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RESEARCH ARTICLE

Two families of values for global cooperative games

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Abstract

A global (cooperative) game describes the utility that the whole set of players generates depending on the coalition structure they form. These games were introduced by Gilboa and Lehrer (Int J Game Theory 20:129–147, 1991) who proposed and characterized a generalization of the Shapley value. We introduce two families of point valued solutions that contain the Gilboa–Lehrer value. We characterize each family by means of reasonable properties, which are related to the ones used by Gilboa and Lehrer.

Keywords Global games · Shapley value · Anonymity · Null player property · Lattice

JEL Classification C71 · D62

1 Introduction

A cooperative game is built around the assumption that each possible coalition of agents can make binding agreements and operate as a single entity. One of the main research questions is how to share the worth generated by the agents that participate in a game. There is vast literature on the topic when the information available is the (transferable) utility that each subset of players generates. The axiomatic approach initiated by Shapley (1953) is the most common way to address the question. It is

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based on discussing desirable principles, described by formal properties that a sharing rule should reasonably satisfy.

In some cases, the available information is the utility that the whole set of players can generate depending on how they are organized in coalitions. This is precisely a global (cooperative) game introduced by Gilboa and Lehrer (1991).[1] A global game does not specify the worth of every possible coalition but the overall utility generated by the coalition structure. This can be interesting when the focus is on the public good side of the cooperation rather than on the incentives of agents or coalitions. Think for instance on the climate change problem. A lot of effort has been put in order to analyze the incentives of the countries to implement carbon reduction policies. It is well known that even if the cooperation of all countries increases the social welfare, agents have incentives to free-ride (Barrett 1994). Most of the contributions regarding the formation and stability of international environmental agreements use non-cooperative games (Finus 2008). Nonetheless, cooperative games have shown to be useful, for instance to study the side payments between countries (Chander and Tulkens 2006). We believe that global cooperative games can be used to study the consequences of the potentially different commitment levels of all the countries. A follow up question is how to asses the importance of each agent's participation in the eventual formation of the grand coalition. This is our main objective. Other papers that study problems closely related to global games include Caulier et al. (2015) and Rossi (2019).

Gilboa and Lehrer (1991) propose and characterize a generalization of the Shapley value to global games. To illustrate this point valued solution consider the toy model in which there are six players and the global utility is either 0 or 1. In order to be successful (utility equal to 1) agents 2 and 3 must join forces as well as agents 4, 5, and 6. That is, the global game assigns 1 to $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$ and to any other coarser partition and 0 to the rest. The Gilboa-Lehrer value of this simple game proposes the allocation vector $(0, 0.2, 0.2, 0.2, 0.2, 0.2)$. That is, it gives a zero payoff to the first player, whose participation in a coalition is not necessary, and equal payoff to the remaining five players. It seems to us that this is not a satisfactory answer as the participation in a two players coalition is rewarded as much as the participation in a larger one. Clearly, in many contexts it is more difficult to reach an agreement among a larger set of players. Four axioms characterize this value, linearity, efficiency, symmetry, and the null player property. The first two are very standard and we also impose them to any sensible solution. Nonetheless, we consider different and weaker versions of the remaining two properties. According to Gilboa and Lehrer (1991) two players are symmetric if their desertion from a coalition to remain alone has the same impact on the worth of any coalition structure. Then, the property requires that two such players get the same payoff. In our toy model all players but 1 are symmetric according to this definition. Even if this could be desirable in some situations we argue that leaving a small coalition may not be the same as leaving a larger one. We study the implications of replacing symmetry by the weaker anonymity in their characterization result. Anonymity states that the payoffs in the permuted game should be equal to

[1] Not to be mixed up with the non-cooperative games with incomplete information introduced by Carlsson and Van Damme (1993).

the permuted payoffs in the original game. This is our second main contribution, the characterization of the family of values that satisfy linearity, efficiency, anonymity, and the null player property. Our first main contribution is the characterization of the larger family of values that satisfy efficiency, linearity, and anonymity. We provide instances of values in each of these families to illustrate the differences between them and also with respect to the Gilboa-Lehrer value. Finally, we explore the possibility of weakening the null player property. Gilboa and Lehrer (1991) consider that a player is null if the global worth is not affected by her leaving a coalition to remain alone and their property states that such players should get a zero payoff. That is, they only consider movements of players from being alone to participating in a coalition. We define complete null players as those involved in more general movements, like the merging of two coalitions that do not affect the overall utility. We show that the complete null player property is implied by efficiency and anonymity. Even if in a different framework, this can be considered parallel to the variety of null player properties that exist for games with externalities (see Section 8.8.3 of Kóczy et al. 2018, for a comprehensive survey on the topic).

Our results rely on the well known lattice of partitions. Formally, a global game is just a real valued function on the set of partitions of a finite set and the set of such games with a fixed player set is a vector space. Using the finer (or coarser) relation among partitions of a finite set it is easy to identify a basis of the vector space, parallel to the well known unanimity basis of classic cooperative games. And using the Möbius inversion formula of the lattice of partitions we explicitly write the coefficients of any game in this basis. Then, the linearity property that we impose allows us to focus on the payoffs of the games in the basis. This facilitates the construction of the two families of values that we propose. To conclude, we identify other values in these families besides the one proposed by Gilboa and Lehrer (1991). We pin down some of them and illustrate their behavior by means of examples.

