag{3.11}$$

for all $z \in \varphi(x)$.

Define as $B_\perp$ the matrix in $\mathbb{R}^{n\times(n-m)}$ whose $n-m$ columns are linearly independent and normal to the subspace generated by columns of B. Then, $B_\perp$ is such that

$$B_\perp^T B = 0_{n-m,m}.$$

Define, moreover the left-side inverse of matrix B, that is

$$\hat{B} = (B^T B)^{-1} B^T,$$

that exists since, from assumption of linearly independence of columns of B, matrix $B^T B$ is nonsingular, hence invertible. Clearly $\hat{B} \in \mathbb{R}^{m\times n}$ is such that

$$\hat{B}\,B = (B^T B)^{-1} B^T B = I_m.$$

Notice that the subspace spanned by columns of B is the same subspace spanned by those of $\hat{B}^T$, being the latter a linear transformation of B. Therefore, also the columns of $\hat{B}^T$ are linearly independent and orthogonal to those of $B_\perp$ and then matrix defined

$$T = \begin{bmatrix} \hat{B} \\ B_\perp^T \end{bmatrix} \in \mathbb{R}^{n\times n},$$

is invertible and such that

$$TB = \begin{bmatrix} \hat{B}\,B \\ B_\perp^T B \end{bmatrix} = \begin{bmatrix} I_m \\ 0_{n-m,m} \end{bmatrix}. \tag{3.12}$$

Thus, matrix T is a square nonsingular matrix and it determines the linear transformation $l_T : \mathbb{R}^n \to \mathbb{R}^n$ defined as $l_T(x) = Tx$ and its inverse $l_T^{-1}(y) = T^{-1}y$. Define the state variable in the mapped space $y = Tx$, then $x = T^{-1}y$, which leads to the dynamic system in the mapped space

$$\begin{aligned} y^+ = Tx^+ \quad &\in TAx + TB\varphi(x) = TAT^{-1}y + TB\varphi(T^{-1}y) = \\ &= TAT^{-1}y + \begin{bmatrix} I_{m,m} \\ 0_{n-m,m} \end{bmatrix} \varphi(T^{-1}y) = \begin{bmatrix} \hat{B}AT^{-1}y + \varphi(T^{-1}y) \\ B_\perp^T AT^{-1}y \end{bmatrix}, \end{aligned}$$

from (3.12). Therefore, the obtained system is

$$y^+ \in \zeta(y) = TAT^{-1}y + \begin{bmatrix} \varphi_1(T^{-1}y) \\ \dots \\ \varphi_m(T^{-1}y) \\ 0 \\ \dots \\ 0 \end{bmatrix}. \tag{3.13}$$

The set valued function $\zeta(\cdot)$ is such that Assumption 3.1 holds and the concave and convex bounding functions can be recovered from the concave and convex functions bounding $\varphi(\cdot)$, i.e., from $\check{\varphi}_j(\cdot)$ and $\hat{\varphi}_j(\cdot)$, for $j \in \mathbb{N}_n$. In fact, for every $j \in \mathbb{N}_n$ and every $y \in \mathbb{R}^n$,

$$\hat{\zeta}_j(y) \le z_j \le \check{\zeta}_j(y), \quad \forall z \in \zeta(y), \tag{3.14}$$

where, for $j \in \mathbb{N}_m$ (i.e., the first m elements of functions $\zeta(\cdot)$), we have

$$\begin{aligned} \check{\zeta}_j(y) &= T_j A T^{-1} y + \check{\varphi}_j(T^{-1}y), \\ \hat{\zeta}_j(y) &= T_j A T^{-1} y + \hat{\varphi}_j(T^{-1}y), \end{aligned} \tag{3.15}$$

while, for $j \in \mathbb{N}_{[m+1,n]}$, we have

$$\check{\zeta}_j(y) = \hat{\zeta}_j(y) = T_j A T^{-1} y, \tag{3.16}$$

which means that, for the last $n-m$ components, function $\zeta(\cdot)$ is linear.

Example 3.6 *Consider the following nonlinear system*

$$x^+ = \begin{bmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \psi(x)$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a nonlinear function defined on $\mathbb{R}^2$ as

$$\psi(x) = (x_1^2 + x_2^2)\sin(x_1 + x_2), \tag{3.17}$$

depicted in Figure 3.3.

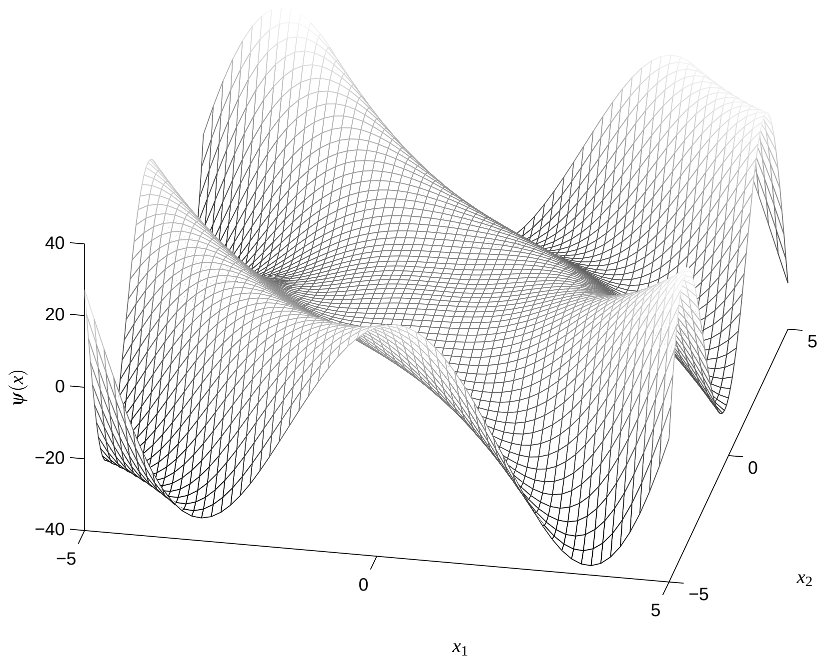

Figure 3.3: Function $\psi(\cdot)$ defined in (3.17).

The set valued map $\varphi(\cdot)$ defined through the convex and concave functions

$$\check{\varphi}(x) = (x_1^2 + x_2^2), \qquad \hat{\varphi}(x) = -(x_1^2 + x_2^2),$$

satisfies Assumption 3.1 and overbounds function $\psi(\cdot)$, that is $\psi \in S_\varphi$, see (2.12). Then system (3.10) with $\varphi(\cdot)$ defined above overbounds the nonlinear system. Altough the nonlinearity affects both states of the systems, it can be confined in a subspace of dimension one. Defining

$$B_\perp = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; T = \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} 1 & -0.5 \\ 1 & 0.5 \end{bmatrix}, \; T^{-1}y = \begin{bmatrix} y_1 - 0.5y_2 \\ y_1 + 0.5y_2 \end{bmatrix},$$

it follows that

$$\begin{aligned} \psi(T^{-1}y) &\geq \hat{\varphi}(T^{-1}y) = -(y_1 - 0.5y_2)^2 - (y_1 + 0.5y_2)^2 = -2y_1^2 - 0.5y_2^2, \\ \psi(T^{-1}y) &\leq \check{\varphi}(T^{-1}y) = (y_1 - 0.5y_2)^2 + (y_1 + 0.5y_2)^2 = 2y_1^2 + 0.5y_2^2. \end{aligned}$$

We have a CCDI system in the mapped state variable $y = Tx$ of the form (3.13), given by

$$\begin{aligned} y_1^+ &\in \zeta_1(y) = 0.8y_1 + 0.05y_2 + \varphi(T^{-1}y), \\ y_2^+ &= \zeta_2(y) = -0.4y_1 + 0.9y_2. \end{aligned}$$

The graphs of concave and convex bounding functions $\hat{\zeta}_1(\cdot)$ and $\hat{\zeta}_1(\cdot)$ and of linear function $\zeta_2(\cdot)$ are represented in Figure 3.4.

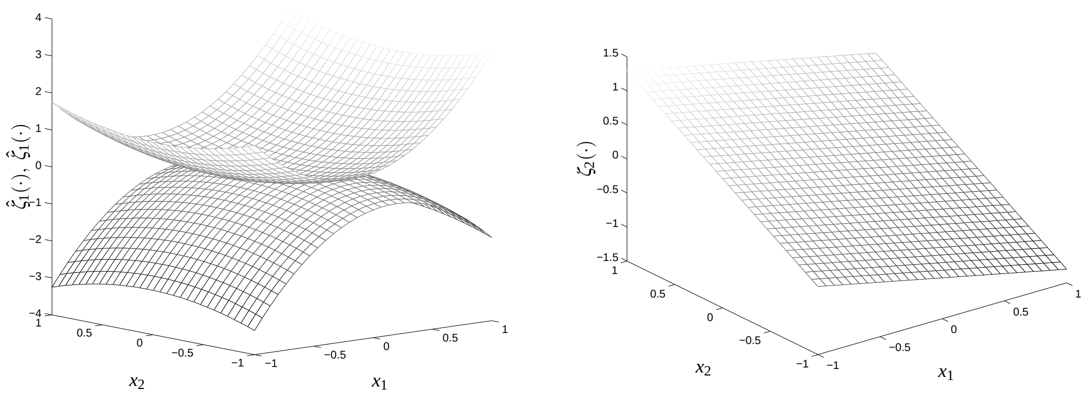

Figure 3.4: Graphs of functions $\hat{\zeta}_1(\cdot)$ and $\check{\zeta}_1(\cdot)$ (*left*) and $\zeta_2(\cdot)$ (*right*).

3.2 Lur'e systems

Lur'e problem, mainly in the continuous-time case, is a classical problem in control theory and nonlinear systems theory, see (Vidyasagar, 1993). The Lur'e problem, whose name is due to Russian scientist A.I. Lur'e, involves a class of nonlinear systems composed by a linear system in closed-loop with static nonlinearities.

The importance of Lur'e systems in the context of control theory stems from the fact that different control schemes appearing in practical applications can be formulated using the Lur'e systems structure (Wada, Ikeda, Ohta and Siljak, 1998; Slotine and Li, 1991; Chu, Huang and Wang, 2001). The particular case of saturation nonlinearity is widely treated in literature, see for example (da Silva and Tarbouriech, 2001).

The stability analysis of Lur'e systems can be performed, for example, by means of Popov and circle criteria, see (Weissenberger, 1968; Vidyasagar, 1993; Khalil, 2002). Particular approaches are available for Lur'e systems with piecewise affine nonlinearities. In this case, the domain of attraction can be estimated by means of a piecewise quadratic Lyapunov function (Johansson and Rantzer, 1998). Also, a novel result to deal with this class of Lur'e systems can be found in (Hu, Huang and Lin, 2004), where a procedure to compute invariant ellipsoids is presented.

In particular, assume that the continuous-time system

$$\begin{cases} \dot{x}(t) = & Ax(t) + Bu(t), \\ y(t) = & Cx(t), \end{cases}$$

is controlled in closed-loop through a static nonlinear function of the output

$$u(t) = -\varphi(y(t)),$$

with $x \in \mathbb{R}^n$ and $u,\ y \in \mathbb{R}^m$ with $m < n$. The only assumption on the nonlinear function is that $\varphi(0) = 0$ and its values at $y \in \mathbb{R}^m$ lie in a sector determined by two scalars $a, b \in \mathbb{R}$ with $a < b$ as in the following

$$[\varphi(y) - ay]^T [by - \varphi(y)] \geq 0, \quad \forall y \in \mathbb{R}^m. \tag{3.18}$$

For the case of $\varphi : \mathbb{R} \to \mathbb{R}$, condition (3.18) is satisfied if

$$\begin{cases} ay \leq \varphi(y) \leq by & \text{if } y \geq 0, \\ by \leq \varphi(y) \leq ay & \text{otherwise}, \end{cases}$$

with $a \leq b$, for every $y \in \mathbb{R}$.

The classical Lur'e problem concerns the conditions under which the system is globally uniformly asymptotically convergent to the origin, the absolute stability problem.

Thus, for the classical approach to Lur'e problem (i.e., considering condition (3.18)) we might consider nonlinear functions whose graph is contained in a sector bounded by linear functions. In our case, sectors determined by convex functions will be considered and it will be shown that, in many case, this will permit to reduce the conservativeness.

Remark 3.7 *The study of Lur'e system, as defined above, involves implicitly the analysis of a family of dynamic systems characterized by a common overbounding system. Once the sector on the space* $\mathbb{R}^2$ *containing* $\varphi(y)$ *for all* $y \in \mathbb{R}$ *is defined, a minimal overbounding system is determined. Every Lur'e system whose nonlinear feedback function is contained in the sector, shares the stability results and any invariant set for the overbounding system is invariant also for any system of the family.*

In this thesis we consider the discrete-time Lur'e system

$$\begin{cases} x_{k+1} &= Ax_k - B\varphi(y_k) \\ y_k &= Fx_k, \end{cases} \tag{3.19}$$

where $x_k \in \mathbb{R}^n$ represents the state vector and $y_k = Fx_k \in \mathbb{R}$ the output of the system and the nonlinear feedback function $\varphi(\cdot)$ satisfies the following conditions.

Assumption 3.8 *Assume that the nonlinear function* $\varphi : \mathbb{R} \to \mathbb{R}$ *determining the system dynamics (3.19) is such that, for every* $y \in \mathbb{R}$*, the following conditions hold:*

(i) $\varphi(y)$ *is piecewise affine.*

(ii) $\varphi(y)$ *is a continuous odd function.*

(iii) $\varphi(y)$ *is concave in* $\mathbb{R}_+$*.*

Since the function $\varphi(\cdot)$ is odd, we have that $\varphi(-y) = -\varphi(y)$ and then it is convex if restricted to $\mathbb{R}_-$. The following property characterizes all the functions $\varphi(\cdot)$ that satisfy Assumption 3.8.

Property 3.9 *(Hu et al., 2004) The piecewise affine function* $\varphi(y)$ *is concave in* $\mathbb{R}_+$ *if and only if it can be expressed as*

$$\varphi(y) = \begin{cases} k_0 y & \text{if } y \in [0, b_1) \\ k_1 y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_N y + c_N & \text{if } y \in [b_N, \infty) \end{cases}, \quad \forall y \geq 0, \tag{3.20}$$

where the scalars k_i, for $i \in \mathbb{N}_{[0,\ N]}$, b_i, for $i \in \mathbb{N}_N$ and c_i, for $i \in \mathbb{N}_N$ satisfy

$$0 < b_1 < b_2 < \ldots < b_N,$$

$$k_0 > k_1 > k_2 > \ldots > k_N,$$

$$c_i = \begin{cases} (k_0 - k_1)b_1, & \text{if } i = 1, \\ c_{i-1} + (k_{i-1} - k_i)b_i, & \text{if } 2 \leq i \leq N. \end{cases}$$

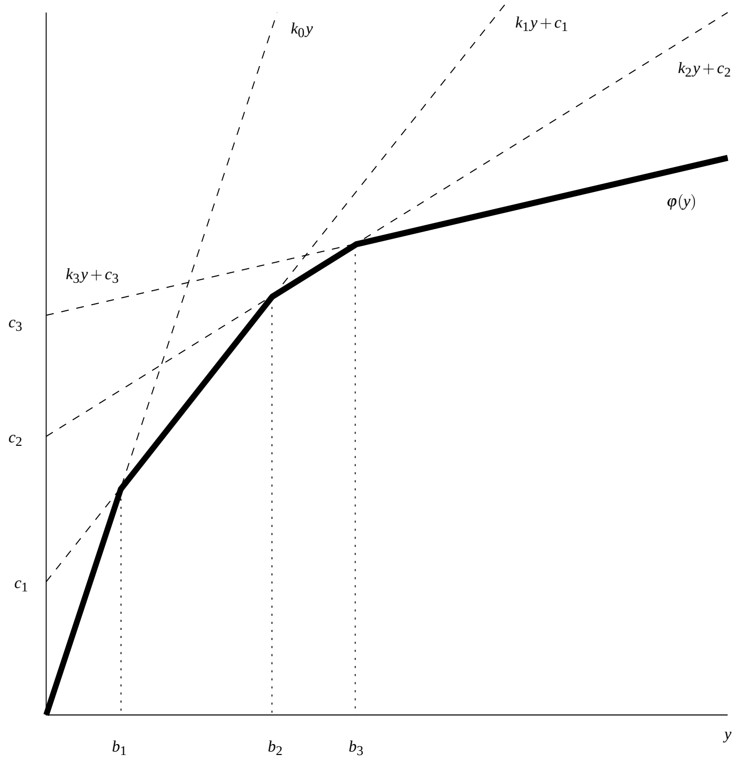

Figure 3.5: An example of a piecewise affine function $\varphi(\cdot)$ concave in $\mathbb{R}_+$.

See Figure 3.5 for an example of piecewise affine concave function in $\mathbb{R}_+$ ($N = 3$). It is easy to determine the convex bounding functions defined on $\mathbb{R}$ for a function $\varphi(\cdot)$ satisfying Assumption 3.8. First we define

$$\check{\varphi}(y) = \max\{k_0 y, \varphi(y)\} = \begin{cases} k_0 y, & \text{if } y \geq -b_1, \\ \varphi(y), & \text{otherwise}, \end{cases} \tag{3.21}$$

which is, then, linear on the set $y \in [-b_1, \infty)$ and convex on $y \in (-\infty. -b_1)$, then convex on $\mathbb{R}$. Define also

$$\hat{\varphi}(y) = \min\{k_0 y, \varphi(y)\} = \begin{cases} k_0 y, & \text{if } y \leq b_1, \\ \varphi(y), & \text{otherwise}, \end{cases} \tag{3.22}$$

concave on $\mathbb{R}$, for analogy. From this we have that

$$\hat{\varphi}(y) \leq \varphi(y) \leq \check{\varphi}(y),$$

for every $y \in \mathbb{R}$.

Hence, in the Lur'e problem under analysis, we consider nonlinearities for which the sector is determined by concave and convex functions, rather than only by linear ones as for the classical approach. In some common cases, this permits to reduce the conservatism introduced by the overbounding process, as shown in the following example.

Example 3.10 *Consider a Lur'e system whose static nonlinearity is given by (3.20) with* $N = 2$, $k_0 = 2$, $k_1 = 1$, $k_2 = 0.5$, $c_1 = 1$, $c_2 = 2$, $b_1 = 1$ *and* $b_2 = 2$. *It can be seen, by geometric inspection, that considering the classical Lur'e approach, the sector containing* $\varphi(\cdot)$ *is delimited by functions* $2y$ *and* $0.5y$*, while, for our approach, the sector is determined by* $2y$ *and* $\varphi(y)$*, as shown in Figure 3.6.*

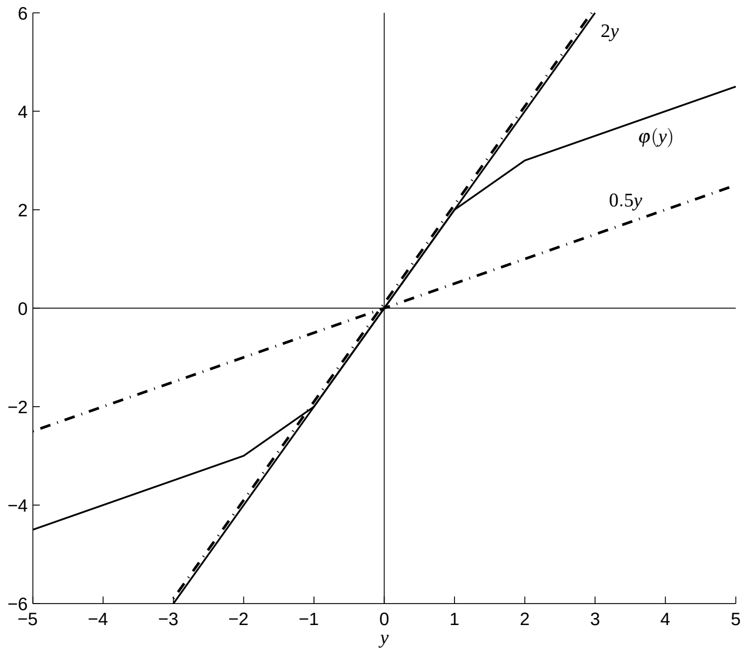

Figure 3.6: Bonding sectors for function $\varphi(\cdot)$.

An immediate way of determining the CDI system bounding the discrete-time Lur'e system (3.19), that is, the set valued map $\mathscr{F}(\cdot)$ such that $Ax - B\varphi(Fx) \subseteq \mathscr{F}(x)$ for all $x \in \mathbb{R}^n$, is by defining

$$\check{f}_\eta(x) = \begin{cases} \eta^T Ax - \eta^T B\check{\varphi}(Fx), & \text{if } \eta^T B \le 0, \\ \eta^T Ax - \eta^T B\hat{\varphi}(Fx), & \text{if } \eta^T B > 0, \end{cases} \tag{3.23}$$

for all $\eta \in \mathbb{R}^n$. Functions $\check{f}_\eta(\cdot)$ defined above are convex on $\mathbb{R}^n$ and the CDI system characterized by the set valued map $\mathscr{F}(\cdot)$ for which Assumption 2.5 holds with $\check{f}_\eta(\cdot)$ given by (3.23), overbounds the Lur'e system. In fact we have that $\eta^T Ax - \eta^T B\varphi(Fx) \le \check{f}_\eta(x)$ for all $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

Moreover, the nonlinearity can be confined in a subspace of dimension one through a proper linear transformation, as illustrated in Section 3.1.1, and the system in the mapped space can be bounded by a CCDI system, for which only the first dynamic function is a set valued map.

Remark 3.11 *Note the analogies between classical Lur'e systems and CDI systems. Consider in fact the minimal CDI system overbounding a Lur'e system, that is system (3.1) with $\mathscr{F}(\cdot)$ satisfying Assumption 2.5 and bounding functions (3.23). While the bounding functions determining the sector for the classical Lur'e problem are linear, for CDI systems the region is defined through convex functions. Conceptually, in CDI systems we exploit properties of convex functions and convex sets, rather than features of linearity, as for classical Lur'e systems.*

Note that the results presented can be extended to systems of the form

$$x_{k+1} = \bar{A}x_k - \bar{B}\bar{\varphi}(y_k),$$

where $\bar{\varphi}(\cdot)$ is an odd piecewise affine function convex in $\mathbb{R}_+$ (it suffices to define $\varphi(\cdot) = -\bar{\varphi}(\cdot)$, $A = \bar{A}$ and $B = -\bar{B}$).

3.3 Generalized saturated systems

We present here a family of nonlinear systems enclosing a wide range of common static nonlinearities, such as saturation, dead-zone, hysteresis, etc. Any generalized saturated system is easily overbounded by a CDI system, as illustrated below.

First the definition of generalized saturated functions is introduced. The dynamic systems composed by one of these functions in closed-loop with a linear system are referred to as generalized saturated systems and determine a class of nonlinear systems which encloses many common and popular nonlinear models.

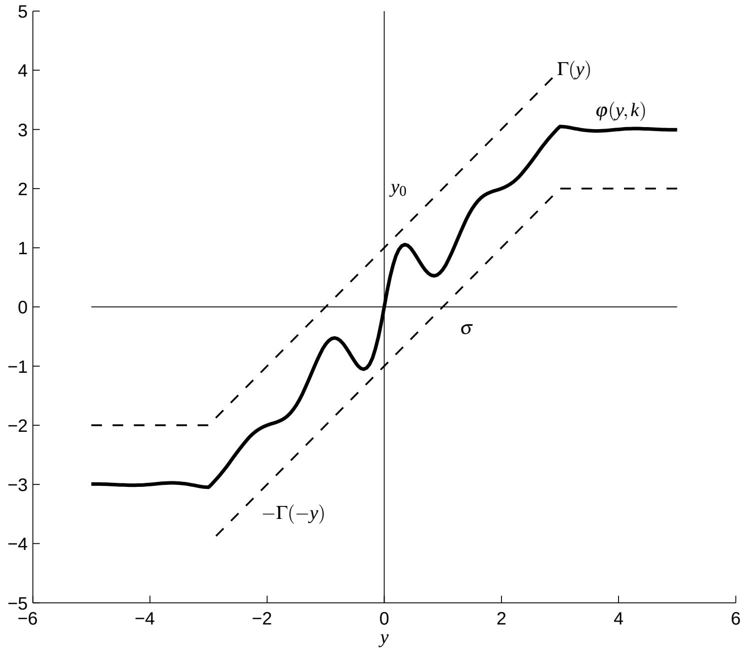

Figure 3.7: Example of generalized saturated function.

We recall here the definition of generalized saturated functions, introduced in (Tarbouriech, Queinnec, Álamo, Fiacchini and Camacho, 2009).

Definition 3.12 *The scalar function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be a generalized saturated function with saturation level $y_0 \in \mathbb{R}$, $y_0 > 0$, dead-zone $\sigma \in \mathbb{R}^n$, $\sigma \geq 0$, and linear slope $\mu \in \mathbb{R}$, $\mu > 0$, if*

$$-\Gamma(-y) \leq \varphi(y,k) \leq \Gamma(y), \quad \forall y \in \mathbb{R}, \forall k \in \mathbb{N}, \tag{3.24}$$

where $\Gamma(y) = \max\{\mu(y+\sigma), -y_0\}$ and $k \in \mathbb{N}$ is the discrete-time instant.

Notice that a generalized saturated function can be time-varying, while function $\Gamma(\cdot)$ is assumed time-invariant. We provided here the definition for the scalar case of generalized saturated functions. The vectorial definition can also be stated, see (Tarbouriech et al., 2009). We considered that the scalar case is expressive enough to introduce an example related to practical control problems.

In Figure 3.7, the geometrical concept of generalized saturated function is illustrated. A generalized saturated function is a function of $y \in \mathbb{R}$, and possibly of time $k \in \mathbb{N}$, whose graph is contained in the region of $\mathbb{R}^2$ enclosed between the graph of $\Gamma(y)$ and $-\Gamma(-y)$.

Some examples of common static nonlinear functions which can be represented as generalized saturated functions are saturation plus dead-zone and hysteresis, depicted in Figure 3.8. It is clear that also saturation, one of the most common nonlinearity affecting real systems, is a particular case of generalized saturated functions.

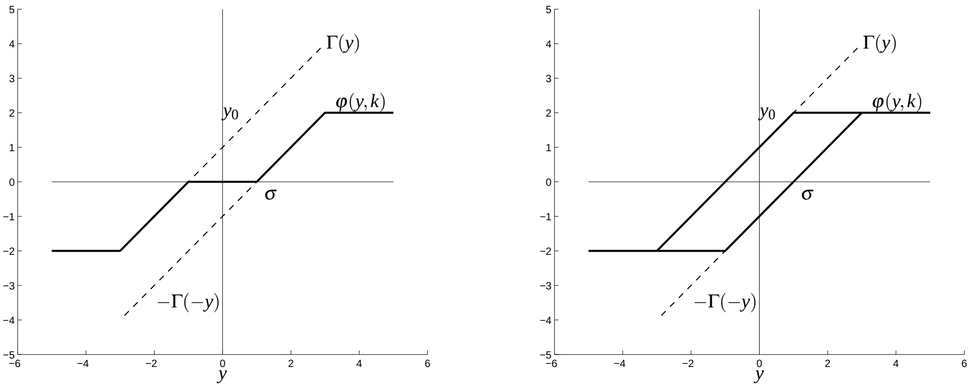

Figure 3.8: Examples: saturation plus dead-zone (*left*) and hysteresis (*right*).

Hence, given the generalized saturated function $\varphi(\cdot,\cdot)$, a dynamic system of the form

$$x_{k+1} = Ax_k + B\varphi(Fx_k, k), \tag{3.25}$$

where $F \in \mathbb{R}^{1\times n}$, is called generalized saturated system.

Analogously to the case of Lur'e systems, a CDI system overbounding a generalized saturated one can be determined. In fact, notice that from

$$\begin{aligned}\Gamma(y) &= \max\{\mu(y+\sigma), -y_0\} = \max\{\mu y + \mu\sigma, -y_0 - \mu\sigma + \mu\sigma\} = \\ &= \max\{\mu y, -y_0 - \mu\sigma\} + \mu\sigma,\end{aligned}$$

we have that (3.24) is equivalent to

$$-\Gamma^0(-y) - \mu\sigma \le \varphi(y,k) \le \Gamma^0(y) + \mu\sigma,$$

with

$$\Gamma^0(y) = \max\{\mu y, -y_0 - \mu\sigma\}.$$

For every $\eta \in \mathbb{R}^n$ and every $x \in \mathbb{R}^n$, we can define the following convex functions

$$f_\eta(x) = \begin{cases} \eta^T Ax + \eta^T B\Gamma(Fx) = \eta^T Ax + \eta^T B\Gamma^0(Fx) + \eta^T B\mu\sigma, & \text{if } \eta^T B \ge 0, \\ \eta^T Ax - \eta^T B\Gamma(-Fx) = \eta^T Ax - \eta^T B\Gamma^0(-Fx) - \eta^T B\mu\sigma, & \text{if } \eta^T B < 0, \end{cases} \tag{3.26}$$

which are such that $\eta^T Ax + \eta^T B\varphi(Fx,k) \leq f_\eta(x)$ for all $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, but do not satisfy condition $f_\eta(0) = 0$. Then they do not fulfill Assumption 2.5. As a matter of fact, we have to define an uncertain CDI system, rather than a CDI system, to bound the generalized saturated one.

Define the convex bounding functions as

$$\check{f}_\eta(x) = \begin{cases} \eta^T Ax + \eta^T B\Gamma^0(Fx), & \text{if } \eta^T B \geq 0, \\ \eta^T Ax - \eta^T B\Gamma^0(-Fx), & \text{if } \eta^T B < 0, \end{cases} \tag{3.27}$$

for all $\eta \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$. The set valued map $\mathscr{F}(\cdot)$ with convex bounding functions given by (3.27) satisfies Assumption 2.5. We prove that the uncertain CDI system whose set valued map is $\mathscr{F}(\cdot) \oplus W$ with convex bounding functions (3.27) and

$$W = \{w = Bv : -\mu\sigma \leq v \leq \mu\sigma\},$$

overbounds the generalized saturated system. In fact, for every $\eta \in \mathbb{R}^n$, we have

$$\phi_W(\eta) = \sup_{w \in W} \eta^T w = \sup_{-\mu\sigma \leq v \leq \mu\sigma} \eta^T Bv = \begin{cases} \eta^T B\mu\sigma, & \text{if } \eta^T B \geq 0, \\ -\eta^T B\mu\sigma, & \text{if } \eta^T B < 0, \end{cases}$$

and then, for every $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have

$$\phi_{\mathscr{F}(x)\oplus W}(\eta) = \sup_{z \in \mathscr{F}(x)\oplus W} \eta^T z = \sup_{z \in \mathscr{F}(x)} \eta^T z + \sup_{w \in W} \eta^T w = \check{f}_\eta(x) + \sup_{w \in W} \eta^T w = f_\eta(x),$$

where $f_\eta(\cdot)$ are defined in (3.26) and $\check{f}_\eta(\cdot)$ in (3.27). Thus, the uncertain CDI system overbounds the generalized saturated one.

3.4 Difference-of-convex (DC) systems

In the previous section we considered as a possible modelling framework for nonlinear and/or uncertain systems a class of systems characterized by set valued maps bounded by convex functions. The particular structure of CDI systems permits to exploit properties of convexity.

We introduce here the concept of DC functions, which allows us to model a very wide family of nonlinear functions and then can be employed to model many nonlinear systems. A function is DC if it can be expressed as the difference of two convex functions, the formal definition of DC function follows, see (Adjiman and Floudas, 1996; Horst and Thoai, 1999; Carrizosa, 2001).

Definition 3.13 *A function $\alpha : \mathbb{R}^p \to \mathbb{R}$ defined on a convex set $D \subseteq \mathbb{R}^p$ is a DC function if there exist two convex functions $\beta, \gamma : \mathbb{R}^p \to \mathbb{R}$ defined on D and such that $\alpha(x) = \beta(x) - \gamma(x)$ for all $x \in D$.*

From continuity of convex functions on any open convex set contained in its effective domain, see Theorem (B.10), it follows that, given functions $\beta(\cdot)$ and $\gamma(\cdot)$ such that $\alpha(x) = \beta(x) - \gamma(x)$, with $\alpha(\cdot)$ DC function defined on D, open, convex and such that $D \subseteq \operatorname{dom} \beta \cap \operatorname{dom} \gamma$, then $\alpha(\cdot)$ is continuous on D.

Remark 3.14 *Note that, unlike the case of convex functions which can be assumed defined on the whole space $\mathbb{R}^p$ considering its extension, see Remark B.6, it is not trivial to determine an extension for DC functions.*

In fact, assume for example that both functions $\beta(\cdot)$ and $\gamma(\cdot)$ defining $\alpha(\cdot)$, as in Definition 3.13, are such that $\operatorname{dom} \beta = \operatorname{dom} \gamma = D$. The extension of any convex function outside of its effective domain is $+\infty$ and, if in this case we employed the same criterion to extend $\beta(\cdot)$ and $\gamma(\cdot)$, function $\alpha(\cdot)$ would not be defined outside D.

For this reason, when dealing with DC functions we expressly state that they are defined on a convex set D, no extension is considered.

In the following we refer to a function $\alpha : \mathbb{R}^p \to \mathbb{R}^q$ as a DC function if $\alpha_j(\cdot)$ is a DC function for all $j \in \mathbb{N}_q$. Recall that, similarly, a function $\beta : \mathbb{R}^p \to \mathbb{R}^q$ is denominated convex if $\beta_j(\cdot)$ is convex for all $j \in \mathbb{N}_q$. Moreover, we claim that a function $\alpha : \mathbb{R}^{p_x} \times \mathbb{R}^{p_u} \to \mathbb{R}$ is a DC function with respect to variables $x \in D \subseteq \mathbb{R}^{p_x}$ and $u \in E \subseteq \mathbb{R}^{p_u}$, meaning that $\alpha(x,u) = \beta(x,u) - \gamma(x,u)$ where $\alpha(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are convex with respect to $(x,u) \in D \times E$.

3.4.1 Brief overview on DC functions

We provide here some important properties of DC functions. First, it is worth mentioning that the set of DC functions defined on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions of this set. Therefore, every function defined on a compact convex set can be approximated by a DC function with any desired precision. Moreover, given a twice differentiable function, i.e., a $\mathscr{C}^2$-function, it is always possible to obtain a DC representation. In effect, suppose that $f_{DC} : D \to \mathbb{R}$ satisfies $\frac{\partial^2}{\partial x^2} f_{DC}(x) \geq -2aI$, for all $x \in D$ with $a > 0$. Recall now that a $\mathscr{C}^2$-function is convex in D if and only if $\frac{\partial^2}{\partial x^2} f_{DC}(x) \geq 0$, for all $x \in D$. Bearing this in mind, it is easy to see that $f_{DC}(x) = g_c(x) - h_c(x)$, with $g_c(x) = f_{DC}(x) + ax^T x$ and $h_c(x) = ax^T x$ constitutes a DC representation of $f_{DC}(x)$. A systematic method to obtain (by means of interval arithmetic) an appropriate value of a for a given $\mathscr{C}^2$-function can be found in (Adjiman and Floudas, 1996).

The following example illustrates this idea. Consider the function $f_{DC}(x) = x^3 + x^2 + 1$ in the domain $x \in [-1, 1]$. Since $\frac{\partial^2}{\partial x^2} f_{DC}(x) = 6x + 2$, it results that $\frac{\partial^2}{\partial x^2} f_{DC}(x) \geq -4$, for all

$x \in [-1,1]$. Thus, $f_{DC}(x)+2x^2$ satisfies $\frac{\partial^2}{\partial x^2}(f_{DC}(x)+2x^2) \geq 0$ for all $x \in [-1,1]$. Defining $g_c(x) = f_{DC}(x)+2x^2$ and $h_c(x) = 2x^2$, the equivalent function $f_{DC}(x) = g_c(x) - h_c(x)$ is a DC function in $x \in [-1,1]$.

Some properties of DC functions, formally presented and proved in (Horst and Thoai, 1999), are listed below.

Property 3.15 *DC functions satisfy the following properties:*

(i) Every function $f : \mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives are continuous everywhere is DC.

(ii) Let D be a compact convex subset of $\mathbb{R}^n$. Then, every continuous function on D is the limit of a sequence of DC functions which converges uniformly on D; i.e., for any continuous function $c : D \to \mathbb{R}$ and for any $\varepsilon > 0$, there exists a DC function $f : D \to \mathbb{R}$ such that $|c(x) - f(x)| \leq \varepsilon$, for all $x \in D$.

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and let $g : \mathbb{R} \to \mathbb{R}$ be convex. Then, the composite function $(g \circ f)(x) = g(f(x))$ is DC.

Operations between functions that preserve the DC nature, presented in (Horst and Thoai, 1999), are summarized in the following property. Proofs of the property (except for part (ii) that is proved in (Horst and Thoai, 1999)) are given here. We have to point out that notation $f^j(\cdot)$ in the following property denotes the j-th element of an ordered set of functions, rather than the power of such function. We employ, in this property, notation $(f(x))^j$ to express j-th power of value $f(x)$.

Property 3.16 *Let $f : D \to \mathbb{R}$ and $f^j : D \to \mathbb{R}$, for $j \in \mathbb{N}_m$, be DC functions defined on $D \subseteq \mathbb{R}^n$ convex. Then the following functions are also DC:*

(i) Any affine combination of DC functions, i.e. for any $\lambda \in \mathbb{R}^m$, function $\sum_{j=1}^{m} \lambda_j f^j(x)$.

(ii) The pointwise maximum and minimum of DC functions, $\max_{j\in\mathbb{N}_m}\{f^j(x)\}$ and $\min_{j\in\mathbb{N}_m}\{f^j(x)\}$.

(iii) The absolute value $|f(x)|$, functions $\max\{0, f(x)\}$ and $\min\{0, f(x)\}$.

(iv) The product of DC functions, $\prod_{j=1}^{m} f^j(x)$.

Proof:

(i) Recall that, if $g(\cdot)$ is convex, then function $\alpha g(\cdot)$ is convex for $\alpha \geq 0$ and concave for $\alpha < 0$ and the fact that the sum of convex functions is convex while the sum of concave functions is concave. For any $\lambda \in \mathbb{R}^m$, denote with $k_- = k_-(\lambda)$ the set of indexes $j \in \mathbb{N}_m$ such that $\lambda_j < 0$ and with $k_+ = k_+(\lambda)$ the set of indexes $j \in \mathbb{N}_m$ such that $\lambda_j \geq 0$. That is $k_-(\lambda) = \{j \in \mathbb{N}_m : \lambda_j < 0\}$ and $k_+(\lambda) = \{j \in \mathbb{N}_m : \lambda_j \geq 0\}$. Then, denoting with $g^j(\cdot)$ and $h^j(\cdot)$ the convex functions such that $f^j(x) = g^j(x) - h^j(x)$, for $j \in \mathbb{N}_m$, we have

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j f^j(x) = \sum_{j=1}^{m} \lambda_j g^j(x) - \sum_{j=1}^{m} \lambda_j h^j(x) = \\ &= \left(\sum_{j\in k_+} \lambda_j g^j(x) - \sum_{j\in k_-} \lambda_j h^j(x) \right) - \left(\sum_{j\in k_+} \lambda_j h^j(x) - \sum_{j\in k_-} \lambda_j g^j(x) \right), \end{aligned}$$

which is the difference of two convex functions, since the terms in brackets are both convex.

(ii) This proof can be found in (Horst and Thoai, 1999).

(iii) Functions in case (iii) are subcases of (ii). In fact, $|f(x)| = \max\{f(x), -f(x)\}$, then a DC function. Denoting $f^0(x) = 0$, which is trivially a DC function, $\max\{0, f(x)\} = \max\{f^0(x), f(x)\}$ is the pointwise maximum of two DC functions and $\min\{0, f(x)\} = \min\{f^0(x), f(x)\}$ is the pointwise minimum of two DC functions.

(iv) The proof is based on last point of Property 3.15. We prove it for $m = 2$, generalization to any positive m follows. Consider two DC functions $f^1(x)$ and $f^2(x)$ defined on D. For every $a, b \in \mathbb{R}$ we have that $(a+b)^2 - a^2 - b^2 = 2ab$ and then $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. This means that,

$$f^1(x)f^2(x) = \frac{(f^1(x) + f^2(x))^2 - (f^1(x))^2 - (f^2(x))^2}{2}.$$

Then if we prove that $(f(x))^2$ is a DC function for every $f(\cdot)$ DC, we have also that the product of two DC function is DC, being $f^1(x)f^2(x)$ expressible as the sum of DC functions in that case. This is a straightforward application of point (iii) of Property 3.15, posing $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = y^2$, clearly convex on $\mathbb{R}$.

■

Finally, note that there exist infinitely many DC representations for every DC function $f_{DC}(x)$. In fact, given the DC function $f_{DC}(x)$ and convex functions $g_c(x)$ and $h_c(x)$ such that $f_{DC}(x) = g_c(x) - h_c(x)$, also functions $\tilde{g}_c(x) = g_c(x) + k(x)$ and $\tilde{h}_c(x) = h_c(x) + k(x)$, with $k(x)$ convex, are convex functions and $f_{DC}(x) = \tilde{g}_c(x) - \tilde{h}_c(x)$.

Examples of applications of DC functions in the field of system analysis can be found in (Álamo, Bravo, Redondo and Camacho, 2007), where DC functions are employed to determine a set-membership estimation algorithm, and in (Bravo, Álamo, Fiacchini and Camacho, 2007), in which the authors proposes a system identification method based on DC functions properties.

3.4.2 DC systems and DCDI systems

Consider the nonlinear discrete-time autonomous system

$$x^+ = f(x), \tag{3.28}$$

where $x \in D \subseteq \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state and function $f : D \to \mathbb{R}^n$ is nonlinear. The system is said to be a DC system if the dynamic function $f(\cdot)$ is a DC function, that is, if the following assumption holds.

Assumption 3.17 *Assume that $f : D \to \mathbb{R}^n$ in (3.28) is a DC function defined on $D \subseteq \mathbb{R}^n$ convex with $0 \in int(D)$ and differentiable at the origin. Denote $g(\cdot)$ and $h(\cdot)$ the convex functions such that $f(x) = g(x) - h(x)$, for all $x \in D$ and assume that $g(0) = 0$ and $h(0) = 0$.*

Hence, considering the autonomous discrete-time system (3.28), the DC system has the following form

$$x^+ = f(x) = g(x) - h(x) = \begin{bmatrix} g_1(x) - h_1(x) \\ \cdots \\ g_n(x) - h_n(x) \end{bmatrix}, \tag{3.29}$$

where $g_i(\cdot)$ and $h_i(\cdot)$ are convex functions in $\mathbb{R}^n$, for all $i \in \mathbb{N}_n$.

DC systems are a very wide class of nonlinear systems, since, as shown in Section 3.4.1, many nonlinear functions admit a DC representation.

Similarly to the case of Example 2.8, every DC system admits an overbounding CDI system. Remarkably useful is the fact that the CDI system overbounding the DC one is implicitly defined by the linearization of functions $h_j(\cdot)$ and $g_j(\cdot)$, for $j \in \mathbb{N}_n$ at the origin, as illustrated below. This implies that the overbounding CDI system has not to be explicitly calculated and that it can be employed in simple computational procedures. The following property provides the convex bounding functions determining a CDI system overbounding a DC system.

Property 3.18 *Given the DC function $f : D \to \mathbb{R}^n$ as in (3.28) such that Assumption 3.17 holds and a $\eta \in \mathbb{R}^n$, consider the set valued map $\mathscr{F}(\cdot)$ defined by the following convex* *bounding functions*

$$\check{f}_\eta(x) = \sum_{j\in k_+} \eta_j \left(g_j(x) - h_j^L(x)\right) + \sum_{j\in k_-} \eta_j \left(g_j^L(x) - h_j(x)\right), \tag{3.30}$$

where $g_j^L(x) = \nabla_x^T g_j(0)x$ *and* $h_j^L(x) = \nabla_x^T h_j(0)x$, *for* $j \in \mathbb{N}_n$ *and* $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n : \eta_j \geq 0\}$ *and* $k_- = k_-(\eta) = \{j \in \mathbb{N}_n : \eta_j < 0\}$.

Then, Assumption 2.5 holds for $\mathscr{F}(\cdot)$ *and* $f \in S_{\mathscr{F}}$.

Proof: Proving that $\mathscr{F}(\cdot)$ satisfies Assumption 2.5 is analogous to the proof of Property 3.3. We have to prove that the induced CDI system overbounds the DC one.

By definition, a CDI system overbounds the DC system, that is $f \in S_{\mathscr{F}}$, see Remark 2.7, if

$$\eta^T f(x) \leq \check{f}_\eta(x), \qquad \forall x \in D, \quad \forall \eta \in \mathbb{R}^n, \tag{3.31}$$

where $\mathscr{F}(\cdot)$ is defined by (3.30).

Since $g_j^L(\cdot)$ and $h_j^L(\cdot)$ are, by definition, the linearizations at the origin of the convex functions $g_j(\cdot)$ and $h_j(\cdot)$ respectively, for $j \in \mathbb{N}_n$, it follows

$$\begin{aligned} g_j^L(x) &\leq g_j(x), \quad \forall j \in \mathbb{N}_n,\ \forall x \in D, \\ h_j^L(x) &\leq h_j(x), \quad \forall j \in \mathbb{N}_n,\ \forall x \in D, \end{aligned} \tag{3.32}$$

from convexity of functions $g_j(\cdot)$ and $h_j(\cdot)$ on D. Thus,

$$\begin{aligned} \eta_j \left(h_j^L(x) - h_j(x)\right) &\leq 0 \quad \forall j \in k_+,\ \forall x \in D, \\ \eta_j \left(g_j(x) - g_j^L(x)\right) &\leq 0 \quad \forall j \in k_-,\ \forall x \in D. \end{aligned} \tag{3.33}$$

for every $\eta \in \mathbb{R}^n$.

Hence, from this and (3.30),

$$\begin{aligned} \eta^T f(x) - \check{f}_\eta(x) &= \sum_{j=1}^{n} \eta_j \left(g_j(x) - h_j(x)\right) - \sum_{j\in k_+} \eta_j (g_j(x) - h_j^L(x)) - \\ &- \sum_{j\in k_-} \eta_j (g_j^L(x) - h_j(x)) = \sum_{j\in k_+} \eta_j (h_j^L(x) - h_j(x)) + \sum_{j\in k_-} \eta_j (g_j(x) - g_j^L(x)) \leq 0. \end{aligned}$$

for all $\eta \in \mathbb{R}^n$ and for any $x \in D$. ■

Analogous considerations can be given for non-autonomous systems. Consider the nonlinear discrete-time time-invariant dynamic system

$$x^+ = f(x,u), \tag{3.34}$$

where $x \in D \subseteq \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state, $u \in E \subseteq \mathbb{R}^m$ is the control action and function $f : D \times E \to \mathbb{R}^n$ is nonlinear. If the dynamic function is DC then the system is a non-autonomous DC system.

Assumption 3.19 *Assume that $f : D \times E \to \mathbb{R}^n$ in (3.34) is a DC function defined on $D \times E \subseteq \mathbb{R}^{n+m}$, with $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}^m$ convex with $(0,0) \in int\,(D \times E)$, and differentiable at the origin. Denote $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ the convex functions such that $f(x,u) = g(x,u) - h(x,u)$, for all $(x,u) \in D \times E$ and assume that $g(0,0) = 0$ and $h(0,0) = 0$.*

Hence, a non-autonomous system is DC if there exist two functions $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ convex on $D \times E$, such that $f(x,u) = g(x,u) - h(x,u)$, for all $x \in D$ and $u \in E$, and the system has the following form

$$x^+ = f(x,u) = g(x,u) - h(x,u) = \begin{bmatrix} g_1(x,u) - h_1(x,u) \\ \cdots \\ g_n(x,u) - h_n(x,u) \end{bmatrix}. \tag{3.35}$$

Note that in both cases of autonomous and non-autonomous DC systems, we assumed that the nonlinear function is differentiable at the origin. This allows to obtain lower bounding functions of convex terms of the DC functions, through a proper linearization at the origin. This assumption can be removed. In fact, since the origin belongs to the interior of the domain of the DC function, the subdifferential of any convex term of the DC function at the origin is not empty and then a linear lower bounding function can be obtained also for any function which is not differentiable at origin.

Also a CDI system overbounding the DC system, for non-autonomous case, can be easily obtained, as illustrated below. No proof have been included since analogous to that of Property 3.18.