The rest of the paper is organized as follows. Section 2 presents the basics of coalitional games and the Shapley value. Section 3 revises the existing results on global games. In Sect. 4 we introduce and characterize the family of linear, efficient, and anonymous values. In Sect. 5 we study the implications of imposing the null player property to the previous family of values. Section 6 concludes.

2 Preliminaries

A *coalitional game* is a pair (N, v) where N is a finite non-empty set of players and $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function* satisfying $v(\emptyset) = 0$. A coalitional game describes the worth, $v(S)$, that each coalition $S \subseteq N$ can guarantee for itself. The worth is assumed to be transferable and infinitely divisible. If the set of players N is fixed then we identify each coalitional game with its characteristic function. We denote by $\mathcal{CG}^N$ the set of coalitional games over N. A coalitional game is said to be *zero normalized* if $v(\{i\}) = 0$ for every $i \in N$. The set of zero normalized coalitional games with set of players N is denoted by $\mathcal{CG}_0^N$. Given $T \subseteq N$ with $T \neq \emptyset$, the *coalitional unanimity game* u_T is defined by $u_T(S) = 1$ for every $S \supseteq T$ and $u_T(S) = 0$, otherwise. Unanimity games constitute a basis of the vector space $\mathcal{CG}^N$ and the coefficients of any $v \in \mathcal{CG}^N$, *Harsanyi dividends*, are given by[2] $\Delta_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S)$, for every $T \subseteq N$ with $T \neq \emptyset$.

A *value* on $\mathcal{CG}^N$, f, is a mapping that assigns to every $v \in \mathcal{CG}^N$ a payoff vector $f(v) \in \mathbb{R}^N$. The *Shapley value* (Shapley 1953) is defined for every $v \in \mathcal{CG}^N$ and $i \in N$ by[3]

$$Sh_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \left[v(S \cup i) - v(S) \right].$$

For completeness, we recall the classic characterization of the Shapley value. A player $i \in N$ is a *null player* in $v \in \mathcal{CG}^N$ if $v(S \cup i) = v(S)$ for every $S \subseteq N \setminus i$. Two players $i, j \in N$ are *symmetric* in $v \in \mathcal{CG}^N$ if $v(S \cup i) = v(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. A *permutation* of the set of players N is a bijection $\theta : N \to N$. We denote by Θ^N the set of permutations of N. Let $\theta \in \Theta^N$ and $v \in \mathcal{CG}^N$, the *permuted game* $\theta v \in \mathcal{CG}^N$ is defined by $\theta v(S) = v(\theta(S))$, for every $S \subseteq N$. Consider the following properties of a value on $\mathcal{CG}^N$, f.

Linearity: $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$, for every $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{CG}^N$.

Efficiency: $\sum_{i \in N} f_i(v) = v(N)$, for every $v \in \mathcal{CG}^N$.

Symmetry: $f_i(v) = f_j(v)$, for every i and j symmetric players in $v \in \mathcal{CG}^N$.

Anonymity: $f_i(\theta v) = f_{\theta(i)}(v)$, for every $v \in \mathcal{CG}^N$, $\theta \in \Theta^N$, and $i \in N$.

Null player property: $f_i(v) = 0$, for every i null player in $v \in \mathcal{CG}^N$.

It is well known that the Shapley value is the only efficient, linear, symmetric value on $\mathcal{CG}^N$ that has the null player property. Moreover, the symmetry property can be replaced by anonymity and the characterization result still holds.

3 Global games

A partition, or coalition structure, of the set of players N is a collection of disjoint subsets such that every $i \in N$ belongs to one of them. We denote by Π^N the set of all coalition structures of N. Let $P, Q \in \Pi^N$, we say that P is finer than Q, or that Q is coarser than P, and write $P \preceq Q$ if for all $S \in P$ there is $T \in Q$ such that $S \subseteq T$. We write $P \prec Q$ when $P \preceq Q$ but $P \neq Q$. The coarsest partition, where all players belong to the grand coalition is denoted by $\lceil N \rceil = \{N\}$ whereas the finest one, where each player forms a coalition is denoted by $\lfloor N \rfloor = \{\{i\} : i \in N\}$.

A *global game* is a pair (N, V) where N is a finite non-empty set of players and $V : \Pi^N \to \mathbb{R}$ such that $V(\lfloor N \rfloor) = 0$. A global game describes the worth generated by the whole set of players when they are organized according to a coalition structure. The worth is assumed to be transferable and infinitely divisible. We omit the reference to N if no confusion arises. We denote by $\mathcal{G}^N$ the set of global games with player set N. An interesting subclass of $\mathcal{G}^N$ arises naturally from the zero normalized coalitional games. Let $v \in \mathcal{CG}_0^N$, then the global game associated with v is defined for every

[2] We use lowercase letters to denote the cardinality of a finite set.

[3] We abuse notation slightly and write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively, for $S \subseteq N$ and $i \in N$.

$P \in \Pi^N$ by

$$V^v(P) = \sum_{S \in P} v(S). \tag{1}$$

A *value* on $\mathcal{G}^N$, f, is a mapping that assigns to every global game $V \in \mathcal{G}^N$ a payoff vector $f(V) \in \mathbb{R}^N$. The value of a player in a global game is a measure of the importance of her participation in the game. Gilboa and Lehrer (1991) introduced global games and, among other things, proposed a value on $\mathcal{G}^N$ using the following transformation of a global game into a zero normalized coalitional game. Let $V \in \mathcal{G}^N$, then the associated (zero-normalized) coalitional game v^V is defined for every $S \in 2^N \backslash \{\emptyset\}$ by

$$v^V(S) = V(\lceil S \rceil \cup \lfloor N \setminus S \rfloor). \tag{2}$$

That is, the worth attached to a coalition S is the worth of the coalition structure in which S forms and the rest of players are organized in singleton coalitions. Obviously, $v^V \in \mathcal{CG}_0^N$. The value on $\mathcal{G}^N$ that they propose is obtained by applying the Shapley value to the associated coalitional game. Even if the definition of the associated game is a natural way to assess the utility that the formation of a coalition generates it implies a big loss of information. Notice that the worth generated by any coalition structure with more than one non-singleton coalition is discarded. The *Gilboa-Lehrer value*, GL, is the value on $\mathcal{G}^N$ defined for every $V \in \mathcal{G}^N$ by

$$GL(V) = Sh(v^V). \tag{3}$$

Gilboa and Lehrer (1991) characterized GL by means of four properties that can be considered parallel to the classic ones of the Shapley value. In order to present them we need some additional notations and definitions. Given a coalition structure $P \in \Pi^N$ and a player $i \in N$, we denote by P_{-i} the partition obtained from P when i leaves the coalition in which she is participating to form a singleton coalition. That is, $P_{-i} = \{S \backslash \{i\} : S \in P\} \cup \{\{i\}\}$. Let $V \in \mathcal{G}^N$. We call i a *null player in* the global game V if for every $P \in \Pi^N$, $V(P) = V(P_{-i})$. We say that i and j are *symmetric players in* the global game V if for every $P \in \Pi^N$, $V(P_{-i}) = V(P_{-j})$. Let f be a value on $\mathcal{G}^N$. Next, we present the four properties.

- LIN $f(\alpha V + \beta W) = \alpha f(V) + \beta f(W)$, for every $\alpha, \beta \in \mathbb{R}$ and $V, W \in \mathcal{G}^N$.
- EFF $\sum_{i \in N} f_i(V) = V(\lceil N \rceil)$, for every $V \in \mathcal{G}^N$.
- SYM $f_i(V) = f_j(V)$, for every i and j symmetric players in $V \in \mathcal{G}^N$.
- NPP $f_i(V) = 0$, for every i null player in $V \in \mathcal{G}^N$.

The first property is linearity. A linear value is invariant under a change in the utility scale and is an additive function on $\mathcal{G}^N$. Efficiency is a very sensible property to impose when the grand coalition is the coalition structure that maximizes the global worth. An efficient value, proposes a way to share the worth that the coalition structure $\lceil N \rceil$ generates. The third property, symmetry, is an equal treatment property. It requires that the value gives the same payoff to two players whose impact to every partition when they abandon the coalition in which they participate to form a singleton coalition is equal. The last property states that null players should get a zero payoff. Note that a player is null if her movement from being alone in the structure to participating in any existing coalition does not affect the global worth.

4 The family of LEA values

Let $\theta \in \Theta^N$ and $V \in \mathcal{G}^N$, the *permuted game* $\theta V \in \mathcal{G}^N$ is defined by $\theta V(P) = V(\theta(P))$, for every $P \in \Pi^N$, where $\theta(P) = \{\theta(S) : S \in P\}$. Let f be a value on $\mathcal{G}^N$. Next, we reformulate the anonymity property to our setting.

ANO $f_i(\theta V) = f_{\theta(i)}(V)$, for every $V \in \mathcal{G}^N$, $\theta \in \Theta^N$, and $i \in N$.

It states that payoffs should not depend on the labeling of players. As Gilboa and Lehrer (1991) pointed out, in their characterization result SYM cannot be replaced by ANO, because even in the presence of LIN, EFF, and NPP, ANO is strictly weaker than SYM.

In order to introduce the family of linear, efficient, and anonymous values on $\mathcal{G}^N$ we need some machinery.

The set of global games with a fixed player set, $\mathcal{G}^N$, is an $(B_n - 1)$-dimensional vector space, where B_n is the Bell number that counts the possible partitions of a set of n elements. To define the values we need a basis of this vector space. Given $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$, the *global unanimity game* U_Q is defined for every $P \in \Pi^N$ by

$$U_Q(P) = \begin{cases} 1, & \text{if } Q \preceq P \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Gilboa and Lehrer (1991) showed that the set of unanimity games, $\left\{U_Q : Q \in \Pi^N, Q \neq \lfloor N \rfloor\right\}$ is a basis of $\mathcal{G}^N$. Next, we provide an explicit expression of the coefficients of any global game in this basis, which can be considered parallel to the well-known Harsanyi dividends of a coalitional game.

Proposition 4.1 *Let $V \in \mathcal{G}^N$. The coefficients of V in the unanimity basis are given by*

$$\delta_Q(V) = \sum_{M \preceq Q} (-1)^{|M|-|Q|} \binom{Q}{M} V(M),$$

where for every $M \preceq Q$ and $T \in Q$, $\binom{Q}{M} = \prod_{T \in Q} (m_T - 1)!$, and m_T is the number of subsets in which T is divided in M, i.e., $m_T = |\{S \in M : S \subseteq T\}|$.