Property 3.20 *Given the DC function $f : D \times E \to \mathbb{R}^n$ as in (3.34) such that Assumption 3.19 holds and a $\eta \in \mathbb{R}^n$, consider the set valued map $\mathscr{F}(\cdot,\cdot)$ defined by the following convex bounding functions*

$$\check{f}_\eta(x,u) = \sum_{j \in k_+} \eta_j \left(g_j(x,u) - h_j^L(x,u) \right) + \sum_{j \in k_-} \eta_j \left(g_j^L(x,u) - h_j(x,u) \right), \tag{3.36}$$

where $g_j^L(x,u) = \nabla_x g_j(0,0)x + \nabla_u g_j(0,0)u$ and $h_j^L(x,u) = \nabla_x h_j(0,0)x + \nabla_u h_j(0,0)u$, for $j \in \mathbb{N}_n$ and $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n :\ \eta_j \geq 0\}$ and $k_- = k_-(\eta) = \{j \in \mathbb{N}_n :\ \eta_j < 0\}$.

Then, Assumption 2.14 holds for $\mathscr{F}(\cdot,\cdot)$ and $f \in S_{\mathscr{F}}$.

3.4.3 Difference-of-Convex Difference Inclusion: DCDI systems

We define here also a class of systems whose dynamics are given by set valued maps rather than real valued functions, analogously to the case of CDI system. This framework is useful for those cases in which the uncertainty is parametric, that is, when the system's dynamic function depends on a parameter.

Consider the discrete-time autonomous nonlinear system given by

$$x^+ \in \text{co}\ (f^j(x) : \ j \in \mathbb{N}_{n_j}), \tag{3.37}$$

where $x \in D$ is the state, x^+ is the successor state and functions $f^j(\cdot)$, for $j \in \mathbb{N}_{n_j}$ with $n_j \geq 1$, are DC functions. System (3.37) is referred to with the term Difference-of-Convex Difference Inclusion (DCDI) system if every function $f^j(\cdot)$ is a DC function defined over a common domain $D \subseteq \mathbb{R}^n$, for $j \in \mathbb{N}_{n_j}$, as formally stated in the following.

Assumption 3.21 *Assume that for every function $f^j : D \to \mathbb{R}^n$ determining the system (3.37), Assumption 3.17 holds, with $D \subseteq \mathbb{R}^n$ common convex domain for every $f^j(\cdot)$, for all $j \in \mathbb{N}_{n_j}$. Denote with $g^j(\cdot)$ and $h^j(\cdot)$ the convex functions such that $f^j(x) = g^j(x) - h^j(x)$, for all $x \in D$.*

In case that the dynamic function of the system is given by a single DC function, i.e. case in which $n_j = 1$, the DCDI system is a DC system. The family of DC systems is a subclass of DCDI systems.

The definition can be extended to controlled systems, that is, considering the following discrete-time non-autonomous system

$$x^+ \in \text{co}\ (f^j(x,u) : \ j \in \mathbb{N}_{n_j}), \tag{3.38}$$

where $x \in D$ is the state, $u \in \mathbb{R}^m$ is the control input and functions $f^j(\cdot,\cdot)$ for $j \in \mathbb{N}_{n_j}$ with $n_j \geq 1$, are nonlinear functions, which satisfy the following assumption.

Assumption 3.22 *Assume that for every nonlinear function $f^j : D \times E \to \mathbb{R}^n$ determining the system (3.38), Assumption 3.19 holds, with $D \times E \subseteq \mathbb{R}^{n+m}$ common convex domain of $f^j(\cdot,\cdot)$, for every $j \in \mathbb{N}_{n_j}$. Denote with $g^j(\cdot,\cdot)$ and $h^j(\cdot,\cdot)$ the convex functions such that $f^j(x,u) = g^j(x,u) - h^j(x,u)$, for all $(x,u) \in D \times E$.*

As for the case of autonomous DCDI systems, if we have $n_j = 1$, the non-autonomous DC system is recovered.

3.4.3.1 Uncertain DCDI systems

A first modelling framework considering the uncertainty is a DC system with additive uncertainty. That is, the uncertain autonomous DC system is given by

$$x^+ = f(x) + w, \tag{3.39}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$ is the bounded additive uncertainty $w \in W$ and $f : D \to \mathbb{R}^n$ fulfills the Assumption 3.17. For the non-autonomous case we have that the DC system has the following dynamics

$$x^+ = f(x,u) + w, \tag{3.40}$$

where now $f : D \times E \to \mathbb{R}^n$ fulfills the Assumption 3.19.

A more general framework is given by the uncertain discrete-time autonomous DC system given by

$$x^+ \in \text{co}\ (f^j(x,w) : \ j \in \mathbb{N}_{n_j}), \tag{3.41}$$

where $x \in D$ is the state and $w \in W$ is the unknown but bounded uncertainty and the following assumption holds for functions $f^j(\cdot,\cdot)$.

Assumption 3.23 *Assume that functions $f^j : D \times W \to \mathbb{R}^n$ in (3.41) are such that:*

- *$f^j(\cdot,w)$ satisfies Assumption 3.17, for every $w \in W$,*
- *$f^j(x,\cdot)$ is affine in w, for every $x \in D$,*

for every $j \in \mathbb{N}_{n_j}$ with $n_j \geq 1$.

Similarly, uncertain non-autonomous DCDI systems are defined as

$$x^+ \in \text{co}\ (f^j(x,u,w) : \ j \in \mathbb{N}_{n_j}), \tag{3.42}$$

where $x \in \mathbb{R}^n$ is the state, $u \in E$ is the control input and $w \in W$ is the unknown but bounded uncertainty and functions $f^j(\cdot,\cdot,\cdot)$ satisfy the following assumption.

Assumption 3.24 *Assume that functions $f^j : D \times E \times W \to \mathbb{R}^n$ in (3.42) are such that:*

- *$f^j(\cdot,\cdot,w)$ satisfies Assumption 3.19, for every $w \in W$,*
- *$f^j(x,u,\cdot)$ is affine in w, for every $(x,u) \in D \times E$,*

for every $j \in \mathbb{N}_{n_j}$ *with* $n_j \geq 1$.

The case of DCDI systems where the uncertainty appears as an additive term, is included in the class of systems (3.41) or (3.42).

3.5 Linear parametric uncertain systems

Another framework which permits to analyze discrete-time nonlinear systems as well as uncertain systems is given by linear systems with parametric uncertainty, mentioned in Section 2.1.1.2. This is the case in which the system is assumed linear with state-transition matrix $A \in \mathbb{R}^n$ depending on a parameter. The parameter can evolve in time, for instance in case that the dynamics depends on external signals, or can be unknown but bounded. Different scenarios raise, depending on the assumptions on the parameter nature and on the degree of knowledge of such parameter.

It will be shown that this framework entails a subclass of CDI systems, in particular those CDI systems whose convex bounding functions are linear. Hence, computing a linear parametric uncertain system approximating a nonlinear system is one of the way of generating an overbounding CDI system.

In the general case, we consider the dynamic system given by

$$x(k+1) = A(k)x(k), \tag{3.43}$$

where $x(k) \in \mathbb{R}^n$ is state and $A(k) \in \mathbb{R}^{n\times n}$ is an element of a set of the space $\mathbb{R}^{n\times n}$, for every $k \in \mathbb{N}$. Denoting such set $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$, the set valued map (2.5) determines bounds on the system evolution, that is

$$x(k+1) = A(k)x(k) \in \{Ax(k) : \ A \in \mathscr{A}\}. \tag{3.44}$$

Analogously, in case of non-autonomous linear parametric uncertain systems, we have that dynamics is given by

$$x(k+1) = A(k)x(k) + B(k)u(k) \in \{Ax(k) + Bu(k) : \ [A,\ B] \in \mathscr{M}\}, \tag{3.45}$$

where $x(k) \in \mathbb{R}^n$ is the current state, $u(k) \in \mathbb{R}^m$ is the control input and matrix $[A(k),\ B(k)] \in \mathscr{M} \subseteq \mathbb{R}^{n\times(n+m)}$.

In both cases, the uncertainties affecting the systems are parametric. Also additive uncertainty can be considered and the systems take the form

$$x(k+1) = A(k)x(k) + w(k) \in \{Ax(k) + w : \ A \in \mathscr{A},\ w \in W\}, \tag{3.46}$$

for autonomous systems, and

$$x(k+1) = A(k)x(k) + B(k)u(k) + w(k) \in \{Ax(k) + Bu(k) + w : \ [A,\ B] \in \mathscr{M},\ w \in W\}, \tag{3.47}$$

for the non-autonomous case, where $w \in W$ and $\mathscr{M} \subseteq \mathbb{R}^{n\times(n+m)}$.

Different assumptions on the knowledge of the dynamic matrices, yield to different frameworks. The following two scenarios are of interest:

- $A(k)$ (and possibly $B(k)$) is unknown for any $k \in \mathbb{N}$, only the bounding set $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ (or $\mathscr{M} \subseteq \mathbb{R}^{n\times(n+m)}$) is known. In the analysis and synthesis it must be taken into account any element of $\mathscr{A}$ (or $\mathscr{M}$): Linear Difference Inclusions (LDI) systems.
- matrix $A(k)$ (and $B(k)$, eventually) is known at any time instant $k \in \mathbb{N}$. Matrix can be assumed dependent on a parameter γ, that can be function of the state x, input u and/or an external signal r. The system is called, in that case, a Linear Parameter Varying (LPV) system.

Particularly relevant are the linear parametric uncertain systems for which the set $\mathscr{A}$ is polytopic subset of the space $\mathbb{R}^{n\times n}$. A linear parametric uncertain system is said to be polytopic if the following assumption holds for the set valued map determining its dynamics.

Assumption 3.25 *Assume that, for a given $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$, there exists a set of n_a elements $A^j \in \mathbb{R}^{n\times n}$ such that*

$$\mathscr{A} = co\ (A^j \in \mathbb{R}^{n\times n} : \ j \in \mathbb{N}_{n_a}). \tag{3.48}$$

Matrices A^j, for $j \in \mathbb{N}_{n_a}$ are the vertices of polytope $\mathscr{A}$.

Remark 3.26 *Given $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ for a linear parametric uncertain system, the set valued map determining the dynamics is*

$$\mathscr{A}(x) = \{Ax \in \mathbb{R}^n : \ A \in \mathscr{A}\}, \tag{3.49}$$

then, in case of polytopic linear uncertain system we have that

$$\mathscr{A}(x) = co\ (A^j x \in \mathbb{R}^n : \ \ j \in \mathbb{N}_{n_a}). \tag{3.50}$$

Notice that the set valued map $\mathscr{A}(\cdot)$ satisfies the Assumption 2.5 being $\mathscr{A}(x) \in \mathscr{K}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$, and the convex bounding functions linear, hence convex. This means that, any *polytopic linear parametric uncertain system is a CDI system. In fact, for every $\eta \in \mathbb{R}^n$, the function of x given by $\eta^T Ax$ is a linear function of $x \in \mathbb{R}^n$, hence convex in x, for any $A \in \mathscr{A}$. Therefore, function*

$$\check{f}_\eta(x) = \max_{A \in \mathscr{A}} \eta^T Ax \tag{3.51}$$

is the pointwise maximum of a family of convex functions, then convex, see (Boyd and Vandenberghe, 2004), and defines the CDI representation of the system whose set valued map is $\mathscr{A}(\cdot)$.

Analogous considerations are valid for linear parametric uncertain systems that are non-autonomous and/or affected by additive uncertainty, also leading to CDI systems since the related set valued maps satisfy Assumptions 2.14.

In many cases, if we are able to ensure a property for every linear dynamic system given by A^j, with $j \in \mathbb{N}_{n_a}$, the property holds also for the LDI system, no matter the real realization of $A(k)$.

3.5.1 Linear difference inclusions: LDI systems

A classical way of representing the effect of lack of knowledge on the systems, as parametric uncertainty approach, is through Linear Difference Inclusion, see Section 2.1.1.

LDI systems are of the form (3.43), for which the state-transition matrix $A(k) \in \mathbb{R}^{n \times n}$ is assumed to be an unknown element of $\mathscr{A} \subseteq \mathbb{R}^{n \times n}$. When $\mathscr{A}$ is a polytope in the space $\mathbb{R}^{n \times n}$, the system is called polytopic LDI.

Such framework finds its justification in the fact that nonlinear functions can be approximated on a bounded set of the space by a linear function, exploiting the Taylor expansion and, in particular the Lagrange form of Remainders, which is based on the Mean value theorem.

Conceptually, these theorems state that any real valued function, $f : \mathbb{R} \to \mathbb{R}$, differentiable at $x \in [a, b]$, can be written as a constant term plus a linear one as in the following:

$$f(x) = f(x_0) + \frac{df(c)}{dx}(x - x_0) \tag{3.52}$$

for a proper $c \in [x_0, x]$, for every $x_0 \in [a, b]$.

This means that, under the assumptions of differentiability of $f(\cdot)$ on a segment (which can be often relaxed to simple convexity) the value of a generic nonlinear function $f(\cdot)$ at $x \in [a, b]$ can be expressed as a linear function whose parameters are given by the value of $f(\cdot)$ at $x_0 \in [a, b]$ and the derivative of $f(\cdot)$ at a point $c \in [a, b]$.

The practical problem is that, in general, once fixed x_0, the point c depends on the evaluation point x, i.e. $c = c(x)$, and such relation is unknown. On the other hand equation (3.52) permits to determine an overbounding function of $f(\cdot)$, that is, a set valued function such that its image of any point $x \in [a, b]$ contains the value $f(x)$.

In fact, given a nonlinear function $f : [a, b] \to \mathbb{R}$ and a point $x_0 \in [a, b]$ we have, from (3.52), that

$$f(x_0) + \min_{c\in[a,b]} \frac{df(c)}{dx}(x-x_0) \le f(x) \le f(x_0) + \max_{c\in[a,b]} \frac{df(c)}{dx}(x-x_0), \tag{3.53}$$

for all $x \in [a, b]$ if $x \ge x_0$ and

$$f(x_0) + \max_{c\in[a,b]} \frac{df(c)}{dx}(x-x_0) \le f(x) \le f(x_0) + \min_{c\in[a,b]} \frac{df(c)}{dx}(x-x_0), \tag{3.54}$$

for all $x \in [a, b]$ if $x < x_0$. Then, denoting

$$d_m = \min_{c\in[a,b]} \frac{df(c)}{dx}, \qquad d_M = \max_{c\in[a,b]} \frac{df(c)}{dx},$$

the set valued function

$$\mathscr{F}(x) = \{(f(x_0) - dx_0) + dx : \ d \in [d_m, d_M]\}, \tag{3.55}$$

is such that $f(x) \in \mathscr{F}(x)$, for all $x \in [a, b]$. Note that, the set valued function $\mathscr{F}(\cdot)$ defined as in (3.55), is given by a set of linear functions whose slopes are those lying in the segment $[d_m, d_M]$.

Methods based on interval arithmetic allow to compute guaranteed bounds d_m and d_M, and then the LDI system overbounding the nonlinear one, see (Bravo et al., 2005).

For instance, for nonlinear system (3.28) with $f : D \to \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ convex and compact, an overbounding LDI system can be determined knowing the bounds on the gradient of every component $f_i(\cdot)$ of the dynamic function, i.e., $\nabla f_i(\cdot)$, at any point in $x \in D$. Assuming that the origin is an equilibrium for the nonlinear dynamic system (3.28), i.e., $f(0) = 0$, and $f(\cdot)$ is differentiable on the compact, convex set $D \subset \mathbb{R}^n$, with $0 \in D$, the LDI system (3.44) with matrix $A(k) \in \mathscr{A}$, where $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ is defined as

$$\mathscr{A} = \left\{ A \in \mathbb{R}^{n\times n} : \ \min_{x\in D} \frac{\partial f_i(x)}{\partial x_j} \le A_{i,j} \le \max_{x\in D} \frac{\partial f_i(x)}{\partial x_j}, \ \forall i, j \in \mathbb{N}_n \right\}, \tag{3.56}$$

is such that $f(x) \in \mathscr{A}(x)$, with $\mathscr{A}(x)$ defined in (3.49). This means that the LDI system with $A(k) \in \mathscr{A}$ is an overbounding system of (3.28).

Remark 3.27 *The set $\mathscr{A}$ is a polytope on $\mathbb{R}^{n\times n}$ if D is compact on $\mathbb{R}^n$. Hence, it can be determined by a finite set of vertices, that is, by a finite set of matrices in $\mathbb{R}^{n\times n}$. Such matrices* *are given by all the possible combinations of maximal and minimal values of the elements of* $\mathscr{A}$*, any other element can be expressed as convex combination of such vertices. The vertices of* $\mathscr{A}$ *are, at most,* 2^{n^2}.

Also in this case the CDI representation is given by convex bounding functions (3.51). From the practical point of view, computing such maximum can be reduced to checking the values for the vertices of $\mathscr{A}$, whose number can be very high.

Nevertheless, in our case, knowing η and x, it is straightforward to compute the maximum in (3.51). In fact, for any given $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have that

$$\check{f}_\eta(x) = \max_{A \in \mathscr{A}} \eta^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} \eta_i \tilde{A}_{i,j} x_j,$$

where

$$\tilde{A}_{i,j} = \begin{cases} \max\limits_{x \in D} \frac{\partial f_i(x)}{\partial x_j}, & \text{if } \eta_i x_j \geq 0, \\ \min\limits_{x \in D} \frac{\partial f_i(x)}{\partial x_j}, & \text{if } \eta_i x_j < 0, \end{cases}$$

for every $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_n$.

3.5.2 Linear parameter varying systems: LPV systems

A discrete-time autonomous LPV system has the following form

$$x(k+1) = A(k)x(k) = A_{\gamma(k)}x(k), \tag{3.57}$$

where $x(k) \in \mathbb{R}^n$ is the current state, for $k \in \mathbb{N}$, the time-varying state-transition matrix $A(k) = A_{\gamma(k)} \in \mathbb{R}^{n \times n}$ depends on the parameter $\gamma = \gamma(k) \in \Gamma \subseteq \mathbb{R}^p$, hence $A_\gamma : \mathbb{R}^p \to \mathbb{R}^{n \times n}$.

Assumption 3.28 *Assume that matrix* $A_\gamma \in \mathbb{R}^{n \times n}$ *depends affinely on the parameter* $\gamma \in \Gamma$*, that is, there exist p martices* $\hat{A}^j \in \mathbb{R}^{n \times n}$*, for* $j \in \mathbb{N}_p$*, and* $\hat{A}^0 \in \mathbb{R}^{n \times n}$ *such that*

$$A_\gamma = \hat{A}^0 + \sum_{j=1}^{p} \hat{A}^j \gamma_j,$$

and $\Gamma \subseteq \mathbb{R}^p$ *is a polytope.*

Remark 3.29 *Under Assumption 3.28 it can be proved that the set*

$$\mathscr{A}(\Gamma) = \{A_\gamma \in \mathbb{R}^{n \times n}, \forall \gamma \in \Gamma\}, \tag{3.58}$$

is polytopic.

It can be proved that, under Assumption 3.28, any vertex of $\mathscr{A}(\Gamma)$ is determined by an extremal realization of Γ. Roughly speaking, we have that A_γ is a vertex of $\mathscr{A}(\Gamma)$ if and only if there exists $\bar{\gamma}$ vertex of Γ such that $A_{\bar{\gamma}} = A_\gamma$,

Analogous definition is given for discrete-time non-autonomous LPV system

$$x^+ = A(k)x(k) + B(k)u(k) = A_{\gamma(k)}x(k) + B_{\gamma(k)}u(k), \tag{3.59}$$

where $x(k) \in \mathbb{R}^n$ is the current state, $u(k) \in \mathbb{R}^m$ is the input, for $k \in \mathbb{N}$, and both the time-varying dynamic matrices $A(k) = A_{\gamma(k)} \in \mathbb{R}^{n\times n}$ and $B(k) = B_{\gamma(k)} \in \mathbb{R}^{n\times m}$ depend on the parameter $\gamma = \gamma(k) \in \Gamma \subseteq \mathbb{R}^p$, then $A_\gamma : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ and $B_\gamma : \mathbb{R}^p \to \mathbb{R}^{n\times m}$. Additive uncertainty can be also considered for LPV.

3.6 Conclusions

In this chapter, modelling frameworks related to CDI systems have been presented. Since many important results will be provided in what follows for CDI systems, it is necessary to point out that such analytical scenario is quite general and strongly related to many common nonlinear and uncertain systems. Methods to obtain a CDI representation of a system and to compute a CDI approximation have been exposed.

First, presenting CCDI systems, it has been shown that, given a nonlinear system, it can be sufficient to determine a finite number of bounding functions to obtain a CDI system and that the CCDI framework encloses common nonlinear and uncertain systems.

Then, classical nonlinear systems, such as Lur'e ones, have been presented in discrete-time. It has been provided a direct method to recover the CDI representation for particular Lur'e systems. This makes Lur'e systems a particular subset of CDI systems.

Another class of nonlinear systems related to CDI ones are the generalized saturated systems. They are systems whose dynamic function is given by a linear system in closed-loop with a generalized saturated function. It has been shown that an overbounding CDI system can be easily obtained.

Then, DC systems have been introduced. Their dynamic functions are related to DC functions, i.e., functions expressible as the difference of convex functions. Also in this case, convexity is the central ingredient that relates DC systems with the overbounding CDI systems.

Finally, linear systems affected by parametric uncertainty have been illustrated. LDI and LPV systems, particularization of linear parametric uncertain systems, have been presented.

This framework, well known in the field of systems theory and control design, is considered here being another particular subclass of CDI systems.

Chapter 4

Set-theory and invariance for CDI systems

In what follows we present the results related to set-theory, especially focused on invariance, for CDI systems. First we provide a necessary and sufficient condition for a convex set to be an invariant set or a λ-contractive set for generic CDI systems, together with the characterization of other aspects involved in invariance. It is worth recalling that an invariant set for a CDI system turns out to be an invariant set also for any system overbounded by the CDI one. This justifies the interest devoted to the analysis of CDI systems, providing a deep insight on a wide class of systems. Another important concept strongly related to invariance, as the one-step operator for CDI systems, is considered in this chapter.

Computational issues on how to obtain an invariant and a λ-contractive set for a CDI system are dealt with in the last section of the chapter. In the case of polytopic potential invariant sets, the necessary and sufficient condition for invariance reduces to checking the satisfaction of a finite number of convex constraints at the vertices of the polytope. Then, the computational burden required is affordable and the condition can be used to design an efficient algorithmic procedure.

The results presented in this chapter are based on the characteristics of particular functions related to any CDI system and denoted as $\check{F}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$. We will refer to $\check{F}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$ as directional upper and lower bounding functions, respectively. The results presented, such as necessary and sufficient conditions for invariance and λ-contractiveness of convex sets for CDI systems, as well as other useful properties, will be posed in terms of such functions.

4.1 Convex invariant sets for CDI systems

We consider here a generic discrete-time autonomous CDI system, that is a system the dynamics of which is given by (2.7), i.e.,

$$x^+ \in \mathscr{F}(x), \tag{4.1}$$

under Assumption 2.5. Recall that this means that the dynamic function is a set valued map $\mathscr{F}(\cdot)$ defined by particular convex bounding functions $\check{f}_\eta(\cdot)$, one for every $\eta \in \mathbb{R}^n$.

Remark 4.1 *The results presented in what follows are valid for CDI and CCDI systems. We will refer to Assumption 2.5 defining CDI systems. For what concerns CCDI systems, we recall that the convex bounding functions for the CCDI framework are derived in Property 3.3.*

It could be also useful to remind that we refer to $\check{f}_\eta(\cdot)$ as convex *bounding functions, for all $\eta \in \mathbb{R}^n$, while $\check{F}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$ are called* directional *bounding functions, upper and lower respectively.*

Remark 4.2 *The definitions of directional bounding functions $\check{F}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$ have been introduced to make explicit their dependence on $\eta \in \mathbb{R}^n$ as well as their condition of convex upper bound and concave lower bound, respectively, although all the results presented could have been posed in terms of $\check{f}_\eta(\cdot)$.*

First we define the directional upper bounding functions for CDI systems. The fact that it is an upper bounding function of $\eta^T z$ for all $z \in \mathscr{F}(x)$ and for all $x \in \mathbb{R}^n$, is a direct consequence of Assumption 2.5 and Assumption 3.1.

Definition 4.3 *Let Assumption 2.5 hold for a given map $\mathscr{F}(\cdot)$. Define the directional upper bounding function $\check{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as*

$$\check{F}(x,\eta) = \check{f}_\eta(x), \tag{4.2}$$

where functions $\check{f}_\eta(\cdot)$, for every $\eta \in \mathbb{R}^n$, determine $\mathscr{F}(\cdot)$.

It is straightforward to prove that $\check{F}(\cdot,\eta)$ provides an upper bounding function of $\eta^T f(\cdot)$, for any $\eta \in \mathbb{R}^n$, and any function $f \in S_{\mathscr{F}}$, where $S_{\mathscr{F}} = \{f : \ f(x) \in \mathscr{F}(x), \ \forall x \in \mathbb{R}^n\}$.

Directional bounding functions $\check{F}(\cdot,\cdot)$ for CDI systems, defined in Definition 4.3, are convex with respect to x and provide upper bounds on $\eta^T z$, for every $z \in \mathscr{F}(x)$ and every $x \in \mathbb{R}^n$. The property follows directly from the characteristics of the convex bounding functions $\check{f}_\eta(\cdot)$.

Property 4.4 *Let Assumption 2.5 hold for a given map $\mathscr{F}(\cdot)$. Function $\check{F}(\cdot,\cdot)$ as in Definition 4.3, is a convex function in $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^n$, such that*

$$\max_{z\in\mathscr{F}(x)} \eta^T z = \check{F}(x,\eta), \tag{4.3}$$

for every $x \in \mathbb{R}^n$ and $\check{F}(0,\eta) = 0$, for all $\eta \in \mathbb{R}^n$.

Proof: If Assumption 2.5 holds and $\check{F}(\cdot,\cdot)$ is defined in Definition 4.3, convexity and satisfaction of (4.3) are implied directly by definition. In fact Assumption 2.5 ensures that $\check{f}_\eta(\cdot)$ is convex and satisfies (4.3), where the supremum can be replaced with maximum since $\mathscr{F}(x)$ is supposed compact, i.e.,

$$\check{F}(x,\eta) = \check{f}_\eta(x) = \sup_{z\in\mathscr{F}(x)} \eta^T z = \max_{z\in\mathscr{F}(x)} \eta^T z.$$

From (4.3) and the fact that $\check{f}_\eta(0) = 0$ for all $\eta \in \mathbb{R}^n$, we have $\check{F}(0,\eta) = 0$. ■

Similarly, the directional lower bounding functions can be defined such that, given a vector $\eta \in \mathbb{R}^n$, a lower bound on $\eta^T z$ with $z \in \mathscr{F}(x)$ is easily obtained.

Definition 4.5 *Let Assumption 2.5 hold for a given map $\mathscr{F}(\cdot)$. Define the directional lower bounding function $\hat{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as*

$$\hat{F}(x,\eta) = -\check{f}_{-\eta}(x). \tag{4.4}$$

Properties analogous to those of directional upper bounding functions $\check{F}(\cdot,\cdot)$ can be given for directional lower bounding functions $\hat{F}(\cdot,\cdot)$, simply applying a sort of duality process, where convexity is replaced by concavity, upper bounding by lower bounding, maximum by minimum, and so on.

We provide the dual of Property 4.4, whose proof is avoided here, since it follows the same lines as that of Property 4.4, recalling that

$$\min_{z\in\mathscr{F}(x)} \eta^T z = -\max_{z\in\mathscr{F}(x)} -\eta^T z = -\check{f}_{-\eta}(x).$$

Property 4.6 *Let Assumption 2.5 hold for a given map $\mathscr{F}(\cdot)$. Function $\hat{F}(\cdot,\cdot)$ as in Definition 4.5, is a concave function in $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^n$, and such that*

$$\min_{z\in\mathscr{F}(x)} \eta^T z = \hat{F}(x,\eta), \tag{4.5}$$

for every $x \in \mathbb{R}^n$ and $\hat{F}(0,\eta) = 0$ for all $\eta \in \mathbb{R}^n$.

From the Properties 4.4 and 4.6, the following corollary, whose proof is avoided because straightforward, stems directly. The corollary ensures that the directional lower and upper bounding functions $\check{F}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$ provide guaranteed bounds on the image of any function $f:\mathbb{R}^n \to \mathbb{R}^n$ overbounded by the set valued $\mathscr{F}(\cdot)$. This permits to use directional upper and lower bounding functions of the CDI system as bounds for generic nonlinear, possibly noncontinuous, dynamic systems.

Corollary 4.7 *Let Assumption 2.5 hold for a given map $\mathscr{F}(\cdot)$. Function $\check{F}(\cdot,\cdot)$ as in Definition 4.3, and function $\hat{F}(\cdot,\cdot)$ as in Definition 4.5 are such that*

$$\hat{F}(x,\eta) \leq \eta^T f(x) \leq \check{F}(x,\eta), \tag{4.6}$$

for every $x \in X$ and $\eta \in \mathbb{R}^n$ and every $f \in S_{\mathscr{F}}$.

Continuity of directional bounding functions $\check{F}(\cdot,\eta)$ and $\hat{F}(\cdot,\eta)$, for every $\eta \in \mathbb{R}^n$, on open subsets of their effective domains stems directly from continuity of convex functions on (relatively) open subsets of the domain, see Theorem B.10.

Remark 4.8 *Note that, given the vector $\eta \in \mathbb{R}^n$, the directional upper and lower overbounding functions $\check{F}(\cdot,\eta)$ and $\hat{F}(\cdot,\eta)$ are continuous on the relative interior of their effective domain, $ri(dom\check{F}(\cdot,\eta))$ and $ri(dom\hat{F}(\cdot,\eta))$, since, from Assumption 2.5, any element $\check{f}_\eta(\cdot)$ for CDI systems is convex on $\mathbb{R}^n$. It is worth recalling that the effective domain of convex function is given by those points of the space at which the function takes values different from $+\infty$.*

4.1.1 Necessary and sufficient condition for invariance for CDI systems

As illustrated in the previous section, directional lower and upper bounding functions, $\hat{F}(\cdot,\cdot)$ and $\check{F}(\cdot,\cdot)$, share the properties of convexity and concavity, respectively, continuity, etc. All the assumptions concerning a CDI system are summarized in the following assumption, to which we will refer when dealing with one of those dynamic systems, for sake of simplicity.

Assumption 4.9 *Let Assumption 2.5 hold for the set valued map $\mathscr{F}:\mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics (4.1), and be $\check{F}:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\hat{F}:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as in Definition 4.3 and Definition 4.5, respectively.*

We consider at first a generic convex, compact set $\Omega \subseteq X$ with $0 \in \text{int}(\Omega)$, recalling that $x \in \Omega$ if and only if

$$\eta^T x \leq \phi_\Omega(\eta), \tag{4.7}$$

for all $\eta \in \mathbb{R}^n$, see Appendix C.

Remark 4.10 *Since invariance and set-theoretic methods find one of their main justifications for control in their capability to deal with hard constraints satisfaction, we introduce in what follows state constraints* $x \in X \subseteq \mathbb{R}^n$*, and input constraints* $u \in U \subseteq \mathbb{R}^m$ *for non-autonomous systems. Clearly the case of unconstrained system is enclosed, simply given by* $X = \mathbb{R}^n$ *and* $U = \mathbb{R}^m$*.*

Constraints on the state, represented by the subset of the state space $X \subseteq \mathbb{R}^n$, will be considered in many cases. The following assumption on the set $X \subseteq \mathbb{R}^n$ will be referred to when constraints on the state are taken into account.

Assumption 4.11 *Assume that the constraint set on the state* $X \subseteq \mathbb{R}^n$*, is closed, convex and with* $0 \in int(X)$*.*

In (Kolmanovsky and Gilbert, 1998) a characterization of invariance for linear systems in terms of support functions is given, as well as some properties of the support functions.

The necessary and sufficient condition for invariance for an autonomous linear uncertain system, presented in (Kolmanovsky and Gilbert, 1998), is adapted below to formulate the condition for λ-contractiveness and invariance of a set for system (4.1), when Assumption 4.9 holds, that is, when the system is a CDI system.

First, we present a property characterizing λ-contractive sets and invariant sets for generic systems whose dynamic function is a set valued map, as in (2.1), although we enunciate the property here for CDI systems (4.1). Recall that, given the set valued map $\mathscr{F}(\cdot)$, map $\mathscr{M}_{\mathscr{F}} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is defined as

$$\mathscr{M}_{\mathscr{F}}(\Omega) = \bigcup_{x \in \Omega} \mathscr{F}(x). \tag{4.8}$$

for all $\Omega \in \mathscr{S}(\mathbb{R}^n)$, see (2.2).

Property 4.12 *Let Assumptions 4.9 and 4.11 hold for the set valued map* $\mathscr{F}(\cdot)$ *determining the system dynamics (4.1) and the state constraint set* X*. Given* $\lambda \in [0,1]$*, a convex, compact set* $\Omega \in \mathscr{K}^0(X)$ *is a* λ*-contractive set (an invariant set if* $\lambda = 1$*) for system (4.1) and constraints* $x \in X$ *if and only if*

$$\eta^T z \leq \lambda \phi_\Omega(\eta), \quad \forall z \in \mathscr{F}(x), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{4.9}$$

Proof: Recall that a convex, compact set $\Omega \subseteq \mathbb{R}^n$ with $0 \in \text{int}\,(\Omega)$ is a λ-contractive set for a dynamic system if the image of any point $x \in \Omega$ through the dynamic function is a subset of $\lambda\Omega$. In particular, for a system (4.1) condition for invariance is

$$\mathscr{M}_{\mathscr{F}}(\Omega) = \bigcup_{x\in\Omega} \mathscr{F}(x) \subseteq \lambda\Omega, \tag{4.10}$$

where $\mathscr{M}_{\mathscr{F}}(\cdot)$ is the set map defined in (4.8), related to $\mathscr{F}(\cdot)$. Condition (4.10) is equivalent to

$$\mathscr{F}(x) \subseteq \lambda\Omega, \quad \forall x \in \Omega,$$

and, from convexity of Ω and by Property C.4, it can be expressed in terms of support function as

$$\phi_{\mathscr{F}(x)}(\eta) \leq \phi_{\lambda\Omega}(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

which, from Property C.5, is equivalent to

$$\phi_{\mathscr{F}(x)}(\eta) \leq \lambda\phi_{\Omega}(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{4.11}$$

By definition of support function we have that invariance condition can be rewritten as condition

$$\sup_{z\in\mathscr{F}(x)} \eta^T z \leq \lambda\phi_{\Omega}(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

which is equivalent to condition (4.9), then the property is proved. ■

Notice that the condition for λ-contractiveness and invariance (4.9) involves any point $x \in \Omega$. Necessary and sufficient condition for invariance can be restricted to the boundary of the set Ω and can be posed as a set of convex constraints, through the employment of the directional upper bounding function $\check{F}(\cdot,\cdot)$, defined in the previous section for CDI systems.

Theorem 4.13 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the system dynamics (4.1) and the state constraint set X. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \in \mathscr{K}^0(X)$ is a λ-contractive set for system (4.1) and constraints $x \in X$ if and only if*

$$\check{F}(x,\eta) \leq \lambda\phi_{\Omega}(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{4.12}$$

Proof: First we prove that from Property 4.4, we have that condition (4.12) evaluated at any element of Ω rather than only on the boundary, that is

$$\check{F}(x,\eta) \leq \lambda\phi_{\Omega}(\eta), \quad \forall x \in \Omega, \quad \eta \in \mathbb{R}^n, \tag{4.13}$$

is equivalent to contractiveness condition (4.9). In fact, from (4.13), we have that

$$\eta^T z \leq \max_{z\in\mathscr{F}(x)} \eta^T z = \check{F}(x,\eta) \leq \lambda\phi_{\Omega}(\eta), \quad \forall z \in \mathscr{F}(x), \quad \forall x \in \Omega, \quad \eta \in \mathbb{R}^n,$$

meaning that condition (4.13) is sufficient for contractiveness, since it implies (4.9). We prove necessity by contradiction supposing that Ω is λ-contractive and that there exist $\bar{x} \in \Omega$ and $\bar{\eta} \in \mathbb{R}^n$ such that

$$\check{F}(\bar{x}, \bar{\eta}) > \lambda \phi_\Omega(\bar{\eta}).$$

This means that, denoting with $\bar{z}$ the element of $\mathscr{F}(\bar{x})$ such that $\bar{\eta}^T \bar{z} = \max_{z \in \mathscr{F}(\bar{x})} \bar{\eta}^T z$, which exists by compactness of $\mathscr{F}(\bar{x})$, we have that

$$\bar{\eta}^T \bar{z} = \check{F}(\bar{x}, \bar{\eta}) > \lambda \phi_\Omega(\bar{\eta}),$$

hence condition (4.9) is violated and Ω is not λ-contractive, which is a contradiction.

Then we have to demonstrate that condition (4.12), involving only the boundary of Ω, holds if and only if (4.13) is satisfied, concerning every element of Ω. Necessity is trivial, since $\partial\Omega \subseteq \Omega$. Sufficiency has to be proved. From compactness and convexity of Ω it follows that given $\bar{x} \in \Omega$ there exists a set of points of $\partial\Omega$ such that $\bar{x}$ is their convex combination (see Theorem 18.5 of (Rockafellar, 1970)). That means that there exist a non-empty set of p points $x^j(\bar{x}) \in \partial\Omega$, with $p = p(\bar{x}) \in \mathbb{N}$, and a set of p real numbers $\theta^j(\bar{x}) \in \mathbb{R}$, for $j \in \mathbb{N}_p$, such that $\bar{x} = \sum_{j=1}^p \theta^j(\bar{x})\, x^j(\bar{x})$, $\theta^j(\bar{x}) \geq 0$ for all $j \in \mathbb{N}_p$, and $\sum_{j=1}^p \theta^j(\bar{x}) = 1$. By convexity of function $\check{F}(\cdot, \eta)$ on the convex, closed set X and equation (4.12), we have that

$$\begin{aligned}
&\check{F}(\bar{x}, \eta) = \check{F}\left(\sum_{j=1}^{p} \theta^j(\bar{x})\, x^j(\bar{x}), \eta\right) \leq \sum_{j=1}^{p} \theta^j(\bar{x})\, \check{F}(x^j(\bar{x}), \eta) \leq \\
&\leq \sum_{j=1}^{p} \theta^j(\bar{x})\, \lambda \phi_\Omega(\eta) = \lambda \phi_\Omega(\eta), \quad \forall \bar{x} \in \Omega,\ \forall \eta \in \mathbb{R}^n.
\end{aligned}$$

This means that condition (4.12) implies condition (4.13) and then it is an also sufficient condition for λ-contractiveness of Ω for system (4.1). ■

Recalling that any λ-contractive set for a given dynamic system and constraint set, is also an invariant set, then the corollary below follows with no need of proof.

Corollary 4.14 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the system dynamics (4.1) and the state constraint set X. A convex, compact set $\Omega \in \mathscr{K}^0(X)$ is an invariant set for system (4.1) if and only if*

$$\check{F}(x, \eta) \leq \phi_\Omega(\eta), \quad \forall x \in \partial\Omega,\ \eta \in \mathbb{R}^n.$$

Theorem 4.13 and Corollary 4.14, provide necessary and sufficient conditions for λ-contractiveness and invariance of a set Ω based on a set of convex constraints.

It is worth recalling the fact that conditions for λ-contractiveness and invariance for nonlinear discrete-time systems can be restricted to the boundary of the set only for particular cases, such as linear and positively homogeneous systems, see (Blanchini and Miani, 2008), usually the analysis has to involve the whole set Ω. However, conditions stated in Theorem 4.13 and Corollary 4.14 concern only the boundary of set Ω.

We show here the strong relation between λ-contractive sets and Lyapunov stability theory. The following important property, useful to prove asymptotic convergence for CDI systems, can be stated.

Property 4.15 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the system dynamics (4.1) and the state constraint set X. For every convex, compact λ-contractive set $\Omega \in \mathscr{K}^0(X)$ for system (4.1) with contracting factor $\lambda \in [0,1]$, also the set $\alpha\Omega \subseteq X$, with $\alpha \in [0,1]$, is a convex, compact λ-contractive set for system (4.1) with contracting factor λ.*

Proof: Compactness and convexity of $\alpha\Omega$ for all $\alpha \in [0,1]$ follows by definition. We have to prove that $\mathscr{M}_{\mathscr{F}}(\alpha\Omega) \subseteq \lambda(\alpha\Omega)$ for all $\alpha \in [0,1]$.

From Theorem 4.13, λ-contractiveness condition of set Ω can be expressed in terms of function $\check{F}(\cdot,\cdot)$ as

$$\check{F}(x,\eta) \leq \lambda\phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{4.14}$$

with $\lambda \in [0,1]$ contracting factor for system (4.1), with condition (4.14) to be satisfied for every $x \in \Omega$ (which is equivalent to be satisfied only on the boundary, as proved in the proof of Theorem 4.13).

Recall that, by definition, $\bar{x} \in \alpha\Omega$ if and only if there exists a $x \in \Omega$ such that $\bar{x} = \alpha x$. This means that every element of the set $\alpha\Omega$ can be written as αx with $x \in \Omega$. We consider the point $\alpha x \in \alpha\Omega$ and we point out that $\alpha x = \alpha x + (1-\alpha)0$. This means that any point of set $\alpha\Omega$ can be expressed as the convex combination of the origin and an element x of Ω with $(1-\alpha)$ and α as convex parameters. From convexity of function $\check{F}(\cdot,\eta)$ for every $\eta \in \mathbb{R}^n$ and since $\check{F}(0,\eta) = 0$ by assumption, we have

$$\begin{aligned} &\check{F}(\alpha x,\eta) = \check{F}(\alpha x + (1-\alpha)0,\eta) \leq \\ &\leq \alpha\check{F}(x,\eta) + (1-\alpha)\check{F}(0,\eta) \leq \alpha\lambda\phi_\Omega(\eta) = \lambda\phi_{\alpha\Omega}(\eta), \quad \forall \eta \in \mathbb{R}^n, \end{aligned}$$

for every $\alpha x \in \alpha\Omega$, which proves the property. ■

We claimed that strong relations link asymptotic stability of a system and λ-contractive sets. In fact, the most common approach employed to ensure asymptotic stability for a dynamic system is based on Lyapunov functions. It will be shown that a λ-contractive set for a CDI system induces a Lyapunov function for the system. Analogous results are known in the context of linear and particular nonlinear systems, see (Blanchini, 1995; Blanchini and Miani, 2008).

We first recall that the Minkowski function of set $\Omega \subseteq \mathbb{R}^n$ convex, compact and with $0 \in \mathrm{int}(\Omega)$, is defined as

$$\Psi_\Omega(x) = \min_{\alpha \geq 0}\{\alpha : \ x \in \alpha\Omega\}. \tag{4.15}$$

Based on the concept of Minkowski function (4.15), we introduce function $\mathscr{V}_\Omega(\cdot)$, which can be used to define a Lyapunov function for systems characterized by set valued maps. We recall here that, given the initial set $X_0 \in \mathscr{S}(X)$, the trajectory for a CDI system (4.1) with dynamic function $\mathscr{F}(\cdot)$ is obtained through iteration

$$X_{k+1} = \mathscr{M}_{\mathscr{F}}(X_k), \tag{4.16}$$

where operator $\mathscr{M}_{\mathscr{F}}(\cdot)$ is defined in (4.8).

Definition 4.16 *Given $\Omega \in \mathscr{K}^0(\mathbb{R}^n)$, define the function $\mathscr{V}_\Omega : \mathscr{S}(\mathbb{R}^n) \to \mathbb{R}$ as*

$$\begin{aligned} \mathscr{V}_\Omega(D) \quad &= \max_{x \in D} \Psi_\Omega(x) = \max_{x \in D} \min_{\alpha \geq 0}\{\alpha : \ x \in \alpha\Omega\} = \\ &= \min_{\alpha \geq 0}\{\alpha : \ x \in \alpha\Omega, \ \forall x \in D\} = \min_{\alpha \geq 0}\{\alpha : \ D \subseteq \alpha\Omega\}. \end{aligned} \tag{4.17}$$

for all $D \in \mathscr{S}(\mathbb{R}^n)$.

Thus $\mathscr{V}_\Omega(\cdot)$ associates a value to any set $D \in \mathscr{S}(\mathbb{R}^n)$.

Corollary 4.17 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1). Every convex, compact λ-contractive set $\Omega \in \mathscr{K}^0(X)$ with contracting factor $\lambda \in [0,1)$ induces a local Lyapunov function on the set Ω.*

Proof: The function $\mathscr{V}_\Omega(\cdot)$ defined in (4.17) is a local Lyapunov function in $\mathscr{S}(\Omega)$. In fact, from $0 \in \mathrm{int}(\Omega)$, function (4.17) is positive definite, i.e., $\mathscr{V}_\Omega(D) \geq 0$ for all $D \in \mathscr{S}(\mathbb{R}^n)$, with $\mathscr{V}_\Omega(D) = 0$ if and only if $D = \{0\}$ and it decreases along the system trajectories as shown in what follows.

Notice that, from Definition 4.16, it follows directly that

$$\mathscr{V}_\Omega(\alpha\Omega) = \alpha, \tag{4.18}$$

for all $\alpha \geq 0$ and $\mathscr{V}_\Omega(D) \leq \mathscr{V}_\Omega(C)$ for all $D, C \in \mathscr{S}(\mathbb{R}^n)$ such that $D \subseteq C$.

From Property 4.15, monotonicity of $\mathscr{M}_{\mathscr{F}}(\cdot)$, see Property 2.2, and λ-contractiveness of Ω, given $D \in \mathscr{S}(\Omega)$ such that $\mathscr{V}_{\Omega}(D) = \alpha$ with $\alpha \in (0, 1]$, we have that $D \subseteq \alpha\Omega \subseteq \Omega$ and then

$$\mathscr{M}_{\mathscr{F}}(D) \subseteq \mathscr{M}_{\mathscr{F}}(\alpha\Omega) \subseteq \lambda\alpha\Omega,$$

which implies

$$\mathscr{V}_{\Omega}(\mathscr{M}_{\mathscr{F}}(D)) \leq \mathscr{V}_{\Omega}(\mathscr{M}_{\mathscr{F}}(\alpha\Omega)) \leq \mathscr{V}_{\Omega}(\lambda\alpha\Omega) = \lambda\alpha < \alpha = \mathscr{V}_{\Omega}(D), \tag{4.19}$$

with $\alpha \in (0,1]$. Notice that, if $\alpha = 0$, then $D = \{0\}$ and the inequalities in (4.19) become equalities. Hence we proved that $\mathscr{V}_{\Omega}(\mathscr{M}_{\mathscr{F}}(D)) < \mathscr{V}_{\Omega}(D)$, for all $D \in \mathscr{S}(\Omega)$ different from the set $\{0\}$, that is, that the value of function $\mathscr{V}_{\Omega}(\cdot)$ decreases along the trajectories of system (4.1). ■

A consequence of Corollary 4.17 is that $\lambda \in [0, 1)$ induces a bound on the decreasing rate of the Lyapunov function along the trajectories. That is, given $X_0 \in \mathscr{S}(\Omega)$ (with $X_0 \neq \{0\}$), which implies $X_0 \subseteq \Omega$ and $\mathscr{V}_{\Omega}(X_0) \leq 1$, we have that

$$\mathscr{V}_{\Omega}(X_{k+1}) \leq \lambda\mathscr{V}_{\Omega}(X_k) < \mathscr{V}_{\Omega}(X_k),$$

and then

$$\mathscr{V}_{\Omega}(X_k) \leq \lambda^k,$$

for all $k \in \mathbb{N}$. Geometrically, it means that $X_0 \subseteq \Omega$ implies

$$X_k \subseteq \lambda^k\Omega$$

for all $k \in \mathbb{N}$. Hence given any set in $\mathscr{S}(\Omega)$ as initial condition, the set valued trajectory converges to the compact set composed by only the origin and the system is asymptotically (exponentially, in fact) stable.