Proof It is well known that the set Π^N of partitions of a finite set N endowed with the ordering $\preceq$ is a lattice (see for instance, Stanley 2011). Then, the coefficients are given by

$$\delta_Q(V) = \sum_{M \preceq Q} \mu(M, Q) V(M),$$

where μ is the Möbius function of $\left(\Pi^N, \preceq\right)$ defined for every $M \preceq Q$ by

$$\mu(M, Q) = (-1)^{|M|-|Q|} \binom{Q}{M},$$

where $\binom{Q}{M} = \prod_{T \in Q} (m_T - 1)!$. □

It is easy to check that these coefficients can also be defined recursively for every $Q \in \Pi^N$ by

$$\delta_Q(V) = V(Q) - \sum_{P \prec Q} \delta_P(V). \tag{5}$$

The coefficients of global games associated with zero normalized coalitional games by Eq. (1) are particularly simple.

Lemma 4.1 *Let $v \in \mathcal{CG}_0^N$. If $Q = \lceil T \rceil \cup \lfloor N \backslash T \rfloor$ for some $T \subseteq N$ with $|T| > 1$, then $\delta_Q(V^v) = \Delta_T(v)$. Otherwise, $\delta_Q(V^v) = 0$.*

Proof Let $v \in \mathcal{CG}_0^N$. By definition, Eq. (1), for every $P \in \Pi^N$,

$$V^v(P) = \sum_{S \in P} v(S) = \sum_{S \in P} \sum_{\substack{T \subseteq N \\ |T|>1}} \Delta_T(v) u_T(S) = \sum_{\substack{T \subseteq N \\ |T|>1}} \Delta_T(v) \sum_{S \in P} u_T(S).$$

Finally, observe that for every $T \subseteq N$ with $|T| > 1$,

$$\sum_{S \in P} u_T(S) = U_{\lceil T \rceil \cup \lfloor N \backslash T \rfloor}(P) = \begin{cases} 1 & \text{if } T \subseteq S \text{ for some } S \in P \\ 0 & \text{otherwise,} \end{cases}$$

which together with Proposition 4.1 concludes the proof. □

Assuming linearity as a desirable condition for a value on $\mathcal{G}^N$, in order to define a value we only need to determine the payoffs in global unanimity games as defined in Eq. (4).

Let $n \in \mathbb{N}$ be the number of players. A *partition of the integer n* is a tuple, $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$, satisfying

1. $t_1, \ldots, t_p, \lambda_1, \ldots, \lambda_p \in \mathbb{N}$,
2. $t_1 < \cdots < t_p$,

3. $\sum_{k=1}^{p} \lambda_k t_k = n$.

The tuple $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$ represents a decomposition of n as a sum of the integers $t_1, \ldots, t_p$, where for each $k \in \{1, \ldots, p\}$, t_k is repeated λ_k times. We omit writing λ_k when it equals one. The set of partitions of n is denoted by $\Pi(n)$. For instance,

$$\Pi(4) = \left\{(4), (1, 3), \left(2^2\right), \left(1^2, 2\right), \left(1^4\right)\right\}.$$

Let $Q \in \Pi^N$. The *norm* of Q is defined as the partition of n:

$$\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \tag{6}$$

where Q consists of λ_k coalitions of cardinality t_k for every $k = 1, \ldots, p$.

As the reader may anticipate, in order to obtain an anonymous value, the payoffs in a global unanimity game can only depend on the norm of the underlying coalition structure, as it describes the sizes of all the coalitions in the partition. The next definition formalizes this idea.

Definition 4.1 A sharing function over $\Pi(n)$ is a mapping α satisfying for all $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n)\backslash\{(1^n)\}$

1. $\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \mathbb{R}^p$,
2. $\displaystyle\sum_{k=1}^{p} \alpha_k \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = 1$.

The set of sharing functions is denoted by F_n.

By convenience, we do not assign weights to $(1^n) \in \Pi(n)$ because the game $U_{\lfloor N \rfloor}$ does not belong to the unanimity basis of $\mathcal{G}^N$. The first condition allows us to associate a coefficient to coalitions of a given size. The second is a normalization condition that will be useful to obtain an efficient value. Let $Q \in \Pi^N$ and $\alpha \in F_n$. Then, for each $k \in \{1, \ldots, p\}$, $\alpha_k(\|Q\|) = \alpha_k\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$ describes the importance of each coalition of size t_k.

We are now in the position to introduce the family of LEA (linear, efficient, and anonymous) values. It formalizes the idea that in global unanimity games, players that belong to coalitions of a given size get the same payoff.

Definition 4.2 Let $\alpha \in F_n$, $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$, and $i \in N$. The α-value, Φ^α, is the linear extension of the value defined for unanimity games by

$$\Phi_i^\alpha\left(U_Q\right) = \frac{\alpha_{k(i)}(\|Q\|)}{\lambda_{k(i)} \cdot t_{k(i)}},$$

where $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$ and $k(i) \in \{1, \ldots, p\}$ is such that $|T| = t_{k(i)}$, with $i \in T \in Q$.

Hence, if $V \in \mathcal{G}^N$ and $\alpha \in F_n$ then Φ^α is determined by

$$\Phi^\alpha(V) = \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V)\Phi^\alpha(U_Q). \tag{7}$$

To illustrate the above definition, consider the old example of an exchange economy introduced by Shafer (1980), see also Scafuri and Yannelis (1984) and Faigle and Grabisch (2012).

Example 4.1 There are four agents, $N = \{1, 2, 3, 4\}$, and two commodities. The utility functions are given by

$$u_1(x, y) = u_2(x, y) = \left(\frac{1}{2}x^\rho + \frac{1}{2}y^\rho\right)^{1/\rho}$$
$$u_3(x, y) = u_4(x, y) = \left(\frac{1}{2}x^\beta + \frac{1}{2}y^\beta\right)^{1/\beta},$$

the endowments are $w_1 = (1, 0)$, $w_2 = (0, 1)$, $w_3 = w_4 = (0, 0)$, and the weights given to each agent are $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $0 \leq \gamma_4 \leq \left(\frac{1}{2}\right)^{1/\rho - 1/\beta}$.

If we allow agents to cooperate, a coalitional game (N, v) can be defined (see Faigle and Grabisch 2012, for the details). Then, the global game associated with the zero-normalization of (N, v), see Eq. (1), is given by

$$\begin{aligned}
&\bar{V}(\{\{1,2\},\{3\},\{4\}\}) = 1 - 2(\tfrac{1}{2})^{1/\rho}, \quad \bar{V}(\{\{1,3\},\{2\},\{4\}\}) = (\tfrac{1}{2})^{1/\beta} - (\tfrac{1}{2})^{1/\rho},\\
&\bar{V}(\{\{2,3\},\{1\},\{4\}\}) = (\tfrac{1}{2})^{1/\beta} - (\tfrac{1}{2})^{1/\rho} \quad \bar{V}(\{\{1,2\},\{3,4\}\}) = 1 - 2(\tfrac{1}{2})^{1/\rho},\\
&\bar{V}(\{\{1,2,3\},\{4\}\}) = \bar{V}(\{\{1,2,4\},\{3\}\}) = 1 - 2(\tfrac{1}{2})^{1/\rho},\\
&\bar{V}(\{\{1,3\},\{2,4\}\}) = \bar{V}(\{\{2,3\},\{1,4\}\}) = \bar{V}(\{\{1,3,4\},\{2\}\}) = (\tfrac{1}{2})^{1/\beta} - (\tfrac{1}{2})^{1/\rho},\\
&\bar{V}(\{\{2,3,4\},\{1\}\}) = (\tfrac{1}{2})^{1/\beta} - (\tfrac{1}{2})^{1/\rho}, \quad \bar{V}(\lceil N \rceil) = 1 - 2(\tfrac{1}{2})^{1/\rho},\\
&\text{and } \bar{V}(P) = 0, \text{ otherwise.}
\end{aligned}$$

Using Lemma 4.1 we can write it as

$$\begin{aligned}
\bar{V} = &\left(1 - 2\left(\tfrac{1}{2}\right)^{1/\rho}\right) U_{\{\{1,2\},\{3\},\{4\}\}} + \left(\left(\tfrac{1}{2}\right)^{1/\beta} - \left(\tfrac{1}{2}\right)^{1/\rho}\right)(U_{\{\{1,3\},\{2\},\{4\}\}} + U_{\{\{2,3\},\{1\},\{4\}\}})\\
&-2\left(\left(\tfrac{1}{2}\right)^{1/\beta} - \left(\tfrac{1}{2}\right)^{1/\rho}\right) U_{\{\{1,2,3\},\{4\}\}}.
\end{aligned}$$

Then, for every $\alpha \in F_n$, Φ^α prescribes the following payoffs:

$$\begin{aligned}
\Phi_1^\alpha(\bar{V}) = &\frac{\alpha_2(1^2, 2)}{2}\left(1 - 3\left(\frac{1}{2}\right)^{1/\rho} + \left(\frac{1}{2}\right)^{1/\beta}\right) + \frac{\alpha_1(1^2, 2)}{2}\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right)\\
&-\frac{2\alpha_2(1, 3)}{3}\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right),\\
\Phi_2^\alpha(\bar{V}) = &\Phi_1^\alpha(\bar{V}),
\end{aligned}$$

$$\Phi_3^\alpha(\bar{V}) = \frac{\alpha_1(1^2,2)}{2}\left(1-2\left(\frac{1}{2}\right)^{1/\rho}\right) + \alpha_2(1^2,2)\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right)$$
$$-\frac{\alpha_2(1,3)}{3}2\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right),$$
$$\Phi_4^\alpha(\bar{V}) = \frac{\alpha_1(1^2,2)}{2}\left(1-2\left(\frac{1}{2}\right)^{1/\rho}\right) + \alpha_1(1^2,2)\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right)$$
$$-2\alpha_1(1,3)\left(\left(\frac{1}{2}\right)^{1/\beta} - \left(\frac{1}{2}\right)^{1/\rho}\right).$$

Notice that agent 4 can get a non-zero payoff in case either $\alpha_1(1^2, 2)$ or $\alpha_1(1, 3)$ is non-null. This is not the case for GL which yields the Shapley value of the zero normalization of (N, v).