Property 4.18 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1). Given two convex, compact λ-contractive sets $\Omega_1 \in \mathscr{K}^0(X)$ and $\Omega_2 \in \mathscr{K}^0(X)$ for the system (4.1) and contracting factors $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$, respectively, their convex hull $\Omega_3 = co\ (\Omega_1,\ \Omega_2)$ is a convex, compact λ-contractive set with contracting factor $\lambda_3 = \max\{\lambda_1,\ \lambda_2\}$, for system (4.1) and $0 \in int(\Omega_3)$.*

Proof: Compactness of Ω_3 follows directly from the fact the convex hull of two compact sets is compact too (see (Rockafellar, 1970) Th. 17.2). Convexity of Ω_3 and the fact that $0 \in \text{int}(\Omega_3)$ follow by definition of convex hull. Moreover $\Omega_3 \subseteq X$ since X is convex, $\Omega_1 \subseteq X$ and $\Omega_2 \subseteq X$ which implies that any convex combination of elements of Ω_1 and Ω_2 belongs to X.

We prove now that Ω_3 is a λ-contractive set with contracting factor $\lambda_3 = \max\{\lambda_1,\ \lambda_2\}$. Condition of λ-contractiveness of Ω_1 and Ω_2 is equivalent to

$$\begin{aligned} \check{F}(x,\eta) &\leq \lambda_1 \phi_{\Omega_1}(\eta), \quad \forall x \in \Omega_1,\ \forall \eta \in \mathbb{R}^n, \\ \check{F}(x,\eta) &\leq \lambda_2 \phi_{\Omega_2}(\eta), \quad \forall x \in \Omega_2,\ \forall \eta \in \mathbb{R}^n, \end{aligned}$$

from Property 4.13. Consider $x^3 \in \Omega_3$, there exist $x^1 \in \Omega_1$, $x^2 \in \Omega_2$ and $\alpha \in [0,1]$ such that $x^3 = \alpha x^1 + (1-\alpha)x^2$, by definition of convex hull. Notice that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. We have

$$\begin{aligned} &\check{F}(x^3,\eta) = \check{F}(\alpha x^1 + (1-\alpha)x^2,\eta) \leq \alpha \check{F}(x^1,\eta) + (1-\alpha)\check{F}(x^2,\eta) \leq \\ &\leq \alpha\lambda_1\phi_{\Omega_1}(\eta) + (1-\alpha)\lambda_2\phi_{\Omega_2}(\eta) \leq \alpha\lambda_3\phi_{\Omega_1}(\eta) + (1-\alpha)\lambda_3\phi_{\Omega_2}(\eta) \leq \\ &\leq \alpha\lambda_3\phi_{\Omega_3}(\eta) + (1-\alpha)\lambda_3\phi_{\Omega_3}(\eta) = \lambda_3\phi_{\Omega_3}(\eta), \end{aligned}$$

for all $\eta \in \mathbb{R}^n$, and where we employed the following property of support functions

$$\phi_U(\eta) \leq \phi_V(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

for all $U,V \subseteq \mathbb{R}^n$, with V closed and convex, such that $U \subseteq V$, and $\Omega_1 \subseteq \Omega_3$ and $\Omega_2 \subseteq \Omega_3$. Moreover we used the fact that, given a convex set U, $\phi_U(\eta) > 0$ for all $\eta \neq 0$ if and only if $0 \in \mathrm{int}(U)$, which leads to $\lambda_1\phi_{\Omega_1}(\eta) \leq \lambda_3\phi_{\Omega_1}(\eta)$ and $\lambda_2\phi_{\Omega_2}(\eta) \leq \lambda_3\phi_{\Omega_2}(\eta)$ for all $\eta \in \mathbb{R}^n$. We obtained that

$$\check{F}(x^3,\eta) \leq \lambda_3\phi_{\Omega_3}(\eta), \quad \forall x^3 \in \Omega_3, \quad \forall \eta \in \mathbb{R}^n,$$

hence the property is proved. ■

In previous properties and results, we often assume convexity of the invariant set under analysis. It can appear restrictive to consider only convex subsets of the state space. The following corollary, direct consequence of Property 4.18, shows that no loss of generality is due to convexity assumption, since the convex hull of any invariant set, is an invariant set itself.

Corollary 4.19 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1). Given a compact invariant set $\Omega \subseteq X$ with $0 \in int\ (\Omega)$, for the system (4.1), the set $\underline{\Omega} = co\ (\Omega)$ is a convex, compact invariant set.*

Proof: Proof of the corollary is simply obtained considering $\Omega_1 = \Omega_2 = \Omega$ in Property 4.18 and $\lambda = 1$. ■

A direct consequence of Corollary 4.19, and then of Property 4.18, is that the maximal invariant set contained in $X \subseteq \mathbb{R}^n$ is convex. We recall that a set $\Omega_M \subseteq X$ is said to be the maximal invariant set if, beside of being invariant, is such that for any invariant set $\Omega \subseteq X$ we have that $\Omega \subseteq \Omega_M$.

Corollary 4.20 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1) and state constraint set $X \subseteq \mathbb{R}^n$. The maximal invariant set $\Omega_M \subseteq X$ is convex.*

Proof: Suppose, by contradiction, that the maximal invariant set $\Omega_M \subseteq X$ is not convex, which means that there exists at least one point $x \in \text{co}\,(\Omega_M)$ such that $x \notin \Omega_M$. From Corollary 4.19, we have that if Ω_M is an invariant then also its convex hull is an invariant. Moreover, since, by definition of convex hull, for every $\Omega_M \subseteq \mathbb{R}^n$ we have

$$\Omega_M \subseteq \text{co}\,(\Omega_M), \tag{4.20}$$

then the set co (Ω_M) is convex, invariant and strictly contains Ω_M, since $x \notin \Omega_M$ but $x \in$ co (Ω_M). Finally, since by convexity of X it follows that co $(\Omega_M) \subseteq X$, we have that Ω_M is not the maximal invariant set in X, which contradicts the hypothesis of maximality of Ω_M. ∎

4.1.2 Robust invariance for uncertain CDI systems

Results presented in the previous section can be extended to CDI systems of the form (2.27), that is, systems presenting additive uncertainty, see Section 2.2.1. We recall here that, for the autonomous case, we are considering systems whose dynamics is given by

$$x^+ \in \mathscr{F}(x) \oplus W, \tag{4.21}$$

where $\mathscr{F}(\cdot)$ is the set valued map characterizing CDI systems and $W \subseteq \mathbb{R}^n$ is the bounding set of the unknown but bounded uncertainty.

We recall here that we distinguish CDI systems with additive uncertainty referring to them as uncertain CDI system. Remind that also CDI system as in (4.1) could have been considered uncertain, since the set valued nature of dynamic function can be viewed as uncertainty representation, see Remark 2.15.

Remark 4.21 *Notice that, given $\mathscr{F}(\cdot)$ in (4.21), we could have defined the set valued function*

$$\mathscr{F}_W(x) = \{z \in \mathbb{R}^n : \; z \in \mathscr{F}(x) \oplus W\}, \tag{4.22}$$

leading to a system $x^+ \in \mathscr{F}_W(x)$, as in (4.1). Nevertheless, such system would not be a CDI system, although characterized by a set valued map, since from Assumption 2.5 we must have $\mathscr{F}_W(0) = \{0\}$ for the system to be a CDI system, and such condition does not hold for non-trivial cases of $W \neq \{0\}$.

First we provide a characterization of robust λ-contractive sets and robust invariant sets for an uncertain CDI system. This will allow us to provide a necessary and sufficient condition for λ-contractiveness and invariance in terms of directional bounding functions $\check{F}(\cdot,\cdot)$, analogously to the case of CDI systems. We recall here that with Assumption 2.3, used below, we suppose that the set W is compact and the origin lies in the interior of its convex hull.

Property 4.22 *Let Assumptions 2.3, 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21) and the state constraint set X. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \in \mathscr{K}^0(X)$ is a robust λ-contractive set (a robust invariant set if $\lambda = 1$) for system (4.21) and constraints $x \in X$ if and only if*

$$\eta^T z \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall z \in \mathscr{F}(x), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{4.23}$$

Proof: A convex set $\Omega \subseteq \mathbb{R}^n$ is a robust λ-contractive set (a robust invariant set if $\lambda = 1$), for an uncertain CDI system (4.21) if

$$\mathscr{M}_{\mathscr{F}_W}(\Omega) = \bigcup_{x \in \Omega} (\mathscr{F}(x) \oplus W) \subseteq \lambda \Omega, \tag{4.24}$$

where $\mathscr{M}_{\mathscr{F}_W} : \mathscr{K}(\mathbb{R}^n) \to \mathscr{K}(\mathbb{R}^n)$ is the set map defined in (4.8), related to $\mathscr{F}_W(\cdot)$ in (4.22).

Condition (4.24) is equivalent to

$$\mathscr{F}(x) \oplus W \subseteq \lambda \Omega, \quad \forall x \in \Omega.$$

In terms of support function we have that Ω is a robust λ-contractive set with contracting factor λ if and only if

$$\phi_{\mathscr{F}(x) \oplus W}(\eta) \leq \phi_{\lambda\Omega}(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

and from Property C.6, we have

$$\phi_{\mathscr{F}(x)}(\eta) \leq \phi_{\lambda\Omega}(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

which is equivalent to

$$\phi_{\mathscr{F}(x)}(\eta) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n.$$

By definition of support function, λ-contractiveness can be formulated as

$$\sup_{z \in \mathscr{F}(x)} \eta^T z \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

which is equivalent to condition (4.23). This proves the property. ■

Notice that, as for the case of CDI systems with no additive uncertainty, condition for robust λ-contractiveness and invariance (4.23) involves any element $x \in \Omega$. We can employ directional upper bounding functions $\check{F}(\cdot,\cdot)$ to obtain an equivalent condition concerning only the elements belonging to the boundary of the candidate robust λ-contractive set.

Theorem 4.23 *Let Assumptions 2.3, 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21) and the state constraint set X. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \in \mathscr{K}^0(X)$ is a robust λ-contractive set for system (4.21) and constraints $x \in X$ if and only if*

$$\check{F}(x,\eta) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{4.25}$$

Proof: The proof is analogous to the proof of Theorem 4.13, that is for necessary and sufficient condition for λ-contractiveness for CDI systems, in absence of additive uncertainty. The main difference is that the bound for the directional upper bounding function is now $\lambda \phi_\Omega(\eta) - \phi_W(\eta)$, for all $\eta \in \mathbb{R}^n$, hence it depends also on the support function of W with respect to $\eta \in \mathbb{R}^n$. ∎

The case of $\lambda = 1$ yields, trivially, to a condition for robust invariance of Ω, which is stated in the following corollary.

Corollary 4.24 *Let Assumptions 2.3, 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21) and the state constraint set X. A convex, compact set $\Omega \in \mathscr{K}^0(X)$ is a robust invariant set for system (4.21) and constraints $x \in X$ if and only if*

$$\check{F}(x,\eta) \leq \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n.$$

Notice that no assumption on convexity of set W has been required, see Assumption 2.3.

Remark 4.25 *It is evident that a necessary condition on a convex, compact set $\Omega \subseteq \mathbb{R}^n$ with $0 \in int\ (\Omega)$ to be a robust invariant set is that*

$$\phi_W(\eta) \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n. \tag{4.26}$$

Indeed $\check{F}(0,\eta) = 0$ for all $\eta \in \mathbb{R}^n$ by assumption and hence, if condition (4.25) is violated, condition (4.23) does not hold.

The geometrical meaning is clear. In fact condition (4.26) is equivalent to the fact that

$$co\ (W) \subseteq co\ (\Omega),$$

and, from convexity of $\Omega \in \mathbb{R}^n$*, we have*

$$W \subseteq co\,(W) \subseteq co\,(\Omega) = \Omega,$$

that is $W \subseteq \Omega$*. If it is not fulfilled, then we have, for* $x_0 = \{0\}$*,*

$$x_1 \in \mathscr{F}(0) \oplus W = W,$$

which violates the geometric condition for robust invariance (4.24) with $\lambda = 1$.

4.2 One-step operator and domain of attraction

An important operator which is widely employed in iterative computation of invariant sets, is the one-step operator. Consider an autonomous dynamic system and a subset D of the state space. The one-step operator associates to D the set of points whose image through the dynamic function is contained in D. We provide the definition for CDI systems.

Definition 4.26 *Let Assumptions 4.9 and 4.11 hold for the set valued map* $\mathscr{F}(\cdot)$ *determining the dynamic system (4.1) and the state constraint set* $X \subseteq \mathbb{R}^n$*. The one-step operator is defined as*

$$Q(\Omega) = \{x \in X : \ \mathscr{F}(x) \subseteq \Omega\}, \tag{4.27}$$

for all $\Omega \in \mathscr{K}(X)$*.*

The one-step operator for autonomous CDI systems satisfies the following property.

Property 4.27 *Let Assumptions 4.9 and 4.11 hold for the set valued map* $\mathscr{F}(\cdot)$ *determining the dynamic system (4.1) and the state constraint set* $X \subseteq \mathbb{R}^n$*. Suppose that* $X \subseteq ri\,(dom\,\check{F}(\cdot,\eta))$ *for any* $\eta \in \mathbb{R}^n$*. We have that the one-step operator is*

$$Q(\Omega) = \bigcap_{\eta \in \mathbb{R}^n} \{x \in X : \ \check{F}(x,\eta) \leq \phi_\Omega(\eta)\}, \tag{4.28}$$

and is convex and closed for every set $\Omega \in \mathscr{K}(X)$*.*

Proof: First we prove that $Q(\Omega)$ defined in (4.27) is equal to (4.28). Given a convex compact set Ω, a point $x \in X$ is mapped by $\mathscr{F}(\cdot)$ inside Ω if and only if

$$\eta^T z \leq \phi_\Omega(\eta), \quad \forall z \in \mathscr{F}(x), \quad \forall \eta \in \mathbb{R}^n,$$

which is equivalent to
$$\max_{z\in\mathscr{F}(x)} \eta^T z \le \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

and therefore, by definition of $\check{F}(\cdot,\cdot)$, we have that necessary and sufficient condition for $x \in X$ to be mapped inside Ω, is that

$$\check{F}(x,\eta) \le \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n. \tag{4.29}$$

The set of elements $x \in X$ mapped inside Ω by $\mathscr{F}(\cdot)$, is composed by those elements $x \in X$ fulfilling condition (4.29). Notice that for any $\eta \in \mathbb{R}^n$ we have an inequality as in (4.29) determining a set of points fulfilling it. Then, the points mapped inside Ω is the set given by the intersection of such sets, see (4.28).

We use condition (4.29) to prove convexity of the set $Q(\Omega) \subseteq X$, assumed Ω convex. Suppose that $x^1,\ x^2 \in X$ are mapped in Ω , i.e., $\check{F}(x^1,\eta) \le \phi_\Omega(\eta)$ and $\check{F}(x^2,\eta) \le \phi_\Omega(\eta)$, for all $\eta \in \mathbb{R}^n$. Then, by convexity of function $\check{F}(\cdot,\eta)$ and convexity of X (see Assumption 4.11), for all $\eta \in \mathbb{R}^n$, for every $\alpha \in [0,1]$, point $x^3 = x^3(\alpha) = \alpha x^1 + (1-\alpha)x^2$ is such that $x^3 \in X$ and

$$\begin{aligned}\check{F}(x^3,\eta) &= \check{F}(\alpha x^1 + (1-\alpha)x^2,\eta) \le \alpha\check{F}(x^1,\eta) + (1-\alpha)\check{F}(x^2,\eta) \le \\ &\le \alpha\phi_\Omega(\eta) + (1-\alpha)\phi_\Omega(\eta) = \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n.\end{aligned}$$

This means that if $x^1, x^2 \in Q(\Omega)$, then any of their convex combinations is an element of $Q(\Omega)$ too, which is equivalent to convexity of $Q(\Omega)$.

To prove the closure of $Q(\Omega)$, some technicalities are required. Since the intersection of an arbitrary collection of closed sets is closed, see (Rockafellar, 1970), if we are able to prove that set $\{x \in X :\ \check{F}(x,\eta) \le \phi_\Omega(\eta)\}$ is closed for any η, closure of $Q(\Omega)$ is proved. From (Rockafellar, 1970) Th.7.1 and related considerations, it can be proved that a (proper) convex function $f(\cdot)$ is closed (i.e., its epigraph is closed), if and only if set

$$\{x \in X :\ f(x) \le \alpha\}, \tag{4.30}$$

is closed for every $\alpha \in \mathbb{R}$, and that a proper convex function $f(\cdot)$ agrees with its closure except perhaps at the boundary of the effective domain, Th.7.4. Since the points of boundary of the effective domain are not contained in X by hypothesis, then replacing $\check{F}(\cdot,\eta)$, convex by construction, with its closure does not affect the level sets (4.30). This means that level sets (4.30) are closed and then their intersection is closed too, hence $Q(\Omega)$ is closed. ■

In Property 4.27 we proved that the one-step operator for a CDI dynamic system (4.1) maps compact convex sets, in closed, convex sets. Compactness is not preserved in general through the one-step operation. This can be pointed out by means of an illustrative example.

Example 4.28 *Recalling that a subset of $\mathbb{R}^n$ is compact if it is closed and bounded, we have to prove that, for a CDI system, $Q(\Omega)$ is not bounded for a compact Ω. Consider the one-dimensional discrete-time linear systems*

$$x^+ = 0,$$

with $X = \mathbb{R}$. The dynamic function can be considered a particular set valued function whose images are points (the origin in fact), that is

$$\mathscr{F}(x) = \{0\}, \quad \forall x \in \mathbb{R}.$$

Also the convex bounding functions, for $\eta \in \partial \mathbf{B}_2^1 = [-1, 1]$, can be recovered trivially, in fact they are given by

$$\check{f}_{-1}(x) = \check{f}_1(x) = 0.$$

Evidently Assumption 4.9 is satisfied. Now considering $\Omega = \{0\}$ and applying the one-step operator, we have that

$$Q(\{0\}) = \{x \in \mathbb{R} : \mathscr{F}(x) = \{0\} \in \{0\}\} = \mathbb{R},$$

that is the whole space $\mathbb{R}$, hence closed and convex but not bounded.

If we add the hypothesis of boundedness of state constraint X, we have that the one-step operator maps convex, compact subsets into convex, compact subsets, that is $Q : \mathscr{K}(X) \to \mathscr{K}(X)$.

Corollary 4.29 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1) and suppose that X is bounded and such that $X \subseteq ri\ (dom\ \check{F}(\cdot, \eta))$ for any $\eta \in \mathbb{R}^n$. Then, the set $Q(\Omega)$ is convex and compact for every set $\Omega \in \mathscr{K}(X)$.*

Proof: The result follows from Property 4.27 and since, by definition, $Q(\Omega)$ is bounded for every Ω, provided X is bounded. ■

Property and corollary stated above mean that the set of points mapped through $\mathscr{F}(\cdot)$ inside a $\Omega \in \mathscr{K}(X)$ is a convex, closed set, compact if X is bounded.

Remark 4.30 *We give here a conceptual definition of domain of attraction.*

The domain of attraction of an asymptotically stable (to the origin) system is given by the set of points of the state space converging to the origin.

For a CDI system with dynamic function $\mathscr{F}(\cdot)$, the domain of attraction of the origin is the set of points of the state space such that the Hausdorff distance between the elements of the trajectory and origin converges to zero. That is, points $x \in X$ such that sets $X_k \in \mathscr{S}(\mathbb{R}^n)$ generated by

$$X_{k+1} = \mathscr{M}_{\mathscr{F}}(X_k),$$

with $X_0 = \{x\}$, are such that $X_k \subseteq X$ for all $k \in \mathbb{N}$ and

$$d_H(X_k, \{0\}) \to 0,$$

as $k \to \infty$, where $d_H : \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathbb{R}$ is the Hausdorff distance defined as

$$d_H(D,C) = \max\{\sup_{x\in D} \inf_{y\in C} d(x,y),\ \sup_{x\in C} \inf_{y\in D} d(x,y)\},$$

for any $D, C \in \mathscr{S}(\mathbb{R}^n)$ and $d(\cdot,\cdot)$ is a distance in $\mathbb{R}^n$.

While the domain of attraction of a continuous-time nonlinear system is open, see lemma below, we have that the domain of attraction for a CDI (discrete-time) system is compact, besides convex, under assumption of boundedness of X. This is proved below, and results useful to approximate the domain of attraction for a CDI system are given.

First we report Lemma 45 of (Vidyasagar, 1993), stating that the domain of attraction of continuous-time nonlinear systems is open.

Lemma 4.31 (L. 45 (Vidyasagar, 1993)) *Suppose that 0 is an attractive equilibrium of a continuous-time nonlinear system, then the domain of attraction of the origin is open, connected and invariant.*

Characterization of the domain of attraction for CDI systems is provided in the following theorem.

Theorem 4.32 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1) and suppose that X is bounded and such that $X \subseteq ri\ (\text{dom}\ \check{F}(\cdot,\eta))$ for any $\eta \in \mathbb{R}^n$. Given any λ-contractive set $\Omega \in \mathscr{K}^0(X)$ with contracting factor $\lambda \in [0,1)$, the sequence of sets Ω_k obtained with $\Omega_0 = \Omega$ and*

$$\Omega_{k+1} = Q(\Omega_k), \tag{4.31}$$

for $k \in \mathbb{N}$, is such that:

(i) Ω_k is invariant for all $k \in \mathbb{N}$,

(ii) *is a nested sequence, that is $\Omega_k \subseteq \Omega_{k+1}$, for all $k \in \mathbb{N}$,*

(iii) *Ω_k is convex, compact and contains the origin in its interior, i.e., $\Omega_k \in \mathscr{K}^0(X)$,*

(iv) *the sequence converges to the domain of attraction $\tilde{\Omega}$,*

(v) *the domain of attraction is convex, compact and contains the origin in its interior, i.e., $\tilde{\Omega} \in \mathscr{K}^0(X)$.*

Proof: First recall that, from Corollary 4.17, any λ-contractive set induces a Lyapunov function and then the CDI system is asymptotically (exponentially, in fact) stable.

(i) (ii) We prove both points recursively. Suppose that $\Omega_k \subseteq X$ is invariant. This means, by definition, that

$$x \in \Omega_k, \quad \Rightarrow \quad \mathscr{F}(x) \subseteq \Omega_k \quad \forall x \in \Omega_k.$$

Since, by definition of one-step operator, see (4.27), and iteration (4.31), we have that

$$\Omega_{k+1} = Q(\Omega_k) = \{x \in X : \mathscr{F}(x) \subseteq \Omega_k\},$$

and it follows that

$$x \in \Omega_k, \quad \Rightarrow \quad x \in \Omega_{k+1},$$

which means that $\Omega_k \subseteq \Omega_{k+1}$. From this inclusion it follows that

$$x \in \Omega_{k+1}, \quad \Rightarrow \quad \mathscr{F}(x) \subseteq \Omega_k \subseteq \Omega_{k+1},$$

that is equivalent to

$$\bigcup_{x \in \Omega_{k+1}} \mathscr{F}(x) \subseteq \Omega_{k+1}$$

which is the definition of invariance for Ω_{k+1}, see (4.10). Then invariance of Ω_k implies $\Omega_k \subseteq \Omega_{k+1}$ and invariance of Ω_{k+1}. Since Ω_0 is assumed invariant, the statements are proved.

(iii) Compactness and convexity of Ω_k, for $k \in \mathbb{N}$, follow directly form Corollary 4.29 and $0 \in \text{int}\,(\Omega_k)$ since $\Omega_0 \in \mathscr{K}^0(X)$, by assumption, and $\Omega_k \subseteq \Omega_{k+1}$.

(iv) Suppose that $x \in X$ is an element of the domain of attraction, i.e., $x \in \tilde{\Omega}$. By definition of domain of attraction, see Remark 4.30, it means that posing $X_0 = \{x\}$ and iterating through (4.16), from asymptotic stability, we have that there exists a $\tilde{k}(x) \in \mathbb{N}$ such that $X_{\tilde{k}(x)} \subseteq \Omega$. This is equivalent to say that for any x in the domain of attraction $\tilde{\Omega}$, we have $x \in \Omega_{\tilde{k}}$, for a proper $\tilde{k} \in \mathbb{N}$.

(v) Convexity of $\tilde{\Omega}$ is due to the fact that, given two points $x^1, x^2 \in \tilde{\Omega}$, there exist two values $k^1 = k^1(x^1) \in \mathbb{N}$ and $k^2 = k^2(x^2) \in \mathbb{N}$ such that $x^1 \in \Omega_{k^1}$ and $x^2 \in \Omega_{k^2}$, see the proof of the previous point. Hence $x^1, x^2 \in \Omega_{k^3}$, with $k^3 = \max\{k^1, k^2\}$ since, from point (ii), $\Omega_{k^1} \subseteq \Omega_{k^3}$ and $\Omega_{k^2} \subseteq \Omega_{k^3}$, and then any convex combination of x^1 and x^2 is an element of Ω_{k^3}, hence also element of $\tilde{\Omega}$. This means that $\tilde{\Omega}$ is convex. The origin is contained in the interior of $\tilde{\Omega}$ since $0 \in \text{int}\,(\Omega_0)$ and the sequence is nested. Finally, $\tilde{\Omega}$ is compact because the space of compacts subsets of the compact X, i.e., $\mathscr{K}(X)$, equipped with Hausdorff distance is a complete metric space and any converging sequence in a complete space has its limits in the space itself.

■

Similar results have been proved for saturated systems and Lur'e systems, in (Álamo, Cepeda, Limón and Camacho, 2006*b*; Álamo, Cepeda, Fiacchini and Camacho, 2009). We notice here that such cases can be easily overbounded by a CDI system.

4.2.1 One-step operator for uncertain CDI systems

Results analogous to the case of CDI systems can be recovered for uncertain CDI systems, that is, in presence of additive uncertainty. In particular, convexity and closure of sets given by the one-step operator is proved.

Property 4.33 *Let Assumptions 2.3, 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21) and the state constraint set X and suppose that $X \subseteq ri\ (dom\ \check{F}(\cdot,\eta))$ for any $\eta \in \mathbb{R}^n$. Given a convex, compact set $\Omega \in \mathscr{K}(X)$, the set*

$$\begin{aligned} Q_W(\Omega) &= \{x \in X :\ \mathscr{F}(x) \oplus W \subseteq \Omega\} = \{x \in X :\ \mathscr{F}(x) \subseteq \Omega \ominus W\} = \\ &= \bigcap_{\eta \in \mathbb{R}^n} \{x \in X :\ \check{F}(x,\eta) \leq \phi_\Omega(\eta) - \phi_W(\eta)\}, \end{aligned} \tag{4.32}$$

is convex and closed, and compact if X is bounded.

Proof: The proof of Property 4.33 is a straightforward modification of the proof of Property 4.27 and Corollary 4.29. Substantially, the proof can be recovered considering the effect of additive uncertainty, W, for instance, replacing $\phi_\Omega(\eta)$ with $\phi_\Omega(\eta) - \phi_W(\eta)$ in inequalities of the proof of Property 4.27. ■

The one-step operator for uncertain CDI systems can be used to design an iterative procedure to obtain a sequence of nested robust invariant sets converging to the domain of attraction, as illustrated in Theorem 4.32.

4.2.2 One-step operator complement

It has been shown in the previous sections that directional bounding functions $\check{F}(\cdot,\cdot)$ provide an analytical tool for invariance and λ-contractiveness, particularly suitable due to its convexity. Convexity of the directional bounding functions allows us, in fact, to infer convergence and stability for an entire set simply posing convex conditions on the boundary.

Moreover, it has been shown that the directional upper bounding function can be used to characterize the one-step operator, that is the set of point mapped inside a given set Ω through the set valued map $\mathscr{F}(\cdot)$, see (4.28). In this section we show that directional lower bounding functions $\hat{F}(\cdot,\cdot)$ can be used to characterize convex regions belonging to the complement of the one-step operator. First we define the basic geometric concept of halfspace, useful in what follows.

Definition 4.34 *Given a vector $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}$, we define as $\mathscr{H}(\eta,d)$ the halfspace containing the elements $x \in \mathbb{R}^n$ such that $\eta^T x \leq d$, i.e.,*

$$\mathscr{H}(\eta,d) = \{x \in \mathbb{R}^n : \ \eta^T x \leq d\}. \tag{4.33}$$

Notice that the scalar d is related to the support function of the halfspace, i.e.,

$$\phi_{\mathscr{H}(\eta,d)}(\eta) = d, \tag{4.34}$$

and that, clearly

$$\begin{aligned} \mathscr{H}(\eta,\alpha d) &= \mathscr{H}(\tfrac{1}{\alpha}\eta,d), \qquad && \forall \alpha > 0, \\ \mathscr{H}(\eta,\alpha d) &= \mathscr{H}(-\tfrac{1}{\alpha}\eta,-d), \qquad && \forall \alpha < 0. \end{aligned} \tag{4.35}$$

Then an halfspace is defined by a direction $\eta \in \mathbb{R}^n$ and a scalar $d \in \mathbb{R}$ which can be interpreted as a sort of distance between the boundary of the halfspace and the origin.

Property 4.35 *Let Assumption 4.9 hold for the set valued map $\mathscr{F}(\cdot)$. Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}$, we have that $\mathscr{F}(x) \cap \mathscr{H}(\eta,d) = \emptyset$ if and only if*

$$\hat{F}(x,\eta) > d. \tag{4.36}$$

Proof: By definition of halfspace, a point $x \in \mathbb{R}^n$ is such that $x \notin \mathscr{H}(\eta,d)$ if and only if $\eta^T x > d$. Then, $\mathscr{F}(x) \cap \mathscr{H}(\eta,d) = \emptyset$ if and only if $\eta^T z > d$, for all $z \in \mathscr{F}(x)$, which is equivalent to

$$\min_{z \in \mathscr{F}(x)} \eta^T z > d.$$

From Property 4.6, the claim follows. ■

The meaning of Property 4.35 is that, given a point $x \in \mathbb{R}^n$ the image of $x \in \mathbb{R}^n$ through $\mathscr{F}(\cdot)$ does not intersect the halfspace $\mathscr{H}(\eta, d)$, which means that $z \notin \mathscr{H}(\eta, d)$ for all $z \in \mathscr{F}(x)$, if and only if condition (4.36) is satisfied.

We prove here that, given a direction $\eta \in \mathbb{R}^n$ and $d \in \mathbb{R}$, the elements of $\mathbb{R}^n$ satisfying condition (4.36) forms a convex set.

Property 4.36 *Let Assumption 4.9 hold for the set valued map $\mathscr{F}(\cdot)$. Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}$, the set of points $x \in \mathbb{R}^n$ such that $\mathscr{F}(x) \cap \mathscr{H}(\eta, d) = \emptyset$ is convex.*

Proof: From Property 4.35, we have to prove that, if $x^1 \in \mathbb{R}^n$ and $x^2 \in \mathbb{R}^n$ satisfy condition (4.36), then every point $x^3 \in \text{co}\ (x^1, x^2)$ satisfies condition (4.36).

By assumption we have that $-\hat{F}(x^1, \eta) < -d$ and $-\hat{F}(x^2, \eta) < -d$ and, for every $x^3 \in \text{co}\ (x^1, x^2)$, there exists a $\theta = \theta(x^3) \in [0, 1]$ such that $x^3 = \theta x^1 + (1-\theta)x^2$. From concavity of function $\hat{F}(\cdot, \eta)$, it follows that

$$\begin{aligned}
&-\hat{F}(x^3, \eta) = -\hat{F}(\theta x^1 + (1-\theta)x^2, \eta) \leq \\
&\leq -\theta \hat{F}(x^1, \eta) - (1-\theta)\hat{F}(x^2, \eta) \leq -\theta d - (1-\theta)d = -d,
\end{aligned}$$

which proves the statement. ∎

From Properties 4.35 and 4.36, given the set $\Omega \in \mathscr{K}(X)$, a condition to determine whether a convex set is contained in the complement of the one-step set $Q(\Omega)$ can be given.

We give here a formal definition of the set of points $x \in \mathbb{R}^n$ whose image through the set valued map $\mathscr{F}(\cdot)$ do not intersect the hyperplane $\mathscr{H}(\eta, d)$. Note that we employ in the definition the directional lower bounding function $\hat{F}(\cdot, \cdot)$, to stress the dependence on the set valued function $\mathscr{F}(\cdot)$.

Definition 4.37 *Let Assumption 4.9 hold for the set valued map $\mathscr{F}(\cdot)$. Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}^n$, we define the set*

$$\mathscr{C}(\eta, d) = \{x \in \mathbb{R}^n :\ \hat{F}(x, \eta) > d\}. \tag{4.37}$$

Moreover, given a set $\Omega \in \mathscr{K}(\mathbb{R}^n)$, we define

$$\mathscr{C}_\Omega(\eta) = \mathscr{C}(\eta, \phi_\Omega(\eta)) = \{x \in \mathbb{R}^n :\ \hat{F}(x, \eta) > \phi_\Omega(\eta)\}, \tag{4.38}$$

and, finally

$$\begin{aligned}
\mathscr{C}_\Omega &= \bigcup_{\eta \in \mathbb{R}^n,\ \eta \neq 0} \mathscr{C}_\Omega(\eta) = \bigcup_{\eta \in \mathbb{R}^n,\ \eta \neq 0} \{x \in \mathbb{R}^n :\ \hat{F}(x, \eta) > \phi_\Omega(\eta)\} = \\
&= \{x \in \mathbb{R}^n :\ \exists \eta \in \mathbb{R}^n,\ \eta \neq 0 :\ \hat{F}(x, \eta) > \phi_\Omega(\eta)\}.
\end{aligned} \tag{4.39}$$

Although trivial, it is worth stressing the fact that the set $\mathscr{C}_\Omega(\eta)$ is the set of points in the state space whose image through $\mathscr{F}(\cdot)$ does not intersect the halfspace

$$\mathscr{H}(\eta,\phi_\Omega) = \{x \in \mathbb{R}^n : \ \eta^T x > \phi_\Omega(\ \eta)\},$$

from Property 4.35 and by definition of $\mathscr{C}_\Omega(\eta)$. Hence, set $\mathscr{C}_\Omega$ is composed by all those points of the state space whose image through the set valued map $\mathscr{F}(\cdot)$ does not intersect the set Ω. Conceptually, it can be viewed as a sort of complement of the one-step set of Ω. As a matter of fact it is not the complement of $Q(\Omega)$, since, given a set Ω and a point $x \in \mathbb{R}^n$, there are three possibility:

1. $\mathscr{F}(x) \subseteq \Omega$,
2. $\mathscr{F}(x) \cap \Omega = \emptyset$,
3. $\mathscr{F}(x) \cap \Omega \neq \emptyset$ and $\mathscr{F}(x) \nsubseteq \Omega$.

Points for which the first possibility holds belong to the one-step set of Ω, those for which either the second or the third option holds form the complement of the one-step set. The set $\mathscr{C}_\Omega$ is composed by those x satisfying the second condition, hence it is a subset of the complement of the one-step operator.

Notice that checking whether a point of the state space belongs to $\mathscr{C}_\Omega(\eta)$ can be reduced to a convex constraint to be tested, hence easy to be checked.

Remark 4.38 *Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ we have that $\mathscr{C}(\eta,d) = \mathscr{C}(\alpha\eta,\alpha d)$, for all $\alpha \in \mathbb{R}$ with $\alpha \geq 0$. Moreover, from homogeneity of order one of support function, i.e., since $\phi_\Omega(\alpha\eta) = \alpha\phi_\Omega(\eta)$, for all $\alpha \geq 0$ and for any $\Omega \subseteq \mathbb{R}^n$, we have that $\mathscr{C}_\Omega(\eta) = \mathscr{C}_\Omega(\alpha\eta)$, for all $\alpha \geq 0$. For this reason it is sufficient to consider $\mathscr{C}_\Omega(\eta)$ for any $\eta \in \partial\mathbf{B}_2^n$ to have a full characterization of sets $\mathscr{C}_\Omega(\eta)$ for all $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and we have that*

$$\mathscr{C}_\Omega = \bigcup_{\eta\in\partial\mathbf{B}_2^n} \mathscr{C}_\Omega(\eta).$$

With the following theorem we make apparent the relation between the one-step set of $\Omega \in \mathscr{K}(\mathbb{R}^n)$ and set $\mathscr{C}_\Omega(\eta)$.

Theorem 4.39 *Let Assumption 4.9 hold for the set valued map $\mathscr{F}(\cdot)$ determining the dynamic system (4.1). Given $\Omega \in \mathscr{K}(\mathbb{R}^n)$, the set $\mathscr{C}_\Omega(\eta) \subseteq \mathbb{R}^n$ defined as in (4.38) is a convex set such that*

$$\mathscr{C}_\Omega(\eta) \cap Q(\Omega) = \emptyset. \tag{4.40}$$

for all $\eta \in \mathbb{R}^n$ with $\eta \neq 0$, where $Q(\cdot)$ is defined in (4.27) and

$$\mathscr{C}_\Omega \cap Q(\Omega) = \emptyset. \tag{4.41}$$

Proof: Convexity of the set $\mathscr{C}_\Omega(\eta) = \mathscr{C}(\eta, \phi_\Omega(\eta))$ follows from Property 4.36. Given a compact, convex set $\Omega \in \mathscr{K}(\mathbb{R}^n)$, condition (4.40) follows directly from Property 4.27 and definition (4.38). Finally condition (4.41) follows from definition (4.39). ■

The result presented for a CDI system can be easily adapted to the case of uncertain CDI systems, that is in presence of an additive term of the uncertainty, whose dynamics is given by (4.21). The objective is to determine a condition at any point of the state space, $x \in \mathbb{R}^n$, ensuring that its image through the dynamic set valued map $\mathscr{F}_W(x) = \mathscr{F}(x) \oplus W$ has empty intersection with a given halfspace. Recall that the image of x through the set valued map $\mathscr{F}_W(\cdot)$ is the successor of x for the uncertain CDI system, where $\mathscr{F}_W(\cdot)$ is defined explicitly in (4.22).

Property 4.40 *Let Assumptions 2.3 and 4.9 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21). Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}$, we have that $(\mathscr{F}(x) \oplus W) \cap \mathscr{H}(\eta, d) = \emptyset$ if and only if*

$$\hat{F}(x, \eta) > d + \phi_W(-\eta). \tag{4.42}$$

Proof: By definition, a point $x \in \mathbb{R}^n$ is not an element of a halfspace, i.e. $x \notin \mathscr{H}(\eta, d)$ if and only if $\eta^T x > d$. Then, $(\mathscr{F}(x) \oplus W) \cap \mathscr{H}(\eta, d) = \emptyset$ if and only if $\eta^T z > d$, for all $z \in \mathscr{F}(x) \oplus W$, which means

$$\min_{z \in \mathscr{F}(x) \oplus W} \eta^T z > d.$$

From Property 4.6, we have that

$$\begin{aligned}
\min_{z \in \mathscr{F}(x) \oplus W} \eta^T z &= \min\{\eta^T z : \ z = y + w, \ y \in \mathscr{F}(x), \ w \in W\} = \\
&= \min\{\eta^T y : \ y \in \mathscr{F}(x)\} + \min\{\eta^T w : \ w \in W\} = \\
&= \min_{y \in \mathscr{F}(x)} \eta^T y + \min_{w \in W} \eta^T w = \hat{F}_\eta(x, \eta) - \max_{w \in W} (-\eta)^T w = \hat{F}(x, \eta) - \phi_W(-\eta),
\end{aligned}$$

where we use the fact that $\min_{x \in D} f(x) = -\max_{x \in D} - f(x)$, for any set D and any function $f(\cdot)$, and supremum is equal to the maximum on W since W is compact. The claim follows. ■

Also convexity is preserved for the uncertain CDI systems case. Since the proof is analogous to that of Property 4.36 it is not provided here.

Property 4.41 *Let Assumptions 2.3 and 4.9 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21). Given $\eta \in \mathbb{R}^n$ with $\eta \neq 0$ and $d \in \mathbb{R}$, the set of points $x \in \mathbb{R}^n$ such that $(\mathscr{F}(x) \oplus W) \cap \mathscr{H}(\eta, d) = \emptyset$ is convex.*

Then, from Properties 4.40 and 4.41, the points of the state space whose image through the set valued map $\mathscr{F}(\cdot) \oplus W$ defining an uncertain CDI system do not intersect a given halfspace is a convex set characterized by condition (4.42).

Remark 4.42 *Alternatively, condition (4.42) can be written as*

$$\hat{F}(x, \eta) > d + \phi_{-W}(\eta), \tag{4.43}$$

since

$$\phi_W(-\eta) = \max_{w \in W} (-\eta)^T w = \max_{w \in W} \eta^T(-w) = \max_{w \in -W} \eta^T w = \phi_{-W}(\eta). \tag{4.44}$$

Although, maybe, trivial, it is important to point out the fact that $\Omega \ominus W$ and $\Omega \oplus (-W)$ (used below) are not the same set, in general. In fact, given $W \subseteq \mathbb{R}^n$, we have

$$-W = \{-v \in \mathbb{R}^n : \ v \in W\} = \{v \in \mathbb{R}^n : \ -v \in W\},$$

which is, geometrically, the mirror image of W with respect to the origin. This means that $\Omega \oplus (-W)$ is the set obtained as the Minkowski sum of Ω and the mirror image of W, usually not equal to the Minkowski difference of Ω and W. In fact

$$\begin{aligned} \Omega \oplus (-W) &= \{z = x + v : \ x \in \Omega, \ -v \in W\}, \\ \Omega \ominus W &= \{z \in \mathbb{R}^n : \ z + w \in \Omega, \ \forall w \in W\}, \end{aligned}$$

which are equal only if $W = \{0\}$.

The relation between the one-step set $Q_W(\Omega)$ and the convex set $\mathscr{C}_{\Omega \oplus (-W)}(\eta)$ is stated below. By definition (4.38), we have that

$$\begin{aligned} \mathscr{C}_{\Omega \oplus (-W)}(\eta) \ &= \{x \in \mathbb{R}^n : \ \hat{F}(x, \eta) > \phi_{\Omega \oplus (-W)}(\eta)\} = \\ &= \{x \in \mathbb{R}^n : \ \hat{F}(x, \eta) > \phi_{\Omega}(\eta) + \phi_{-W}(\eta)\} = \\ &= \{x \in \mathbb{R}^n : \ \hat{F}(x, \eta) > \phi_{\Omega}(\eta) + \phi_W(-\eta)\}. \end{aligned} \tag{4.45}$$

We recall here that $\mathscr{C}_{\Omega \oplus (-W)}(\eta)$ is the set of points in the state space whose image through $\mathscr{F}_W(\cdot)$, defined in (4.22), does not intersect the halfspace $\mathscr{H}(\eta, \phi_{\Omega}(\eta))$, from Property 4.40 and by definition of $\mathscr{C}_{\Omega \oplus (-W)}(\eta)$. We can consider $\mathscr{C}_{\Omega \oplus (-W)}$ as a sort of complement of the one-step set $Q_W(\Omega)$ for an uncertain CDI system, defined in (4.32).

Theorem 4.43 *Let Assumptions 2.3 and 4.9 hold for the set valued map $\mathscr{F}(\cdot)$ and uncertainty bounding set W determining the uncertain CDI system (4.21). Given $\Omega \in \mathscr{K}(\mathbb{R}^n)$, the set $\mathscr{C}_{\Omega \oplus (-W)}(\eta) \subseteq \mathbb{R}^n$ defined as in (4.38) is a convex set such that*

$$\mathscr{C}_{\Omega \oplus (-W)}(\eta) \cap Q_W(\Omega) = \emptyset. \tag{4.46}$$

for all $\eta \in \mathbb{R}^n$ with $\eta \neq 0$, where $Q_W(\cdot)$ is defined in (4.32) and

$$\mathscr{C}_{\Omega \oplus (-W)} \cap Q_W(\Omega) = \emptyset. \tag{4.47}$$

Proof: Convexity stems directly from Property 4.41, while condition (4.46) follows from (4.45) and Property 4.33. In fact if $x \in \mathscr{C}_{\Omega\oplus(-W)}(\eta)$ then

$$\phi_{\Omega}(\eta) - \phi_W(\eta) \leq \phi_{\Omega}(\eta) + \phi_W(-\eta) < \hat{F}(x,\eta) \leq \check{F}(x,\eta), \tag{4.48}$$

since, by assumption of $0 \in \text{int}(\text{co}(W))$, then $\phi_W(\eta) \geq 0$ for all $\eta \in \mathbb{R}^n$. Then $x \notin Q_W(\Omega)$. Condition (4.47) follows from definition (4.39). ■

The results presented in this section can be used to design an algorithm to compute invariant sets for a CDI system. Consider the strategy presented in (Bravo et al., 2005). The algorithm proposed is a branch and bound procedure. At any step, a set of boxes in the state space are considered. For every box of the set, it is checked whether its image is contained in the union of boxes. An approximation method based on interval arithmetic is used to compute a bound of such image. If the image of a box (or, better, its approximation) is contained in the union of boxes, then such box is maintained in the set of boxes for the next step. If its image does not intersect the union, then the box is removed from the set of boxes. Finally if the image intersects the union of boxes, then such box is split and its parts are added to the set of boxes for the next steps. Now, the procedure to check whether the image of a box is contained in a set can be reduced in our case to check condition (4.12) (or (4.25) for the uncertain case) at its vertices (with $\lambda = 1$), while the condition for empty intersection is given by testing at the vertices if condition (4.36) (or (4.42) in presence of uncertainty) is satisfied. Recall that in our case, no approximations are needed, since the conditions presented are necessary and sufficient, hence no conservatism is introduced in the process.

4.3 Computational issues

One of the main purposes of this thesis, beside of characterizing theoretically invariance and set-theory for nonlinear systems, is to provide computational procedures to obtain an invariant or a λ-contractive set for nonlinear systems. The aim of this section is to illustrate how the theoretical results concerning CDI systems, presented and proved in this chapter, can be used to define algorithms for numerical issues. It is worth recalling, in fact, that a relevant motivation of our research is to contribute to fill the gap between the great, and increasing, importance of invariance and set-theoretic methods in control for nonlinear systems and the practical applicability of the computational techniques presented in literature.

The method for computing invariant and λ-contractive sets for CDI systems that we propose here is based on the following scheme.

- First, we obtain an ellipsoidal invariant set for an LDI system which locally overbounds the CDI one. By means of an LDI approximation, valid in a neighborhood of the origin, we are able to determine an ellipsoid which is invariant also for the CDI system, applying procedures based on powerful and well known computational tools, like convex programming and LMI. Since in practice it can be often assumed that the mismatch between the CDI system and the overbounding LDI one is small, the behavior of the two systems can be expected to be close, at least within the neighborhood. Then the resulting invariant ellipsoid captures the local behavior of the CDI system and it is obtained by means of simple linearity-based computational techniques.

- The resulting ellipsoidal invariant set is then employed to determine a polytopic invariant set for the LDI system, denote it Ω_L. This is an important computational step, since, as shown in the next step, the conditions for invariance and λ-contractiveness for CDI systems, provided in the previous sections, entail low computational burden when applied to polytopic sets. The procedure to obtain the polytopic invariant set from the ellipsoidal one is based on an iterative algorithm, finitely determined and whose determination index is provided below.
- Given the polytopic invariant set Ω_L for the LDI system, the elements of the family of sets obtained scaling Ω_L, that is $\alpha\Omega_L$ for $\alpha \geq 0$, are used as potential invariant sets for the CDI system. Roughly speaking, the shape of Ω_L is used to determine a larger set Ω for which the condition of invariance holds. In fact, necessary and sufficient conditions for invariance and λ-contractiveness can be employed to determine whether a polytope in the state space is an invariant set for the CDI system, by means of a finite number of convex constraints, as proved below. This implies that procedures for checking the condition for invariance (or λ-contractiveness) of a polytope, characterized by affordable computational effort, can be defined and used to design an algorithm to obtain an invariant set or a λ-contractive set for a generic CDI system.
- Eventually, further techniques which permit to enlarge a given invariant set Ω, or a λ-contractive one, are proposed. The main benefit of such techniques, a sketch of which will be presented here, is that the basic shape of the invariant set can be modified, and adapted to the particular nonlinear nature of the CDI system.

The steps which lead to the definition of the algorithm for computing a polytopic invariant or λ-contractive set for a CDI system are detailed below.

4.3.1 LDI system locally overbounding a CDI system

First, it is worth presenting some considerations on the relation between CDI and LDI systems, to show that the latters provide overbounds of the formers.

Recall that CDI systems, as stated above, find one of their main justification in their capability to approximate nonlinear systems. Hence, an LDI system overbounding a CDI one entails a further level of conservatism introduced, when used to bound a nonlinear system. On the other hand, it has to be noticed that

- the LDI overbounding system will be defined in a neighborhood of the origin, where the mismatches with the original CDI system (and also with the possible nonlinear systems whose bound is the CDI one) are small;
- the LDI system is used to obtain a preliminary invariant set for the CDI system, whose shape and geometry matter more than its size. Then the size of the invariant set is enlarged by means of the other, subsequent, steps.

Thus, the degree of conservatism introduced by the use of an overbound of the CDI system is compensated by the computational benefits provided by the linearity properties of an LDI system.

Here we provide methods for obtaining an overbounding LDI representation of a CDI system, which justifies the use of such modelling framework to obtain a first invariant set for generic systems.

4.3.1.1 LDI systems overbounding CDI and CCDI systems

Given a CDI system, it is trivial to obtain an overbounding CCDI system. Recall that a CDI systems is a dynamic system (4.1) whose dynamics are determined by the set valued map

$$\mathscr{F}_{CDI}(x) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{f}_\eta(x), \forall \eta \in \mathbb{R}^n\},$$

with $\check{f}_\eta(\cdot)$ convex functions. Consider the system (4.1) given by the set valued map

$$\mathscr{F}_{CCDI}(x) = \{z \in \mathbb{R}^n : \ \hat{f}_j(x) \leq z_j \leq \check{f}_j(x), \forall j \in \mathbb{N}_n\},$$

with

$$\check{f}_j(x) = \check{f}_{e^j}(x), \quad \hat{f}_j(x) = -\check{f}_{-e^j}(x), \quad \forall j \in \mathbb{N}_n,$$

where $e^j \in \mathbb{R}^n$ is the vector with all entries equal to 0 but the j-th, which is 1. Trivially, the latter is a CCDI system, see Assumption 3.1, and overbounds the CDI one. In fact, for every $x \in \mathbb{R}^n$, the set $\mathscr{F}_{CCDI}(x)$ is defined by a subset of those convex constraints determining $\mathscr{F}_{CDI}(x)$.

Hence we can consider the problem of generating an LDI system overbounding a CCDI system. For that purpose the reader is referred to Section 3.5.1, where a method to obtain an LDI system overbounding a nonlinear one is provided. In particular, consider a nonlinear system

$$x^+ = f(x),$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ defined on a neighborhood of the origin $D \subseteq \mathbb{R}^n$ and such that $f(0) = 0$. The LDI system

$$x(k+1) = A(k)x(k),$$

with $A(k) \in \mathscr{A}$ and $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ defined in (3.56), overbounds the nonlinear system within the set D. Clearly the computational procedures presented in this section and based on an LDI overbounding system, can be directly applied to nonlinear systems, using the approximation method illustrated in Section 3.5.1.

For a CCDI system, an analogous procedure yields to an overbounding LDI system. In fact, recall that the dynamic function for a CCDI system is determined by a set of convex and concave functions. If everyone of those functions, $\check{f}_j(\cdot)$ and $\hat{f}_j(\cdot)$, are overbounded by an LDI function, then the related LDI system overbounds the CCDI one. Hence, the polytope in the space $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ obtained as

$$\mathscr{A} = \left\{ A \in \mathbb{R}^{n\times n} : \ \underline{A}_{i,j} \leq A_{i,j} \leq \overline{A}_{i,j}, \ \forall i, \ j \in \mathbb{N}_n \right\}, \tag{4.49}$$

with

$$\begin{array}{l} \underline{A}_{i,j} = \min\left\{ \min_{x\in D} \frac{\partial \hat{f}_i(x)}{\partial x_j}, \ \min_{x\in D} \frac{\partial \check{f}_i(x)}{\partial x_j} \right\}, \\ \overline{A}_{i,j} = \max\left\{ \max_{x\in D} \frac{\partial \hat{f}_i(x)}{\partial x_j}, \ \max_{x\in D} \frac{\partial \check{f}_i(x)}{\partial x_j} \right\}, \end{array} \tag{4.50}$$

for all $i, j \in \mathbb{N}_n$, provides the LDI system overbounding the CCDI system. Clearly, such LDI system overbounds also any other system overbounded by the CCDI system.

Assuming that the analytical expression of partial derivatives of functions $\check{f}_j(\cdot)$ and $\hat{f}_j(\cdot)$, with $j \in \mathbb{N}_n$, are available, the problem of computing their maximal and minimal values, or at least bounds of them, can be solved by applying interval arithmetic, see (Bravo et al., 2005).

Particular classes of CDI systems whose elements are easily overbounded by LDI systems, at least locally, are Lur'e systems and systems presenting generalized saturated functions. Lur'e systems, in fact, are linear in a neighborhood of the origin, see (3.19). Then, the linear system given by

$$\begin{cases} x_{k+1} &= (A - Bk_0F)x_k, \\ y_k &= Fx_k, \end{cases}$$

is equal to the Lur'e system for any $x \in \mathbb{R}^n$ such that $|Fx| \leq b_1$, which is a band in the state space containing the origin, since $b_1 > 0$, see Assumption 3.8 and Property 3.9.

Similarly, for a system presenting a generalized saturated function in feedback, see Definition 3.12, we have that there exists a neighborhood of the origin within which the system matches a linear system presenting an additive bounded uncertainty. In fact, the system

$$x_{k+1} = (A + B\mu F)x_k + w(k),$$

where $w(k) \in W = \{w = Bv \in \mathbb{R}^n : \ -\mu\sigma \leq v \leq \mu\sigma\}$, defined in the set $D = \{x \in \mathbb{R}^n : |Fx| \leq \frac{y_0}{\mu} + \sigma\}$ overbounds the generalized saturated system (3.25). Notice that in this case asymptotic convergence to the origin can not be proved, the concept of ultimately boundedness should be employed. For sake of simplicity, in the following the case of LDI system in absence of additive uncertainty is considered, see (Boyd et al., 1994; Gurvits, 1995).

4.3.2 Polytopic local invariant set

In what follows, we suppose to know a polytopic LDI system overbounding the original CDI one, which is defined by the set valued map $\mathscr{F}(\cdot)$. That is, suppose that there exists a polytopic set $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ for which Assumption 3.25 holds and such that $\mathscr{F} \subseteq \mathscr{A}$. Then the system (see Section 3.5)

$$x^+ \in \mathscr{A}(x), \tag{4.51}$$

overbounds the CDI system. Recall that, from Assumption 3.25, the set $\mathscr{A}$ can be expressed as the convex hull of a finite number $n_a \in \mathbb{N}$ of matrices $A^j \in \mathbb{R}^{n\times n}$, with $j \in \mathbb{N}_{n_a}$.

A method for capturing the geometry of λ-contractive invariant sets with high contracting factor, for the LDI system, is proposed here. This contractive set is used to enhance the results of the proposed methodology. Recall that, if a quadratic Lyapunov function is defined for an LDI system, its level sets are λ-contractive ellipsoids and then invariant sets for the system.

One of the major benefit of quadratic Lyapunov functions is the fact there are methods and computational tools to compute them and obtain, among all the possible quadratic function, the optimal one with respect a to certain criterion. A quadratic function $V(x) = x^T Px$ is a Lyapunov function for the LDI system (4.51) if matrix $P \in \mathbb{R}^{n\times n}$ is such that

$$\begin{aligned} &P = P^T > 0, \\ &(A^j)^T PA^j - P < 0, \quad \forall j \in \mathbb{N}_{n_a}. \end{aligned} \tag{4.52}$$

where $A^j \in \mathbb{R}^{n\times n}$ are the n_a matrices whose convex hull defines the LDI system.

Hence, any square matrix $P \in \mathbb{R}^{n\times n}$ satisfying (4.52) determines a Lyapunov function for the LDI system (4.51). This well known result and the fact that the level sets of $V(\cdot)$ are λ-contractive sets for the LDI system (4.51) can be proved similarly to Corollary 4.17.

Here we propose as an optimality criterion to select among every possible Lyapunov function, the minimization of the induced contraction factor of the level sets. Solve the following LMI optimization problem:

$$\begin{aligned}
\max_{P=P^T,\ \gamma>0} \quad & \gamma \\
\text{s.t.} \quad & P > I, \\
& P < \mu I, \\
& (A^j)^T P A^j - P \leq -\gamma P, \quad \forall j \in \mathbb{N}_{n_a},
\end{aligned} \tag{4.53}$$

where $A^j \in \mathbb{R}^{n\times n}$ are the vertices of polytope $\mathscr{A}$.

The optimization variable $\gamma \in \mathbb{R}$ is related to the contraction factor of the level sets. In fact, condition $(A^j)^T P A^j \leq (1-\gamma)P$ is equivalent to

$$(x^+)^T P(x^+) \leq \max_{A\in\mathscr{A}} x^T A^T P A x = \max_{j\in\mathbb{N}_{n_a}} x^T (A^j)^T P (A^j) x \leq (1-\gamma) x^T P x,$$

where x^+ is given by the LDI system (4.17). Then given the ellipsoid $\mathscr{E}(P) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$, from $x \in \mathscr{E}(P)$ it follows that $x^+ \in \sqrt{1-\gamma}\mathscr{E}(P)$, or, in terms of set inclusion, $A\mathscr{E}(P) \subseteq \sqrt{1-\gamma}\mathscr{E}(P)$ for every $A \in \mathscr{A}$. That is equivalent to say that, for every $x \in \mathbb{R}^n$, we have

$$x^T P x \leq \alpha \quad \Rightarrow \quad (x^+)^T P (x^+) \leq (1-\gamma)\alpha, \quad \forall x^+ \in \mathscr{A}(x), \tag{4.54}$$

for all $\alpha \geq 0$.

The contraction factor of the ellipsoid is minimized (and then the decreasing rate of the Lyapunov function is maximized) maximizing γ, subject to condition number constraints, as the inclusion of the constraints $I < P < \mu I$, with $\mu > 1$, guarantees that the condition of matrix P is bounded by μ.

The following definition and lemma allow the determination of a λ-contractive polytope ensuring a contracting factor arbitrarily chosen but greater then the contracting factor of the ellipsoidal invariant set, i.e. $\sqrt{1-\gamma}$.

Definition 4.44 *Given a matrix $H \in \mathbb{R}^{m\times n}$ define the polytope*

$$\mathscr{L}(H) = \{x \in \mathbb{R}^n : \|Hx\|_\infty \leq 1\}. \tag{4.55}$$

The proof of the following lemma can be found in (Álamo, Cepeda, Limón and Camacho, 2006*a*).

Lemma 4.45 *Consider the ellipsoid $\mathscr{E}(P) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$, with $P = P^T > 0$. Suppose that λ_i, $i = 1,\ldots,n$ are the eigenvalues of matrix P and $p^i \in \mathbb{R}^n$, $i = 1,\ldots,n$ the corresponding* *orthonormal eigenvectors. Denote*

$$\Gamma(P) = \mathscr{L}\left(\begin{bmatrix} \sqrt{n\lambda_1}(p^1)^T \\ \dots \\ \sqrt{n\lambda_n}(p^n)^T \end{bmatrix}\right). \tag{4.56}$$

Then

$$\frac{1}{\sqrt{n}}\mathscr{E}(P) \subseteq \Gamma(P) \subseteq \mathscr{E}(P). \tag{4.57}$$

Consider $\Omega = \Gamma(P)$ as defined in (4.56). In the following it will be shown that, choosing an appropriate value of λ, a λ-contractive polytopic set can be iteratively obtained, applying a finitely determined algorithm similar to those presented in (Gilbert and Tan, 1991; Kolmanovsky and Gilbert, 1998). Furthermore, an upperbounding value of the determination index as a function of the contraction factor $\sqrt{1-\gamma}$ of the ellipsoid and the required contraction factor λ for the polytope is provided. Such value allows one to compute the λ-contractive set avoiding the iterative algorithm.

It is useful to recall the concept of reachable sets and in particular to give here the expression for an LDI system characterized by the polytope $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$. Given the initial set $X_0 \subseteq \mathbb{R}^n$, the reachable sets are given by

$$X_{k+1}(X_0) = \mathscr{M}_{\mathscr{A}}(X_k(X_0)) = \bigcup_{x\in X_k(X_0)} \mathscr{A}(x), \quad k \in \mathbb{N}, \tag{4.58}$$

where the dependence on the initial condition is expressed explicitly. Clearly X_0 can be a singleton in $\mathbb{R}^n$.

Theorem 4.46 *Consider P and γ obtained solving the optimization problem (4.53) and $\Omega = \Gamma(P) = \{x \in \mathbb{R}^n : Hx \leq 1\}$ defined in (4.56). Given the contraction factor*

$$\lambda = (\sqrt{1-\gamma}, 1), \tag{4.59}$$

and the integer $i \geq 0$, $C_i^\lambda(\Omega)$ is defined as

$$C_i^\lambda(\Omega) = \{x \in \mathbb{R}^n : X_j(x) \subseteq \lambda^j\Omega,\ \forall j \in \mathbb{N}_{[0,i]}\},$$

where $X_j(x) \subseteq \mathbb{R}^n$ are the reachable sets generated by recursion (4.58) with $X_0(x) = \{x\}$.

Then $C_\infty^\lambda(\Omega)$ is a non-empty λ-contractive set for the LDI system (4.51). Moreover, $C_i^\lambda(\Omega) = C_\infty^\lambda(\Omega)$ for all $i \geq i^*$, where

$$i^* = \frac{\ln n}{\ln\left(\frac{\lambda^2}{1-\gamma}\right)} - 1. \tag{4.60}$$

Proof: Given the function $\Phi(x)$ defined as

$$\Phi(x) = \max_{i\in\mathbb{N}_n} |\sqrt{n\lambda_i}(p^i)^T x|,$$

we have that $x \in \Omega$ if and only if $\Phi(x) \leq 1$, by definition (4.56).

Notice that, given Ω, the set $C_i^\lambda(\Omega)$ is the set of points $x \in \Omega$ such that all the j-th reachable sets, with $j \in \mathbb{N}_{[0,i]}$, generated by (4.58) with initial condition $X_0 = \{x\}$ are subsets of the elements of the sequence of contracted sets $\lambda^j\Omega$, then

$$C_i^\lambda(\Omega) = \{x \in \mathbb{R}^n : Hx_j \leq \lambda^j 1,\ \forall x_j \in X_j(x),\ \forall j \in \mathbb{N}_{[0,i]}\}.$$

Thus, we have that $x \in C_i^\lambda(\Omega)$ if and only if $\Phi(x_j) \leq \lambda^j$, for all $x_j \in X_j(x)$, for all $j \in \mathbb{N}_{[0,i]}$. Clearly $x \in C_\infty^\lambda(\Omega)$ if and only if $\Phi(x_j) \leq \lambda^j$, for all $x_j \in X_j(x)$, for all $j \in \mathbb{N}$. We provide a condition on index i such that, if satisfied, then $x \in C_i^\lambda(\Omega)$ implies $x \in C_\infty^\lambda(\Omega)$. Note that

$$x^T Px = \sum_{i=1}^{n} \lambda_i((p^i)^T x)^2,$$

and from

$$\Phi^2(x) = \max_{i\in\mathbb{N}_n} n\lambda_i((p^i)^T x)^2,$$

it follows that

$$\frac{\Phi^2(x)}{n} = \max_{i\in\mathbb{N}_n} \lambda_i((p^i)^T x)^2 \leq x^T Px.$$

By the definition of $C_i^\lambda(\Omega)$ and (4.57) we have that $x_0 \in C_i^\lambda(\Omega)$ implies $x_0^T Px_0 \leq 1$. Then, as P is the solution to (4.53) and from (4.54), it follows

$$\Phi^2(x_{i+1}) \leq nx_{i+1}^T Px_{i+1} \leq n(1-\gamma)x_i^T Px_i \leq n(1-\gamma)^{i+1}x_0^T Px_0 \leq n(1-\gamma)^{i+1},$$

for all $x_j \in X_j(x_0)$, with $j \in \mathbb{N}_{[0,i]}$, for $i \geq 0$. From the former inequality, it follows that, given $x_0 \in C_i^\lambda(\Omega)$, the condition

$$\Phi(x_{i+1}) \leq \lambda^{i+1}, \quad \forall x_{i+1} \in X_{i+1}(x_0), \quad \forall i \geq i^*,$$

is fulfilled if $n(1-\gamma)^{i+1} \leq \lambda^{2(i+1)}$ for $i \geq i^*$. Notice that this yields $x_0 \in C_\infty^\lambda(\Omega)$ and hence $C_i^\lambda(\Omega) = C_\infty^\lambda(\Omega)$.

Proving that $n(1-\gamma)^{i+1} \leq \lambda^{2(i+1)}$ for all $i \geq i^*$ is equivalent to prove that $n\left(\frac{1-\gamma}{\lambda^2}\right)^{i+1} \leq 1$ for all $i \geq i^*$. A necessary condition for this to be fulfilled is that $\frac{1-\gamma}{\lambda^2} \leq 1$, and it is satisfied by every value of λ defined in (4.59). Then, we have

$$n\left(\frac{1-\gamma}{\lambda^2}\right)^{i+1} \leq 1 \quad \Leftrightarrow \quad \ln n + (i+1)\ln\left(\frac{1-\gamma}{\lambda^2}\right) \leq 0 \quad \Leftrightarrow \quad i+1 \geq \frac{\ln n}{\ln\left(\frac{\lambda^2}{1-\gamma}\right)}. \tag{4.61}$$

Hence, we obtain that if

$$i \geq i^* = \frac{\ln n}{\ln\left(\frac{\lambda^2}{1-\gamma}\right)} - 1,$$

then $n(1-\gamma)^{i+1} \leq \lambda^{2(i+1)}$. ■

It is clear that the greater is γ, the smaller is the admissible contracting factor of the obtained λ-contractive polytopic set, see (4.59). Set $C_\infty^\lambda(\Omega)$ is polytopic and it captures the geometry of the highly contractive ellipsoidal invariant set.

Remark 4.47 *Another positive consequence of computing the maximal γ is that the determination index for computing $C_\infty^\lambda(\Omega)$ is reduced. In fact, from (4.60), the bigger is γ the smaller is i^*.*

4.3.3 Algorithmic computation of a λ-contractive set for CDI systems

We provide here the algorithm to obtain a λ-contractive set for a CDI system. The procedure is based on the steps mentioned at the beginning of this section.

The algorithm is based on the necessary and sufficient condition for invariance and λ-contractiveness of a set for a CDI system, stated in Theorem 4.13. We recall here that, from convexity of the directional bounding functions determining a CDI system, such necessary and sufficient condition is a boundary condition.

Computationally, it is not possible to check the condition for generic sets $\Omega \in \mathscr{K}(\mathbb{R}^n)$, since it can involve an infinite number of constraints, one for every $x \in \partial\Omega$ and for every $\eta \in \mathbb{R}^n$. On the other hand, for the particular case of polytopic sets Ω, the number of constraints is equal to $n_v n_h$, where n_v and n_h are the numbers of vertices and facets of Ω, as proved in the following.

Property 4.48 *Let Assumptions 4.9 and 4.11 hold for the set valued map $\mathscr{F}(\cdot)$ determining the system dynamics (4.1) and the state constraint set X. A polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, with $H \in \mathbb{R}^{n_h \times n}$ and whose vertices are $v^j \in \mathbb{R}^n$ for $j \in \mathbb{N}_{n_v}$, is a λ-contractive set for a $\lambda \in [0,1]$ and constraints $x \in X$ if and only if $\Omega \subseteq X$ and*

$$\check{F}(v^j, H_i^T) \leq \lambda, \quad \forall j \in \mathbb{N}_{n_v}, \quad \forall i \in \mathbb{N}_{n_h}. \tag{4.62}$$

Proof: Since (4.12) is a necessary and sufficient condition for a generic $\Omega \in \mathscr{K}(X)$ to be a λ-contractive set for a CDI system, then the equivalence between (4.12) and (4.62) provides the proof of the property.

We recall that for a polytope $\Gamma = \{x \in \mathbb{R}^n : Hx \leq b\}$, with $H \in \mathbb{R}^{n_h \times n}$ and $b \in \mathbb{R}^{n_h}$, we have that

$$x \in \Gamma \quad \Leftrightarrow \quad \eta^T x \leq \phi_\Gamma(\eta),\ \forall \eta \in \mathbb{R}^n \quad \Leftrightarrow \quad H_i x \leq b_i = \phi_\Gamma(H_i^T),\ \forall i \in \mathbb{N}_{n_h},$$

that is, the condition of set-membership for a point x and a polytope Γ in terms of support function is reduced to a finite number of constraints, concerning only directions H_i^T, for $i \in \mathbb{N}_{n_h}$, see Appendix C. Then from Property 4.12 and Theorem 4.13, we have that the necessary and sufficient condition for λ-contractiveness of polytope Ω is given by

$$\check{F}(x, H_i^T) \leq \lambda, \quad \forall x \in \partial\Omega, \quad \forall i \in \mathbb{N}_{n_h}. \tag{4.63}$$

which involves all the points of the boundary but only n_h directions. We prove in what follows that (4.62) and (4.63) are equivalent. Notice that (4.63) implies (4.62) since the vertices are elements of the boundary, i.e., $v^j \in \partial\Omega$, for all $j \in \mathbb{N}_{n_v}$. We prove the inverse implication.

Assume that (4.62) holds. Any element $\hat{x} \in \partial\Omega$ can be expressed as the convex combination of vertices of Ω, that is there exist a set of n_v real numbers $\theta^j(\hat{x}) \in \mathbb{R}$, for $j \in \mathbb{N}_{n_v}$, such that $\hat{x} = \sum_{j=1}^{n_v} \theta^j(\hat{x})\, v^j$, $\theta^j(\hat{x}) \geq 0$ for all $j \in \mathbb{N}_{n_v}$, and $\sum_{j=1}^{n_v} \theta^j(\hat{x}) = 1$. Then, from (4.62) we have that

$$\check{F}(\hat{x}, H_i^T) = \check{F}(\sum_{j=1}^{n_v} \theta^j(\hat{x})\, v^j, H_i^T) \leq \sum_{j=1}^{n_v} \theta^j(\hat{x})\, \check{F}(v^j, H_i^T) \leq \sum_{j=1}^{n_v} \theta^j(\hat{x})\lambda = \lambda, \quad \forall \hat{x} \in \partial\Omega,$$

for all $i \in \mathbb{N}_{n_h}$, from convexity of functions $\check{F}(\cdot, H_i^T)$. Then condition (4.62) implies (4.63). ■

Then, Property 4.48 provides a necessary and sufficient condition for a polytope to be a λ-contractive set, consisting in $n_v n_h$ constraints. The results presented in such property can be used to check whether a polytopic set is a λ-contractive set, an iterative procedure can be designed. Given a λ-contractive set Ω_k at k-th step, a set $\hat{\Omega}$ such that $\Omega_k \subseteq \hat{\Omega}$ is generated and invariance or contractiveness is checked. If the set $\hat{\Omega}$ fulfils conditions of Property 4.48, then Ω_{k+1} is posed equal to $\hat{\Omega}$ and an enlarged λ-contractive set is obtained.

In what follows we present a result that will permit to generate a point $x^k \in X$ such that $x^k \notin \Omega_k$ and the related set $\hat{\Omega} = \mathrm{co}\ (x^k \cup \Omega_k)$ satisfies Property 4.48.

Property 4.49 *Let Assumptions 4.9 and 4.11 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n :\ Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, and $\lambda \in [0,1]$, such that hypothesis of Property 4.48 holds for Ω, and, given $\hat{x} \in X$, define the set $\hat{\Omega} = co(\Omega \cup \hat{x})$. If $\hat{x} \in X$ is such that $\check{F}(\hat{x}, H_i^T) \leq \lambda$, for every $i \in \mathbb{N}_{n_h}$, then $\hat{\Omega}$ is a λ-contractive set (a control invariant set if $\lambda = 1$) for system (4.1) and constraints $x \in X$.*

Proof: Under the hypothesis of the property, we have that, from Property 4.48,

$$\check{F}(v^j, H_i^T) \leq \lambda, \quad \forall j \in \mathbb{N}_{n_v}, \quad \forall i \in \mathbb{N}_{n_h}, \tag{4.64}$$

where $v^j \in \mathbb{R}^n$, with $j \in \mathbb{N}_{n_v}$, are the vertices of Ω, and

$$\check{F}(\hat{x}, H_i^T) \leq \lambda, \quad \forall i \in \mathbb{N}_{n_h}. \tag{4.65}$$

Recall that any element of a polytope can be expressed as the convex combination of its vertices and notice that vertices of $\hat{\Omega}$ are given by a subset of the vertices of Ω and, eventually, point $\hat{x}$. Moreover, for any element $x \in \hat{\Omega}$, there exists $\theta \in \mathbb{R}^{n_h+1}$ with $\theta_j \geq 0$, for all $j \in \mathbb{N}_{n_h+1}$, such that $\sum_{j=1}^{n_h+1} \theta_j = 1$ and

$$x = \sum_{j=1}^{n_h} \theta_j v^j + \theta_{n_h+1}\hat{x}.$$

Hence, as in the proof of Property 4.48 and from (4.64) and (4.65), it can be proved that

$$\check{F}(x, H_i^T) \leq \lambda, \quad \forall i \in \mathbb{N}_{n_h}, \tag{4.66}$$

for all $x \in \hat{\Omega}$. Since, from Theorem 4.13, condition (4.66) is equivalent to

$$\mathscr{F}(x) \subseteq \lambda\Omega,$$

then, for every $x \in \hat{\Omega}$ we have that $\mathscr{F}(x) \subseteq \lambda\Omega \subseteq \lambda\hat{\Omega}$ and this means that $\hat{\Omega}$ is a λ-contractive set for system (4.1). ■

Now, given a direction $\eta^k \in \mathbb{R}^n$ (assumed generated randomly in the algorithm) and the set $\Omega_k = \{x \in \mathbb{R}^n : \; H^k x \leq 1\}$ with $H^k \in \mathbb{R}^{n_h^k \times n}$, we compute the point x^k as the solution of the following convex programming problem:

$$\begin{aligned} \max_{x^k \in X} \quad & \eta^k x^k \\ \text{s.t.} \quad & \check{F}(x^k, (H_i^k)^T) \leq \lambda, \quad \forall i \in \mathbb{N}_{n_h^k}. \end{aligned} \tag{4.67}$$

The optimizer of problem (4.67) is such that $x^k \in X$ and either $x^k \in \partial\Omega_k$ or $x^k \notin \Omega_k$ and satisfies condition of Property 4.49. Then co $(\Omega_k \cup x^k)$ is a λ-contractive set. Hence, it has to be only checked if x^k lies on the boundary or not.

Finally, we provide the algorithm to compute a λ-contractive polytopic set for a CDI system.

Algorithm 1 Computing a λ-contractive set for a CDI system (4.1).

Given the CDI system (4.1) under Assumption 4.9:

(1) Obtain the polytopic LDI system locally overbounding the CDI one. Denote with A^j the n_a matrices determining the LDI system.

(2) Obtain P and γ from optimization problem (4.53) for the overbounding LDI system.

(3) Choose $\hat{\lambda} \in (\sqrt{1-\gamma}, 1)$ and obtain $\Omega_L = C_i^{\hat{\lambda}}(\Gamma(P))$ where

$$i = \left\lceil \frac{\ln n}{\ln\left(\frac{\hat{\lambda}^2}{1-\gamma}\right)} \right\rceil - 1.$$

(4) Choose $\lambda \in (\hat{\lambda}, 1]$ and compute α_L, the maximal $\alpha > 0$ such that $\Omega = \alpha\Omega_L$ fulfils condition (4.62). Pose $\Omega_0 = \alpha_L \Omega_L$ and $k = 0$.

(5) Generate $\eta^k \in \mathbb{R}^n$ and compute $x^k \in X$ as an optimizer of the convex problem (4.67).

(6) If $x^k \notin \Omega_k$, then $\Omega_{k+1} = \mathrm{co}(\Omega_k \cup x^k) = \{x \in \mathbb{R}^n : \ H^{k+1}x \leq 1\}$, with $H^{k+1} \in \mathbb{R}^{n_h^{k+1} \times n}$ and a proper $n_h^{k+1} \in \mathbb{N}$, otherwise go to (5).

(7) Pose $k = k+1$. If $k \geq k_{max}$ stop, otherwise go to (5).

(8) Return Ω_k, λ-contractive set for system (4.1) with contracting factor λ.

Remark 4.50 *In case that the nonlinear system is overbounded by an uncertain CDI system (for generalized saturated systems, for instance, see Section 3.3), slight modifications of the algorithm have to be introduced. That is, in case that the CDI system bounding the nonlinear one has the form*

$$x^+ \in \mathscr{F}(x) \oplus W,$$

one possible way to proceed, is to consider the CDI system

$$x^+ \in \mathscr{F}(x),$$

for the first three steps of the algorithm. That is, the procedure to generate an LDI system and to compute first an ellipsoidal invariant set and then a polytopic one for such LDI system, might be performed neglecting the additive uncertainty bounded by W. Alternatively, methods, present in literature, for computing ellipsoidal robust invariant sets can be applied.

Then, in step (4), the uncertainty can be considered by replacing condition (4.62) with the following

$$\check{F}(v^j, H_i^T) \leq \lambda - \phi_W(H_i^T), \quad \forall j \in \mathbb{N}_{n_v}, \quad \forall i \in \mathbb{N}_{n_h}, \tag{4.68}$$

which, can be proved to be the necessary and sufficient condition for invariance (λ-contractiveness, in fact) for uncertain CDI systems and polytopic Ω.

For what concerns the enlarging procedure, in step (5) the convex optimization problem (4.67) to be solved to obtain x^k, has to be replaced with

$$\begin{array}{ll} \max\limits_{x^k \in X} & \eta^k x^k \\ s.t. & \check{F}(x^k, (H_i^k)^T) \leq \lambda - \phi_W(H_i^T), \quad \forall i \in \mathbb{N}_{n_h^k}, \end{array} \tag{4.69}$$

in which the effect of the additive uncertainty is taken into account. The following steps are not affected by the presence of the additive uncertainty.

The resulting sequence of sets are polytopic λ-contractive sets for the uncertain CDI system with contraction factor λ.

4.3.4 Numerical example

We provide here an example of the application of Algorithm 1 for computing an invariant set for CDI systems.

We consider a generalized saturated system (3.25), see Section 3.3, that is

$$x_{k+1} = Ax_k + B\varphi(Fx_k, k),$$

with matrices

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1.1 \end{bmatrix}, \quad F = \begin{bmatrix} -0.5236 & -1.1264 \end{bmatrix},$$

and the generalized saturated function bounded as

$$-\Gamma(-y) \leq \varphi(y,k) \leq \Gamma(y), \quad \forall y \in \mathbb{R}, \forall k \in \mathbb{N},$$

where $\Gamma(y) = \max\{\mu(y+\sigma), -y_0\}$ with $\mu = 1$, $\sigma = 0.2$ and $y_0 = 1.8$. The bounding sector characterizing the generalized saturated function is represented in Figure 4.1.

We recall that, as shown in Section 3.3, the generalized saturated function can be overbounded by the uncertain CDI system

$$x^+ \in \mathscr{F}(x) \oplus W,$$

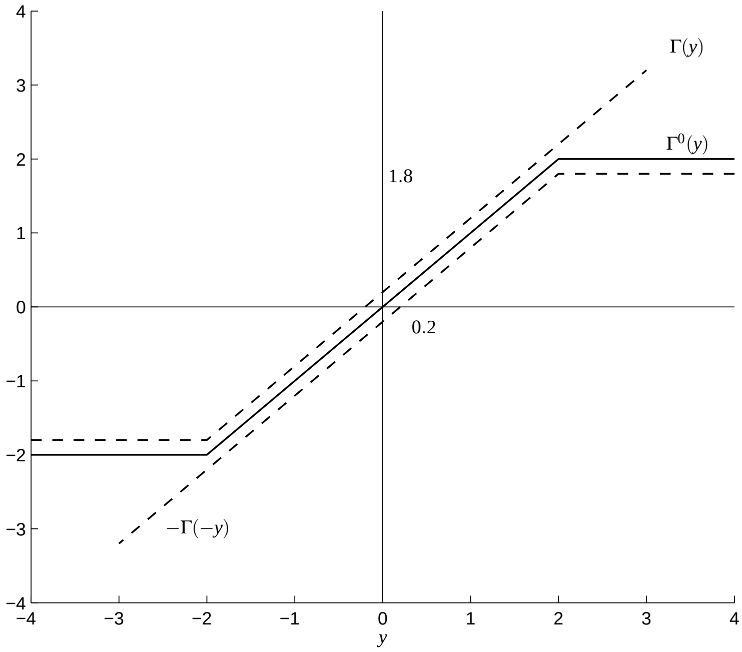

Figure 4.1: Bounding sector for the generalized saturated function.

where the set valued map $\mathscr{F}(\cdot)$ satisfies Assumption 2.5 with convex bounding functions given by (3.27), that is

$$\check{f}_\eta(x) = \begin{cases} \eta^T Ax + \eta^T B\Gamma^0(Fx), & \text{if } \eta^T B \geq 0, \\ \eta^T Ax - \eta^T B\Gamma^0(-Fx), & \text{if } \eta^T B < 0, \end{cases}$$

for all $\eta \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$ and with

$$\Gamma^0(y) = \max\{\mu y, -y_0 - \mu\sigma\} = \max\{y, -2\},$$

and the bounds on the additive uncertainty are

$$W = \{w = Bv : -\mu\sigma \leq v \leq \mu\sigma\} = \{w = Bv : -0.2 \leq v \leq 0.2\}.$$

The state are assumed to be constrained in the region

$$X = \{x \in \mathbb{R}^2 : \ -15 \leq x_1 \leq 15, \ \ -6 \leq x_2 \leq 6\}.$$

As illustrated in Remark 4.50, we consider first the CDI system, neglecting the contribution of the additive uncertainty for the first steps of the algorithm. Notice that a local LDI

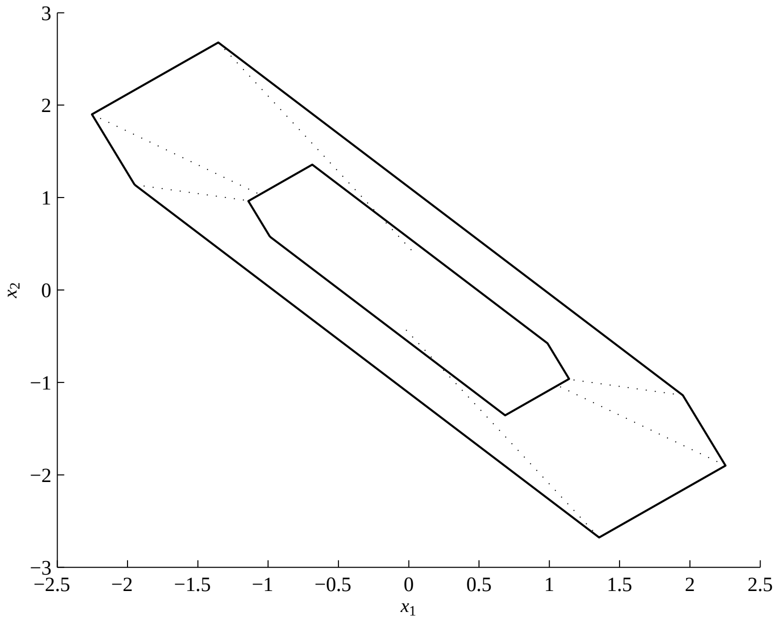

Figure 4.2: Sets Ω_L and $\hat{\lambda}\Omega_L$ and evolutions of the vertices of Ω_L.

system overbounding the CDI one is given by

$$x_{k+1} = (A+BF)x_k = \begin{bmatrix} 0.8382 & 0.4368 \\ -0.5760 & -0.1390 \end{bmatrix} x_k,$$

whose eigenvalues are $0.3496 \pm 0.1133i$, lying in the unitary circle. Notice also that, actually, the LDI system is a linear one.

As a matter of fact, in the region of the state space given by

$$D = \{x \in \mathbb{R}^n : \ |Fx| \leq \frac{y_0}{\mu} + \sigma\} = \{x \in \mathbb{R}^n : \ |Fx| \leq 2\},$$

the CDI system (in absence of additive uncertainty) and the linear one are exactly the same.

Hence, we solve the optimization problem (4.53) which provides the ellipsoidal invariant set for the linear system. The ellipsoid is defined by matrix

$$P = \begin{bmatrix} 9.5583 & 7.4099 \\ 7.4099 & 7.4183 \end{bmatrix},$$

with $\gamma = 0.7693$, which means a contracting factor $\sqrt{1-\gamma} = 0.4803$. In step (3) we compute the polytopic λ-contractive set Ω_L for the linear system with $\hat{\lambda} = 0.5063$, that yields to

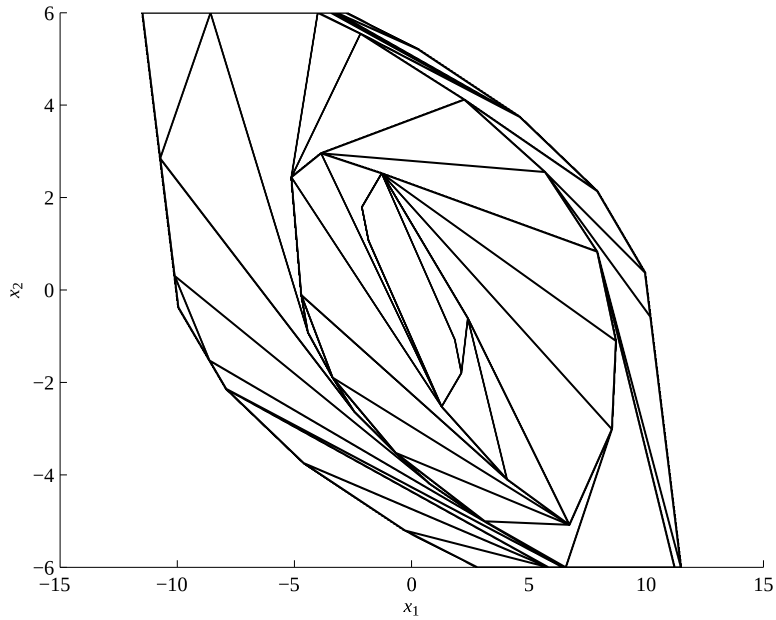

Figure 4.3: Sequence of robust invariant sets Ω_k for the CDI system, for $k \in \mathbb{N}_{[0,k_{max}]}$, generated by the enlarging process.

determination index $i = 7$. It is worth pointing out that, since the overbounding system is a linear one, standard algorithms for computation of polytopic invariant sets could have been applied in this case. Figure 4.2 shows the polytopic invariant set Ω_L as well as the polytope $\hat{\lambda}\Omega_L$. Moreover, in dotted lines, the evolutions of the vertices of Ω_L through the linear system are depicted. It can be notice that every vertex of Ω_L is mapped inside the set $\hat{\lambda}\Omega_L$, which is a graphical confirmation of λ-contractiveness of Ω_L.

Now, since we are interested in a robust invariant set for the uncertain CDI system, we choose $\lambda = 1$ and apply the following steps of the algorithm. Due to the presence of additive uncertainty, the modification of the algorithm exposed in Remark 4.50 are applied. In Figure 4.3, the sequence of robust invariant sets generated by the enlarging process are depicted. The inner set is Ω_0 computed at step (4) of the algorithm, obtained by means of a dichotomic procedure.

Finally, the biggest robust invariant set $\Omega_{k_{max}}$, with $k_{max} = 100$, computed through Algorithm 1 is shown in Figure 4.4. Notice that the state constraints are satisfied, as the robust invariant set is contained in the set X.

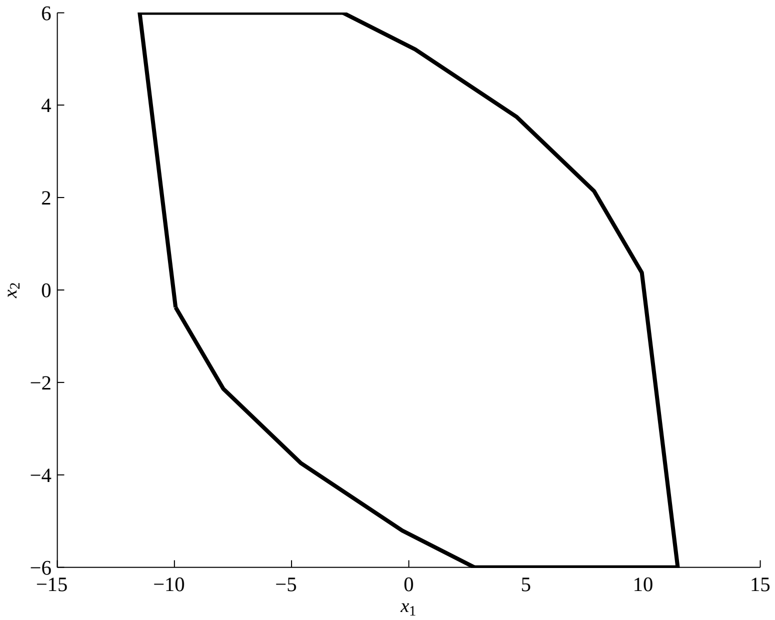

Figure 4.4: Robust invariant set $\Omega_{k_{max}}$ for the CDI system, generated through Algorithm 1.

4.4 Conclusions

In this chapter invariance and λ-contractiveness of convex, compact sets for CDI systems have been characterized. First, directional lower and upper bounding functions, denoted $\hat{F}(\cdot,\cdot)$ and $\check{F}(\cdot,\cdot)$ respectively, have been introduced and their properties illustrated.

Conditions for invariance and λ-contractiveness are posed as a sets of constraints involving directional upper bounding functions. Thanks to convexity of such constraints, they can be imposed only at the boundary of the potential invariant (λ-contractive) set, unlike generic nonlinear systems. This will lead, in next chapters, to define procedures for computing polytopic invariant sets based on convex constraints satisfaction, hence with affordable computational requirements, for particular classes of nonlinear systems.

Also the classical tool for iterative computation of invariant sets, the one-step operator, has been analyzed for CDI systems. It has been proved that the one-step operator for CDI systems preserves convexity and compactness, under mild assumptions. The related results can be applied to generate a sequence of invariant sets for the CDI system, converging to the domain of attraction, provided asymptotic stability of the origin. Properties of concave directional lower bounding functions allow us to determine convex regions of the state space contained in the complement of the one-step set, for a given set Ω.

The chapter ends with a section treating some computational aspects. In particular, a procedure to obtain an invariant or a λ-contractive set for a CDI system is illustrated. The procedure is based on the necessary and sufficient condition for invariance and λ-contractiveness for CDI systems.

Chapter 5

Convex invariant sets for nonlinear systems

In this chapter the practical problem of how to obtain an invariant set for a nonlinear system is addressed. The attention is focused on two particular classes of nonlinear systems, that is, DC and Lur'e systems, whose dynamics are given by single valued functions rather than set valued maps. Recall that both frameworks enclose a wide family of nonlinearities frequently encountered when dealing with real systems. The main objective of this chapter is to provide algorithmic procedures for obtaining an invariant set or a λ-contractive set for a given nonlinear system.

Summarizing, in this chapter we particularize the results presented in Chapter 4 to a very wide class of nonlinear systems. The particular nature of DC systems and Lur'e systems and the generality of properties presented in Chapter 4, allow us to provide relevant results valid for a wide family of nonlinear systems and to define practical algorithmic procedures for the computation of λ-contractive and invariant sets.

In the first section, a sufficient condition for invariance for constrained DC systems is given. In particular, exploiting some properties of DC functions, a condition for invariance of a polytopic set is provided. Both cases of deterministic and uncertain systems are considered. An algorithm for computing a local invariant set, possibly a λ-contractive set, for a nonlinear deterministic DC system is presented. It will be shown that, under mild conditions, the algorithm always provide a non-empty invariant set.

Similarly, a condition for robust invariance for the uncertain nonlinear DC systems is proposed. A relation between the contraction factor of the λ-contractive set for the nominal nonlinear system and a measure of the maximal uncertainty that can be tolerated before the set loses invariance in case of uncertainty is provided.

In the second section, an analysis method to estimate the domain of attraction of a class of discrete-time Lur'e systems is presented. A new notion of invariance, denoted *LNL*-invariance is introduced. This new concept generalizes the notion of *SNS*-invariance introduced in (Álamo, Cepeda, Limón and Camacho, 2006*b*) for saturated systems. Although a discrete-time Lur'e system can be considered as a particular case of CDI system, as shown in Chapter 3, we present a specific method to address the particular problem. Then we will show that the problem can be solved also from the point of view of CDI systems. In fact, we will see that the definition of a CDI system overbounding the Lur'e one and the application of properties of CDI systems, will lead to the same results obtained through the *ad hoc* method for Lur'e systems.

5.1 Convex invariant sets for DC systems

An autonomous DC system is a nonlinear system whose dynamic function fulfills Assumption 3.17. We recall here that with Assumption 3.17 we suppose that $f(\cdot)$ is a DC function defined on the convex set $D \subseteq \mathbb{R}^n$ with $0 \in \text{int}(D)$, it is differentiable at the origin, and the convex functions $g(\cdot)$ and $h(\cdot)$, such that $f(x) = g(x) - h(x)$, satisfy $g(0) = 0$ and $h(0) = 0$.

Summarizing, we consider the autonomous nonlinear discrete-time system

$$x^+ = f(x), \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is a DC function satisfying Assumption 3.17.

Assumption 3.17 implies that the origin is a root of the DC function, that is $f(0) = 0$. Note that if $f(0) = 0$, then condition $g(0) = 0$ and $h(0) = 0$ yields no loss of generality. In fact, if $g(0) = r$ and $h(0) = s$, then $r = s$ from $f(0) = 0$. Denoting $\hat{g}(x) = g(x) - r$ and $\hat{h}(x) = h(x) - r$, we have that $f(\cdot)$ admits a different DC representation as $f(x) = \hat{g}(x) - \hat{h}(x)$ with $\hat{g}(0) = 0$ and $\hat{h}(0) = 0$. Furthermore, in what follows, we will assume that function $f(\cdot)$ is twice differentiable, see Assumption 5.1.

It is worth recalling that any convex function defined on an open set D is continuous on any (relatively) open subset $X \subseteq D$. Since a DC function is representable as the difference of two convex functions, Assumption 3.17 implies local continuity of $f(\cdot)$, see Theorem B.10.

Assumption 5.1 *Assume that a given DC function $f : D \to \mathbb{R}^n$ is twice differentiable at the* *origin and the eigenvalues of the Jacobian of $f(\cdot)$ at the origin,*

$$J_x(0) = \begin{bmatrix} \nabla_x^T f_1(0) \\ \nabla_x^T f_2(0) \\ \cdots \\ \nabla_x^T f_n(0) \end{bmatrix},$$

lie in the interior of the unitary circle.

Assumption 5.1 implies that the origin is an exponentially stable equilibrium for the linearized system

$$x^+ = J_x(0)x. \tag{5.2}$$

From classic stability theory for nonlinear systems, see for instance the center manifold theory, it follows that there exists $\varepsilon > 0$ such that for any initial condition $x_0 \in \varepsilon \mathbf{B}_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq \varepsilon\} \subseteq D$ the corresponding trajectory converges to the origin, that is $\lim\limits_{k\to\infty} x_k = 0$, where $x_{k+1} = f(x_k)$.

Moreover, twice differentiability of the dynamic DC function is useful to prove that the proposed procedure guarantees the computation of a non-empty invariant set. In fact, as shown in what follows, if the dynamic DC function is twice differentiable, then a λ-contractive set, hence invariant, for the linearized system is also invariant for the DC system, if appropriately scaled.

Remark 5.2 *With Assumption 5.1, function $f(\cdot)$ is supposed to be twice differentiable at the origin. Then, under this assumption, there exist two constants $\rho > 0$ and $\sigma > 0$ such that $\sigma \mathbf{B}_2^n \subseteq D$ and*

$$\|f(x) - J_x(0)x\|_\infty \leq \rho x^T x, \quad \forall x \in \sigma \mathbf{B}_2^n,$$

where, by definition, $\mathbf{B}_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$.

In fact, from the Lagrange form of the Remainders, we have that the Taylor expansion of $f_i(\cdot)$, for every $i \in \mathbb{N}_n$, is defined in $\sigma \mathbf{B}_2^n$ as

$$f_i(x) = \nabla_x^T f_i(0)x + x^T H_{f_i}(c^i(x))x$$

for a $c^i(x) \in \sigma \mathbf{B}_2^n$ and a proper $\sigma > 0$, where $H_{f_i}(\cdot)$ is the Hessian of function $f_i(\cdot)$, defined in a neighborhood of the origin, for all $i \in \mathbb{N}_n$. Then a finite bound on the linearization error can be found, yielding to an overbounding CDI system, see Example 2.8.