Next we show that the members of this family are characterized by means of linearity, efficiency, and anonymity.

Theorem 4.1 *A value on $\mathcal{G}^N$ satisfies* LIN*,* EFF*, and* ANO *if and only if it is an α-value for some $\alpha \in F_n$.*

Proof On the one hand, we prove that all the values in the family satisfy the properties. Let $\alpha \in F_n$.
LIN: Φ^α is linear by construction.
EFF: Let $V \in \mathcal{G}^N$, by Definition 4.2

$$\sum_{i\in N}\Phi_i^\alpha(V) = \sum_{i\in N}\sum_{\substack{Q\in\Pi^N\\ Q\neq\lfloor N\rfloor}} \delta_Q(V)\Phi_i^\alpha(U_Q) = \sum_{\substack{Q\in\Pi^N\\ Q\neq\lfloor N\rfloor}} \delta_Q(V)\sum_{i\in N}\Phi_i^\alpha(U_Q)$$

Given $Q \in \Pi^N$ with $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$, for every $k \in \{1, \ldots, p\}$ we define

$$T_k = \bigcup_{T\in Q:|T|=t_k} T.$$

Obviously, $\{T_k : k = 1, \ldots, p\} \in \Pi^N$. Then,

$$\sum_{i\in N}\Phi_i^\alpha(U_Q) = \sum_{k=1}^{p}\sum_{i\in T_k}\Phi_i^\alpha(U_Q) = \sum_{k=1}^{p}\sum_{i\in T_k}\frac{\alpha_k(\|Q\|)}{\lambda_k\cdot t_k} = \sum_{k=1}^{p}\alpha_k(\|Q\|) = 1,$$

where the third equality holds because $|T_k| = \lambda_k \cdot t_k$ and last equality is by point 2. of Definition 4.1.

Finally, by Proposition 4.1

$$\sum_{i\in N}\Phi_i^\alpha(V)=\sum_{\substack{Q\in\Pi^N\\ Q\neq\lfloor N\rfloor}}\delta_Q(V)=V(\lceil N\rceil)$$

ANO: Let $\theta\in\Theta^N$ and $i\in N$. We first show that Φ^α satisfies anonymity on global unanimity games. Let $Q\in\Pi^N$ with $Q\neq\lfloor N\rfloor$. First we prove that

$$\delta_Q(\theta V)=\delta_{\theta Q}(V). \tag{8}$$

In fact, as $|\theta(Q)|=|Q|$ y $\binom{\theta(Q)}{\theta(M)}=\binom{Q}{M}$ for all $M\preceq Q$ we have

$$\begin{aligned}\delta_Q(\theta V)&=\sum_{M\preceq Q}(-1)^{|M|-|Q|}\binom{Q}{M}\theta V(M)\\&=\sum_{\theta(M)\preceq\theta(Q)}(-1)^{|\theta(M)|-|\theta(Q)|}\binom{\theta(Q)}{\theta(M)}V(\theta(M))\\&=\sum_{M\preceq\theta(Q)}(-1)^{|M|-|\theta(Q)|}\binom{\theta(Q)}{M}V(M)=\delta_{\theta(Q)}(V).\end{aligned}$$

Also, since $\|Q\|=\|\theta(Q)\|$ and $k(i)=k(\theta(i))$, by Definition 4.2

$$\Phi_i^\alpha(\theta U_Q)=\Phi_{\theta(i)}^\alpha(U_Q). \tag{9}$$

Notice that $\theta V(P)=V(\theta(P))$, for every $P\in\Pi(N)$. Then,

$$\begin{aligned}\theta V(P)=V(\theta(P))&=\sum_{Q\in\Pi(N)\setminus\{\lfloor N\rfloor\},Q\preceq\theta(P)}\delta_Q(V)U_Q(\theta(P))\\&=\sum_{Q\in\Pi(N)\setminus\{\lfloor N\rfloor\},\theta^{-1}(Q)\preceq P}\delta_Q(V)U_Q(\theta(P))\\&=\sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\},M\preceq P}\delta_{\theta(M)}(V)U_{\theta(M)}(\theta(P))\\&=\sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\},M\preceq P}\delta_{\theta(M)}(V)\theta U_{\theta(M)}(P)\end{aligned}$$

That is,

$$\theta V=\sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}}\delta_{\theta(M)}(V)\theta U_{\theta(M)}.$$

Hence, using LIN and Eq. (9)

$$\begin{aligned}\phi_i^\alpha(\theta V) &= \sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}} \delta_{\theta(M)}(V)\phi_i^\alpha(\theta U_{\theta(M)}) \\ &= \sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}} \delta_{\theta(M)}(V)\phi_{\theta(i)}^\alpha(U_{\theta(M)}) \\ &= \phi_{\theta(i)}^\alpha(V)\end{aligned}$$

On the other hand, let f be a value on $\mathcal{G}^N$ satisfying the three properties. By LIN we only need to find a sharing function α such that for every $i \in N$ and $Q \in \Pi^N$, $Q \neq \lfloor N\rfloor$,

$$f_i(U_Q) = \frac{\alpha_{k(i)}(\|Q\|)}{\lambda_{k(i)} \cdot t_{k(i)}}. \tag{10}$$