5.1.1 Contractiveness and invariance condition for DC systems

The theorems presented in this section provide sufficient conditions for checking whether a set is an invariant set, or a λ-contractive set, for a DC dynamic system. The main feature of these conditions is the affordable computational burden required, for the polytopic case. This allows one to employ them to design a simple algorithm for computing a convex λ-contractive invariant set for the DC system.

Property 5.3 *Given a DC function $f : D \to \mathbb{R}^n$, function $\eta^T f(\cdot)$ is a DC function defined on $D \subseteq \mathbb{R}^n$, for every $\eta \in \mathbb{R}^n$.*

Proof: The claim follows directly from the fact that DC functions are closed under the sum operator and since

$$\eta^T f(x) = \sum_{i=1}^{n} \eta_i f_i(x),$$

for all $x \in D$ and $\eta \in \mathbb{R}^n$. ■

In the following, the function $\check{F}(\cdot, \eta)$ related to a generic vector $\eta \in \mathbb{R}^n$ is defined for the CDI system overbounding the DC system, see Definition 4.3. We have already proved, see Proposition 3.18, that for any DC system there exists an overbounding CDI system directly determined by the convex bounding functions $\check{f}_\eta(\cdot)$, for $\eta \in \mathbb{R}^n$, defined as

$$\check{f}_\eta(x) = \sum_{j \in k_+} \eta_j \left(g_j(x) - h_j^L(x) \right) + \sum_{j \in k_-} \eta_j \left(g_j^L(x) - h_j(x) \right), \tag{5.3}$$

for every $x \in \mathbb{R}^n$ and every $\eta \in \mathbb{R}^n$, where $g_j^L(x) = \nabla_x^T g_j(0)x$ and $h_j^L(x) = \nabla_x^T h_j(0)x$, for $j \in \mathbb{N}_n$ and $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n : \ \eta_j \geq 0\}$ and $k_- = k_-(\eta) = \{j \in \mathbb{N}_n : \ \eta_j < 0\}$. Notice that $k_+(\eta)$ is the set of indexes of non-negative elements of vector η, and $k_-(\eta)$ the set of indexes of negative elements of η.

Definition 5.4 *Given the DC function $f : D \to \mathbb{R}^n$ as in (5.1) such that Assumption 3.17 holds and a $\eta \in \mathbb{R}^n$, define the directional upper bounding function $\check{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as*

$$\check{F}(x, \eta) = \check{f}_\eta(x), \tag{5.4}$$

where functions $\check{f}_\eta(\cdot)$, for every $\eta \in \mathbb{R}^n$, are given by (5.3).

Convexity of directional bounding functions $\check{F}(\cdot, \eta)$, for all $\eta \in \mathbb{R}^n$, stems directly from convexity of functions $\check{f}_\eta(\cdot)$.

Property 5.5 *Given the DC function $f : D \to \mathbb{R}^n$ as in (5.1) such that Assumption 3.17 holds, for every $\eta \in \mathbb{R}^n$, function $\check{F}(\cdot,\eta)$ defined in (5.4) is convex with respect to $x \in D$.*

Proof: Although it is a direct consequence of (5.4), a sketch of the proof is recalled here. By Definition 5.4, function $\check{F}(\cdot,\eta)$ is the sum of elements composed by the sum of a convex term and a linear one. In fact, if $j \in k_+$ then $\eta_j \geq 0$, $\eta_j g_j(x)$ is convex and $-\eta_j h_j^L(x)$ is linear; for $j \in k_-$, since $\eta_j < 0$ and $-h_j(x)$ is concave, $-\eta_j h_j(x)$ is convex and $\eta_j g_j^L(x)$ is linear. From the fact that a linear function is convex (and concave), and since the sum of two convex functions is a convex function, then $\check{F}(\cdot,\eta)$ is convex with respect to $x \in D$, for all $\eta \in \mathbb{R}^n$. ■

The following theorem states that the approximation error $|\eta^T f(x) - \check{F}(x,\eta)|$, due to the overbounding process, vanishes quadratically as x tends to the origin. This property will be used in the following for proving that the proposed algorithm always provides an invariant set for the deterministic DC system.

Theorem 5.6 *Let Assumptions 3.17 and 5.1 hold. There exist $\delta > 0$ and $\rho > 0$ such that*

$$|\eta^T f(x) - \check{F}(x,\eta)| \leq \|\eta\|_\infty \rho x^T x,$$

for all $\eta \in \mathbb{R}^n$ and $x \in \delta \mathbf{B}_2^n$, where function $\check{F}(\cdot,\cdot)$ is defined in (5.4).

Proof: By Assumption 5.1, $g(\cdot)$ and $h(\cdot)$ are twice differentiable at the origin. Then, there exists a $\delta > 0$ such that $g(\cdot)$, and $h(\cdot)$ are twice differentiable in $\delta \mathbf{B}_2^n$. From twice differentiability of $g(\cdot)$ and $h(\cdot)$, it follows that there are $\rho_{g,j} > 0$ and $\rho_{h,j} > 0$, $j \in \mathbb{N}_n$, such that

$$\begin{aligned} |g_j(x) - g_j^L(x)| &\leq \rho_{g,j} x^T x, \\ |h_j^L(x) - h_j(x)| &\leq \rho_{h,j} x^T x, \end{aligned} \tag{5.5}$$

for all $j \in \mathbb{N}_n$ and for all $x \in \delta \mathbf{B}_2^n$, see Remark 5.2. From (5.5) and by definition of function $\check{F}(\cdot,\cdot)$ we have that

$$\begin{aligned} &|\eta^T f(x) - \check{F}(x,\eta)| = |\sum_{j \in k_+} \eta_j (h_j^L(x) - h_j(x)) + \sum_{j \in k_-} \eta_j (g_j(x) - g_j^L(x))| \leq \\ &\leq \sum_{j \in k_+} |\eta_j| \, |h_j^L(x) - h_j(x)| + \sum_{j \in k_-} |\eta_j| \, |g_j(x) - g_j^L(x)| \leq \\ &\leq \|\eta\|_\infty \left(\sum_{j \in k_+} |h_j^L(x) - h_j(x)| + \sum_{j \in k_-} |g_j(x) - g_j^L(x)| \right) \leq \|\eta\|_\infty \sum_{j=1}^{n} \left(\rho_{g,j} + \rho_{h,j}\right) x^T x, \end{aligned}$$

for every $\eta \in \mathbb{R}^n$ and $x \in \delta \mathbf{B}_2^n$. Making $\rho = \sum_{j=1}^{n} \left(\rho_{g,j} + \rho_{h,j}\right)$, the theorem is proved. ■

Note that the constant ρ does not depend on the vector η.

In the following it is proved that, given a vector $\eta \in \mathbb{R}^n$, the function $\check{F}(\cdot,\eta)$ provides an upper bound of the DC function $\eta^T f(\cdot)$.

Property 5.7 *Given the DC function $f: D \to \mathbb{R}^n$ as in (5.1) such that Assumption 3.17 holds, for every $\eta \in \mathbb{R}^n$ we have*

$$\eta^T f(x) \leq \check{F}(x,\eta), \quad \forall x \in D, \tag{5.6}$$

where $\check{F}(\cdot,\eta)$ is defined in (5.4).

Proof: The theorem follows directly from Property 3.18 and Definition 5.4. In fact, the convex bounding functions $\check{f}_\eta(\cdot)$, for $\eta \in \mathbb{R}^n$, defined in Property 3.18 determine an overbounding CDI system, which means

$$\eta^T f(x) \leq \check{f}_\eta(x) = \check{F}(x,\eta), \quad \forall x \in D$$

for all $\eta \in \mathbb{R}^n$, where the equality is given by Definition 5.4. ■

We now address the analysis of λ-contractiveness and invariance for nonlinear systems, particularizing the results of Chapter 4 to the case of DC systems. In (Kolmanovsky and Gilbert, 1998) a characterization of invariance for linear systems in terms of support functions is given, as well as some properties of the support functions.

The necessary and sufficient condition for invariance for an autonomous linear uncertain system, presented in (Kolmanovsky and Gilbert, 1998), is adapted below to formulate the condition for λ-contractiveness and invariance of a set for any nonlinear system (5.1). State constraints, $x \in X$, are considered. We recall that with Assumption 4.11, used in what follows, we suppose that the constraint set on the state $X \subseteq \mathbb{R}^n$, is closed, convex and with $0 \in \text{int}(X)$.

Property 5.8 *Let Assumption 4.11 hold for constraints $x \in X$. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \in \mathscr{K}^0(D)$ is a λ-contractive set (an invariant set if $\lambda = 1$) for the nonlinear system (5.1) and constraints $x \in X$ if and only if*

$$\eta^T f(x) \leq \lambda \phi_\Omega(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{5.7}$$

Proof: Recall that a set $\Omega \in \mathscr{K}^0(X)$ is a λ-contractive set for system (5.1) if $\Omega \subseteq X$ and $f(\Omega) \subseteq \lambda\Omega$, see (A.9), and, from convexity of set Ω and Property C.4, we have that Ω is λ-contractive if and only if

$$\phi_{f(\Omega)}(\eta) \leq \phi_{\lambda\Omega}(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{5.8}$$

which, from Property C.5, is equivalent to

$$\phi_{f(\Omega)}(\eta) \leq \lambda \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n.$$

Since Ω is a compact set then, for any $\eta \in \mathbb{R}^n$, and by definition of support function we have

$$\phi_{f(\Omega)}(\eta) = \max_{z \in f(\Omega)} \eta^T z = \max_{x \in \Omega} \eta^T f(x).$$

and therefore conditions (5.7) and (5.8) are equivalent and the claim follows. ■

The meaning of Property 5.8 is that the image of any element $x \in \Omega$ through the nonlinear function $f(\cdot)$ has to be contained inside the set $\lambda\Omega \subseteq X$, definition of λ-contractiveness. Hence the necessary and sufficient condition for a set Ω to be λ-contractive (5.7) is a condition involving any element of the set Ω and any direction $\eta \in \mathbb{R}^n$. Such condition is given, then, by an infinite number of non-convex constraints, one for every $x \in \Omega$ and every $\eta \in \mathbb{R}^n$.

First we present a convex relaxation of condition for λ-contractiveness, in order to obtain an only sufficient condition but composed by convex constraints involving only the elements on the boundary of the convex set $\Omega \subseteq D$.

Theorem 5.9 *Let Assumptions 3.17 and 4.11 hold for the system dynamics (5.1) and the state constraint set X. A compact, convex set $\Omega \in \mathscr{K}^0(X)$ such that*

$$\check{F}(x,\eta) \leq \lambda \phi_\Omega(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n \tag{5.9}$$

where function $\check{F}(\cdot,\cdot)$ is defined in (5.4) and $\lambda \in [0,1]$, is a λ-contractive set (an invariant set if $\lambda = 1$) for system (5.1) and constraints $x \in X$ with contraction factor λ.

Proof: From Property 5.7, it follows that

$$\check{F}(x,\eta) \leq \lambda \phi_\Omega(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n, \tag{5.10}$$

implies satisfaction of condition (5.7), and then λ-contractiveness of Ω. Since the inverse is not true in general, as $f \in S_{\mathscr{F}}$ with $\mathscr{F}(\cdot)$ set valued map of the overbounding CDI system, the condition is only sufficient. To finish the proof, we have to show that (5.9) is satisfied if and only if condition (5.10) is fulfilled. We refer to the proof of Theorem 4.13, in which an analogous property is proved for CDI systems. ■

Theorem 5.9 provides a sufficient condition for λ-contractiveness of a set Ω based on a set of convex constraints. Remind that the condition for invariance for nonlinear discrete-time systems can be restricted to the boundary of the set only for particular cases, such as linear and positively homogeneous systems, see (Blanchini and Miani, 2008), while inequalities (5.9) provide a condition for λ-contractiveness and invariance involving only the boundary of Ω. Furthermore, in the particular case in which Ω is a polytope, the condition will be given by a finite number of convex constraints.

Before that, results regarding the case of nonlinear uncertain systems, with additive uncertainty, are presented.

5.1.1.1 Robust invariant set for DC systems

In the previous section the necessary and sufficient condition of λ-contractiveness for a compact set Ω for a nonlinear deterministic DC system is given in Property 5.8. A convex relaxation yielding to a sufficient condition for λ-contractiveness and invariance has been provided in Theorem 5.9. Analogous conditions for uncertain nonlinear systems will be provided in this section, in particular, the results presented in the previous section can be directly extended to DC systems presenting additive uncertainties.

Consider the uncertain autonomous nonlinear system

$$x^+ = f(x) + w, \tag{5.11}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$ is the bounded additive uncertainty $w \in W$ and $f : D \to \mathbb{R}^n$ fulfills the Assumption 3.17.

First we provide the necessary and sufficient condition for a compact, convex set $\Omega \in \mathscr{K}^0(X)$ to be a robust λ-contractive set and a robust invariant set for the uncertain nonlinear system (5.11). We recall here that with Assumption 2.3 we suppose that $W \subseteq \mathbb{R}^n$ is a compact set in the state space with $0 \in \text{int}\,(\text{co}\,(W))$.

Property 5.10 *Let Assumptions 2.3 and 4.11 hold. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \in \mathscr{K}^0(X)$ is a λ-contractive set (a robust invariant set if $\lambda = 1$) for system (5.11) and constraints $x \in X$ if and only if*

$$\eta^T f(x) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{5.12}$$

Proof: By definition of Minkowski summation, the image of set Ω through dynamic function of system (5.11) is given by $f(\Omega) \oplus W$. As proved for Property 5.8, condition of λ-contractiveness of Ω (invariance if $\lambda = 1$) in terms of set inclusion is that the image of Ω through the dynamic function is contained in the set $\lambda\Omega \in \mathscr{K}^0(X)$, which means $f(\Omega) \oplus W \subseteq \lambda\Omega$. In terms of support function, we have that Ω is a λ-contractive set with contraction factor λ if and only if

$$\phi_{f(\Omega)\oplus W}(\eta) \leq \phi_{\lambda\Omega}(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{5.13}$$

and from Property C.6, it follows that this is equivalent to

$$\phi_{f(\Omega)}(\eta) \leq \phi_{\lambda\Omega}(\eta) - \phi_W(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

and therefore, equivalence between λ-contractiveness and condition (5.12) follows. ■

Notice that no assumption on convexity of set W has been required. As for the case of deterministic DC system, condition is given by a constraint, nonconvex in principle, that has to be proved for every point $x \in \Omega$ and every vector $\eta \in \mathbb{R}^n$. Analogously, exploiting convexity of function $\check{F}(\cdot,\cdot)$ defined in Definition 5.4, a sufficient condition can be formulated such that its fulfillment at the points of the boundary of Ω implies λ-contractiveness for the uncertain nonlinear system (5.11).

We recall here the meaning of the assumptions involved in the following theorem, characterizing a λ-contractive set for a DC dynamic systems in presence of additive uncertainties and state constraints. Assumption 3.17 concerns the DC nature of the dynamic function $f(\cdot)$, Assumption 4.11 the hypothesis on the state constraints X and Assumption 2.3 the uncertainty bounding set W.

Theorem 5.11 *Let Assumptions 2.3, 3.17 and 4.11 hold. A compact, convex set $\Omega \in \mathscr{K}^0(X)$, such that*

$$\check{F}(x,\eta) \leq \lambda_w \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n, \tag{5.14}$$

where function $\check{F}(\cdot,\cdot)$ is defined in (5.4) and $\lambda_w \in [0,1]$, is a λ-contractive set (an invariant set if $\lambda = 1$) for system (5.11) and constraints $x \in X$ with contraction factor λ_w.

Proof: From Properties 5.7 and 5.10, it follows that if condition

$$\check{F}(x,\eta) \leq \lambda_w \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n, \tag{5.15}$$

is fulfilled, then Ω is a λ-contractive set for the uncertain autonomous DC system (5.11). Thus, the condition is only sufficient, as the converse is not true in the general case. As for the proof of Theorem 5.9, we have to prove that condition (5.15) involving every element of Ω is satisfied if and only if condition on the boundary (5.14) is fulfilled. As in the case of proofs of Theorems 4.23 and 5.9, the result can be proved following the line of the proof of Theorem 4.13. ■

Notice that, as for the case of deterministic nonlinear DC system, the condition for λ-contractiveness and robust invariance in presence of additive uncertainty (5.15) involves only points on the boundary of the set Ω.

Given a λ-contractive set Ω with contraction factor λ_n for the deterministic system (5.1), consider the uncertain system (5.11) with same DC function $f(\cdot)$. An explicit relation between the contraction factor λ_n and the uncertainty bounding set W such that Ω is λ-contractive also for the uncertain system (5.11) can be inferred.

Property 5.12 *Let Assumptions 2.3, 3.17 and 4.11 hold, where $f(\cdot)$ in (5.1) and (5.11) is the same. Suppose that the convex, compact set $\Omega \in \mathscr{K}^0(X)$ satisfies condition (5.9) with $\lambda = \lambda_n$, for a $\lambda_n \in [0,1]$. If there exists a $\lambda_w \in [0,1]$ such that*

$$\phi_W(\eta) \leq (\lambda_w - \lambda_n)\phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{5.16}$$

then Ω is a λ-contractive set with contraction factor λ_w (a robust control invariant set if $\lambda_w = 1$) for the uncertain system (5.1) and constraints $x \in X$.

Proof: From condition (5.9), supposed fulfilled, and (5.16), it follows immediately

$$\check{F}(x,\eta) \leq \lambda_n \phi_\Omega(\eta) \leq \lambda_w \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n,$$

and then, from Theorem 5.11, Ω is a λ-contractive set with contraction factor λ_w (a robust invariant set if $\lambda_w = 1$) for the uncertain DC system (5.11). ■

The relation between the contraction factor of a compact invariant set Ω for the deterministic system (5.1) and the set W bounding the uncertainty in (5.11), provided in Property 5.12, can be employed to design a robust invariant set for the uncertain system.

In the following section we focus on computational issues related to the deterministic system, the result can be extended to the uncertain case considering the sufficient condition for robust invariance (5.14) and using the relation (5.16).

5.1.2 Polytopic λ-contractive and invariant set

Previous sections provide sufficient condition for λ-contractiveness and invariance of compact set Ω for deterministic DC systems, condition (5.10), and for λ-contractiveness and robust invariance for uncertain nonlinear systems, (5.15). Both conditions are given by a set of convex constraints involving only the elements of boundary of Ω, for all $\eta \in \mathbb{R}^n$.

In the case that the set Ω is a polytope, λ-contractiveness and invariance conditions are represented by a finite set of convex constraints to be checked only at the vertices of Ω. As shown in the following, this allows one to design an algorithm to check whether a polytope is a λ-contractive set for the deterministic DC system, algorithm which ensures to obtain a non-empty λ-contractive set or an invariant set, under mild assumptions.

We consider here the deterministic DC system (5.1), and we assume that the candidate set Ω is a polytope containing the origin in its interior, i.e., $\Omega \in \mathscr{K}^0(X)$. It is worth recalling that, given a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq p\}$ with $H \in \mathbb{R}^{n_h \times n}$ and $p \in \mathbb{R}^{n_h}$, we have that $\phi_\Omega(H_i^T) = p_i$ for all $i \in \mathbb{N}_{n_h}$ and then

$$\Omega = \{x \in \mathbb{R}^n : H_i x \leq \phi_\Omega(H_i^T), \ \forall i \in \mathbb{N}_{n_h}\},$$

see Property C.3. For every polytope Ω with $0 \in \text{int}(\Omega)$, there exist a finite integer n_h and a matrix $H \in \mathbb{R}^{n_h \times n}$ such that

$$\Omega = \{x \in \mathbb{R}^n : \; Hx \leq 1\}.$$

The sufficient condition for polytope $\Omega \subseteq D$ to be a λ-contractive set with contraction factor λ for DC system (5.1) follows.

Theorem 5.13 *Let Assumptions 3.17 and 4.11 hold. Given a $\lambda_n \in [0,1]$, a polytopic set $\Omega = \{x \in \mathbb{R}^n : \; Hx \leq 1\} \subseteq D$, with $H \in \mathbb{R}^{n_h \times n}$, whose n_v vertices are denoted v^j with $j \in \mathbb{N}_{n_v}$, with $n_v \in \mathbb{N}$, such that*

$$\check{F}(v^j, H_i^T) \leq \lambda_n, \qquad \forall j \in \mathbb{N}_{n_v},\; \forall i \in \mathbb{N}_{n_h}, \tag{5.17}$$

where function $\check{F}(\cdot,\cdot)$ is defined in (5.4), is a λ-contractive set with contraction factor λ_n (an invariant set if $\lambda_n = 1$) for the nonlinear DC system (5.1) and constraints $x \in X$.

Moreover, if $x \in \Omega$, then x_k obtained through (5.1), for $k \in \mathbb{N}$, with $x_0 = x$, satisfies $x_k \in \lambda_n^k \Omega$, for all $k \in \mathbb{N}$.

Proof: By definition, Ω is a λ-contractive set if and only if $f(\Omega) \subseteq \lambda_n \Omega$, and then, from considerations analogous to those of proof of Theorem 5.9, it is sufficient to prove that (5.17) is necessary and sufficient condition for fulfillment of condition (5.10). This implies the claim of λ-contractiveness of the polytope Ω. For that, it is sufficient to recall that any element of a polytope is the convex combination of its vertices. That is for any $\hat{x} \in \Omega$, there exists n_v values $\theta^j(\hat{x}) \geq 0$, with $j \in \mathbb{N}_{n_v}$ such that $\hat{x} = \sum_{j=1}^{n_v} \theta^j(\hat{x}) v^j$ and $\sum_{j=1}^{n_v} \theta^j(\hat{x}) = 1$. Therefore the claim can be proved in a way similar to the prove of Theorem 5.9.

The second part of the theorem is proved next. Consider $\varepsilon \in [0, 1]$. From the convexity of $\check{F}(\cdot, H_i^T)$, for all $i \in \mathbb{N}_{n_h}$, and (5.17), it is inferred that

$$\begin{aligned}
&\check{F}(\varepsilon v^j, H_i^T) - \varepsilon\lambda_n \leq \max_{\varepsilon \in [0,1]} \left\{\check{F}(\varepsilon v^j, H_i^T) - \varepsilon\lambda_n\right\} = \\
&= \max\left\{\check{F}(0, H_i^T) - 0;\; \check{F}(v^j, H_i^T) - \lambda_n\right\} \leq \max\left\{0, \check{F}(v^j, H_i^T) - \lambda_n\right\} = 0,
\end{aligned}$$

for all $j \in \mathbb{N}_{n_v}$ and $i \in \mathbb{N}_{n_h}$, that is,

$$\check{F}(\varepsilon v^j, H_i^T) \leq \varepsilon\lambda_n, \quad \forall j \in \mathbb{N}_{n_v}\; \forall i \in \mathbb{N}_{n_h}, \tag{5.18}$$

for all $\varepsilon \in [0, 1]$. From convexity of Ω, and then convexity of $\varepsilon\Omega$, and the fact that εv^j, with $j \in \mathbb{N}_{n_v}$ are the vertices of $\varepsilon\Omega$, any $\hat{x} \in \varepsilon\Omega$ can be expressed as $\hat{x} = \sum_{j=1}^{n_v} \theta^j(\hat{x})(\varepsilon v^j)$, for proper $\theta^j(\hat{x}) \geq 0$, for $j \in \mathbb{N}_{n_v}$ and $\sum_{j=1}^{n_v} \theta^j(\hat{x}) = 1$. Then,

$$\begin{aligned}
&H_i f(\hat{x}) = H_i f\left(\sum_{j=1}^{n_v} \theta^j(\hat{x})(\varepsilon v^j)\right) \leq \check{F}\left(\sum_{j=1}^{n_v} \theta^j(\hat{x})\varepsilon v^j, H_i^T\right) \leq \\
&\leq \sum_{j=1}^{n_v} \theta^j(\hat{x})\check{F}(\varepsilon v^j, H_i^T) \leq \sum_{j=1}^{n_v} \theta^j(\hat{x})\varepsilon\lambda_n = \varepsilon\lambda_n,
\end{aligned} \tag{5.19}$$

for all $i \in \mathbb{N}_{n_h}$. This means that $x_k \in \varepsilon\Omega$ implies $x_{k+1} = f(x_k) \in \varepsilon\lambda_n\Omega$, for all $\varepsilon \in [0,1]$. Hence, Ω is a λ-contractive set with contraction factor λ_n for the nonlinear DC system and $x_0 \in \Omega$ implies $x_k \in \lambda_n^k\Omega$. ■

The former theorem provides a criterion for checking whether a polytopic set $\Omega \subseteq D$ is a λ-contractive set for a nonlinear system (5.1). Thus, it suffices to check $n_v \cdot n_h$ inequalities to determine if the sufficient condition for λ-contractiveness in fulfilled.

Another important feature of Theorem 5.13 is that any polytopic set Ω fulfilling condition (5.17) induces a Lyapunov function and implicitly proves exponential stability of the origin for the deterministic nonlinear system (5.1), as proved for CDI systems in Corollary 4.17.

Analogously, condition for a polytope Ω to be a λ-contractive set or a robust control invariant set for the uncertain nonlinear system (5.11) can be formulated as a finite set of convex constraints to be checked for every vertex and every row of matrix H.

Theorem 5.14 *Let Assumptions 2.3, 3.17 and 4.11 hold. Given a $\lambda_w \in [0,1]$, a polytopic set $\Omega = \{x \in \mathbb{R}^n :\ Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, whose n_v vertices are denoted v^j with $j \in \mathbb{N}_{n_v}$, with $n_v \in \mathbb{N}$, such that*

$$\check{F}(v^j, H_i^T) \leq \lambda_w - \phi_W(H_i^T), \qquad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h}, \tag{5.20}$$

where function $\check{F}(\cdot,\cdot)$ is defined in (5.4), is a λ-contractive set with contraction factor λ_w (a robust invariant set if $\lambda_w = 1$) for the uncertain DC system (5.11) and constraint $x \in X$.

Proof: The proof is a direct adaptation of proof of Theorems 5.11 and 5.13. ■

Note that in case of uncertain nonlinear system, λ-contractiveness of polytopic set Ω does not imply exponential stability; it is not possible to guarantee asymptotic stability for systems in form (5.11), affected by additive unknown but bounded uncertainty. Convergence to a set can be proved in that case, that is a Lyapunov function outside a set can be induced proving that the system is ultimately bounded, see (Blanchini and Miani, 2008).

The relation between contraction factor of a λ-contractive set Ω for the deterministic system and the measure of the bounding set W such that Ω preserves λ-contractiveness and invariance in presence of uncertainty, inequality (5.16), reduces to a finite number of convex inequalities when Ω is a polytope.

Property 5.15 *Let Assumptions 2.3, 3.17 and 4.11 hold, where $f(\cdot)$ in (5.1) and (5.11) is the same. Suppose the polytopic set $\Omega = \{x \in \mathbb{R}^n :\ Hx \leq 1\} \subseteq D$, with $H \in \mathbb{R}^{n_h \times n}$ satisfies condition (5.17) for a $\lambda_n \in [0,1]$. If there exists a $\lambda_w \in [0,1]$ such that*

$$\max_{i \in \mathbb{N}_{n_h}} \phi_W(H_i^T) \leq \lambda_w - \lambda_n, \tag{5.21}$$

then Ω is a λ-contractive set with contraction factor λ_w (a robust control invariant set if $\lambda_w = 1$) for the uncertain DC system (5.11) and constraint $x \in X$.

Proof: From condition (5.17) and supposition of existence of a $\lambda_w \in [0,1]$ such that (5.21) is satisfied then

$$\check{F}(x, H_i^T) \leq \lambda_n \leq \lambda_w - \max_{i \in \mathbb{N}_{n_h}} \phi_W(H_i^T) \leq \lambda_w - \phi_W(H_i^T), \quad \forall i \in \mathbb{N}_{n_h},$$

recalling that, from Assumption 2.3, the origin is an element of co (W) and then the support function of W with respect to any vector $\eta \in \mathbb{R}^n$ is positive, i.e., $\phi_W(H_i^T) > 0$, for all $i \in \mathbb{N}_{n_h}$. Then also condition (5.20) is satisfied and Ω is a λ-contractive set with contraction factor λ_w for the uncertain DC system (5.11). ■

The relation between the contraction factor of a compact invariant set Ω for the deterministic system (5.1) and the set W bounding the uncertainty in (5.11), provided in Property 5.12, can be employed to design a robust invariant set for the uncertain system.

5.1.3 Computational issues

In this section a method, based on DC functions properties, for computing a λ-contractive set for a nonlinear system is presented. The method is based on the determination of a λ-contractive set for the linear system obtained as the linearization of system (5.1) at the origin. Recall that, by Assumption 5.1 the linear system is asymptotically stable and hence λ-contractive sets can be obtained.

For that purpose, the approach illustrated in Section 4.3 for general CDI systems and overbounding LDI systems can be applied. It has to be noticed that a linear system can be considered as an LDI system whose characterizing polytope in the space of square matrices, $\mathscr{A} \subseteq \mathbb{R}^{n \times n}$, is a singleton, that is, a single matrix. Hence, solving the optimization problem (4.53) with $n_a = 1$ and $A^1 = J_x(0)$ provides a contracting ellipsoid for the linearized system, determined by the optimal P and with contraction factor $\sqrt{(1-\gamma)}$. The polytope Ω capturing the geometry of the ellipsoid $\mathscr{E}(P)$ is obtained as $\Gamma(P)$, see (4.56). Then, the results of Theorem 4.46 provide a bound of the determination index i^* and a polytopic λ-contractive set can be determined computing iteratively the set $C_i^\lambda(\Omega) = C_\infty^\lambda(\Omega)$ with $i \geq i^*$. Alternatively, set $C_\infty^\lambda(x, \Omega)$ can be obtained explicitly by choosing $i \geq i^*$ and removing redundant inequalities from the system

$$\begin{cases} Hx \leq 1, \\ HAx \leq \lambda 1, \\ \dots \\ H(A)^i x \leq \lambda^i 1, \end{cases}$$

where $A = J_x(0)$.

Remark 5.16 *Suppose that a λ-contractive set for the linearized system, denoted $\hat{\Omega}$, is obtained. From Remark 5.2 and λ-contractiveness of $\hat{\Omega}$, it is easy to infer that there is $\alpha > 0$ such that $f(x) \in \alpha\hat{\Omega}$, $\forall x \in \alpha\hat{\Omega}$. This means that, the set $\alpha\hat{\Omega}$ is an invariant set for the nonlinear system.*

The following procedure permits to obtain an invariant set for the nonlinear autonomous system (5.1):

- A polytope λ-contractive invariant set for the linearized system (5.2) is first computed, employing the results from Section 4.3 as illustrated above. This set will be denoted $\hat{\Omega}$.
- Exploiting the sufficient condition for λ-contractiveness and invariance (5.9), the greatest value of α guaranteeing that $\Omega = \alpha\hat{\Omega}$ is an invariant set for the nonlinear system is computed.

A similar strategy for linear saturated systems was proposed in (Tarbouriech and Gomes Da Silva Jr., 1997) without using DC functions. The main feature of our criterion is the affordable computational burden needed and the generality of the approach, since it is applicable to a wide class of nonlinear functions.

Algorithm 2 for computing an invariant set for a nonlinear DC system is given below.

Note that the $\hat{\lambda}$ admissible is greater than $\sqrt{1-\gamma}$ to make finite the upper bound i^* on the determination index defined in (4.60). The proposed algorithm always yields a non-empty invariant set, as the following theorem states.

Theorem 5.17 *Let Assumptions 3.17 and 5.1 hold for system (5.1). Then Algorithm 2 converges to a non-empty λ-contractive set $\Omega = \alpha\hat{\Omega}$, with contraction factor λ, for the DC system (5.1).*

Proof: From twice differentiability of functions $g(\cdot)$ and $h(\cdot)$, there exists $\delta > 0$ such that $g(\cdot), h(\cdot) \in \mathscr{C}^2$ for all $x \in \delta\mathbf{B}_2^n$. From Assumption 5.1 and the results of previous section, it is clear that $\hat{\Omega}$ is non-empty and constitutes a λ-contractive set with contraction factor $\hat{\lambda}$ for the linear system. That is, $\hat{\Omega} = \{x \in \mathbb{R}^n : \hat{H}x \leq 1\}$, with $\hat{H} \in \mathbb{R}^{n_{\hat{h}} \times n}$, satisfies $\hat{H}Ax \leq \hat{\lambda}$, $\forall x \in \hat{\Omega}$. First, the set $\alpha\hat{\Omega}$ has to be contained in $\mathbf{B}_2^n$, that is, $\alpha \in (0, \breve{\alpha}]$ where $\breve{\alpha} = \max\{\alpha >$

Algorithm 2 Computing a λ-contractive set for nonlinear DC system (5.1).

Given the DC system (5.1):

(1) Obtain the Jacobian at the origin, $J_x(0)$.

(2) Obtain P and γ from optimization problem (4.53) with $n_a = 1$ and $A^1 = J_x(0)$.

(3) Choose $\hat{\lambda} \in (\sqrt{1-\gamma}, 1)$ and obtain $\hat{\Omega} = C_i^{\hat{\lambda}}(\Gamma(P))$ where

$$i = \left\lceil \frac{\ln n}{\ln\left(\frac{\hat{\lambda}^2}{1-\gamma}\right)} \right\rceil - 1.$$

(4) Choose $\lambda \in (\hat{\lambda}, 1]$ and compute the maximal $\alpha > 0$ such that $\Omega = \alpha\hat{\Omega}$ fulfills the assumptions of Theorem 5.9.

(5) Return Ω, λ-contractive set for system (5.1) with contraction factor λ.

$0 : \alpha\hat{\Omega} \subseteq \delta\mathbf{B}_2^n\}$. Note that, from the twice differentiability of $g(\cdot)$ and $h(\cdot)$, it follows that there is a $\gamma > 0$ such that

$$|\hat{H}_i(f(x) - Ax)| \leq \|\hat{H}_i^T\|_\infty \gamma x^T x, \quad \forall x \in \delta\mathbf{B}_2^n,$$

for $i \in \mathbb{N}_{n_{\hat{h}}}$. From this and Theorem 5.6, given a $\alpha \in (0, \breve{\alpha}]$ and denoting v^j, for $j \in \mathbb{N}_{n_v}$ the n_v vertices of $\hat{\Omega}$, we have that

$$\begin{aligned}
&\breve{F}(\alpha v^j, \hat{H}_i^T) = \hat{H}_i A\alpha v^j + \hat{H}_i(f(\alpha v^j) - A\alpha v^j) + \breve{F}(\alpha v^j, \hat{H}_i^T) - \hat{H}_i f(\alpha v^j) \leq \\
&\leq \hat{H}_i A\alpha v^j + |\hat{H}_i(f(\alpha v^j) - A\alpha v^j)| + |\breve{F}(\alpha v^j, \hat{H}_i^T) - \hat{H}_i f(\alpha v^j)| \leq \\
&\leq \alpha\hat{\lambda} + \|\hat{H}_i^T\|_\infty \gamma\alpha^2 (v^j)^T v^j + \|\hat{H}_i^T\|_\infty \rho\alpha^2 (v^j)^T v^j,
\end{aligned}$$

for all $i \in \mathbb{N}_{n_{\hat{h}}}$ and $j \in \mathbb{N}_{n_v}$. Hence, if

$$\alpha\hat{\lambda} + \|\hat{H}_i^T\|_\infty \gamma\alpha^2 (v^j)^T v^j + \|\hat{H}_i^T\|_\infty \rho\alpha^2 (v^j)^T v^j \leq \alpha\lambda,$$

for all $i \in \mathbb{N}_{n_{\hat{h}}}$, $j \in \mathbb{N}_{n_v}$, then the theorem is proved. Such condition is fulfilled for $\alpha \in (0, \min\{\breve{\alpha}, \hat{\alpha}\}]$ where

$$\hat{\alpha} = \min_{i \in \mathbb{N}_{n_{\hat{h}}}, j \in \mathbb{N}_{n_v}} \frac{\lambda - \hat{\lambda}}{\|\hat{H}_i^T\|_\infty (\gamma + \rho)(v^j)^T v^j}.$$

■

5.1.4 Numerical example

Consider the following DC function

$$f(x) = \begin{bmatrix} 1 + 0.1x_1 + 0.5x_2 - e^{0.1x_1^2} \\ 0.1 + 0.9x_1 - 0.1x_2 - 0.1\cos(x_2) + 0.05x_2^2 \end{bmatrix},$$

where $x = [x_1,\, x_2]^T \in \mathbb{R}^2$. Note that it can be expressed in standard DC form $f(x) = g(x) - h(x)$, posing

$$g(x) = \begin{bmatrix} 0.5x_2 \\ 0.1 + 0.9x_1 - 0.1\cos(x_2) + 0.05x_2^2 \end{bmatrix},$$
$$h(x) = \begin{bmatrix} -1 - 0.1x_1 + e^{0.1x_1^2} \\ 0.1x_2 \end{bmatrix}.$$

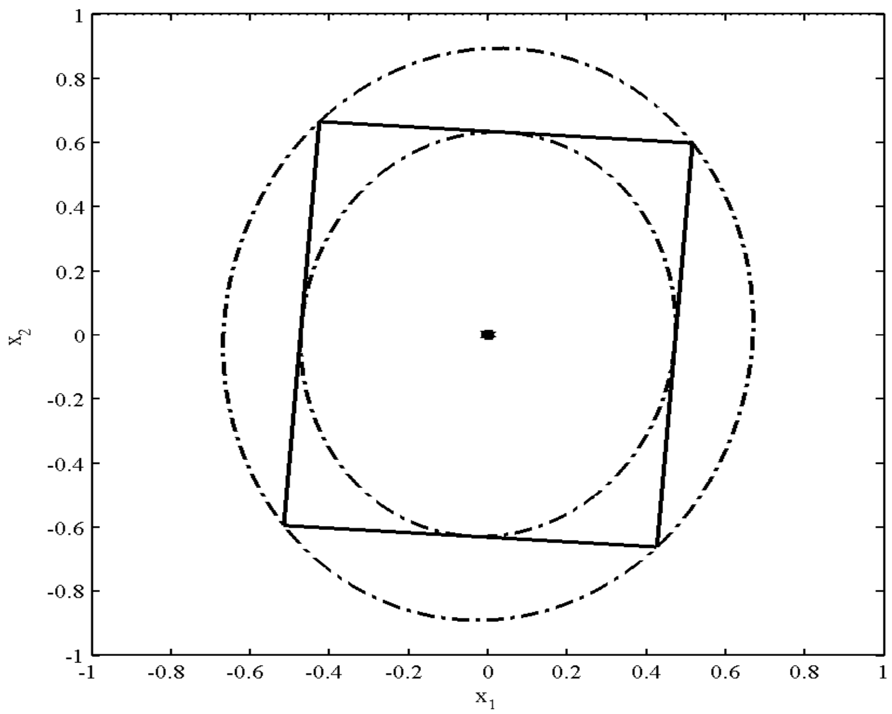

Figure 5.1: λ-contractive invariant set for the linearized system Ω_L (solid line) and ellipsoids $\mathscr{E}(P,1)$ and $\mathscr{E}(P,\frac{1}{n})$ (dashed lines).

Functions $g(\cdot)$ and $h(\cdot)$ are globally convex, in fact the Hessians of functions $g_1(\cdot),\ g_2(\cdot),\ h_1(\cdot),\ h_2(\cdot)$ exist and are positive semi-definite in the whole space $\mathbb{R}^2$. The nonlinear DC system has been linearized around the origin, obtaining the stable dynamic matrix

$$A = \begin{bmatrix} 0.1000 & 0.5000 \\ 0.9000 & -0.1000 \end{bmatrix},$$

whose eigenvalues are -0.6782 and 0.6782.

First, the λ-contractive set for the linear system is computed. Solving the LMI optimization problem (4.53), we obtain

$$P = \begin{bmatrix} 2.2363 & -0.0699 \\ -0.0699 & 1.2579 \end{bmatrix}, \qquad \gamma = 0.54.$$

Hence $\hat{\lambda} \in [0.6783, 1)$ and the contraction factor is set to $\hat{\lambda} = 0.733$. Employing the results of Theorem 4.46, the λ-contractive set for the linearized system is computed and it is represented in Figure 5.1, jointly with the ellipsoids $\mathscr{E}(P, 1)$ and $\mathscr{E}(P, \frac{1}{n})$. Although the bound on the determination index i is $i \geq 4$, one iteration suffices to obtain $\hat{\Omega}$.

Then, the set $\hat{\Omega} = C_\infty^\lambda(\Gamma(P))$ is used in Algorithm 2. The contraction factor has been chosen greater than $\hat{\lambda}$ and close to 1, $\lambda = 0.9973$. Choosing a value of λ close to 1, the algorithm provides a greater λ-contractive set with respect to those obtained employing smaller λ (at the expense of reducing the contractiveness of the obtained invariant set). Hence, the set $\Omega = \alpha\hat{\Omega}$ is a λ-contractive invariant set for the DC system. In particular the value of α is computed by means of a dichotomic-search based procedure.

Finally, the set Ω is compared with a numerical estimation of the domain of attraction of the DC system. Some points in the state space have been chosen randomly as initial conditions. In Fig. 5.2 only the initial conditions which lead to asymptotically stable trajectories have been depicted. The convex hull of such points provides an approximation of the domain of attraction for the DC system. Note that, despite of the strong nonlinearity of the function (an exponential term is present), the λ-contractive and invariant set Ω represents a good portion of the domain of attraction.

A greater invariant set can be obtained by employing an enlarging method analogous to the one presented in Section 4.3 for CDI systems.

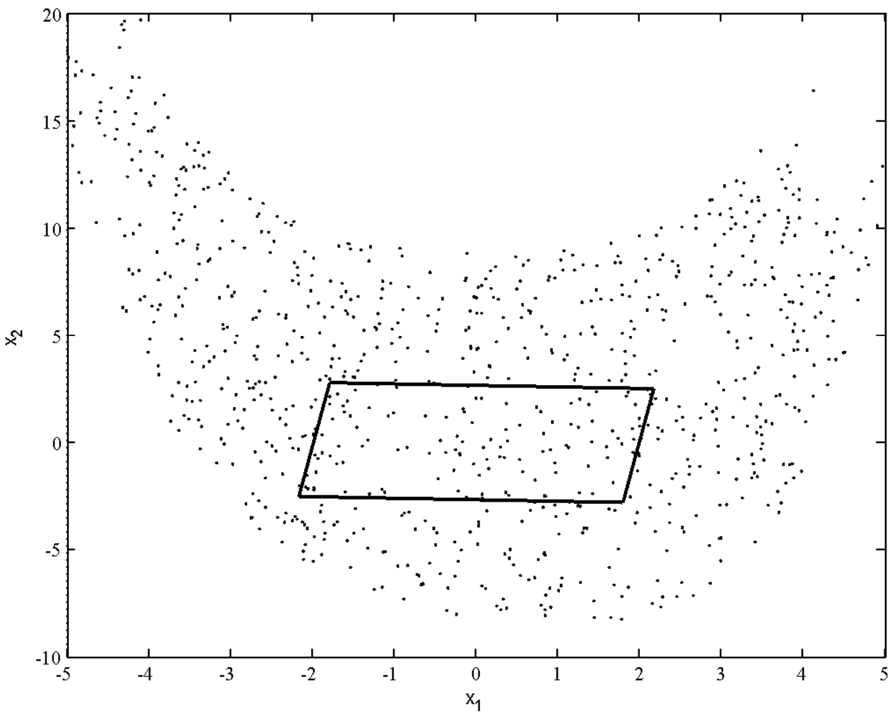

Figure 5.2: Asymptotically stable initial points (dots) and λ-contractive invariant set for the DC system.

5.2 Convex invariant sets for Lur'e systems

In this section, we introduce a new concept of invariance for Lur'e systems, called *LNL*-invariance. Conceptually, a set Ω is an *LNL*-invariant for a Lur'e system if, for every $x \in \Omega$, both the successors obtained through the Lur'e system dynamics and through the dynamics obtained by linearizing it at the origin, are contained in Ω.

An algorithm to determine the largest *LNL*-invariant set for this class of systems is proposed. Moreover, it is proved that the *LNL*-invariant sets provided by this algorithm are polytopes and constitute an estimation of the domain of attraction of the nonlinear system. Based on its geometrical properties, a simple algorithm to obtain the largest *LNL*-invariant set is proposed. *LNL*-invariance is a more conservative concept than traditional invariance but its geometrical properties allows us to obtain a polytopic estimation of the domain of attraction of the nonlinear system. It is shown that any invariant set obtained for an LDI approximation of the Lur'e system is an *LNL*-invariant set which is included into the obtained estimation of the domain of attraction.

We remind that Lur'e systems under analysis, see Section 3.2, are dynamic systems in which a static one-dimensional nonlinearity appears in the feedback path. In particular we assume that such nonlinearity has a piecewise affine nature, as well as concave in $\mathbb{R}$ and odd, as specified in Assumption 3.8. That is, we consider the following discrete-time system

$$\begin{cases} x_{k+1} &= Ax_k - B\varphi(y_k) \\ y_k &= Fx_k, \end{cases}$$

where $x_k \in \mathbb{R}^n$ is the state vector and $y_k = Fx_k \in \mathbb{R}$ the one-dimensional output of the system and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies Assumption 3.8, that is, function $\varphi(\cdot)$ is a continuous, odd, piecewise-affine concave in $\mathbb{R}_+$.

A characterization of functions satisfying Assumption 3.8 is given in Property 3.9, as proved in (Hu et al., 2004). An example of function fulfilling Assumption 3.8, hence defining a Lur'e system, is shown in Figure 3.5. We analyze here some properties of function $\varphi(\cdot)$. For that purpose the following definition is introduced.

Definition 5.18 *Given the piecewise-affine odd function concave in* $\mathbb{R}_+$

$$\varphi(y) = \begin{cases} k_0 y, & \text{if } y \in [0, b_1), \\ k_1 y + c_1, & \text{if } y \in [b_1, b_2), \\ \dots & \\ k_N y + c_N, & \text{if } y \in [b_N, \infty), \end{cases} \qquad \forall y \geq 0,$$

the odd functions $\varphi_i(y)$, $i \in \mathbb{N}_N$ *are defined as:*

$$\varphi_i(y) = \begin{cases} k_0 y, & \text{if } y \in [0, d_i), \\ k_i y + c_i, & \text{if } y \in [d_i, \infty), \end{cases} \qquad \forall y \geq 0, \tag{5.22}$$

where $d_i = \dfrac{c_i}{k_0 - k_i}$, *for* $i \in \mathbb{N}_N$.

It might be useful to provide the expression of a piecewise-affine odd function $\varphi(y)$ for $y < 0$. Recall that a function $\varphi(\cdot)$ is odd if

$$\varphi(y) = -\varphi(-y),$$

for all $y \in \mathbb{R}$. Then we have that, for negative values of y, the function defined in Definition 5.18 is given by

$$\varphi(y) = \begin{cases} k_0 y, & \text{if } y \in (-b_1, 0), \\ k_1 y - c_1, & \text{if } y \in (-b_2, -b_1], \\ \dots & \\ k_N y - c_N, & \text{if } y \in (-\infty, -b_2], \end{cases} \qquad \forall y < 0,$$

and functions $\varphi_i(y)$ are

$$\varphi_i(y) = \begin{cases} k_0 y, & \text{if } y \in (d_i, 0), \\ k_i y - c_i, & \text{if } y \in (-\infty, d_i], \end{cases} \qquad \forall y < 0,$$

for all $i \in \mathbb{N}_N$.

Figure 5.3 shows functions $\varphi_i(\cdot)$, for $i \in \mathbb{N}_3$ corresponding to function $\varphi(\cdot)$ of Figure (3.5).

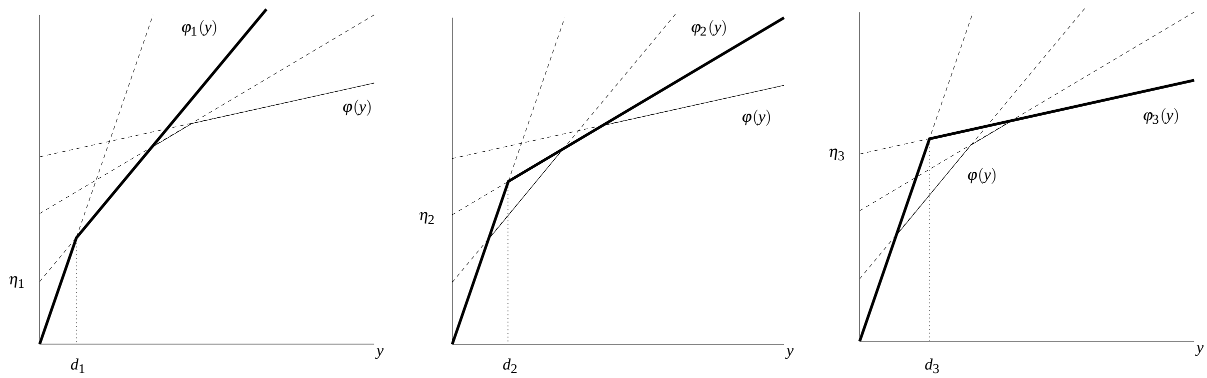

Figure 5.3: Functions $\varphi_i(\cdot)$, $i \in \mathbb{N}_3$ corresponding to function $\varphi(\cdot)$ of Figure 3.5.

It can be observed in Figure 5.3 that $\varphi(y)$ is the pointwise minimum of $\varphi_1(y)$, $\varphi_2(y)$ and $\varphi_3(y)$. The following lemma states a useful relationship between function $\varphi(\cdot)$ and functions $\varphi_i(\cdot)$, $i \in \mathbb{N}_N$.

Lemma 5.19 *(Hu and Lin, 2004; Hu et al., 2004) Suppose that $\varphi(\cdot)$ is an odd piecewise-affine function concave in $\mathbb{R}_+$. Then*

- $\varphi_i(y) \in co\ (k_0 y, \varphi(y))$, *for all* $y \in \mathbb{R}$ *and for all* $i \in \mathbb{N}_N$.
- $\varphi(y) \in co\ (\varphi_1(y), \varphi_2(y), \ldots, \varphi_N(y))$, *for all* $y \in \mathbb{R}$.

Hereafter, the concept of *LNL*-invariance is presented. This notion of invariance is stronger than the classical one. However, the *LNL*-invariance has some geometrical properties that allows one to obtain the greatest *LNL*-invariant set by means of a simple algorithm. Moreover, it will be shown that every λ-contractive set for the nonlinear system is contained into the greatest *LNL*-invariant set provided by the proposed algorithm.

Definition 5.20 *Consider system $x_{k+1} = Ax_k - B\varphi(Fx_k)$ and let function $\varphi(\cdot)$ be defined as in equation (3.20). Functions $f(x)$ and $f_L(x)$ are defined as*

$$\begin{aligned} f(x) &= Ax - B\varphi(Fx), \\ f_L(x) &= Ax - Bk_0Fx. \end{aligned} \tag{5.23}$$

The notion of *LNL*-invariance is introduced in the following definition.

Definition 5.21 *A set Ω is said to be LNL-invariant for system $x_{k+1} = Ax_k - B\varphi(Fx_k)$ if $x \in \Omega$ implies*

$$f(x) = Ax - B\varphi(Fx) \in \Omega,$$
$$f_L(x) = Ax - Bk_0Fx \in \Omega.$$

This implies that, if Ω is *LNL*-invariant it is also invariant but not viceversa. We will see that this leads to obtain the greatest convex λ-contractive invariant set for the Lur'e system, or alternatively seen, that any convex invariant (contractive) set is contained in the *LNL*-domain of attraction.

Remark 5.22 *LNL stands for $Linear$ and $Non-Linear$. Note that the new constraint $f_L(x) \in \Omega$ added to the concept of LNL-invariance is not a very strong constraint as there is a neighborhood of the origin where $f(x)$ equals $f_L(x)$.*

Definition 5.23 *We say that $S_0, S_1, \ldots, S_k$ is an admissible sequence if $S_i \in \{1, -1\}$, $i = 0, \ldots, k$.*

Definition 5.24 *Given x and $S \in \{1, -1\}$, function $G(x,S)$ is defined as follows*

$$G(x,S) = \begin{cases} f(x) & if \quad S = 1, \\ f_L(x) & if \quad S = -1. \end{cases}$$

Definition 5.25 *We say that x belong to the LNL-domain of attraction of system $x_{k+1} = Ax_k - B\varphi(Fx_k)$ if the recursion*

$$x_{k+1} = G(x_k, S_k), \quad x_0 = x,$$

converges to the origin for every admissible infinite sequence $\{S_0, S_1, S_2, \ldots\}$.

As it will be shown, the *LNL*-domain of attraction is a convex set that can be obtained by means of a simple recursion.

In the following definition, some one-step operators related with the notion of *LNL*-invariance are presented.

Definition 5.26 *Given a set Ω and the system $x_{k+1} = Ax_k - B\varphi(Fx_k)$, where $\varphi(\cdot)$ is defined in Definition 5.18, the one-step operators $Q_{NL}(\cdot)$, $Q_L(\cdot)$ and $Q_{LNL}(\cdot)$ are defined as follows*

$$\begin{aligned} Q_{NL}(\Omega) &= \{x \in \mathbb{R}^n : \ Ax - B\varphi(Fx) \in \Omega\}, \\ Q_L(\Omega) &= \{x \in \mathbb{R}^n : \ Ax - Bk_0Fx \in \Omega\}, \\ Q_{LNL}(\Omega) &= Q_L(\Omega) \cap Q_{NL}(\Omega). \end{aligned}$$

As stated in the following property, the operator $Q_{LNL}(\cdot)$ allows one to determine whether a set is an *LNL*-invariant or not.

Property 5.27 *Ω is an LNL-invariant set if and only if $\Omega \subseteq Q_{LNL}(\Omega)$.*

Proof: This result is a direct consequence of Definitions 5.21 and 5.26. ■

Given a convex set Ω, the one-step set $Q_{NL}(\Omega)$ is not necessarily convex due to the nonlinear nature of function $\varphi(\cdot)$. The non-convex nature of $Q_{NL}(\Omega)$ makes it difficult the use of operator $Q_{NL}(\cdot)$ in the computation of invariant sets for the considered Lur'e systems. The most remarkable property of $Q_{LNL}(\cdot)$ is that, given a convex polytopic set Ω, $Q_{LNL}(\Omega)$ is a polytope. To prove it, the following auxiliary operators $Q_{LNL,i}(\cdot)$ are defined.

Definition 5.28 *Given functions $\varphi_i(\cdot)$, for $i \in \mathbb{N}_N$, defined in equation (5.22), the operators $Q_{LNL,i}(\cdot)$ $i \in \mathbb{N}_N$ are defined as*

$$Q_{LNL,i}(\Omega) = Q_L(\Omega) \cap \{x \in \mathbb{R}^n : \ Ax - B\varphi_i(Fx) \in \Omega\}.$$

The following lemma is propaedeutic for the proof of the subsequent theorem.

Lemma 5.29 *Given an odd piecewise-affine function $\varphi(\cdot)$, concave in $\mathbb{R}_+$, consider functions $\varphi_i(\cdot)$, for $i \in \mathbb{N}_N$, defined as in Definition 5.22. Then:*

$$a\varphi_i(y) \leq \max\{ak_0y, ak_iy - |ac_i|\}.$$

Proof: There are two different possibilities, $|y| \leq d_i$ or $|y| > d_i$, that will be analyzed separately.

If $|y| \leq d_i$ then $\varphi_i(y) = k_0 y$ and the inequality holds.

In case that $|y| > d_i$, then $\varphi_i(y) = k_i y + sign(y) c_i$. Note that due to Property 3.9: $k_i < k_0$, $\eta_i > 0$ and $d_i = \frac{c_i}{k_o - k_i} > 0$. There are now four different possibilities:

i. $a > 0$ and $y > d_i$. In this case: $a\varphi_i(y) = ak_i y + ac_i < ak_0 y$.

ii. $a > 0$ and $y < -d_i$. In this case: $a\varphi_i(y) = ak_i y - ac_i = ak_i y - |ac_i|$.

iii. $a < 0$ and $y > d_i$. In this case: $a\varphi_i(y) = ak_i y + ac_i = ak_i y - |ac_i|$.

iv. $a < 0$ and $y < -d_i$. In this case: $a\varphi_i(y) = ak_i y - ac_i < ak_0 y$.

■

The following theorem states that $Q_{LNL,i}(\cdot)$ is a convex operator.

Theorem 5.30 *Let Ω be a polytope given by $\Omega = \{x \in \mathbb{R}^n : Hx \leq g\}$, $Q_{LNL,i}(\Omega)$ is a polytope that can be obtained from the equality*

$$Q_{LNL,i}(\Omega) = P_i(\Omega), \quad i \in \mathbb{N}_N,$$

where

$$P_i(\Omega) = Q_L(\Omega) \cap \{x \in \mathbb{R}^n : H(A - Bk_i F)x \leq g + |c_i HB|\},$$

and $|c_i HB|$ denotes the vector whose entries are equal to the absolute values of the entries of vector $\eta_i HB$.

Proof: Let us suppose that there is $x \in P_i(\Omega)$ such that $x \notin Q_{LNL,i}(\Omega)$. Since $P_i(\Omega) \subseteq Q_L(\Omega)$, it results that $x \notin Q_{LNL,i}(\Omega)$ implies $x \notin \{x : Ax - B\varphi_i(Fx) \in \Omega\}$. That is, there exists j such that

$$H_j(Ax - B\varphi_i(Fx)) > g_j,$$

where H_j and g_j denote the j-th row of H and j-th component of g respectively. Using the inequality $a\varphi_i(y) \leq \max\{ak_0 y, ak_i y - |ac_i|\}$ (see Lemma 5.29) it follows that

$$-H_j B\varphi_i(Fx) \leq \max\{-H_j Bk_0 Fx, -H_j Bk_i Fx - |H_j Bc_i|\}.$$

Two different cases must be considered:

1. $-H_jBk_0Fx \geq -H_jBk_iFx - |H_jBc_i|$. In this case

$$g_j < H_j(Ax - B\varphi_i(Fx)) \leq H_jAx - H_jBk_0Fx = H_j(A - Bk_0F)x.$$

This contradicts the fact that $H(A - Bk_0F)x \leq g$ (recall that $P_i(\Omega) \subseteq Q_L(\Omega)$).

2. $-H_jBk_0Fx < -H_jBk_iFx - |H_jBc_i|$. In this other case

$$g_j < H_j(Ax - B\varphi_i(Fx)) \leq H_jAx - H_jBk_iFx - |H_jBc_i|.$$

This contradicts the fact that $H(A - Bk_iF)x \leq g + |c_iHB|$ (see the definition of $P_i(\Omega)$).

Then, $x \in P_i(\Omega)$ implies $x \in Q_{LNL,i}(\Omega)$. This proves that $P_i(\Omega) \subseteq Q_{LNL,i}(\Omega)$.

To conclude the proof it will be shown that $Q_{LNL,i}(\Omega) \subseteq P_i(\Omega)$. Suppose that $x \in Q_{LNL,i}(\Omega)$. Note that the functions defined in (5.22) can be expressed as

$$\varphi_i(y) = k_iy + c_i\sigma\left(\frac{k_0 - k_i}{c_i}y\right),$$

where $\sigma(y) = \text{sign(y)}\min\{|\text{y}|, 1\}$ is the saturation function. As $-|c_iHB| \leq -c_iHB\sigma(y)$, for every $y \in \mathbb{R}$, it follows that

$$H(Ax - Bk_iFx) - |c_iHB| \leq H(Ax - Bk_iFx) - HBc_i\sigma\left(\frac{k_0 - k_i}{c_i}Fx\right) = H(Ax - B\varphi_i(Fx)) \leq g,$$

where the last inequality follows from $x \in Q_{LNL,i}(\Omega)$. This proves that if $x \in Q_{LNL,i}(\Omega)$ then $x \in P_i(\Omega)$, or equivalently, $Q_{LNL,i}(\Omega) \subseteq P_i(\Omega)$. ∎

In the following theorem it is shown that the operator $Q_{LNL}(\cdot)$ can be obtained from operators $Q_{LNL,i}(\cdot)$, $i \in \mathbb{N}_N$.

Theorem 5.31 *Let Ω be a polytope given by $\Omega = \{x \in \mathbb{R}^n : \; Hx \leq g\}$, then*

$$Q_{LNL}(\Omega) = \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega).$$

Proof: First, it will be shown that $Q_{LNL}(\Omega) \subseteq \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$. Let us suppose that $x \in Q_{LNL}(\Omega)$. Then, by definition

$$\begin{aligned} Ax - B\varphi(Fx) &\in \Omega, \\ Ax - Bk_0Fx &\in \Omega. \end{aligned} \tag{5.24}$$

Lemma 5.19 states that $\varphi_i(Fx) \in \text{co}\ (k_0Fx, \varphi(Fx))$, $i \in \mathbb{N}_N$. Bearing this in mind it is inferred from equation (5.24) and the convexity of Ω that

$$Ax - B\varphi_i(Fx) \in \Omega, \quad i \in \mathbb{N}_N.$$

That is, $x \in Q_{LNL,i}(\Omega)$. This proves that $Q_{LNL}(\Omega) \subseteq \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$.

To finish the proof it is sufficient to show that $\bigcap_{i=1}^{N} Q_{LNL,i}(\Omega) \subseteq Q_{LNL}(\Omega)$. Suppose that $x \in \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$. By definition,

$$\begin{aligned} &Ax - Bk_0Fx \in \Omega, \\ &Ax - B\varphi_i(Fx) \in \Omega, \quad \forall i \in \mathbb{N}_N. \end{aligned}$$

Since Lemma 5.19 states that $\varphi(Fx) \in \text{co}\ (\varphi_1(Fx), \dots, \varphi_N(Fx))$ it is concluded that

$$\begin{aligned} Ax - B\varphi(Fx) \in \Omega, \\ Ax - Bk_0Fx \in \Omega. \end{aligned}$$

Therefore, $x \in Q_{LNL}(\Omega)$ and the statement is proved. ■

Theorem 5.32 *Let Ω be a polytope given by $\Omega = \{x \in \mathbb{R}^n : \ Hx \leq g\}$. Then $Q_{LNL}(\Omega)$ is a polytope that can be obtained from the following equality*

$$Q_{LNL}(\Omega) = \bigcap_{i=1}^{N} P_i(\Omega), \tag{5.25}$$

where $P_i(\Omega) = Q_L(\Omega) \cap \{x \in \mathbb{R}^n : \ H(A - Bk_iF)x \leq g + |c_iHB|\}$.

Proof: The proof is a direct application of Theorems (5.30) and (5.31). ■

Now, the *LNL*-domain of attraction (see Definition 5.25) can be obtained by means of a simple recursion. It is also stated in this section that any λ-contractive set is contained in the *LNL*-domain of attraction. As it will be shown, this implies that the proposed approach outperforms any estimation strategy based on linear difference inclusions. The following theorem provides an important result.

Theorem 5.33 *Denote $L(F)$ the region of linear behavior of system (3.19), that is, $L(F) = \{x \in \mathbb{R}^n : |Fx| \leq b_1\}$. Suppose that $\Phi \subseteq L(F)$ is a polytopic invariant set, with non zero volume, corresponding to the asymptotically stable system $x_{k+1} = (A - Bk_0F)x_k$. Denote now $C_0 = \Phi$ and consider the following recursion*

$$C_{k+1} = Q_{LNL}(C_k).$$

Then:

(i) C_k *is a polytope, for all* $k \in \mathbb{N}$.

(ii) C_k *is an LNL-invariant set, for all* $k \in \mathbb{N}$.

(iii) C_k *belongs to the LNL-domain of attraction of the system, for all* $k \in \mathbb{N}$.

(iv) The sequence $\{C_0, C_1, \ldots\}$ *converges to the LNL-domain of attraction of system (3.19).*

(v) The LNL-domain of attraction of system (3.19) is a convex set.

Proof:

(i) Theorem 5.32 states that if Ω is a polytope then also $Q_{LNL}(\Omega)$ is a polytope. This, and the fact that C_0 is a polytope, prove that the recursion $C_{k+1} = Q_{LNL}(C_k)$ always yields polytopes.

(ii) As C_0 belongs to $L(F)$ it results that $x_{k+1} = Ax_k - Bk_0Fx_k = Ax_k - B\varphi(Fx_k)$, for every $x \in C_0$. Therefore, C_0 is not only an invariant set for the linear system $x^+ = Ax - Bk_0Fx$, but also for the nonlinear system: $x^+ = Ax - B\varphi(Fx)$. This is equivalent to say that C_0 is an LNL-invariant set.

Suppose that C_{k-1} is an LNL-invariant set. Property 5.27 guarantees that $C_{k-1} \subseteq Q_{LNL}(C_{k-1}) = C_k$. Therefore, if $x \in C_k$ then $Ax - Bk_0Fx \in C_{k-1} \subseteq C_k$ and $Ax - B\varphi(Fx) \in C_{k-1} \subseteq C_k$. This proves the claim.

(iii) From the LNL-invariance of $C_0 \subseteq L(F)$ and the asymptotic stability of the non-saturated system it is inferred that C_0 belongs to the LNL-domain of attraction of the system. Note that if C_{k-1} belongs to the LNL-domain of attraction then $C_k = Q_{LNL}(C_{k-1})$ also belongs to the LNL-domain of attraction. This is due to the fact that $G(x,S) \in C_{k-1}$, for all $x \in C_k$ and for all $S \in \{1,-1\}$. Therefore, the recursion $C_{k+1} = Q_{LNL}(C_k)$ with $C_0 = \Phi$ yields LNL-invariant sets that belong to the LNL-domain of attraction.

(iv) Suppose now that x belongs to the LNL-domain of attraction of the system. As Φ is an invariant set with non zero volume, there exists p such that the recursion $x_{k+1} = G(x_k, S_k)$ with $x_0 = x$ satisfies $x_p \in \Phi = C_0$ for all admissible sequence $S_0, S_1, \ldots, S_p$.

This is equivalent to say that x is included in C_p. That is, if x belongs to the LNL-domain of attraction then there exists a finite integer p such that x is included into the p-th LNL-invariant set provided by the algorithm.

(v) It is sufficient to show that given two points x_1 and x_2 belonging to the LNL-domain of attraction, $\lambda x_1 + (1-\lambda)x_2$ belongs to the LNL-domain of attraction for every $\lambda \in [0,1]$. If x_1 and x_2 belong to the LNL-domain of attraction then it is clear from the previous claim that there exist p_1 and p_2 such that $x_1 \in C_{p_1}$, $x_2 \in C_{p_2}$. Denote now $p = \max\{p_1, p_2\}$, taking into account that $C_k \subseteq C_{k+1}$, $\forall k \geq 0$, it is inferred that $x_1 \in C_p$ and $x_2 \in C_p$. From the fact that C_p is a convex set contained in the LNL-domain of attraction of the system it is concluded that $\lambda x_1 + (1-\lambda)x_2$ belongs to C_p and therefore to the LNL-domain of attraction for every $\lambda \in [0,1]$.

■

The recursion presented in the previous theorem requires an invariant set for the linear system $x_{k+1} = (A - Bk_0F)x_k$, included in $L(F)$. This admissible invariant set can be obtained by standard algorithms (see (Gilbert and Tan, 1991; Blanchini, 1999)).

Property 5.34 *Suppose that Ω is a λ-contractive set in the sense defined in (Milani, 2002), that is, there exists $\lambda \in [0, 1)$ such that*

$$x \in \varepsilon\Omega \quad \Rightarrow \quad Ax - B\varphi(Fx) \in \lambda\varepsilon\Omega, \quad \forall \varepsilon \in [0, 1]. \tag{5.26}$$

Then Ω is an LNL-invariant set and it belongs to the LNL-domain of attraction of the system.

Proof: First, it will be proved that if Ω fulfills (5.26) then it is also LNL-invariant, that is, for all $x \in \Omega$,

$$Ax - B\varphi(Fx) \in \Omega, \tag{5.27}$$

and

$$Ax - Bk_0Fx \in \Omega. \tag{5.28}$$

Equation (5.26) and the fact that $\lambda\Omega \subseteq \Omega$ guarantee that equation (5.27) is satisfied. It remains to show that equation (5.28) is fulfilled for every $x \in \Omega$.

Given $x \in \Omega$, there exists $\varepsilon \in [0,1]$ such that $|F\varepsilon x| \leq b_1$. From this,

$$\varphi(F\varepsilon x) = k_0 F\varepsilon x,$$

and $\varepsilon x \in \varepsilon\Omega$.

Hence, from equation (5.26) we have that

$$A\varepsilon x - Bk_0F\varepsilon x = A\varepsilon x - B\varphi(F\varepsilon x) \in \lambda\varepsilon\Omega.$$

It follows that

$$A\varepsilon x - Bk_0F\varepsilon x \in \lambda\varepsilon\Omega.$$

Notice that this is equivalent to $Ax - Bk_0Fx \in \lambda\Omega$. This proves that if Ω is a λ-contractive set, then it is also an *LNL*-invariant set.

In the following it will be proved that Ω belongs to the *LNL*-domain of attraction of the system. This means that, if $x(0) \in \Omega$, then, for every admissible sequence $\{S_1, S_2, \dots, S_k\}$, $\lim\limits_{k\to\infty} x_k = 0$ where $x_{k+1} = G(x_k, S_k)$. Following the same arguments as before, it can be shown that if $x \in \lambda^{k-1}\Omega$, then $G(x,S) \in \lambda^k\Omega$, for all $S \in \{1,-1\}$. That is, $x_k \in \lambda^k\Omega$, for every admissible sequence $\{S_0, S_1, \dots, S_{k-1}\}$. Therefore, $\lim\limits_{k\to\infty} x_k = 0$, for every admissible sequence $\{S_k\}_0^\infty$. This proves the claim. ■

A relevant consequence follows from Property 5.27. Indeed, it is well known that any invariant set obtained using a linear difference inclusion of the nonlinearity yields a λ-contractive invariant set (see (Blanchini, 1994)). From Property 5.27 it follows that any approach based on a linear difference inclusion provides a contractive set that is contained in the one obtained with the proposed result. Another interpretation of the former property is that any estimation of the domain of attraction obtained by means of a Lyapunov function induced by a convex set is contained in the *LNL*-domain of attraction.

5.2.1 CDI approach to invariance computation for Lur'e systems

Now we propose an alternative approach to the problem of characterization and computation of invariant sets and of the domain of attraction for a Lur'e system. This approach is based on the results proved for CDI systems. In fact, we employ a CDI system characterized by the set valued function $\mathscr{F}(\cdot)$ overbounding the Lur'e system, that is such that $Ax - B\varphi(Fx) \subseteq \mathscr{F}(x)$.

In particular, the CDI system is determined by means of its convex bounding functions $\check{f}_\eta(\cdot)$, for $\eta \in \mathbb{R}^n$, as illustrated in Section 3.2, which are recalled here:

$$\check{f}_\eta(x) = \begin{cases} \eta^T Ax - \eta^T B\check{\varphi}(Fx), & \text{if } \eta^T B \le 0, \\ \eta^T Ax - \eta^T B\hat{\varphi}(Fx), & \text{if } \eta^T B > 0, \end{cases}$$

where

$$\check{\varphi}(y) = \max\{k_0 y, \varphi(y)\} = \begin{cases} k_0 y, & \text{if } y \ge -b_1, \\ \varphi(y), & \text{otherwise,} \end{cases}$$

and

$$\hat{\varphi}(y) = \min\{k_0 y, \varphi(y)\} = \begin{cases} k_0 y, & \text{if } y \le b_1, \\ \varphi(y), & \text{otherwise.} \end{cases}$$

Recall moreover that, from Property 4.27, the one-step operator of a set Ω is defined by means of the directional bounding functions $\check{F}(\cdot,\cdot)$. From Definition 4.3, the directional upper bounding function for $x \in X$ and $\eta \in \mathbb{R}^n$ is given by

$$\check{F}(x,\eta) = \check{f}_\eta(x) = \begin{cases} \eta^T A x - \eta^T B \check{\varphi}(Fx), & \text{if } \eta^T B \le 0, \\ \eta^T A x - \eta^T B \hat{\varphi}(Fx), & \text{if } \eta^T B > 0, \end{cases}$$

and the one-step operator is given by

$$Q(\Omega) = \bigcap_{\eta \in \mathbb{R}^n} \{x \in X : \check{F}(x,\eta) \le \phi_\Omega(\eta)\}. \tag{5.29}$$

Below we prove that function $\check{F}(\cdot,\cdot)$ can be alternatively expressed as the pointwise maximum of affine functions. In fact, by geometric inspection we have that

$$\check{\varphi}(y) = \max\{k_0 y,\ k_1 y - c_1,\ k_2 y - c_2,\ \ldots,\ k_N y - c_N\},$$
$$\hat{\varphi}(y) = \min\{k_0 y,\ k_1 y + c_1,\ k_2 y + c_2,\ \ldots,\ k_N y + c_N\},$$

and thus

$$\check{F}(x,\eta) \;=\; \begin{cases} \eta^T A x - \eta^T B \max\{k_0 F x,\ k_1 F x - c_1,\ \ldots,\ k_N F x - c_N\}, & \text{if } \eta^T B \le 0, \\ \eta^T A x - \eta^T B \min\{k_0 F x,\ k_1 F x + c_1,\ \ldots,\ k_N F x + c_N\}, & \text{if } \eta^T B > 0. \end{cases}$$

Since, for every set of p functions $f_i(x)$, with $i \in \mathbb{N}_p$, and every $x \in X$, we have

$$\max\{f_i(x) :\ \forall i \in \mathbb{N}_p\} = -\min\{-f_i(x) :\ \forall i \in \mathbb{N}_p\},$$
$$a \max\{f_i(x) :\ \forall i \in \mathbb{N}_p\} = \max\{a f_i(x) :\ \forall i \in \mathbb{N}_p\}, \quad \text{if } a > 0,$$
$$a \max\{f_i(x) :\ \forall i \in \mathbb{N}_p\} = \min\{a f_i(x) :\ \forall i \in \mathbb{N}_p\}, \quad \text{if } a \le 0,$$

it follows that

$$\check{F}(x,\eta) \;=\; \begin{cases} \eta^T A x + \max\{ & -\eta^T B k_0 F x, & \\ & -\eta^T B k_1 F x + \eta^T B c_1, & \\ & \ldots, & \text{if } \eta^T B \le 0, \\ & -\eta^T B k_N F x + \eta^T B c_N\}, & \\ & & \\ \eta^T A x + \max\{ & -\eta^T B k_0 F x, & \\ & -\eta^T B k_1 F x - \eta^T B c_1, & \\ & \ldots, & \text{if } \eta^T B > 0. \\ & -\eta^T B k_N F x - \eta^T B c_N\}, & \end{cases}$$

Now, from the fact that

$$\begin{aligned} \eta^T B &= -|\eta^T B|, && \text{if } \eta^T B \leq 0, \\ \eta^T B &= |\eta^T B|, && \text{if } \eta^T B > 0, \end{aligned}$$

and from $c_i \geq 0$ for every $i \in \mathbb{N}_N$, it follows that

$$\check{F}(x,\eta) = \eta^T Ax + \max\{ \begin{array}{l} -\eta^T Bk_0 Fx, \\ -\eta^T Bk_1 Fx - |\eta^T Bc_1|, \\ \dots, \\ -\eta^T Bk_N Fx - |\eta^T Bc_N|\}. \end{array} \tag{5.30}$$

Hence, finally we have that $x \in Q(\Omega)$ if and only if

$$\begin{aligned} &\eta^T Ax - \eta^T Bk_0 Fx \leq \phi_\Omega(\eta), \\ &\eta^T Ax - \eta^T Bk_1 Fx - |\eta^T Bc_1| \leq \phi_\Omega(\eta), \\ &\dots, \\ &\eta^T Ax - \eta^T Bk_N Fx - |\eta^T Bc_N| \leq \phi_\Omega(\eta), \end{aligned}$$

which means $\check{F}(x,\eta) \leq \phi_\Omega(\eta)$, for all $\eta \in \mathbb{R}^n$. Such condition, in case of polytopic sets Ω is given by a finite number of linear constraints, as for the case of $Q_{LNL}(\Omega)$, see (5.25). The one-step operator for the CDI system overbounding the Lur'e one can be used in the algorithm illustrates in Theorem 5.33 to compute a sequence of invariant sets for the CDI system. Any invariant set for the CDI system is invariant also for every overbounded system, hence also for the Lur'e one.

Finally, we prove that the one-step operators, for the CDI system and for the Lur'e system, are the same. This means that the algorithm generates the same sequence of invariant sets.

Property 5.35 *Given $\Omega \in \mathscr{K}(\mathbb{R}^n)$ and a Lur'e system (3.19) for which Assumption 3.8 holds, we have that*

$$Q_{LNL}(\Omega) = Q(\Omega), \tag{5.31}$$

where the one-step operators are defined in (5.25) and in (5.29), with $\check{F}(\cdot,\cdot)$ in (5.30).

Proof: The property follows directly from the definitions of one-step operators (5.25) and (5.29), with $\eta^T = H_j$ and $\phi_\Omega(H_j^T) = g_j$, for every j-th row of matrix H. ■

5.2.2 Numerical example

We provide here two numerical examples of application of the results presented in this section. A two-dimensional and a three-dimensional Lur'e systems are considered.

Example 5.36 *Let us consider the system* $x_{k+1} = Ax_k - B\varphi(Fx_k)$ *with*

$$A = \begin{bmatrix} 1.2 & 1 \\ 0 & 1.2 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},\qquad F = [0.6290 \quad 1.2261]\,. \tag{5.32}$$

and the odd function $\varphi(\cdot)$

$$\varphi(y) = \begin{cases} y, & \text{if } y \in [0,2) \\ 0.25y + 1.5, & \text{if } y \in [2,4) \\ 2.5, & \text{if } y \in [4,\infty) \end{cases}, \quad \forall y \geq 0. \tag{5.33}$$

This function is represented in Figure 5.4. The matrix F *is the LQR gain for the linear region computed employing identity matrices as weights.*

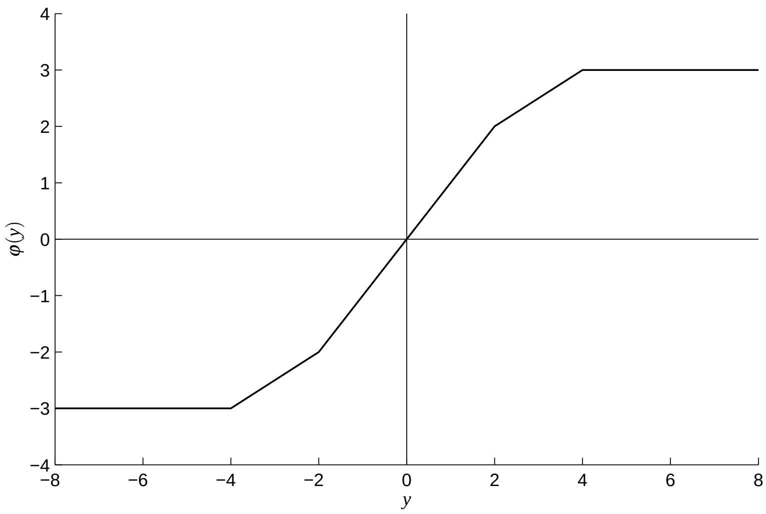

Figure 5.4: Nonlinear function $\varphi(\cdot)$.

Theorem 5.33 shows how to obtain a sequence of LNL-invariant sets that constitutes an estimation of the domain of attraction of the nonlinear system. This sequence has been computed for system (5.32) and it is shown in Figure 5.5.

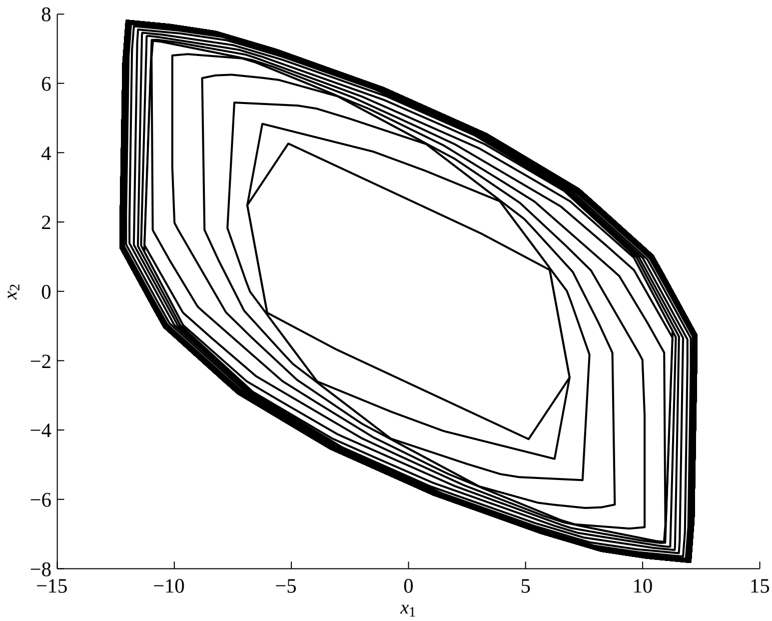

Figure 5.5: Sequence of invariant sets for the Lur'e system.

In that figure, the most inner set is an invariant set of the linear system corresponding to the zone of linear behavior of the system. The sequence C_0, C_1, ... converging to the LNL-domain of attraction is represented in that figure.

This is not the only method to determine invariant sets for piecewise-affine feedback systems. In (Hu and Lin, 2004), the authors propose an algorithm to obtain ellipsoidal invariant sets for saturated feedback systems. Figure 5.6 shows the ellipsoidal invariant set obtained by means of the results presented in (Hu and Lin, 2004), the polytopic LNL-invariant obtained by means of the algorithm proposed in this paper and a numerical approximation of the non-convex maximal invariant set. As can be seen, the polytopic LNL-invariant set provides an improvement with respect to the ellipsoidal one and it is a sharp convex approximation of the maximal invariant set.

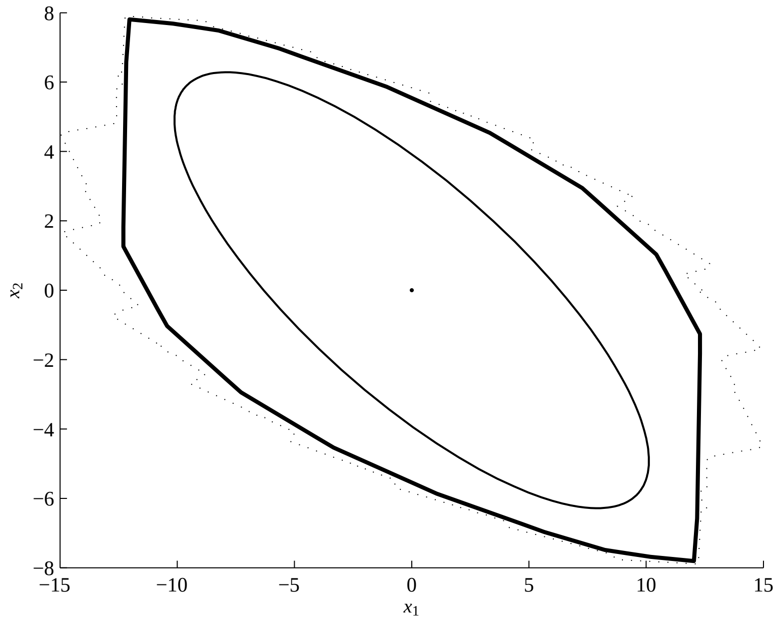

Figure 5.6: *LNL*-invariant set (thick line), ellipsoidal invariant set (thin line) and non-convex invariant set (dotted line).

Example 5.37 *Consider the system* $x^+ = Ax - B\varphi(Fx)$ *with*

$$A = \begin{bmatrix} 1.2 & 1 & 3 \\ 0 & 1.1 & 1 \\ 0 & 0 & 1.2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}, \tag{5.34}$$

$$F = [0.3447 \quad 0.5178 \quad 1.7351].$$

The matrix F is the LQR gain obtained using identity matrices as weights and the nonlinearity is the same as in the previous example, see Figure 5.4.

The resulting tridimensional invariant sets are represented in Figure 5.7. Again, it can be appreciated that the ellipsoidal invariant set obtained through the method presented in (Hu and Lin, 2004) lies inside the LNL-invariant set. This is not surprising because the result of (Hu et al., 2004) are based on the use of linear difference inclusions.

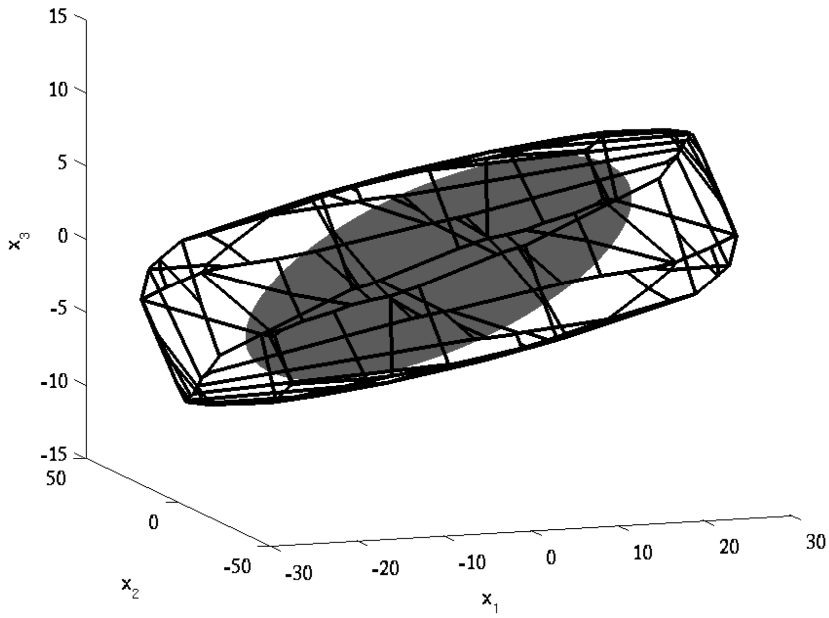

Figure 5.7: LNL-invariant set and ellipsoidal set.

5.3 Conclusions

In this chapter computational issues related to convex invariance for the DC and Lur'e systems are addressed. We proposed here the computational procedure for these particular families of nonlinear systems, since they represent two useful frameworks for dealing with real systems, which often present nonlinear dynamic functions rather than set valued ones.

First DC systems have been considered. Sufficient conditions for invariance and λ-contractiveness of a convex set for DC systems are presented, then the obtained results are particularized to the case of polytopes in the state space. This leads to interesting properties which can be employed in the design of algorithmic procedures for computing invariant sets for nonlinear systems.

Thus, an algorithm for computing a λ-contractive invariant set for discrete-time nonlinear DC systems has been presented. We propose a method which overcomes the main problem of the computation of local invariant sets for a nonlinear system, the computational complexity, often unmanageable. In particular, we exploited properties of DC functions to formulate the algorithm for computing a λ-contractive invariant set.

Then, the problem of estimation of the domain of attraction of Lur'e systems is presented. The results concerning iterative computation of convex invariant sets and the one-step operator is developed for Lur'e systems. The procedure proposed is based on an convex approximation of the one-step operator for a Lur'e system. It has been proved that the algorithm generates a sequence of nested invariant sets converging to the *LNL*-domain of attraction, which represents a convex estimation of the real (possibly non-convex) domain of attraction for the system.

The same problem is treated also from the point of view of CDI systems. It is shown that, for every Lur'e system, an overbounding CDI system can be easily computed and, employing the one-step operator related to the approximated CDI system, the obtained results are the same as those achieved through the ad-hoc Lur'e approach.

The main feature of the proposed algorithms is their simplicity and their affordable computational burden, indeed no global optimization problem has to be solved. Moreover, they have been demonstrated to be quite general, since they can be applied to a large class of nonlinear systems.

Chapter 6

Control invariant sets for nonlinear systems

In this chapter we consider control invariance for nonlinear non-autonomous systems. The main objective is to provide methods for the practical computation of λ-contractive and control invariant sets, and the related control law, for nonlinear non-autonomous systems, possibly uncertain. This means that we are interested in sets which are λ-contractive and (robust) invariant for the considered nonlinear system in closed-loop with a proper control law. Since the chapter is focused on practical and computational issues, we will consider non-autonomous DC system, which, we recall, encloses a very wide class of nonlinear systems.

The chapter deals with deterministic and uncertain non-autonomous DC systems. Conditions for λ-contractiveness and control invariance, based on CDI systems properties, will be first given for generic convex and compact sets, then the attention will be directed to the case of polytopic sets, more suitable for computational aims. Another important concept, useful in the contest of control invariant sets computation, as the one-step operator for DC systems, is analyzed. Then, practical issues concerning algorithmic procedures to obtain control invariant sets and the related control laws are presented and applied to a numerical example.

6.1 Control invariant sets for DC systems

We consider here an uncertain non-autonomous DC system, that is a nonlinear discrete-time time-invariant dynamic system

$$x^+ = f(x,u) + w, \tag{6.1}$$

where $x \in X \subseteq D \subseteq \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state, $u \in U \subseteq E \subseteq \mathbb{R}^m$ is the control action, $w \in W \subseteq \mathbb{R}^n$ is the unknown but bounded uncertainty and $f(\cdot,\cdot)$ is a particular DC function. The state constraint set is X and the input constraint set is U, while sets D and E determine the domain on which function $f(\cdot,\cdot)$ is defined.

In this chapter Assumption 3.19 will be extensively used. We recall here that with Assumption 3.19 we suppose that $f : D \times E \to \mathbb{R}^n$ in (6.1) is a DC function defined on $D \times E \subseteq \mathbb{R}^{n+m}$, with $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}^m$ convex with $(0,0) \in \text{int}\ (D \times E)$, and differentiable at the origin. Moreover, denoting $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ the convex functions such that $f(x,u) = g(x,u) - h(x,u)$, for all $(x,u) \in D \times U$, we assume that $g(0,0) = 0$ and $h(0,0) = 0$. In the sequel we will suppose that Assumption 3.19 holds, to determine the DC nature of the dynamic function $f(\cdot,\cdot)$ in (6.1).

Hypothesis on the state and input constraints, X and U, are expressed in the following assumption, to which we will refer along the chapter.

Assumption 6.1 *Assume that the constraint sets on the state $X \subseteq \mathbb{R}^n$ and on the input $U \subseteq \mathbb{R}^m$, are closed, convex and with $0 \in int(X)$.*

In the following, when dealing with the presence of additive uncertainty, also an assumption on the uncertainty bounding set W is often supposed to hold. In particular Assumption 2.3 is referred to, meaning that $W \subseteq \mathbb{R}^n$ is assumed to be a compact set with $0 \in \text{int}\ (\text{co}\ (W))$ (no convexity is required).

Remark 6.2 *We consider here nonlinear systems with additive uncertainty, although the results presented can be applied to more generic frameworks, see Remark 6.5. Moreover, many of the results presented in this chapter can be extended to DC functions $f(\cdot,\cdot)$ which are not differentiable at the origin, see Remark 6.9.*

A further assumption, not too restrictive in fact, is that the linearization of system (6.1) is stabilizable.

Assumption 6.3 *Assume that the linear system obtained linearizing function $f(\cdot,\cdot)$ in (6.1) at the origin and in absence of uncertainty, is stabilizable.*

We recall here that a set $\Omega \subseteq \mathbb{R}^n$ is a robust control invariant set for system (6.1) and constraints $x \in X$ and $u \in U$ if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u(x) \in U$ such that $f(x,u(x)) + w \in \Omega$, for all $w \in W$, see the Appendix A. Then, a set Ω is a robust control invariant set for the system if there exists an admissible control law $u = u(x) \in U$ defined for all $x \in \Omega$ such that every trajectory of the controlled system (6.1) starting within Ω remains inside it regardless on the uncertainty realization.

Also the definition of λ-contractive set for the uncertain non-autonomous DC system is recalled here. A set $\Omega \subseteq \mathbb{R}^n$ is said to be a λ-contractive set for the uncertain DC system (6.1) and constraints $x \in X$ and $u \in U$ if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u(x) \in U$ such that $f(x,u(x)) + w \in \lambda\Omega$, for all $w \in W$, with $\lambda \in [0, 1]$. Clearly, λ-contractiveness implies control invariance.

Remark 6.4 *In case that the set Ω is polytopic and $0 \in int(\Omega)$, that is if there exist a finite $n_h \in \mathbb{N}$ and a matrix $H \in \mathbb{R}^{n_h \times n}$ such that $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, the condition for the set $\Omega \subseteq X$ to be λ-contractive is the existence of a $u(x) \in U$ such that $Hf(x,u(x),w) \leq \lambda$, for all $x \in \Omega$ and $w \in W$.*

In the following we propose a condition for a convex set Ω, subset of the state space, to be a robust control invariant set for the uncertain non-autonomous DC system. Then the condition is employed to design an algorithm for computation of a robust control invariant set. The affordable computational burden required and the generality are important characteristics of the proposed approach.

Remark 6.5 *The results provided here can be extended to the dynamic systems given by $x^+ = f(x,u,w)$, where $W \subseteq \mathbb{R}^n$ is a polytope and the dependence of function $f(\cdot,\cdot,\cdot)$ with respect to $w \in W$ is affine, i.e., for every $\bar{x} \in X$ and $\bar{u} \in U$, function $f(\bar{x},\bar{u},w)$ is affine in w. In this case the system can be expressed as*

$$x^+ \in co\ (f(x,u,w^i),\ i \in \mathbb{N}_{n_w}),$$

where w^i, with $i \in \mathbb{N}_{n_w}$, are the n_w vertices of polytope W. It can be proved that if a set $\Omega \subseteq X$ and a control law $u(x) \in U$ defined for all $x \in \Omega$, are such that Ω is a control invariant set for every system $x^+ = f(x,u(x),w^i)$, with $i \in \mathbb{N}_{n_w}$, then Ω is a control invariant set also for system $x^+ = f(x,u(x),w)$.

6.1.1 Control invariance condition for DC systems

In this section we present theoretical properties and results on control invariance for a convex set and a DC system, which will be useful to define control strategies for nonlinear systems. In the following, in fact, such properties are employed for the case of polytopic control invariant sets, leading to computational procedures characterized by affordable complexity.

Given the DC function $f(\cdot,\cdot)$ as in Assumption 3.19 the directional bounding functions $\check{F}(\cdot,\cdot,\eta)$ for any $\eta \in \mathbb{R}^n$, is defined below.

First it is worth recalling here that the CDI system overbounding a non-autonomous DC system is characterized by

$$\check{f}_\eta(x,u) = \sum_{j\in k_+} \eta_j \left(g_j(x,u) - h_j^L(x,u)\right) + \sum_{j\in k_-} \eta_j \left(g_j^L(x,u) - h_j(x,u)\right), \tag{6.2}$$

where $g_j^L(x,u) = \nabla_x g_j(0,0)x + \nabla_u g_j(0,0)u$ and $h_j^L(x,u) = \nabla_x h_j(0,0)x + \nabla_u h_j(0,0)u$, for $j \in \mathbb{N}_n$ and $k_+ = k_+(\eta) = \{j \in \mathbb{N}_n : \ \eta_j \geq 0\}$ and $k_- = k_-(\eta) = \{j \in \mathbb{N}_n : \ \eta_j < 0\}$, see Property 3.20. Recall that $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$ denote the j-th components of $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$, $j \in \mathbb{N}_n$, respectively, $k_+ = k_+(\eta)$ is the set of indexes of non-negative elements of vector $\eta \in \mathbb{R}^n$ and $k_- = k_-(\eta)$ the set of indexes of negative elements of η.

Definition 6.6 *Given the DC function $f : D \times E \to \mathbb{R}^n$ as in (6.1) such that Assumption 3.19 holds and $\eta \in \mathbb{R}^n$, define the directional upper bounding function $\check{F} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ as*

$$\check{F}(x,u,\eta) = \check{f}_\eta(x,u), \tag{6.3}$$

where functions $\check{f}_\eta(\cdot,\cdot)$, for every $\eta \in \mathbb{R}^n$, are given by (6.2).

Then we have that the explicit expression of $\check{F}(x,u,\eta)$ is

$$\check{F}(x,u,\eta) = \sum_{j\in k_+} \eta_j \left(g_j(x,u) - h_j^L(x,u)\right) + \sum_{j\in k_-} \eta_j \left(g_j^L(x,u) - h_j(x,u)\right). \tag{6.4}$$

Property 6.7 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given the DC function $f : X \times U \to \mathbb{R}^n$ as in (6.1), for every $\eta \in \mathbb{R}^n$, function $\check{F}(\cdot,\cdot,\eta)$ defined in (6.4) is convex with respect to $(x,u) \in X \times U$.*

Proof: By Definition 6.6, function $\check{F}(\cdot,\cdot,\eta)$ is the sum of elements composed by the sum of a convex term and a linear one. From the fact that a linear function is convex and since also the sum of two convex functions is a convex function, the claim follows. ■

In the following we prove that, for any $\eta \in \mathbb{R}^n$, the function $\check{F}(\cdot,\cdot,\eta)$ provides an upper bound of the function $\eta^T f(\cdot,\cdot)$.