Let $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n)\setminus\{(1^n)\}$ and $Q \in \Pi^N$ such that $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$. For every $k = 1, \ldots, p$, we define

$$\alpha_k\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \lambda_k \cdot t_k \cdot f_i(U_Q)$$

where $i \in T \in Q$ with $|T| = t_k$. Obviously, this function satisfies Eq. (10). It only remains to check that α is a sharing function. By definition $\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \mathbb{R}^p$ for all $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n)\setminus\{(1^n)\}$. Note that since f satisfies ANO, all players that belong to coalitions of a given cardinality get the same payoff in U_Q. By EFF and ANO of f, we have

$$\sum_{k=1}^{p}\alpha_k\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \sum_{k=1}^{p}\lambda_k \cdot t_k \cdot f_i(U_Q) = \sum_{i\in N} f_i(U_Q) = U_Q(\lceil N\rceil) = 1,$$

which concludes the proof. □

Notice that Eq. (10) provides a method to obtain the sharing function associated with a LEA value from the payoffs in global unanimity games. Next, we find the sharing function associated with the Gilboa-Lehrer value.

Example 4.2 Obviously, the Gilboa-Lehrer value belongs to the family because it satisfies LIN, EFF, and ANO. Recall that it is defined as the Shapley value of the (zero-normalized) coalitional game associated with a global game, see Eq. (3). Note that the coalitional game associated with a global unanimity game, see Eq. (2), is a coalitional unanimity game. Indeed, let $Q \in \Pi^N$ with $Q \neq \lfloor N\rfloor$ and define

$$R_Q = \bigcup_{T\in Q:|T|>1} T.$$

Then, by Eq. (2), $v^{U_Q} = u_{R_Q}$ because $Q \preceq \lceil S \rceil \cup \lfloor N \setminus S \rfloor$ if and only if $R_Q \subseteq S$. Using Eq. (3) we can write

$$GL\left(U_Q\right) = Sh\left(u_{R_Q}\right).$$

That is, if $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$. Then, for every $i \in N$,

$$GL_i(U_Q) = \begin{cases} \frac{1}{n} & \text{if } t_1 > 1 \\ 0 & \text{if } t_1 = 1,\ \{i\} \in Q \\ \frac{1}{n-\lambda_1} & \text{if } t_1 = 1,\ \{i\} \notin Q \end{cases}$$

Hence $GL = \Phi^\alpha$ with

$$\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \begin{cases} \frac{1}{n}(\lambda_1 t_1, \ldots, \lambda_p t_p) & \text{if } t_1 > 1 \\ \frac{1}{n-\lambda_1}(0, \lambda_2 t_2, \ldots, \lambda_p t_p) & \text{if } t_1 = 1. \end{cases} \tag{11}$$

Consider, for instance, the global unanimity game of the partition $P = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$. The Gilboa-Lehrer value, gives a zero payoff to player 1 and treats the remaining five players equally. Then,

$$GL\left(U_P\right) = \left(0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

Example 4.3 Another interesting LEA value that does not satisfy NPP is the *Equal division* value, defined for every $V \in \mathcal{G}^N$ and $i \in N$ by

$$ED_i(V) = \frac{V(\lceil N \rceil)}{n}.$$

Note that, when applied to unanimity games, for every $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$,

$$ED_i(U_Q) = \frac{1}{n}.$$

It is easy to check that it is a LEA value. Indeed, $ED = \Phi^\alpha$ where

$$\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \frac{1}{n}\left(\lambda_1 t_1, \ldots, \lambda_p t_p\right).$$

In the framework of coalitional games van den Brink (2007) conducted an axiomatic comparison of the Shapley value and the Equal division value. We conclude by calculating the payoffs in the global unanimity game of the partition $P = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$. Obviously, all players get the same fraction of 1, i.e.,

$$ED\left(U_P\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

To conclude, and thanks to a remark from a referee, we point out that applying a LEA value to the associated global game by Eq. (1) we obtain a linear, efficient, and anonymous value for coaltional games that belongs to the family studied in Hernandez-Lamoneda et al. (2008). Indeed, let $v \in \mathcal{CG}^N$ and consider its zero normalized game $v_0 \in \mathcal{CG}_0^N$. For every $\alpha \in F_n$, we define a value on $\mathcal{CG}^N$ by $\varphi^\alpha(v) = v(i) + \Phi^\alpha\left(V^{v_0}\right)$. It is easy to check that φ^α satisfies linearity, efficiency and anonymity (see page 5) and therefore it is a member of the family of values parameterized in Hernandez-Lamoneda et al. (2008).

5 The family of LEAN values

In this section we characterize the family of values on $\mathcal{G}^N$ that satisfy LIN, EFF, ANO, and NPP. Obviously, this family is different to the LEA values because the Equal division value defined above does not satisfy NPP. In other words, we impose the null player property used by Gilboa and Lehrer (1991), NPP, to the family of values introduced in Definition 4.2. We have already mentioned that not all values in the family satisfy NPP. So, we restrict the unanimity functions presented in Definition 4.1 by requiring one more condition. The condition states that the coefficient associated to a coordinate of cardinality one in a partition of an integer should be equal to zero. We formalize this idea in our second main result, where we characterize the family of LEAN (linear, efficient, anonymous, and null player property) values.