Property 6.8 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given the DC function $f : X \times U \to \mathbb{R}^n$ as in (6.1), for every $\eta \in \mathbb{R}^n$ we have*

$$\eta^T f(x,u) \leq \check{F}(x,u,\eta), \quad \forall (x,u) \in X \times U, \tag{6.5}$$

where $\check{F}(\cdot,\cdot,\eta)$ is defined in (6.4).

Proof: Since $g_j^L(\cdot,\cdot)$ and $h_j^L(\cdot,\cdot)$ are, by definition, the linearizations at the origin of the convex functions $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$, respectively, for $j \in \mathbb{N}_n$, it follows

$$\begin{aligned} g_j^L(x,u) &\leq g_j(x,u), \quad \forall j \in \mathbb{N}_n,\ \forall (x,u) \in X \times U, \\ h_j^L(x,u) &\leq h_j(x,u), \quad \forall j \in \mathbb{N}_n,\ \forall (x,u) \in X \times U. \end{aligned} \tag{6.6}$$

Thus, $\eta_j(h_j^L(x,u) - h_j(x,u)) \leq 0$ if $j \in k_+$ and $\eta_j(g_j(x,u) - g_j^L(x,u)) \leq 0$ if $j \in k_-$, for all $j \in \mathbb{N}_n$ and any $(x,u) \in X \times U$. Hence, from this and (6.4), we have

$$\begin{aligned} &\eta^T f(x,u) - \check{F}(x,u,\eta) = \sum_{j=1}^{n} \eta_j \left(g_j(x,u) - h_j(x,u)\right) - \\ &\sum_{j\in k_+} \eta_j(g_j(x,u) - h_j^L(x,u)) - \sum_{j\in k_-} \eta_j(g_j^L(x,u) - h_j(x,u)) = \\ &= \sum_{j\in k_+} \eta_j(h_j^L(x,u) - h_j(x,u)) + \sum_{j\in k_-} \eta_j(g_j(x,u) - g_j^L(x,u)) \leq 0. \end{aligned}$$

for all $\eta \in \mathbb{R}^n$ and for any $(x,u) \in X \times U$. ■

Remark 6.9 *Notice that differentials at the origin of functions $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$, with respect to x and u, for $j \in \mathbb{N}_n$, are used to determine linear functions $g_j^L(\cdot,\cdot)$ and $h_j^L(\cdot,\cdot)$ fulfilling the inequalities (6.6), see Definition 6.6. In case functions $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$, for $j \in \mathbb{N}_n$, are convex but not differentiable at the origin, linear functions satisfying inequalities (6.6) can be obtained by means of the subdifferential of $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$ at the origin with respect to x and u.*

Another important property of function $\check{F}(\cdot,\cdot,\cdot)$ is presented in what follows.

Property 6.10 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given the DC function $f : X \times U \to \mathbb{R}^n$ as in (6.1) and $\eta \in \mathbb{R}^n$, for every collection of n_k elements $\eta^k \in \mathbb{R}^n$ and $\theta_k \geq 0$, with $k \in \mathbb{N}_{n_k}$, such that $\sum_{k=1}^{n_k} \theta_k = 1$ and $\eta = \sum_{k=1}^{n_k} \theta_k \eta^k$, we have*

$$\check{F}(x,u,\eta) \leq \sum_{k=1}^{n_k} \theta_k \check{F}(x,u,\eta^k), \quad \forall (x,u) \in X \times U, \tag{6.7}$$

where $\check{F}(\cdot,\cdot,\eta)$ is defined in (6.4).

Proof: Both sides of inequality (6.7) are given by a sum of n terms. For the lefthand side term, this stems directly from the definition of $\check{F}(\cdot,\cdot,\cdot)$. For the righthand side term, for every $k \in \mathbb{N}_{n_k}$, we have that

$$\check{F}(x,u,\eta^k) = \sum_{j\in k_+(\eta^k)} \eta_j^k \left(g_j(x,u) - h_j^L(x,u)\right) + \sum_{j\in k_-(\eta^k)} \eta_j^k \left(g_j^L(x,u) - h_j(x,u)\right),$$

and then

$$\begin{aligned}\sum_{k=1}^{n_k}\theta_k\check{F}(x,u,\eta^k) &= \sum_{k=1}^{n_k}\theta_k\left(\sum_{j\in k_+(\eta^k)}\eta_j^k\left(g_j(x,u)-h_j^L(x,u)\right)+\right.\\ &\left.+\sum_{j\in k_-(\eta^k)}\eta_j^k\left(g_j^L(x,u)-h_j(x,u)\right)\right)=\sum_{j=1}^{n}\sum_{k=1}^{n_k}c_{j,k}(x),\end{aligned}$$

where

$$c_{j,k}(x)=\begin{cases}\theta_k\eta_j^k(g_j(x)-h_j^L(x)), & \text{if } \eta_j^k\geq 0,\\ \theta_k\eta_j^k(g_j^L(x)-h_j(x)), & \text{if } \eta_j^k<0,\end{cases}$$

for every $j\in\mathbb{N}_n$ and $k\in\mathbb{N}_{n_k}$.

We prove that the j-th term of the lefthand side is smaller than the j-th term of the righthand side, for every $j\in\mathbb{N}_n$. This, clearly, implies (6.7).

Given a generic $j\in\mathbb{N}_n$, denote $d=d(j)=[\eta_j^1,\eta_j^2,\ldots,\eta_j^{n_k}]^T$ and define $d^+=\sum_{k\in k_+(d)}\theta_k\eta_j^k$ and $d^-=\sum_{k\in k_-(d)}\theta_k\eta_j^k$. We have that $d^+\geq 0$, $d^-\leq 0$, by definition, and the j-th term of the righthand side of (6.7) is given by

$$\sum_{k=1}^{n_k}c_{j,k}(x)=d^+(g_j(x,u)-h_j^L(x,u))+d^-(g_j^L(x,u)-h_j(x,u)).$$

Suppose that $\eta_j\in k_+(\eta)$, the case of $\eta_j\in k_-(\eta)$ is similar. We have to prove that

$$\eta_j(g_j(x,u)-h_j^L(x,u))\leq d^+(g_j(x,u)-h_j^L(x,u))+d^-(g_j^L(x,u)-h_j(x,u)). \tag{6.8}$$

Since $\eta_j=d^++d^-$ by definition, $g_j(x,u)-g_j^L(x,u)\geq 0$, $h_j(x,u)-h_j^L(x,u)\geq 0$ by convexity, and $d^-\leq 0$, we have

$$\begin{aligned}\eta_j(g_j(x,u)-h_j^L(x,u)) &\leq (d^++d^-)(g_j(x,u)-h_j^L(x,u))-\\ &-d^-(g_j(x,u)-g_j^L(x,u))-d^-(h_j(x,u)-h_j^L(x,u)),\end{aligned} \tag{6.9}$$

since the righthand side term in (6.9) is obtained by adding positive quantities to the lefthand side one. Notice that (6.9) is equivalent to (6.8), then the property is proved. ■

The previous properties are the basis for the following results on control invariance and λ-contractiveness of a convex, compact set Ω for a non-autonomous DC system.

Property 6.11 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given $\lambda\in[0,1]$, a convex, compact set $\Omega\subseteq\mathscr{K}^0(X)$ is a λ-contractive set (a robust control invariant set if $\lambda=1$) for system*

(6.1) and constraints $x \in X$ and $u \in U$ if and only if there exists a control law $u = u(x) \in U$ such that

$$\eta^T f(x,u(x)) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{6.10}$$

Proof: By definition, $\Omega \subseteq \mathscr{K}^0(X)$ is a λ-contractive set for system (6.1) and constraints $x \in X$ and $u \in U$ if there exists a control law $u = u(x) \in U$ such that

$$x^+ = f(x,u(x)) + w \in \lambda\Omega, \qquad \forall x \in \Omega,\ \forall w \in W. \tag{6.11}$$

Taking into account that Ω is a convex, compact set and equation (C.2), we have that condition (6.11) is equivalent to

$$\eta^T (f(x,u(x)) + w) \leq \phi_{\lambda\Omega}(\eta), \quad \forall x \in \Omega,\ \forall w \in W,\ \forall \eta \in \mathbb{R}^n. \tag{6.12}$$

Notice that, by definition of support function, we have

$$\phi_{\lambda\Omega}(\eta) = \sup_{x \in \lambda\Omega} \eta^T x = \sup_{x \in \Omega} \eta^T \lambda x = \lambda \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

and therefore (6.12) is satisfied if

$$\eta^T f(x,u(x)) \leq \lambda \phi_\Omega(\eta) - \sup_{w \in W} \eta^T w, \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n,$$

which, in turn, is equivalent to equation (6.10). ■

The necessary and sufficient condition for a set Ω to be a robust control invariant λ-contractive set, given by (6.10), is very hard to be tested since, even if the control law is assumed to be known, checking the condition requires in practice to verify the fulfillment of an infinite number of non-convex constraints, one for all $x \in \Omega$ and all $\eta \in \mathbb{R}^n$.

In what follows we present a convex relaxation of such condition to obtain an only sufficient condition for a convex set $\Omega \subseteq X$ to be λ-contractive. The convex nature of the proposed condition allows one to devise a simple algorithm to check its fulfillment. The sufficient condition for λ-contractiveness of a set Ω is given in the following property. Note that it has to be verified only on the boundary of Ω.

Property 6.12 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given $\lambda \in [0,1]$ and a compact, convex set $\Omega \subseteq \mathscr{K}^0(X)$, if there exists a control law $u = u(x) \in U$ defined on $x \in \partial\Omega$ such that*

$$\check{F}(x,u(x),\eta) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n, \tag{6.13}$$

where function $\check{F}(\cdot,\cdot,\cdot)$ is defined in (6.4), then Ω is a λ-contractive set (a robust control invariant set if $\lambda = 1$) for system (6.1) and constraints $x \in X$ and $u \in U$.

Proof: First note that, from Property 6.8, it follows that

$$\check{F}(x,u,\eta) \leq \lambda \phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n, \tag{6.14}$$

implies fulfillment of equation (6.10), and then λ-contractiveness and robust control invariance of Ω. In general the inverse is not true, for this reason the condition is only sufficient, while it is necessary and sufficient for the CDI system implicitely overbounding the DC one. We prove that there exists a control law $u(x) \in U$ defined on $\partial\Omega$ such that the condition (6.13) is satisfied if and only if there exist a $\hat{u}(x) \in U$ defined on Ω such that condition (6.14) is fulfilled.

Necessity is trivial, since Ω compact implies $\partial\Omega \subseteq \Omega$. Sufficiency has to be proved. Suppose, hence, that there exists a control law $u = u(x) \in U$, for all $x \in \partial\Omega$ such that condition (6.13) is fulfilled. From compactness and convexity of Ω it follows that given $\hat{x} \in \Omega$ there exists a set of points of $\partial\Omega$ such that $\hat{x}$ is their convex combination (see Theorem 18.5 of (Rockafellar, 1970)). This means that there exist a non-empty set of p points $x^j(\hat{x}) \in \partial\Omega$, with $p = p(\hat{x}) \in \mathbb{N}$, and a set of p reals $\theta^j(\hat{x}) \in \mathbb{R}$, for $j \in \mathbb{N}_p$, such that $\hat{x} = \sum_{j=1}^p \theta^j(\hat{x})\, x^j(\hat{x})$, $\theta^j(\hat{x}) \geq 0$ for all $j \in \mathbb{N}_p$, and $\sum_{j=1}^p \theta^j(\hat{x}) = 1$. Since also the set U has been assumed convex and the control law $u(x) \in U$ is defined on the boundary of Ω, then, denoting $u^j(\hat{x}) = u(x^j(\hat{x}))$, for all $j \in \mathbb{N}_p$, we have that $\hat{u}(\hat{x}) = \sum_{j=1}^p \theta^j(\hat{x})\, u^j(\hat{x})$ is such that $\hat{u}(\hat{x}) \in U$. By convexity of function $\check{F}(\cdot,\cdot,\eta)$ on the convex set $X \times U$ and equation (6.13), it follows that

$$\begin{aligned}
&\check{F}(\hat{x},\hat{u}(\hat{x}),\eta) = F\left(\sum_{j=1}^{p} \theta^j(\hat{x})\, x^j(\hat{x}), \sum_{j=1}^{p} \theta^j(\hat{x})\, u^j(\hat{x}), \eta\right) \leq \\
&\leq \sum_{j=1}^{p} \theta^j(\hat{x})\, F(x^j(\hat{x}),u^j(\hat{x}),\eta) \leq \sum_{j=1}^{p} \theta^j(\hat{x})(\lambda\phi_\Omega(\eta) - \phi_W(\eta)) = \\
&= \lambda\phi_\Omega(\eta) - \phi_W(\eta), \quad \forall x \in \Omega,\ \forall \eta \in \mathbb{R}^n.
\end{aligned}$$

This means that condition (6.13) implies condition (6.14) and then it is a sufficient condition for λ-contractiveness of Ω for system (6.1). ■

Property 6.12, besides giving a sufficient condition for λ-contractiveness and robust control invariance based on convex constraints satisfaction, provides a simple way to compute an admissible, nonlinear in general, control law defined on Ω. In fact, a family of control laws defined on $\Omega \subseteq X$ can be obtained from the knowledge of the control determined on the boundary of Ω, as illustrated in the corollary below.

Corollary 6.13 *Let Assumptions 2.3, 3.19 and 6.1 hold and consider $\lambda \in [0,1]$, a compact, convex set $\Omega \subseteq \mathscr{K}^0(X)$ and a control law $u(x)$ defined on $\partial\Omega$ such that equation (6.13) is fulfilled. Any control law $\hat{u}(\hat{x})$ defined at $\hat{x} \in \Omega$ as $\hat{u}(\hat{x}) = \sum_{j=1}^p \theta^j(\hat{x})\, u^j(\hat{x})$, where $p \in \mathbb{N}$, $x^j(\hat{x}) \in \partial\Omega$ and $\theta^j(\hat{x}) \in \mathbb{R}$, for $j \in \mathbb{N}_p$, are such that $\hat{x} = \sum_{j=1}^p \theta^j(\hat{x})\, x^j(\hat{x})$ and $\theta^j(\hat{x}) \geq 0$ and $\sum_{j=1}^p \theta^j(\hat{x}) = 1$, is an admissible robust control law such that Ω is a λ-contractive set (a robust invariant set if $\lambda = 1$) for the nonlinear system (6.1) in closed-loop.*

This means that, given a point $x \in \Omega$, any admissible set of points x^j with $j \in \mathbb{N}_p$ on the boundary such that x is a convex combination of them, determines an admissible control input at x. Any control law defined on Ω as a selection at $x \in \Omega$ among all the admissible control input determined in that way provides an admissible robust control law.

6.1.2 Condition for control invariance for polytopic Ω.

The aim of the section is to propose a condition for control invariance, and λ-contractiveness, simple to be computed. In particular, under the assumption that Ω is a polytope, we provide a condition for control invariance given by a finite number of convex constraints involving only its vertices.

First we consider a sufficient condition for a polytope $\Omega \in \mathscr{K}^0(X)$ to be λ-contractive for the deterministic nonlinear system

$$x^+ = f(x,u), \tag{6.15}$$

where $f(\cdot,\cdot)$ is the DC dynamic function of (6.1). Then the result will be extended to provide a sufficient condition for robust control invariance of a polytope for the uncertain nonlinear system (6.1).

Property 6.14 *Let Assumptions 3.19 and 6.1 hold. Given $\lambda_n \in [0,1]$ and a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, and denoting as $v^j \in \mathbb{R}^n$, for $j \in \mathbb{N}_{n_v}$, its n_v vertices, if there exist control actions defined at the vertices, $u^j = u(v^j) \in U$, for all $j \in \mathbb{N}_{n_v}$, such that*

$$\check{F}(v^j, u^j, H_i^T) \leq \lambda_n, \qquad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h}, \tag{6.16}$$

where function $\check{F}(\cdot,\cdot,\cdot)$ is defined in (6.4), then Ω is a λ-contractive set (a control invariant set if $\lambda_n = 1$) for system (6.15) and constraints $x \in X$ and $u \in U$. Moreover, there exists a control law $u(x) \in U$ defined on Ω such that for any $x_0 \in \Omega$ the trajectory $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.15) with control law $u_k = u(x_k)$, satisfies $x_k \in \lambda_n^k \Omega$, for all $k \in \mathbb{N}$.

Proof: First notice that, from Property 6.8 and equation (C.3), it follows that

$$\check{F}(x,u,H_i^T) \leq \lambda \phi_\Omega(H_i^T) = \lambda, \quad \forall x \in \Omega, \quad \forall i \in \mathbb{N}_{n_h} \tag{6.17}$$

implies fulfillment of equation (6.10) with $W = \{0\}$, and then λ-contractiveness of Ω. In general the inverse is not true, for this reason the condition is only sufficient. Similarly to the case of generic sets $\Omega \in \mathscr{K}^0(X)$, see Property 6.12, we have to demonstrate that there exists a control law $u^j \in U$ defined at vertices v^j, for $j \in \mathbb{N}_{n_v}$ such that the condition (6.16) is satisfied if and only if there exist a $\hat{u}(x) \in U$ defined on Ω such that condition (6.17) is fulfilled. Necessity is trivial, since $v^j \in \Omega$ for all $j \in \mathbb{N}_{n_v}$. Sufficiency has to be proved.

Notice that, since any point of a polytope, $x \in \Omega$, can be expressed as the convex combination of its vertices then there exist $\theta^j(x) \geq 0$, $j \in \mathbb{N}_{n_v}$, such that $x = \sum_{j=1}^{n_v} \theta^j(x)\, v^j$, and $\sum_{j=1}^{n_v} \theta^j(x) = 1$. Moreover $u(x) = \sum_{j=1}^{n_v} \theta^j(x)\, u^j$ is admissible, i.e., $u(x) \in U$, from convexity of U. Consider $\varepsilon \in [0, 1]$. From the convexity of $\check{F}(\cdot,\cdot,H_i^T)$, for any $H_i^T \in \mathbb{R}^n$ and (6.16), it is inferred that, for all $j \in \mathbb{N}_{n_v}$ and $i \in \mathbb{N}_{n_h}$

$$
\begin{aligned}
&\check{F}(\varepsilon v^j, \varepsilon u^j, H_i^T) - \varepsilon\lambda_n \leq \max_{0\leq\varepsilon\leq 1} \left\{ \check{F}(\varepsilon v^j, \varepsilon u^j, H_i^T) - \varepsilon\lambda_n \right\} = \\
&= \max\left\{ \check{F}(0,0,H_i^T) - 0;\ \check{F}(v^j, u^j, H_i^T) - \lambda_n \right\} \leq 0,
\end{aligned}
$$

since $\check{F}(0,0,H_i^T) = 0$ from Assumption 3.19, and that means that $\check{F}(\varepsilon v^j, \varepsilon u^j, H_i^T) \leq \varepsilon\lambda_n$, for all $j \in \mathbb{N}_{n_v}$ and $i \in \mathbb{N}_{n_h}$, for any $\varepsilon \in [0, 1]$. Consider $\hat{x} \in \varepsilon\Omega$ and notice that there exists $x \in \Omega$ such that $\hat{x} = \varepsilon x = \sum_{j=1}^{n_v} \theta^j(x)\varepsilon v^j$, by definition. Define $\hat{u}(\hat{x}) = \varepsilon u(x) = \sum_{j=1}^{n_v} \theta^j(x)\varepsilon u^j$, clearly $\hat{u}(\hat{x}) \in U$. From Property 6.8 and convexity of function $\check{F}(\cdot,\cdot,H_i^T)$, for any $H_i^T \in \mathbb{R}^n$, it follows that if $\hat{x} \in \varepsilon\Omega$ then

$$
\begin{aligned}
&H_i f(\hat{x}, \hat{u}(\hat{x})) \leq \check{F}(\hat{x}, \hat{u}(\hat{x}), H_i^T) = \check{F}\left(\sum_{j=1}^{n_v} \theta^j(x)\varepsilon v^j, \sum_{j=1}^{n_v} \theta^j(x)\varepsilon u^j, H_i^T \right) \leq \\
&\leq \sum_{j=1}^{n_v} \theta^j(x)\check{F}(\varepsilon v^j, \varepsilon u^j, H_i^T) \leq \sum_{j=1}^{n_v} \theta^j \varepsilon\lambda_n = \varepsilon\lambda_n,
\end{aligned}
$$

for all $i \in \mathbb{N}_{n_h}$. This means that condition (6.17) is satisfied posing for $\varepsilon = 1$ and that $\hat{x} \in \varepsilon\Omega$ implies $f(\hat{x}, u(\hat{x})) \in \varepsilon\lambda_n\Omega$, for all $\varepsilon \in [0,1]$. Hence, Ω is a λ-contractive set for the DC system (6.15) and $x_0 \in \Omega$ implies $x_k \in \lambda_n^k\Omega$. ■

The following corollary will be employed to enlarge a (robust) control invariant set.

Corollary 6.15 *Let Assumptions 3.19 and 6.1 hold. Consider a polytopic set $\Omega = \{x \in \mathbb{R}^n :$ $Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, its vertices $v^j \in \mathbb{R}^n$ for $j \in \mathbb{N}_{n_v}$ and admissible control actions defined at the vertices $u^j = u(v^j) \in U$ for all $j \in \mathbb{N}_{n_v}$ such that condition (6.16) is fulfilled. Given $\hat{x} \in X$, define $\hat{\Omega} = co(\Omega \cup \hat{x}) = \{x \in \mathbb{R}^n :\ \hat{H}x \leq 1\}$, where $\hat{H} \in \mathbb{R}^{n_{\hat{h}} \times n}$ and $n_{\hat{h}} \in \mathbb{N}$. If there exists $\hat{u} = \hat{u}(\hat{x}) \in U$ such that $\check{F}(\hat{x}, \hat{u}, \hat{H}_i^T) \leq \lambda_n$, for every $i \in \mathbb{N}_{n_{\hat{h}}}$, then $\hat{\Omega}$ is a λ-contractive set (a control invariant set if $\lambda_n = 1$) for system (6.15) and constraints $x \in X$ and $u \in U$.*

Proof: Consider $\hat{\Omega}$ as candidate λ-contractive set in Property 6.14. If $\hat{x} \in \Omega$, then $\hat{\Omega} = \Omega$, trivial. Consider $\hat{x} \notin \Omega$. We have to check condition (6.16) for $\hat{\Omega}$ and all its vertices, which are given by $\hat{x}$ and a subset of the vertices of Ω. Point $\hat{x}$ fulfills condition (6.16) for all $\hat{H}_i^T$ by hypothesis, we consider now any vertex of Ω.

Since $\Omega \subseteq \hat{\Omega}$ we have that $a_i = \max_x\{\hat{H}_i x : x \in \Omega\} \leq 1$, for every $i \in \mathbb{N}_{n_{\hat{h}}}$. Since strong duality holds, see (Boyd and Vandenberghe, 2004), we have that

$$\begin{aligned} a_i = \min_{\alpha, \theta^i \in \mathbb{R}^{n_h}} \ & \alpha \\ \text{s.t.} \ & \alpha = \sum_{k=1}^{n_h} \theta_k^i, \\ & \hat{H}_i = \sum_{k=1}^{n_h} \theta_k^i H_k, \\ & \theta_k^i \geq 0, \quad \forall k \in \mathbb{N}_{n_h}, \end{aligned}$$

which means that the dual optimizer, denote it $\hat{\theta}^i \in \mathbb{R}^{n_h}$, is such that $\hat{H}_i = \sum_{k=1}^{n_h} \hat{\theta}_k^i H_k$ and $\sum_{k=1}^{n_h} \hat{\theta}_k^i = a_i \leq 1$, for all $i \in \mathbb{N}_{n_{\hat{h}}}$.

From Property 6.10, for all v^j and u^j, $j \in \mathbb{N}_{n_v}$, we have that

$$\check{F}(v^j, u^j, \hat{H}_i^T) \leq \sum_{k=1}^{n_k} \theta_k^i \check{F}(v^j, u^j, H_k^T) \leq \sum_{k=1}^{n_k} \theta_k^i \lambda_n = a_i \lambda_n \leq \lambda_n,$$

for all $i \in \mathbb{N}_{n_{\hat{h}}}$, since vertices of Ω are assumed to satisfy condition (6.16). The result is proved. ■

Any set $\hat{\Omega}$ as in Corollary 6.15 is a control invariant set, with contraction factor λ_n. Property 6.14 provides a criterion to design a control law and to determine whether a polyhedral set $\Omega \subseteq X$ is a robust control invariant set for a discrete-time DC system in absence of uncertainty, the subsequent corollary permits to determine an enlarged control invariant set. These results are easily extended to the uncertain DC system (6.1).

Property 6.16 *Let Assumptions 2.3, 3.19 and 6.1 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, its n_v vertices $v^j \in \mathbb{R}^n$ for $j \in \mathbb{N}_{n_v}$, and the uncertain DC system (6.1). If there exists an admissible control law defined at the vertices $u^j = u(v^j) \in U$ for all $j \in \mathbb{N}_{n_v}$, such that*

$$\check{F}(v^j, u^j, H_i^T) \leq \lambda_w - \phi_W(H_i^T), \qquad \forall j \in \mathbb{N}_{n_v}, \ \forall i \in \mathbb{N}_{n_h}, \tag{6.18}$$

for a $\lambda_w \in [0,1]$, where function $\check{F}(\cdot,\cdot,\cdot)$ is defined in (6.4), then Ω is a λ-contractive set for the DC system (6.1) with contraction factor λ_w.

Moreover, given any $\hat{x} \in X$ and denoting $\hat{\Omega} = co(\Omega \cup \hat{x}) = \{x \in \mathbb{R}^n : \hat{H}x \leq 1\}$ with $\hat{H} \in \mathbb{R}^{n_{\hat{h}}}$, if there exists $\hat{u} = u(\hat{x}) \in U$ satisfying $\check{F}(\hat{x}, \hat{u}, \hat{H}_i^T) \leq \lambda_w - \phi_W(\hat{H}_i^T)$, for all $i \in \mathbb{N}_{n_{\hat{h}}}$, also the set $\hat{\Omega}$ is a λ-contractive set for the DC system (6.1) with contraction factor λ_w.

Proof: Condition on Ω to be a λ-contractive set for the uncertain DC system is that $H(f(x,u(x)) + w) \leq \lambda_w$, for all $x \in \Omega$ and $w \in W$. Inequalities (6.18) implies the fulfillment of the condition at the vertices v^j, $j \in \mathbb{N}_{n_v}$, in fact

$$\begin{aligned} &H_i f(v^j,u^j) + H_i w \leq \check{F}(v^j,u^j,H_i^T) + H_i w \leq \\ &\leq \check{F}(v^j,u^j,H_i^T) + \sup_{w \in W} H_i w = \check{F}(v^j,u^j,H_i^T) + \phi_W(H_i^T) \leq \lambda_w, \end{aligned}$$

for all $j \in \mathbb{N}_{n_v}$ and $i \in \mathbb{N}_{n_h}$. From convexity of function $\check{F}(\cdot,\cdot,\eta)$, for any $\eta \in \mathbb{R}^n$ and convexity of Ω and U, the results can be proved similarly to Property 6.14 and Corollary 6.15. ■

Then, Property 6.16 provides a sufficient condition for a given polytope $\Omega \in \mathscr{K}^0(X)$ to be a robust control invariant and λ-contractive set for the uncertain DC system (6.1). In what follows we demonstrate a convexity related property of the set of polytopes for which the sufficient condition is satisfied. For this purpose, a definition is introduced here.

Definition 6.17 *Let Assumptions 2.3, 3.19 and 6.1 hold. We denote with $\Lambda(\lambda_w) \subseteq \mathscr{K}^0(X)$ the set of polytopes $\Omega \subseteq X$ which satisfy hypothesis of Property 6.16. That is, polytope $\Omega \in \Lambda(\lambda_w)$ if there exists an admissible control law defined at its vertices such that condition (6.18) is fulfilled.*

Roughly speaking, given $\lambda_w \in [0,1]$, the set $\Lambda(\lambda_w)$ is composed by those polytopes in X which satisfy the sufficient condition for robust invariance and λ-contractiveness (6.18) for the non-autonomous uncertain DC system.

Property 6.18 *Let Assumptions 2.3, 3.19 and 6.1 hold. Given a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$ and $\lambda_w \in [0,1]$, the set*

$$\Gamma(\Omega,\lambda_w) = \{\gamma \in \mathbb{R}_+ :\ \gamma\Omega \in \Lambda(\lambda_w)\} \tag{6.19}$$

is an interval in $\mathbb{R}_+$.

Proof: First notice that $0 \notin \Gamma(\Omega,\lambda_w)$, since this would imply the existence of $u \in U$ such that

$$f(0,u) \oplus W \subseteq \{0\},$$

which contradicts the assumption $0 \in \text{int}\ (\text{co}\ (W))$. Then we consider $\gamma > 0$.

Denoting the vertices of Ω as v^j, for $j \in \mathbb{N}_{n_v}$, then the vertices of the polytope $\gamma\Omega$ are $v_\gamma^j = \gamma v^j$, for $j \in \mathbb{N}_{n_v}$, with $\gamma > 0$. Moreover, since for any $\gamma > 0$ we have

$$\gamma\Omega = \{x \in \mathbb{R}^n :\ Hx \leq \gamma\} = \{x \in \mathbb{R}^n :\ \frac{1}{\gamma}Hx \leq 1\},$$

then it can be proved that $\gamma\Omega$ satisfies condition (6.18) if and only if there exist $u_\gamma^j = u^j(\gamma v^j) \in U$ such that

$$\check{F}(\gamma v^j, u_\gamma^j, H_i^T) \leq \gamma\lambda_w - \phi_W(H_i^T), \qquad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h}. \tag{6.20}$$

In fact we have that condition (6.18) for set $\gamma\Omega$ is given by

$$\check{F}(\gamma v^j, u_\gamma^j, \frac{1}{\gamma}H_i^T) \leq \lambda_w - \phi_W(\frac{1}{\gamma}H_i^T), \quad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h},$$

and from

$$\check{F}(\gamma v^j, u_\gamma^j, \frac{1}{\gamma}H_i^T) = \frac{1}{\gamma}\check{F}(\gamma v^j, u_\gamma^j, H_i^T), \quad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h},$$

and

$$\phi_W(\frac{1}{\gamma}H_i^T) = \frac{1}{\gamma}\phi_W(H_i^T), \quad \forall i \in \mathbb{N}_{n_h},$$

then conditions (6.18) and (6.20) are equivalent.

Assume that $\alpha \in \Gamma(\Omega, \lambda_w)$ and $\beta \in \Gamma(\Omega, \lambda_w)$, i.e., there exist two control laws $u_\alpha^j \in U$, defined at the vertices v_α^j of $\alpha\Omega$, and $u_\beta^j \in U$, defined at the vertices v_β^j of $\beta\Omega$, for all $j \in \mathbb{N}_{n_v}$, such that $\alpha\Omega$ and $\beta\Omega$ satisfy the hypothesis of Property 6.16. For any $\theta \in [0,1]$, denote $\delta = \delta(\theta) = \theta\alpha + (1-\theta)\beta$ and define $u_\delta^j = u_\delta^j(\theta) = \theta u_\alpha^j + (1-\theta)u_\beta^j$, for all $j \in \mathbb{N}_{n_h}$. Note that $u_\delta^j \in U$, $j \in \mathbb{N}_{n_v}$, by convexity of U. From convexity of function $\check{F}(\cdot,\cdot,\eta)$ for any $\eta \in \mathbb{R}^n$, it follows that

$$\begin{aligned}\check{F}(\delta v^j, u_\delta^j, H_i^T) &= \check{F}(\theta\alpha v^j + (1-\theta)\beta v^j, \theta u_\alpha^j + (1-\theta)u_\beta^j, H_i^T) \leq \\ &\leq \theta\check{F}(\alpha v^j, u_\alpha^j, H_i^T) + (1-\theta)\check{F}(\beta v^j, u_\beta^j, H_i^T) \leq \delta\lambda_w - \phi_W(H_i^T), \quad j \in \mathbb{N}_{n_v},\ i \in \mathbb{N}_{n_h}\end{aligned} \tag{6.21}$$

which means that $\delta\Omega$ with control law u_δ^j defined at its vertices δv^j, for $j \in \mathbb{N}_{n_h}$, satisfies hypothesis of Property 6.16, then $\delta\Omega \in \Lambda(\lambda_w)$. This implies that $\delta(\theta) \in \Gamma(\Omega, \lambda_w)$ for any $\theta \in [0,1]$, or equivalently, that $\Gamma(\Omega, \lambda_w)$ is a convex set in $\mathbb{R}$, hence an interval in $\mathbb{R}_+$. ■

An explicit relation between the contractive factor λ_n for the deterministic DC system (6.15) and the uncertainty that can be tolerated by the system before Ω looses its condition of invariance is easily inferred. That is, given a control law and a set Ω whose contraction factor is λ_n for system (6.15), if

$$\max_{i \in \mathbb{N}_{n_h}} \phi_W(H_i^T) \leq \lambda_w - \lambda_n, \tag{6.22}$$

then Ω is a λ-contractive set for the uncertain system (6.1), with contraction factor λ_w.

6.1.3 One-step operator for non-autonomous DC systems

We present here some considerations on the one-step operator for the particular case of non-autonomous DC systems and polytopic sets Ω. It has been already pointed out the importance of the one-step operator, on which many iterative algorithms for the computation of invariant and λ-contractive sets are based. See for instance Theorems 4.32 and 5.33, which provide properties of the sequence of invariant sets obtained applying iteratively the one-step operator for CDI systems and Lur'e systems. Similar results can be stated for the case of non-autonomous DC systems.

It is worth pointing out that, in what follows, we consider the non-autonomous DC system (6.15) and that for the one-step operator and its convex approximations defined below the dependence on the contraction factor λ is posed explicitly, for notational clearness.

Definition 6.19 *Let Assumptions 3.19 and 6.1 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, $\lambda \in [0,1]$ and the DC system (6.15). Define*

$$\begin{aligned} Q_F(\Omega,\lambda) &= \{x \in X : \exists u \in U,\ \check{F}(x,u,H_i^T) \leq \lambda, \forall i \in \mathbb{N}_{n_h}\}, \\ Q(\Omega,\lambda) &= \{x \in X : \exists u \in U,\ f(x,u) \in \lambda\Omega\}, \end{aligned} \tag{6.23}$$

where function $\check{F}(\cdot,\cdot,\cdot)$ is defined in (6.4).

Operator $Q_F(\cdot,\cdot)$ provides a convex inner approximation of the exact one-step operator $Q(\cdot,\cdot)$, widely employed in many classical recursive algorithms to compute control invariant and λ-contractive sets. Recall that Property 6.14, to which we refer in the following property, is the sufficient condition for λ-contractiveness for polytopic sets and deterministic non-autonomous DC systems.

Property 6.20 *Let Assumptions 3.19 and 6.1 hold. For any polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, and $\lambda \in [0,1]$, set $Q_F(\Omega,\lambda)$ is convex and closed and such that $Q_F(\Omega,\lambda) \subseteq Q(\Omega,\lambda)$. If Ω is such that hypothesis of Property 6.14 holds, then $\Omega \subseteq Q_F(\Omega,\lambda)$ and $Q_F(\Omega,\lambda)$ is a λ-contractive set with contraction factor λ.*

Proof: Convexity and closure follow from the definition, since for every $i \in \mathbb{N}_{n_h}$ the set

$$P_F(\Omega,\lambda) = \{(x,u) \in X \times U : \check{F}(x,u,H_i^T) \leq \lambda, \forall i \in \mathbb{N}_{n_h}\},$$

is convex and closed, from Property 6.7 and Assumption 6.1, and $Q_F(\Omega,\lambda)$ is the projection on the state space of the set $P_F(\Omega,\lambda) \subseteq X \times U$.

Inclusion $Q_F(\Omega,\lambda)\subseteq Q(\Omega,\lambda)$ follows from Property 6.8. Inclusion $\Omega\subseteq Q_F(\Omega,\lambda)$ is due to the fact that if $x\in\Omega$ then there exists $u\in U$ such that $\check{F}(x,u,H_i^T)\leq\lambda$, for all $i\in\mathbb{N}_{n_h}$, as proved in proof of Property 6.14. Finally, contractiveness is due to the fact that $x\in Q_F(\Omega,\lambda)$ implies $x\in Q(\Omega,\lambda)$ and then the existence of $u(x)\in U$ such that $f(x,u)\in\lambda\Omega\subseteq\lambda Q_F(\Omega,\lambda)$. That is, $x\in Q_F(\Omega,\lambda)$ implies the existence of $u=u(x)\in U$ such that $f(x,u)\in\lambda Q_F(\Omega,\lambda)$, which is the definition of λ-contractiveness for the set $Q_F(\Omega,\lambda)$. ■

Hence, given a polytope $\Omega\in\mathscr{K}^0(X)$ and $\lambda\in[0,1]$, the iterative application of operator $Q_F(\cdot,\cdot)$, provides a sequence of nested λ-contractive sets for the DC system (6.15). The main problem from the computational point of view is that $Q_F(\Omega,\lambda)$ is not a polytope in general case, thus the sequence of sets generated by applying iteratively the one-step operator are not polytopic, not even if the initial set is a polytopic control invariant set. Alternatively, a polytopic inner approximation can be computed. The following property is instrumental to that purpose and provides a deeper insight on the characterization of the convex one-step operator $Q_F(\cdot,\cdot)$.

Property 6.21 *Let Assumptions 3.19 and 6.1 hold. Given any polytope* $\Omega=\{x\in\mathbb{R}^n:\ Hx\leq 1\}\subseteq X$, *with* $H\in\mathbb{R}^{n_h\times n}$, *and* $\lambda\in[0,1]$, *then* $Q_F(\Omega,\lambda)=\hat{Q}_F(\Omega,\lambda)$ *with*

$$\hat{Q}_F(\Omega,\lambda)=\{x\in X:\exists u\in U,\check{F}(x,u,\eta)\leq\lambda,\forall\eta\in\Theta(H)\},\tag{6.24}$$

where function $\check{F}(\cdot,\cdot,\cdot)$ *is defined in (6.4) and*

$$\Theta(H)=\{\eta\in\mathbb{R}^n:\exists\theta\in\mathbb{R}^{n_h},\theta_i\geq 0,\sum_{i=1}^{n_h}\theta_i\leq 1,\eta=\sum_{i=1}^{n_h}\theta_i H_i^T\}.\tag{6.25}$$

Proof: Notice that $\Theta(H)\subseteq\mathbb{R}^n$ is the polytope whose elements are those vectors in $\mathbb{R}^{n_h}$ that can be expressed as convex combinations of the rows of matrix H, that is of H_i^T, with $i\in\mathbb{N}_{n_h}$.

We prove that $Q_F(\Omega,\lambda)\subseteq\hat{Q}_F(\Omega,\lambda)$ and $\hat{Q}_F(\Omega,\lambda)\subseteq Q_F(\Omega,\lambda)$. The latter is trivial, since $H_i^T\in\Theta(H)$, for all $i\in\mathbb{N}_{n_h}$, and then $Q_F(\Omega,\lambda)$ is defined by a set of constraints which are a subset of those defining $\hat{Q}_F(\Omega,\lambda)$. We prove $Q_F(\Omega,\lambda)\subseteq\hat{Q}_F(\Omega,\lambda)$ by reduction to absurd. Suppose that there is a $\hat{x}\in Q_F(\Omega,\lambda)$ such that $\hat{x}\notin\hat{Q}_F(\Omega,\lambda)$. Then, there exists a $\eta\in\Theta(H)$ such that

$$\check{F}(\hat{x},\hat{u},\eta)>\lambda,\quad\forall\hat{u}\in U.$$

From $\hat{x}\in Q_F(\Omega,\lambda)$ and Property 6.10, we have that

$$\lambda<\check{F}(\hat{x},\hat{u},\eta)\leq\sum_{i=1}^{n_h}\theta_i\check{F}(\hat{x},\hat{u},H_i^T)\leq\sum_{i=1}^{n_h}\theta_i\lambda\leq\lambda,$$

for a proper selection of $\theta\in\mathbb{R}^{n_h}$, which is absurd. Then $Q_F(\Omega,\lambda)\subseteq\hat{Q}_F(\Omega,\lambda)$, and the property is proved. ■

The meaning of Property 6.21 is clear: a point $x \in X$ belongs to set $Q_F(\Omega,\lambda)$ if and only if there exists $u \in U$ such that $\check{F}(x,u,\eta) \leq \lambda$, not only for $\eta = H_i^T$, with $i \in \mathbb{N}_{n_h}$, but also for any vector η that belongs to the polytope $\Theta(H) \subseteq \mathbb{R}^n$.

Corollary 6.22 *Let Assumptions 3.19 and 6.1 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, and $\lambda \in [0,1]$, such that hypothesis of Property 6.14 holds for Ω, and $\hat{x} \in Q_F(\Omega,\lambda)$. Then the set $\hat{\Omega} = co(\Omega \cup \hat{x}) = \{x \in \mathbb{R}^n : \hat{H}x \leq 1\}$, where $\hat{H} \in \mathbb{R}^{n_{\hat{h}} \times n}$ and $n_{\hat{h}} \in \mathbb{N}$, is a λ-contractive set (a robust control invariant set if $\lambda_n = 1$) for system (6.15) and constraints $x \in X$ and $u \in U$.*

Proof: We prove that also $\hat{\Omega}$ satisfies the hypothesis of Property 6.14, which, we recall, provides a sufficient condition for a polytope to be a λ-contractive set. Analogously to the proof of Corollary 6.15, we have that $\Omega \subseteq \hat{\Omega}$ implies the existence of $\theta^j \geq 0$ such that $\hat{H}_j^T = \sum_{k=1}^{n_h} \theta_k^j H_k^T$, $\sum_{k=1}^{n_h} \theta_k^j \leq 1$, which is equivalent to $\hat{H}_j^T \in \Theta(H)$, for all $j \in \mathbb{N}_{n_{\hat{h}}}$. Property 6.20 and $\hat{x} \in Q_F(\Omega,\lambda)$ imply that $\hat{\Omega} \subseteq Q_F(\Omega,\lambda)$, because also Ω is a subset of $Q_F(\Omega,\lambda)$ and set $Q_F(\Omega,\lambda)$ is convex. From this and Property 6.21 we have that for every $x \in \hat{\Omega}$ there exists $u \in U$ such that $\check{F}(x,u,\eta) \leq \lambda$, for all $\eta \in \Theta(H)$, and in particular for every vertex of $\hat{\Omega}$ and every $\hat{H}_j^T$, $j \in \mathbb{N}_{n_{\hat{h}}}$. Hence hypothesis of Property 6.14 holds for $\hat{\Omega}$, then $\hat{\Omega}$ is a λ-contractive set. ■

Property 6.22 means that, given a polytopic set Ω that satisfies the sufficient condition for λ-contractiveness, any point $\hat{x}$ in the set $Q_F(\Omega,\lambda)$ determines another λ-contractive polytope for the non-autonomous DC system. Then, this permits to design an algorithmic procedure to generate a sequence of nested control invariant or λ-contractive polytopic sets for the DC system.

6.1.4 Practical issues on design.

The first issue to be tackled in order to apply the results shown in the previous section is how to define the potential control invariant set Ω. Once a suitable guess for Ω is given, the sufficient condition for control invariance can be applied. One possible choice is to select, as initial guess of Ω, a (robust) invariant set for the linear system obtained linearizing the DC system.

Standard iterative algorithms to determine a (robust) invariant set for linear systems have been proposed in literature, see for instance (Gilbert and Tan, 1991; Blanchini, 1999) and (Kolmanovsky and Gilbert, 1998). In case of absence of uncertainty, given a λ-contractive invariant set Ω_L for the linearized system with contraction factor $\lambda \in [0,\ 1)$, a properly scaled set $\alpha\Omega_L$, with $\alpha > 0$, can provide an invariant set also for the nonlinear system.

Once a (robust) control invariant set Ω is given, many approaches can be considered in order to obtain the control law such that set Ω is a robust invariant set for the nonlinear DC system (6.1) in closed-loop. From a practical point of view, it is sufficient to define a control action at any of the vertices, i.e. $u^j = u(v^j)$, to obtain a control law defined over the whole set Ω. In fact, as illustrated above, a proper convex combination of values $u(v^j)$, with $j \in \mathbb{N}_{n_v}$, determines a control value at every $x \in \Omega$ such that invariance of Ω is ensured for the closed-loop system, from convexity of Ω and functions $\check{F}(\cdot,\cdot,\eta)$.

Here we propose an algorithm to generate a control invariant set for the deterministic DC system. We select as initial shape for the control invariant set, a polytope Ω which is a control invariant set for the system linearized at the origin.

Recall that by Assumption 6.3, there exists a linear feedback such that the linearized system is asymptotically stable. A sketch of the algorithm follows, where $k_{max} \in \mathbb{N}$ is the number of maximal iterations performed.

Algorithm 3 Computing a control invariant set for a deterministic DC system.

Given the DC system as in (6.15):

(1) Compute a linear feedback $K \in \mathbb{R}^{m\times n}$ such that the linearized system is asymptotically stable in closed-loop and compute an invariant set Ω for the closed-loop linear system.

(2) Obtain $\hat{\alpha}$, (a lower approximation of) the maximal α such that $\alpha\Omega$ fulfills (6.16). Denote $\Omega_0 = \hat{\alpha}\Omega = \{x \in \mathbb{R}^n :\ H^0 x \leq 1\}$, with $H^0 \in \mathbb{R}^{n_h^0 \times n}$ and a proper $n_h^0 \in \mathbb{N}$, and $k = 0$.

(3) Generate $x^k \in X$ such that $x^k \notin \Omega_k$ and define $\hat{\Omega} = \mathrm{co}(\Omega_k \cup x^k) = \{x \in \mathbb{R}^n :\ H^{k+1} x \leq 1\}$, with $H^{k+1} \in \mathbb{R}^{n_h^{k+1} \times n}$ and a proper $n_h^{k+1} \in \mathbb{N}$.

(4) If there exists $u^k \in U$ such that $\check{F}(x^k, u^k, (H_i^{k+1})^T) \leq \lambda_n$, for every $i \in \mathbb{N}_{n_h^{k+1}}$ then $\Omega_{k+1} = \hat{\Omega}$, otherwise go to (3).

(5) Pose $k = k+1$. If $k \geq k_{max}$ stop, otherwise go to (3).

Once a stabilizing feedback law is determined, we employ the method illustrated in (Fiacchini, Álamo and Camacho, 2007) to obtain Ω, invariant set for the linearized system in closed-loop with the linear feedback. The sufficient condition for control invariance (6.16) for a polytopic set Ω is employed in a dichotomic algorithm to compute an approximation of the maximal $\alpha > 0$ such that $\alpha\Omega$ is a control invariant set for nonlinear system (6.1). Checking if inequality (6.16) is fulfilled by every vertex of $\alpha\Omega$, is achieved by solving a set of convex problems. Once a control invariant set $\alpha\Omega$ has been obtained, Corollary 6.15 is used to enlarge the set. Random points x^k in the state space are generated: if for $x^k \in X$ there exists a $u^k \in U$ fulfilling the hypothesis of Corollary 6.15, then the new control invariant set is obtained as the convex hull of the current control invariant set and point x^k.

It has to be pointed out that the enlargement step often requires a considerable computational effort. An accurate procedure to select properly points x^k should be defined for high dimensional cases, with the aim of reducing the complexity due to the computation of hyperplanes determining sets $\hat{\Omega}$.

Another computational problem involves the choice of $x^k \in X$ such that $x \notin \Omega_k$, that can be a non-trivial procedure for high dimensional problems. We propose an alternative procedure based on the following corollary, which is a direct consequence of Property 6.22.

Corollary 6.23 *Let Assumptions 3.19 and 6.1 hold. Consider a polytope* $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$*, with* $H \in \mathbb{R}^{n_h \times n}$*, and* $\lambda \in [0,1]$*, such that hypothesis of Property 6.14 holds for* Ω*, and, given* $\hat{x} \in X$*, define the set* $\hat{\Omega} = co(\Omega \cup \hat{x})$*. If there exists* $\hat{u} = \hat{u}(\hat{x}) \in U$ *such that* $\check{F}(\hat{x}, \hat{u}, H_i^T) \leq \lambda_n$*, for every* $i \in \mathbb{N}_{n_h}$*, then* $\hat{\Omega}$ *is a* λ*-contractive set (a control invariant set if* $\lambda_n = 1$*) for system (6.15) and constraints* $x \in X$ *and* $u \in U$*.*

Proof: Altough the result can be considered as an application of Property 6.22, the proof is provided. It is worth pointing out that under the hypothesis of the corollary we have that

$$\check{F}(\hat{x}, \hat{u}, H_i^T) \leq \lambda_n, \ \forall i \in \mathbb{N}_{n_h} \quad \Rightarrow \quad H_i f(\hat{x}, \hat{u}) \leq \lambda_n, \ \forall i \in \mathbb{N}_{n_h} \quad \Rightarrow \quad f(\hat{x}, \hat{u}) \in \lambda_n \Omega,$$

and $f(x, u(x)) \in \lambda_n \Omega$ for all $x \in \Omega$ and a proper $u = u(x) \in U$, being Ω a λ-contractive set. This does not imply λ-contractiveness of $\hat{\Omega}$, that is $f(x, u(x)) \in \lambda_n \hat{\Omega}$, for all $x \in \hat{\Omega}$ and for proper $u = u(x) \in U$, which is the claimed result and has to be proved.

We consider the non-trivial case of $x \notin \Omega$. We prove that also $\hat{\Omega}$ satisfies the hypothesis of Property 6.14. Denote $\hat{H} \in \mathbb{R}^{n_{\hat{h}} \times n}$, with $n_{\hat{h}} \in \mathbb{N}$, the matrix such that $\hat{\Omega} = \{x \in \mathbb{R}^n : \hat{H}x \leq 1\}$. Remind that the set of $n_{\hat{v}}$ vertices of $\hat{\Omega}$ is composed by $\hat{x}$ and a subset of vertices of Ω and then, by assumption, every vertex of $\hat{\Omega}$ satisfies (6.16). We prove that satisfaction of condition (6.16) with H_i^T, for every $i \in \mathbb{N}_{n_h}$, implies fulfillment also with $\hat{H}_j^T$, for all $j \in \mathbb{N}_{n_{\hat{h}}}$. Analogously to the proof of Corollary 6.15, we have that $\Omega \subseteq \hat{\Omega}$ implies the existence of $\theta^j \in \mathbb{R}^{n_h}$ with $\theta_k^j \geq 0$, such that $\hat{H}_j^T = \sum_{k=1}^{n_h} \theta_k^j H_k^T$, $\sum_{k=1}^{n_h} \theta_k^j \leq 1$ for all $j \in \mathbb{N}_{n_{\hat{h}}}$. From this and Property 6.10 we have that for vertex $\hat{v}^k$ of $\hat{\Omega}$ there exists $\hat{u}^k \in U$ such that $\check{F}(\hat{v}^k, \hat{u}^k, \hat{H}_j^T) \leq \lambda_n$, for all $j \in \mathbb{N}_{n_{\hat{h}}}$, and every $k \in \mathbb{N}_{n_{\hat{v}}}$. Hence hypothesis of Property 6.14 holds for $\hat{\Omega}$, then $\hat{\Omega}$ is a λ-contractive set. ■

The meaning of the previous corollary is that given a polytope Ω satisfying the sufficient condition for λ-contractiveness for DC systems (6.16), any point $\hat{x} \in X$ for which there exists $\hat{u}(\hat{x}) \in U$ such that $\check{F}(\hat{x}, \hat{u}, H_i^T) \leq \lambda_n$, for every $i \in \mathbb{N}_{n_h}$, determines a set which satisfies the same sufficient condition, hence it is λ-contractive. Thus the corollary can be used to generate a sequence of nested polytopes sharing the property of λ-contractiveness.

Then, a convex problem can be solved to determine points $x^k \in X$ to enlarge the control invariant set. The 3-rd and 4-th steps of Algorithm 3 should be replaced with the following.

(3a) Generate $\eta^k \in \mathbb{R}^n$ and compute $x^k \in X$ and $u^k \in U$ as an optimizer of the convex problem

$$\max_{x\in X,u\in U}\{(\eta^k)^T x:\ \check{F}(x,u,(H_i^k)^T) \leq \lambda_n, \forall i \in \mathbb{N}_{n_h^k}\}.$$

(4a) If $x^k \notin \Omega_k$, then $\Omega_{k+1} = \mathrm{co}(\Omega_k \cup x^k) = \{x \in \mathbb{R}^n :\ H^{k+1}x \leq 1\}$, with $H^{k+1} \in \mathbb{R}^{n_h^{k+1}\times n}$ and a proper $n_h^{k+1} \in \mathbb{N}$, otherwise go to (3a).

Although a random component is still present, in the choice of vector η^k, with this enlarging method point x^k lies in the complement of Ω^k or on its boundary. For this reason, in point (4a), we check if $x^k \notin \Omega$. On the other hand, if for all direction $\eta^k \in \mathbb{R}^n$ the solution x^k is on the boundary, we found the maximal convex set fulfilling Property 6.14.

Remark 6.24 *Notice that, with the modification proposed, two important computational problems are overcome. The first is the fact that the choice of a point $x^k \notin \Omega_k$ and $x^k \in X$, which can be a very demanding computational task for high dimension, is avoided in practice. The second problem is related to the necessity of computing matrix H^{k+1} from point x^k and Ω. Also this computation can be demanding for high dimension, then it is strongly preferable to avoid it, at least for all those points x^k discarded in step (4) of Algorithm 3. Through the modification proposed the computation of matrix H^{k+1} is performed only for points x^k inducing an enlargement.*

Finally, since there exists a relation between the contraction factor for the deterministic system and the maximal uncertainty affordable, see inequality (6.22), the procedure for computing a control invariant for the deterministic case can be employed to obtain a robust control invariant set. Moreover, considerations analogous to those of Remark 4.50 permit to directly adapt the algorithm to the uncertain case.

6.1.5 Numerical example

To illustrate the proposed method to compute a control invariant set and design a control law, we apply it to an example proposed in continuous-time version in (Chen, Ballance and O'Reilly, 2001), where ellipsoidal invariant sets are considered. The same system, discretized, has been used by (Cannon et al., 2003) to test their results on computation of control invariant parallelogram. The example allows us to compare the proposed technique with different methods.

The bilinear discrete-time system, obtained discretizing the continuous-time system presented in (Chen et al., 2001), is

$$x_{k+1} = \begin{bmatrix} 1 & T \\ T & 1 \end{bmatrix} x_k + T \left\{ \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1-\mu) \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} x_k \right\} u_k$$

with the sample time $T = 0.01$ and the parameter $\mu = 0.9$, the constraints on input and state are $U = \{u \in \mathbb{R} : \ |u| \leq 2\}$ and $X = \{x \in \mathbb{R}^2 : \ \|x\|_\infty \leq 4\}$. Note that the assumption of stabilizability is satisfied. The system considered is deterministic, no uncertainty is assumed at first. Since we are interested in a control invariant set, we set $\lambda_n = 1$.

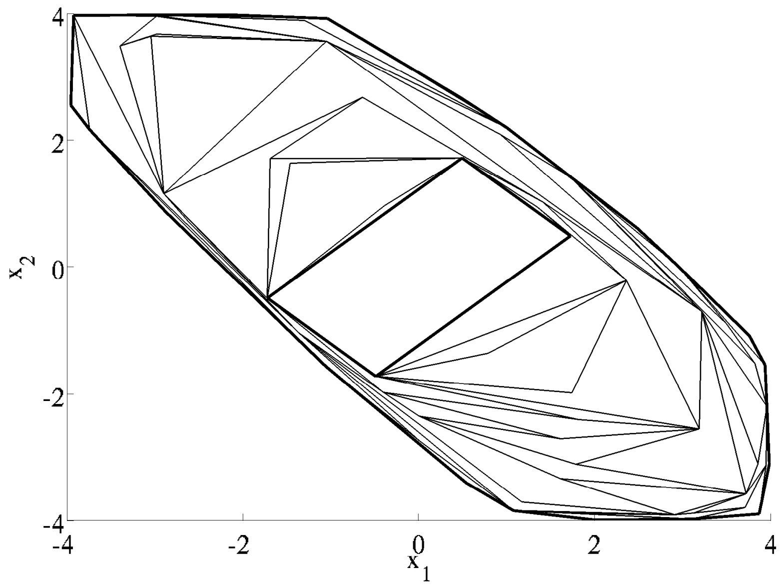

Figure 6.1: Polytopic control invariant sets generated by Algorithm 3.

The algorithm has been applied to the case under analysis, the sequence of control invariant sets is depicted in Figure 6.1. The initial set Ω_0 is the inner polytope drawn in bold line, polytopes Ω_k, $k \in \mathbb{N}_{60}$, are represented in thin lines. The final set obtained with $k_{max} = 60$ is the external polytope, depicted in bold line too.

In Figure 6.2, a comparison between the ellipsoidal control invariant set proposed in (Chen et al., 2001), the parallelogram provided by (Cannon et al., 2003) and the control invariant set obtained applying the proposed algorithm, is provided. It is evident the improvement achieved exploiting the DC structure of the system, allowing the control law to be nonlinear and dependent on the particular geometry of the control invariant set. The price to pay is the complexity increase of the representation of the invariant set.

Finally, also a robust control invariant set is computed. Applying relation (6.22), we found that, if additive uncertainty for the continuous-time system is bounded by $W = \{w \in \mathbb{R}^n : \|w\|_\infty \leq 0.4\}$, the set depicted in dashed line in Figure 6.2 is a robust control invariant set.

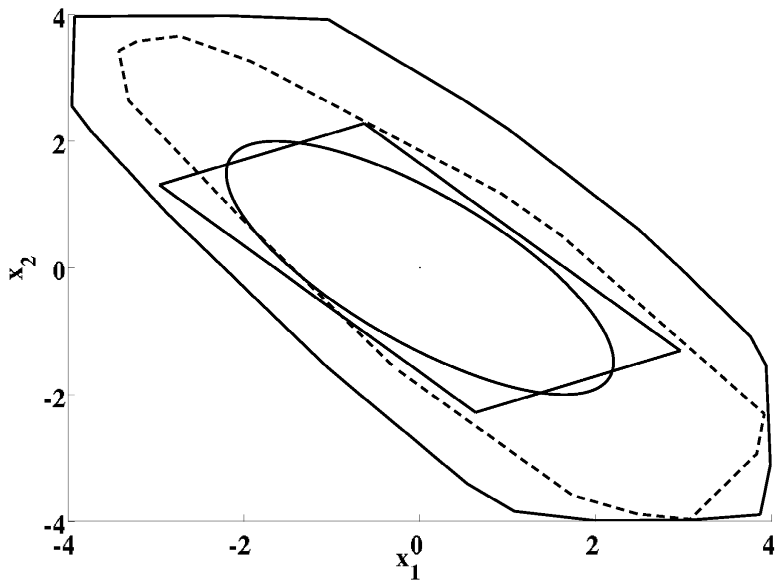

Figure 6.2: Comparison: invariant ellipsoid, invariant parallelogram and polytopic invariant set and robust invariant set generated by Algorithm 3.

6.2 Conclusions

In this chapter the problem of obtaining control invariant sets for nonlinear systems has been tackled. Particular attention has been devoted to computational aspects, with the purpose of proposing algorithmic procedures to obtain λ-contractive and control invariant sets for nonlinear systems. The class of non-autonomous DC systems, deterministic and uncertain, has been considered. Properties of CDI systems have been adapted to such class of systems and to polytopic sets to provide practical applicability to the results presented. A sufficient condition for control invariance of a set has been employed as the basis of the algorithm proposed. Also the one-step operator characterization has been provided, pointing out the beneficial properties for computational purposes.

Chapter 7

Conclusions

In this thesis we dealt with issues related to set-theory and, in particular, invariance for nonlinear systems. The main objective of this thesis is to provide theoretical results to characterize invariance related topics and computational tools to obtain in practice control laws and invariant and λ-contractive sets, for nonlinear and uncertain systems.

As exposed extensively in the introduction and throughout the thesis, invariance has gained in the last years great importance in the field of dynamic systems analysis and control design. In fact, invariant and λ-contractive sets are regions of the state space whose elements can be related to many key properties often required in control theory, such as stability, convergence, hard constraints satisfaction, Lyapunov theory, robustness, etc.

Particularly significant is the case of prediction based control laws, such as MPC. We recall here that MPC-related strategies found their main justification in their capability to cope with hard constraints satisfaction and convergence, also in presence of additive uncertainty and nonlinearity. It is also worth pointing out that many of the beneficial results ensured by MPC are based on the assumption of availability of an invariant set or a control invariant set to be used as terminal region. Moreover, a control law and a Lyapunov function are often required to prove stability and convergence of the predictive control law. Furthermore, in the case of presence of additive uncertainty, many modern MPC techniques, called tube-based, use the concept of reachable sets to guarantee constraints satisfaction and convergence, at least to a bounded set.

Hence, if on one hand invariance, λ-contractiveness and set-theory in general are greatly useful for MPC design, as well as for the design of many robust and nonlinear control schemes, on the other, few general results regarding these topics for nonlinear and uncertain systems have been provided. The main conceptual aim of our research efforts is to contribute to the reduction of this gap.

We noticed that many of the classical approaches to the problem of invariant sets computation, and related issues, have been developed for the case of linear systems, also in presence of additive uncertainty, but many of these results are hardly extendable to the case of nonlinear systems. This is due to the fact that many properties concerning invariance are based on linearity, which permits often to infer features involving an uncountable set of points by means of the analysis of a finite subset of them. This does not happen, in general, when a nonlinearity is present in the system. Furthermore, the computational tools useful to deal with linear systems related topics are often far less complex and more efficient than those which are required to be used when the system is nonlinear, such as nonlinear optimization problems.

It has been shown that the key concept which can permit to adapt linearity based techniques to nonlinear systems, what we called the "missing" ingredient, is convexity. Convexity of sets and functions allows us to cast many important results, well established for linear systems, to the case of invariant sets computation for nonlinear ones. The price is an affordable increase of the computational complexity, and some conservatism in certain cases, but the results are general and can be applied to a very wide class of nonlinear systems.

The main contributions of the thesis are:

- *Unifying modelling framework*. We introduced a novel modelling framework, represented by the CDI systems. The elements of such class of systems are deeply characterized by convexity, as the set valued map determining the systems dynamics, are defined through a set of convex functions. It has been shown that CDI systems are very powerful in order to approximate nonlinear systems. As a matter of fact, the elements of many classes of common nonlinear and uncertain systems admit a CDI representation or, at least, can be easily approximated by a CDI system. We proved that CDI systems are able to represent or approximate Lur'e systems, generalized saturated systems, DC systems, linear parametric uncertain (hence for LDI and LPV) systems. Moreover, with such framework, the analysis is extended to a family of systems, as all the systems whose dynamic function is overbounded by a CDI one share some important properties, invariance for instance.

- *Procedures to obtain CDI representations for nonlinear systems*. We have provided particular classes of nonlinear and uncertain systems for which a CDI representation or an approximation can be determined. The first family of systems are the CCDI one, which are particular CDI systems whose dynamic functions are determined by a finite number of convex and concave functions. This makes such class of systems suitable in order to bound many nonlinear systems. Then the cases of Lur'e and generalized saturated systems have been considered, providing a method to obtain directly a CDI overbounding system for any element of these classes. It has been shown that many common nonlinearities affecting real systems lead to Lur'e or generalized saturated systems. Another important class of systems related to CDI ones are the called DC systems. It has been proved that DC systems can be easily approximated by CDI ones and are highly expressive in the nonlinear context. In fact, any nonlinear system whose dynamic function is twice differentiable has a DC representation and those whose dynamics are given by a continuous one admit an arbitrarily close DC approximation. Also linear parametric uncertain systems (and then LPV and LDI systems) are particular cases of CDI systems. Summarizing, many common nonlinear systems, either deterministic or uncertain, can be cast in CDI form or be approximated by a CDI system.

- *Theoretical results on invariance for CDI systems*. We employed the convexity based characteristics of the elements of this unifying framework to extend some well established and powerful results, valid for linear systems, to CDI ones. One of the main contributions, whose particular adaptations have been used throughout the thesis, is the necessary and sufficient condition for invariance and λ-contractiveness of a convex, compact set, for deterministic and uncertain CDI systems. It has to be stressed that such conditions are boundary type ones, as for linear systems. Their satisfaction can be tested by means of the analysis of a subset, possibly finite, of elements of the potential invariant and λ-contractive set. It has been proved that any convex, compact λ-contractive set containing the origin in its interior induces a Lyapunov function for a CDI system. Also this property does not hold for generic nonlinear systems. Another important tool as the one-step operator, on which many standard procedures for computing invariant and λ-contractive sets are based, has been analyzed. It has also been proved that its iterative application generates a sequence of convex, compact nested invariant sets converging to the domain of attraction of the origin for a CDI system. Finally, some considerations on how to apply the proposed theoretical results has been presented, leading to an algorithmic procedure for invariant and λ-contractive sets computation.

- *Application of theoretical results to autonomous nonlinear systems*. The theoretical results developed for the CDI framework have been applied for practical purposes, to particular classes of common nonlinear autonomous systems. Sufficient conditions for a polytope to be invariant and λ-contractive for a DC system, also in presence of additive uncertainty, have been proposed. Such conditions ensure generality, entailed by the expressive power of DC functions in the nonlinear context, and computational efficiency, being boundary type ones, as for linear systems. Computational issues are considered and an algorithm for obtaining a polytopic invariant and λ-contractive set for nonlinear systems is given. It is proved that the algorithmic procedure ensures to provide a λ-contractive set for the case of DC systems with no additive uncertainty. The problem of convex invariant sets computation for a particular class of Lur'e systems, enclosing many nonlinear systems, is considered. An ad-hoc method to obtain a sequence of nested convex invariant sets converging to a convex approximation of the domain of attraction is presented. It has also been proved that the use of LDI approximation of the systems, commonly employed in this case, leads to an invariant set contained in the approximation of the domain of attraction. Explicit relations with the CDI methods are illustrated.

- *Application of theoretical results to non-autonomous nonlinear systems*. Finally, the theoretical results exposed for CDI framework have been employed for dealing with the problem of convex control invariant sets and λ-contractive sets computation for non-autonomous DC systems, and for the related control laws determination. It has to be recalled the importance of such structures for the application of MPC for nonlinear systems. Sufficient conditions for (robust) control invariance and λ-contractiveness for non-autonomous DC systems have been provided, focussing on computational purposes. Particular attention have been directed to the polytopic case. It has been proved that such conditions induce the implicit computation of (robust) control laws determined by means of a finite number of convex optimization problems. Also the characterization of the one-step operator for non-autonomous DC systems has been addressed. Computational issues on the algorithmic generation of (robust) control invariant sets and the related (robust) control laws for non-autonomous DC systems have been analyzed.

Summarizing, the main objective of the thesis has been to provide theoretical results as well as computational tools useful for dealing with set-theory in general, and invariance related topics in particular, in the context of nonlinear and uncertain systems. For that purpose, the key unifying CDI framework has been introduced. It has been shown that the properties based on convexity characterizing CDI systems permit to extend techniques and methods, proper of the linear case, to the nonlinear one. Based on this, computational procedures to obtain in practice polytopic invariant and λ-contractive sets for nonlinear systems have been proposed and applied to classes of common nonlinear systems.

Finally, we provide a list of possible developments of the results exposed and directions of future research:

- Application of CDI framework to the problem of robust model predictive control for nonlinear uncertain systems, robust NMPC. It has been shown that many nonlinear systems can be overbounded by CDI systems. Hence, given the uncertain nonlinear system, the CDI one overbounding it can be used to compute external approximations of the reachable sets and the nonlinear MPC can be solved imposing that the resulting reachable tube satisfies the hard constraints.

- Synthesis of H_∞ controller for uncertain nonlinear system. It has been proved that a λ-contractive set Ω for a CDI system can induce, in a region of the space, a Lyapunov function whose level sets are given by $\alpha\Omega$, for α in a certain interval. Considering the Minkowski function of Ω as a sort of norm to be minimized, it can be determined an H_∞ control law ensuring ultimate boundedness and performance.

- Numerical algorithms to obtain CDI representations for nonlinear systems. The objective is to extend the applicability of the presented results to a wider class of nonlinear systems as well as to obtain tighter overbounding CDI approximations, reducing the conservatism for the elements of particular families of systems.

- Application of the CDI framework to obtain solutions to LMI-based problems. Convexity characterizing CDI systems can be used to obtain solutions of classical problems of control theory that can be posed in LMI form.

- Set-membership identification for nonlinear systems. Properties of CDI systems can be used to bound the region of the parameter vector consistent with the measurements and the model. Preliminary results in this direction are already available for the DC framework.

- Numerical solution of nonlinear programming problems rising from NMPC through the use of CDI framework.

Appendix A

Definitions and properties of invariance

Here we provide standard definitions related to invariance for discrete-time systems, see (Blanchini and Miani, 2008).

Consider an autonomous discrete-time system

$$x^+ = f(x), \tag{A.1}$$

where $x \in \mathbb{R}^n$ is the state, $x^+ \in \mathbb{R}^n$ is the successor state and $f : D \to \mathbb{R}^n$ is defined on the set $D \subseteq \mathbb{R}^n$.

We say that a set $\Omega \subseteq D$ is a positive invariant set if every trajectory $\{x_k\}_{k\in\mathbb{N}}$ generated by (A.1) and with $x_0 \in \Omega$, is such that $x_k \in \Omega$ for all $k \in \mathbb{N}$. The formal definition follows.

Definition A.1 *A set in $\Omega \subseteq \mathbb{R}^n$ is a positive invariant set for the the discrete-time autonomous system (A.1) if $\Omega \subseteq D$ and $f(x) \in \Omega$ for all $x \in \Omega$.*

An alternative definition of positive invariance for discrete-time autonomous systems can be stated in terms of the image of function $f(\cdot)$. A set $\Omega \subseteq D$ is a positive invariant set if

$$f(\Omega) \subseteq \Omega. \tag{A.2}$$

Remark A.2 *The expression* positive invariant set *is employed in literature to differentiate the concept illustrated in Definition A.1 from the called* invariance set*: a set $\Omega \subseteq D$ is an invariant set if $x_k \in \Omega$ for all $k \in \mathbb{Z}$, for any $x_0 \in \Omega$. This means that also the elements of trajectory at negative instants must be contained in Ω to be an invariant set. Since the concept of invariant set is not employed in the thesis, we refer to positive invariant sets simply as invariant sets.*

In the case of presence of uncertainties affecting the dynamic system, the condition of robust invariance has to be introduced. Consider a discrete-time uncertain autonomous system

$$x^+ = f(x,w), \tag{A.3}$$

where $x \in \mathbb{R}^n$ is the state, $x^+ \in \mathbb{R}^n$ is the successor state, $w \in \mathbb{R}^p$ is the uncertainty, i.e. $w \in W$ with $W \subseteq \mathbb{R}^p$, and $f : D \times W \to \mathbb{R}^n$ is a function defined on the set $D \times W \subseteq \mathbb{R}^{n+p}$.

Definition A.3 *A set $\Omega \subseteq \mathbb{R}^n$ is a robust positive invariant set for the uncertain autonomous system (A.3) if $\Omega \subseteq D$ and $f(x,w) \in \Omega$ for all $x \in \Omega$ and all $w \in W$.*

That is, robust invariance means that any trajectory of the uncertain system starting inside the set Ω, remains confined in it for every possible realization of the uncertainty $w \in W$, at every time instant.

In terms of set relation, and by definition of Minkowski sum of sets, we have that set $\Omega \subseteq D$ is a robust invariant set for system (A.3) if it is such that

$$f(\Omega, W) \subseteq \Omega. \tag{A.4}$$

In the case that the uncertainty is additive, i.e., if the system has the form $x^+ = f(x) + w$, with $f(\cdot)$ defined in $D \subseteq \mathbb{R}^n$, then $\Omega \subseteq D$ is a robust invariant set if

$$f(\Omega) \oplus W \subseteq \Omega, \tag{A.5}$$

by definition of Minkowski summation, or, equivalently, if

$$f(\Omega) \subseteq \Omega \ominus W. \tag{A.6}$$

Analogous definitions can be given for non-autonomous systems, that is, in presence of a manipulable input. Consider the non-autonomous system

$$x^+ = f(x,u), \tag{A.7}$$

where $x \in D$ is the state, $x^+ \in \mathbb{R}^n$ is the successor state, $u \in E$ is the control input and $f : D \times E \to \mathbb{R}^n$ is a function defined on the set $D \times E \subseteq \mathbb{R}^{n+m}$.

A set $\Omega \subseteq D$ is a control invariant set if there exists a control law $u = u(x) \in E$, defined for every $x \in \Omega$, such that every trajectory $\{x_k\}_{k \in \mathbb{N}}$ generated by (A.7), in closed-loop with $u(x)$ and with $x_0 \in \Omega$, is such that $x_k \in \Omega$ for all $k \in \mathbb{N}$.

Definition A.4 *A set $\Omega \subseteq D$ is a control invariant set for the discrete-time system (A.7) if there exists a control law $u = u(x) \in E$ such that $\Omega \subseteq D$ and $f(x, u(x)) \in \Omega$ for all $x \in \Omega$.*

Clearly, a set Ω is a control invariant set for a dynamic system if there exists an admissible control law $u = u(x)$ defined on Ω such that Ω is an invariant set for the system in closed-loop with $u(x)$.

For the case of presence of uncertainty, the concept of robust control invariance has to be introduced. Robust control invariance deals with the uncertain non-autonomous system

$$x^+ = f(x,u,w), \tag{A.8}$$

where $x \in \mathbb{R}^n$ is the state, $x^+ \in \mathbb{R}^n$ is the successor state, $u \in E$ is the control input, $w \in W$, with $W \subseteq \mathbb{R}^p$ is uncertainty and $f : D \times E \times W \to \mathbb{R}^n$ is a function defined on the set $D \times E \times W \subseteq \mathbb{R}^{n+m+p}$.

Definition A.5 *A set $\Omega \subseteq \mathbb{R}^n$ is a robust control invariant set for the uncertain discrete-time system (A.8) if there exists a control law $u = u(x) \in E$ such that $\Omega \subseteq D$ and $f(x,u(x),w) \in \Omega$ for all $x \in \Omega$ and all $w \in W$.*

A concept strongly related with invariance, that is λ-contractiveness, is introduced. Conceptually, if invariance is the property of a set whose elements are mapped inside the set itself, λ-contractiveness entails that the elements are mapped inside a "contraction" of such set. In the following the definition of λ-contractive set is given.

Definition A.6 *A convex, compact set $\Omega \subseteq \mathbb{R}^n$ with the origin in its interior is a λ-contractive set for the discrete-time system (A.1) if there exists $\lambda \in [0,1]$ such that $\Omega \subseteq D$ and $f(x) \in \lambda\Omega$ for all $x \in \Omega$. λ is called the contracting factor of Ω.*

In terms of set relation, set $\Omega \subseteq D$ is a λ-contractive set for system (A.1) if it is such that

$$f(\Omega) \subseteq \lambda\Omega. \tag{A.9}$$

Analogous definitions of λ-contractive set for non-autonomous and uncertain systems can be given. Note that, definition of λ-contractive set for contracting factor $\lambda = 1$ is equivalent to the definition of invariant set and that λ-contractiveness implies invariance. We provide also the definition of robust control invariant set.

Definition A.7 *A convex, compact set $\Omega \subseteq \mathbb{R}^n$ with the origin in its interior is a λ-contractive set for the discrete-time uncertain non-autonomous system (A.1) if there exist $\lambda \in [0,1]$ and a control law $u = u(x) \in E$ such that $\Omega \subseteq D$ and $f(x,u(x),w) \in \lambda\Omega$ for all $x \in \Omega$ and all $w \in W$. λ is called the contracting factor of Ω.*

It is worth providing two important results on the existence of invariant sets, for linear autonomous systems, see (Gilbert and Tan, 1991) and (Kolmanovsky and Gilbert, 1998).

First we recall that, for an (uncertain) autonomous linear system with state constraint $x \in X$, the maximal (robust) invariant set X_∞ is given by all the points in X such that the related trajectories do not leave X, at any time step. It is easy to prove that every invariant set contained in X is contained in X_∞. Computationally, the maximal (robust) invariant set can be obtained iteratively. If the number of iterations required to compute the maximal (robust) invariant set is finite, it is called finitely determined and such number is referred to as the determination index. A result for deterministic linear systems, provided in (Gilbert and Tan, 1991), is recalled here.

Property A.8 *For a given linear asymptotically stable system $x^+ = Ax$, subject to constraint $x \in X$, with X bounded and $0 \in int(X)$, the maximal invariant set is finitely determined.*

The minimal invariant set R_∞ for an uncertain autonomous linear system

$$x^+ = Ax + w,$$

with $w \in W$ and W compact and containing the origin, is given by those points of the state space that can belong to an admissible trajectory starting at the origin. It can be proved that such set is contained in every invariant set for the system. The explicit expression of the minimal robust invariant set is

$$R_\infty = \bigoplus_{i=0}^{\infty} A^i W,$$

which means that, in general, it is obtained by means of a sum with an infinite number of terms. Important results for uncertain linear systems, presented in (Kolmanovsky and Gilbert, 1998), are summarized in the following property.

Property A.9 *Consider a linear uncertain system $x^+ = Ax + w$, subject to constraint $x \in X$ and $w \in W$, with X and W compact and containing the origin. The maximal robust invariant set X_∞ is non-empty if and only if $R_\infty \subseteq X$ and it is finitely determined if $R_\infty \subseteq int(X)$.*

Standard iterative algorithms to determine a (robust) invariant set for uncertain linear systems have been proposed in literature, see the cited references.

Kolmanovsky and Gilbert's algorithm (adapted to the system with no output and constraints on state) is recalled here. Such algorithm is based on the following sets:

$$
\begin{aligned}
R_0 &= \{0\}, \\
R_t &= A^{t-1}W \oplus A^{t-2}W \oplus \dots AW \oplus W,
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= X \ominus (A^{t-1}W \oplus A^{t-2}W \oplus \dots AW \oplus W) = \\
&= X \ominus A^{t-1}W \ominus A^{t-2}W \ominus \dots AW \ominus W) = \\
&= X \ominus R_t,
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
O_t &= \{x \in \mathbb{R}^n :\ A^i x \in Y_i,\ \forall i \in \mathbb{N}_t\} = \\
&= \{x \in \mathbb{R}^n :\ A^i x \in X \ominus R_i,\ \forall i \in \mathbb{N}_t\} = \\
&= \{x \in \mathbb{R}^n :\ A^i x \oplus R_i \subseteq X,\ \forall i \in \mathbb{N}_t\}.
\end{aligned}
$$

An iterative procedure to compute recursively such sets, presented in (Kolmanovsky and Gilbert, 1998), is adapted to the case of absence of output and sketched here.

- Initialization:

$$Y_0 = X, \quad O_0 = X; \tag{A.11}$$

- Iteration:

$$
\begin{aligned}
&Y_{t+1} = Y_t \ominus A^t W, \\
&O_{t+1} = O_t \cap \{x \in \mathbb{R}^n :\ A^{t+1} x \in Y_{t+1}\},
\end{aligned}
\tag{A.12}
$$

and O_∞ is the maximal robust invariant set.

An alternative way to compute the maximal robust invariant set, based on the procedure well presented in (Blanchini and Miani, 2008), is given by the following steps.

- Initialization:

$$\Omega_0 = X, \tag{A.13}$$

- Iteration:

$$
\begin{aligned}
\Omega_{t+1} &= \Omega_t \cap \left(A^{-1}(\Omega_t \ominus W)\right) = \\
&= \Omega_t \cap \{x \in \mathbb{R}^n :\ Ax + w \in \Omega_t,\ \forall w \in W\} = \\
&= \Omega_t \cap \{x \in \mathbb{R}^n :\ Ax \oplus W \subseteq \Omega_t\},
\end{aligned}
\tag{A.14}
$$

with Ω_∞ maximal robust invariant set.

Here we demonstrate that the two methods are essentially the same proving that $O_t = \Omega_t$ for all $t \in \mathbb{N}$.

$$\begin{aligned}
\Omega_0 \quad &= \{x:\ x \in X\},\\
\Omega_1 \quad &= \{x:\ x \in X,\ Ax + w \in X, \forall w \in W\} = \{x:\ x \in X,\ Ax \oplus W \subseteq X\},\\
\Omega_2 \quad &= \{x:\ x \in \Omega_1;\ Ax + w \in \Omega_1,\ \forall w \in W\} =\\
&= \{x:\ x \in X;\ Ax + w \in X,\ \forall w \in W;\\
&\qquad Ax + w \in X,\ \forall w \in W;\ A(Ax + w_1) + w_2 \in X,\ \forall w_1, w_2 \in W\} =\\
&= \{x:\ x \in X,\ Ax \oplus W \subseteq X,\ A^2 x \oplus AW \oplus W \subseteq X\} =\\
&= \{x:\ x \in X,\ Ax \subseteq X \ominus W,\ A^2 x \subseteq X \ominus (AW \oplus W)\} =\\
&= \{x:\ x \in X,\ Ax \subseteq X \ominus R_1,\ A^2 x \subseteq X \ominus R_2\} =\\
&= \{x:\ A^i x \subseteq X \ominus R_i,\ \forall i \in \mathbb{N}_{[0,2]}\} = O_2,\\
\dots \quad &\\
\Omega_{t+1} \quad &= \{x:\ x \in \Omega_t,\ Ax + w \in \Omega_t,\ \forall w \in W\} =\\
&= \{x:\ A^i x \in X \ominus R_i,\ \forall i \in \mathbb{N}_{[0,t]};\ A^i(Ax + w) \in X \ominus R_i,\ \forall w \in W,\ \forall i \in \mathbb{N}_{[0,t]}\} =\\
&= \{x:\ A^i x \in X \ominus R_i;\ \forall i \in \mathbb{N}_{[0,t]};\ A^{t+1} x + A^t w \in X \ominus R_t,\ \forall w \in W, \} =\\
&= \{x:\ A^i x \in X \ominus R_i;\ \forall i \in \mathbb{N}_{[0,t]};\ A^{t+1} x \in X \ominus R_{t+1}\} = O_{t+1}.
\end{aligned} \tag{A.15}$$

Appendix B

Convex sets and convex functions

Many of the following definition and property are based on the works (Rockafellar, 1970; Schneider, 1993; Boyd and Vandenberghe, 2004; Ben-Tal and Nemirovski, 2001), the aim of which is the deep analysis of convexity of sets and functions.

Definition B.1 *A set $S \subseteq \mathbb{R}^n$ is said to be convex if, for every $x \in S$ and $y \in S$, we have that*

$$(1-\lambda)x + \lambda y \in S, \tag{B.1}$$

for all $\lambda \in [0,1]$.

Geometrically, this means that, if two points are elements of a convex set $S \subseteq \mathbb{R}^n$, then the whole segment between them is contained in S, and viceversa.

Definition B.2 *Given a set of point $x^j \in \mathbb{R}^n$, with $j \in \mathbb{N}_m$, the element*

$$x = \sum_{j=1}^{m} \lambda_j x^j,$$

is said to be a convex combination of points x^j, $j \in \mathbb{N}_m$ if parameters $\lambda_j \geq 0$, for all $j \in \mathbb{N}_m$ and $\sum_{j=1}^{m} \lambda_j = 1$.

Convexity of a set and the concept of convex combination of points are strictly related, as shown in the following theorem, see (Rockafellar, 1970).

Theorem B.3 *A set $S \subseteq \mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements.*

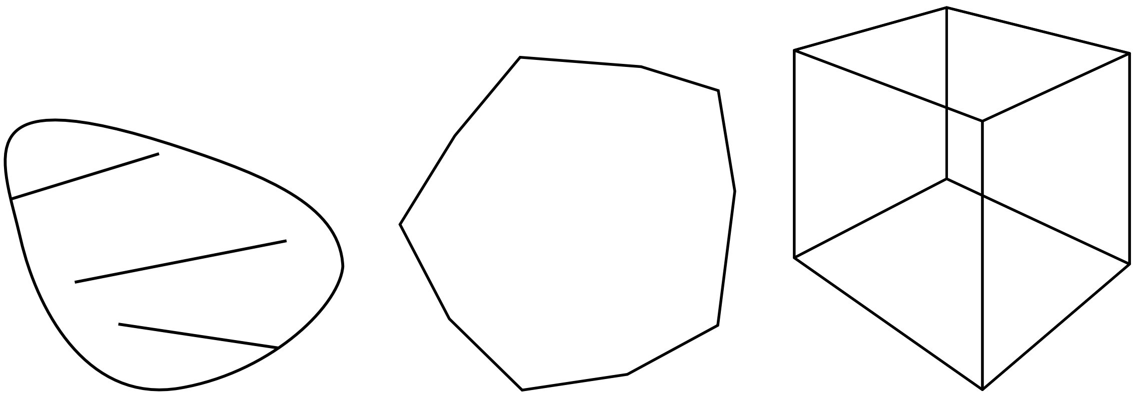

Figure B.1: Convex sets.

Definition B.4 *For any set $S \subseteq \mathbb{R}^n$, the convex hull of S, denoted $co(S)$ is the set of all the convex combinations of elements of S.*

Alternatively, the convex hull of a set $S \subseteq \mathbb{R}^n$ can be defined as the intersection of all convex sets containing S.

It is convenient to shortly recall some properties of convex sets.

- The intersection of an arbitrary collection of convex sets is convex.
- If C and D are two convex sets, then $C \oplus D$ is convex.
- Given a linear map A from $\mathbb{R}^n$ to $\mathbb{R}^m$, the image of every convex set in $\mathbb{R}^n$ is convex and the preimage of every convex set in $\mathbb{R}^m$ is convex.

We provide here the definition of convexity for functions, showing the strong relation with the concept of convex set.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, denote with $\mathrm{dom} f$ the effective domain of f, i.e. the set of points $x \in \mathbb{R}^n$ such that $f(x) < +\infty$, and define its graph as $\{(x, f(x)) \in \mathbb{R}^{n+1} :\ x \in \mathrm{dom}\ f\}$ and its epigraph, meaning "above the graph", as

$$\mathrm{epi}(f) = \{(x,\mu) \in \mathbb{R}^{n+1} :\ x \in D, \mu \geq f(x)\}, \tag{B.2}$$

where $D \subseteq \mathbb{R}^n$ is the domain of f, see (Boyd and Vandenberghe, 2004). Now, simply, a function is convex if its epigraph is a convex subset of $\mathbb{R}^{n+1}$.

Definition B.5 *A function $f : D \to \mathbb{R}$ is convex if its epigraph is a convex subset of $\mathbb{R}^{n+1}$. A function $f(\cdot)$ is concave if $-f(\cdot)$ is convex.*

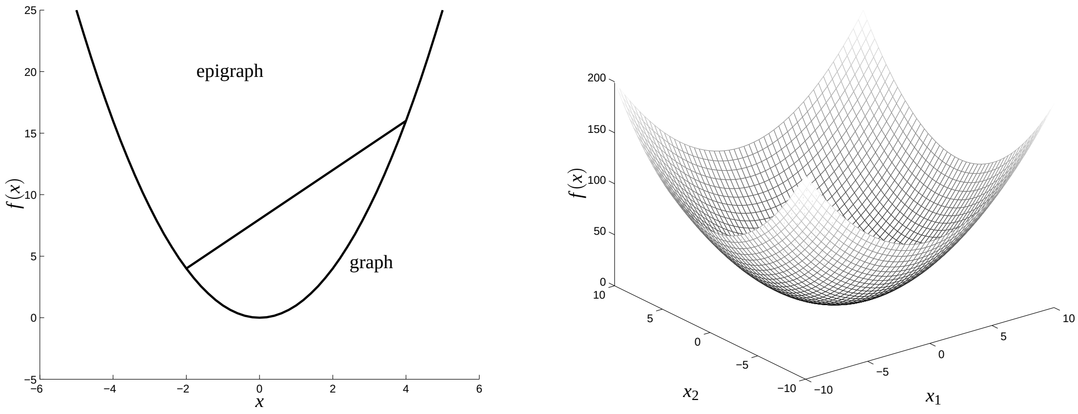

Figure B.2: Two dimensional and three dimensional graphs of convex functions.

Remark B.6 *The effective domain of a convex function is a convex subset of $\mathbb{R}^n$. Hence, given a convex function, we can consider it defined on its effective domain or consider its extension to $\mathbb{R}^n$, by defining $f(x) = +\infty$ for all $x \notin \mathrm{dom} f$, since the effective domain and the epigraph are the same in both cases. The convex functions can be assumed to be defined on the whole space $\mathbb{R}^n$, implicitly considering the extension of f if it is not defined for all $x \in \mathbb{R}^n$.*

An alternative definition of convex function, commonly employed, is presented here as a theorem, see (Rockafellar, 1970).

Theorem B.7 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom} f$ is a convex set of $\mathbb{R}^n$ and for all $x, y \in \mathrm{dom} f$ we have that*

$$f((1-\theta)x + \theta y) \leq (1-\theta) f(x) + \theta f(y), \quad \forall \theta \in [0, 1]. \tag{B.3}$$

Since we are interested, in many cases, in dynamic functions with values on $\mathbb{R}^n$, we extend the definition of convexity to multivalued functions.

Definition B.8 *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is convex if $f_j : \mathbb{R}^n \to \mathbb{R}$ is convex, for all $j \in \mathbb{N}_m$.*

Remark B.9 *Since in the thesis we deal also with non-autonomous systems, i.e. $x^+ = f(x, u)$, we show here the meaning of convexity for a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.*

A function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex if its epigraph is convex. Alternatively, we say that it is convex if, for every $x, y \in \mathbb{R}^n$ and $u, v \in \mathbb{R}^m$ we have

$$g((1-\theta)(x,u)+\theta(y,v)) \leq (1-\theta)g(x,u)+\theta g(y,v), \quad \forall\theta \in [0,1]. \tag{B.4}$$

Notice that, if a function $g(\cdot,\cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^m$, then for every $x_0 \in \mathbb{R}^n$, $g(x_0,\cdot)$ is convex on $\mathbb{R}^m$ and for every $u_0 \in \mathbb{R}^m$, $g(\cdot,u_0)$ is convex on $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, as that one characterizing a non-autonomous system, is convex if $f_j(\cdot,\cdot)$ is convex for every $j \in \mathbb{N}_n$.

Finally we report Thorem 10.1 from (Rockafellar, 1970) which stated a key relation between convexity (and then concavity) and continuity of a function.

Theorem B.10 *A convex function $f(\cdot)$ on $\mathbb{R}^n$ is continuous relative to any relatively open convex set C in its effective domain, in particular relative to $ri(\mathrm{dom} f)$.*

With $\mathrm{ri}(D)$ (and analogously with relatively open) we denote the relative interior of $D \subseteq \mathbb{R}^n$, that is, geometrically, it is the interior of the set once we restrict the analysis to the smaller affine space containing the set. To avoid formal definition, we give here an example: suppose that the effective domain of function $f(\cdot)$ is a segment in $\mathbb{R}^2$. No non-trivial open set is contained in the set. But restricting the analysis to the line containing the segment as a (possibly translated) subspace of dimension 1, then the (relative) interior of $f(\cdot)$ is the greatest open segment contained in the segment $\mathrm{dom} f$.

Appendix C

Support function

We provide here some properties related to support function. Such tool has been extensively developed and analyzed in (Rockafellar, 1970; Schneider, 1993) and has been applied to control, see (Kolmanovsky and Gilbert, 1998). We first provide the definition of support function.

Definition C.1 *Given a set $\Omega \subseteq \mathbb{R}^n$, the support function of Ω evaluated at $\eta \in \mathbb{R}^n$ is defined as*

$$\phi_\Omega(\eta) = \sup_{x \in \Omega} \eta^T x.$$

A geometrical meaning of the support function of a set Ω evaluated at η is the signed "distance" of the point of Ω (or its closure) further from the origin, along the direction η.

If Ω is bounded then its support function is defined for any $\eta \in \mathbb{R}^n$. If $0 \in \mathrm{co}(\Omega)$ then $\phi_\Omega(\eta) \geq 0$ for all $\eta \in \mathbb{R}^n$, if $0 \in \mathrm{int}(\mathrm{co}(\Omega))$ then $\phi_\Omega(\eta) > 0$ for all $\eta \in \mathbb{R}^n$, with $\eta \neq 0$. Moreover, the support function of a non-empty convex set is a positively homogeneous function, which is the meaning of the following property.

Property C.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ is convex. Then*

$$\phi_\Omega(\lambda\eta) = \lambda\phi_\Omega(\eta), \tag{C.1}$$

for all $\eta \in \mathbb{R}^n$ and all $\lambda > 0$.

If set Ω is convex and closed, then it is determined by its support function. That is a point belongs to the set, $x \in \Omega$, if and only if

$$\eta^T x \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

then we have that the set Ω can be defined through its support function.

Property C.3 *If Ω is closed and convex, then*

$$\Omega = \{x \in \mathbb{R}^n : \ \eta^T x \leq \phi_\Omega(\eta), \ \forall \eta \in \mathbb{R}^n\}. \tag{C.2}$$

If Ω is a polytope, i.e. $\Omega = \{x \in \mathbb{R}^n : Hx \leq b\}$, with $H \in \mathbb{R}^{n_h \times n}$ and $b \in \mathbb{R}^{n_h}$ then

$$x \in \Omega \quad \Leftrightarrow \quad H_i x \leq b_i = \phi_\Omega(H_i^T), \quad \forall i \in \mathbb{N}_{n_h}. \tag{C.3}$$

A particularly interesting class of sets for which the value of support function can be easily evaluated at any $\eta \in \mathbb{R}^n$ are the ellipsoids. In fact, given an ellipsoid $\mathscr{E}(P) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$, centered at the origin, we have that

$$\phi_{\mathscr{E}(P)}(\eta) = \sqrt{\eta^T P^{-1} \eta},$$

for every $\eta \in \mathbb{R}^n$. From the fact that, given $x_0 \in \mathbb{R}^n$ and a set $\Omega \subseteq \mathbb{R}^n$, the following relation holds:

$$\phi_{x_0 \oplus \Omega} = \eta^T x_0 + \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n,$$

the support function can be computed directly for every ellipsoid.

Support functions can be also employed to express the condition of set inclusion.

Property C.4 *Suppose $\Omega \subseteq \mathbb{R}^n$ is closed and convex. $\Gamma \subseteq \Omega$ if and only if*

$$\phi_\Gamma(\eta) \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n. \tag{C.4}$$

Other properties of support functions are summarized below.

Property C.5 *Suppose $\Omega \subseteq \mathbb{R}^n$ is convex. Then*

$$\phi_{\alpha\Omega}(\eta) = \alpha \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{C.5}$$

for every $\alpha > 0$.

Proof: By definition of support function, we have

$$\phi_{\alpha\Omega}(\eta) = \sup_{x \in \alpha\Omega} \eta^T x = \sup_{x \in \Omega} \eta^T \alpha x = \alpha \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n.$$

$\blacksquare$

Property C.6 *Suppose* $\Omega,\ \Gamma \subseteq \mathbb{R}^n$ *are convex sets. Then*

$$\phi_{\Omega \oplus \Gamma}(\eta) = \phi_\Omega(\eta) + \phi_\Gamma(\eta), \quad \forall \eta \in \mathbb{R}^n. \tag{C.6}$$

Moreover, for any set $\Omega \subseteq \mathbb{R}^n$ we have that

$$\phi_\Omega(\eta) = \phi_{\mathrm{co}\,(\Omega)}(\eta), \quad \forall \eta \in \mathbb{R}^n.$$

From positive homogeneity of support function, a condition requiring to be checked for every $\eta \in \mathbb{R}^n$ can be, in fact, restricted to the unitary ball, the p-norm ball $\mathbf{B}_p^n$ for instance, centered around the origin. For example, if $x \in \mathbb{R}^n$ is such that

$$\eta^T x \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbf{B}_p^n, \tag{C.7}$$

then

$$\eta^T x \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n, \tag{C.8}$$

which is equivalent to say that x is an element of the convex, closed set Ω, see Property C.3.

In fact, every element $\eta \in \mathbb{R}^n$ can be written as $\eta = \alpha(\eta)\bar{\eta}(\eta)$ with $\bar{\eta}(\eta) \in \mathbf{B}_p^n$ and $\alpha(\eta) \geq 0$ (it is sufficient to pose $\alpha(\eta) = \|\eta\|_p$ and $\bar{\eta}(\eta) = \frac{1}{\|\eta\|_p}\eta$). Therefore, from Property C.2, if condition (C.7) is fulfilled then for any $\eta \in \mathbb{R}^n$

$$\eta^T x = \alpha(\eta)\bar{\eta}^T(\eta)x \leq \alpha(\eta)\phi_\Omega(\bar{\eta}(\eta)) = \phi_\Omega(\eta),$$

and then condition (C.8) follows.

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