Theorem 5.1 *Let $\alpha \in F_n$ be such that for every $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \backslash \{(1^n)\}$ with $t_1 = 1$,*

$$\alpha_1\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = 0.$$

Then, the α-value satisfies NPP. *Moreover, the α-values associated with unanimity functions satisfying this condition are the only values on $\mathcal{G}^N$ satisfying* LIN, EFF, ANO, *and* NPP.

Proof For the existence, take an $\alpha \in F_n$ satisfying the above condition. To show that Φ^α satisfies NPP, the following claim will be useful.
Claim: Let $V \in \mathcal{G}^N$ and $i \in N$ a null player in the global game V. Then, for every $Q \in \Pi^N$ with $\{i\} \notin Q$ and $Q \neq \lfloor N \rfloor$, $\delta_Q(V) = 0$.

We prove the Claim by induction on the rank of the partition. The rank of Q is given by $r(Q) = n - (|Q| - 1)$. Since $Q \neq \lfloor N \rfloor$, take $Q \in \Pi^N$ with $r(Q) = 2$ and $\{i\} \notin Q$. Then, $Q = \lceil \{i, j\} \rceil \cup \lfloor N \backslash \{i, j\} \rfloor$. Since i is a null player in V, $V(Q) = V(Q_{-i})$. But $Q_{-i} = \lfloor N \rfloor$ and by definition $V(\lfloor N \rfloor) = 0$. Then, $V(Q) = 0$. Moreover, $\lfloor N \rfloor$ is the only partition which is finer than Q. Then, using the recursive definition of the coefficients in Eq. (5) and the fact that $\delta_{\lfloor N \rfloor}(V) = 0$,

$$\delta_Q(V) = V(Q) - \delta_{\lfloor N \rfloor}(V) = 0 - 0 = 0.$$

Take $Q \in \Pi^N \backslash \{\lfloor N \rfloor\}$ with $r(Q) = 3$. Then, two cases can arise.[4]

1. $Q = \lceil\{i, j, k\}\rceil \cup \lfloor N \setminus \{i, j, k\}\rfloor$. Then, the partitions $P \prec Q$ are

$$\begin{array}{ll} P_1 = \lfloor N \rfloor, & P_2 = \lceil\{i, j\}\rceil \cup \lfloor N \setminus \{i, j\}\rfloor, \\ P_3 = \lceil\{i, k\}\rceil \cup \lfloor N \setminus \{i, k\}\rfloor, & P_4 = \lceil\{j, k\}\rceil \cup \lfloor N \setminus \{j, k\}\rfloor \end{array}$$

   having rank 2 the partitions P_2, P_3, P_4 and rank 1 the partition P_1. Since $\{i\} \notin P_2$ and $\{i\} \notin P_3$, we have already seen that the coefficients associated with these partitions are equal to zero. The coefficient associated with P_1 is zero by convention. Moreover, note that $Q_{-i} = P_4$ and by the recursive definition of the coefficients, $\delta_{P_4}(V) = V(P_4) = V(Q_{-i})$. Then, using again the recursive definition of the coefficients and the fact that i is a null player in V we can write

$$\delta_Q(V) = V(Q) - \sum_{r=1}^{4} \delta_{P_r}(V) = V(Q) - V(Q_{-i}) = 0.$$

2. $Q = \lceil\{i, j\}\rceil \cup \lceil\{k, l\}\rceil \cup \lfloor N \setminus \{i, j, k, l\}\rfloor$. Then, the partitions $P \prec Q$ are

$$\begin{array}{ll} P_1 = \lfloor N \rfloor, & P_2 = \lceil\{i, j\}\rceil \cup \lfloor N \setminus \{i, j\}\rfloor, \\ P_3 = \lceil\{k, l\}\rceil \cup \lfloor N \setminus \{k, l\}\rfloor & \end{array}$$

   having rank 2 the partitions P_2, P_3 and rank 1 the partition P_1. We have already seen that $\delta_{P_2}(V) = 0$ because $\{i\} \notin P_2$. Recall, that $\delta_{P_1}(V) = \delta_{\lfloor N \rfloor}(V) = 0$. Moreover, note that $Q_{-i} = P_3$ and by the recursive definition of the coefficients, $\delta_{P_3}(V) = V(P_3) = V(Q_{-i})$. Then, using again the recursive definition of the coefficients and the fact that i is a null player in V we can write

$$\delta_Q(V) = V(Q) - \sum_{r=1}^{3} \delta_{P_r}(V) = V(Q) - V(Q_{-i}) = 0,$$

   where the second equality holds by the induction hypothesis.

Let us assume that the result is true for every $Q \in \Pi^N \backslash \{\lfloor N \rfloor\}$ with $1 \leq r(Q) = r < n$. Take $Q \in \Pi^N \backslash \{\lfloor N \rfloor\}$ with $r(Q) = r + 1$. By the induction hypothesis, $\delta_P(V) = 0$ for every P such that $\{i\} \notin P$ and $r(P) \leq r(Q) - 1$. Applying the recursive definition of the coefficients of Eq. (5) twice,

$$\delta_Q(V) = V(Q) - \sum_{\substack{P \prec Q \\ \{i\} \in P}} \delta_P(V) = V(Q) - \sum_{P \preceq Q_{-i}} \delta_P(V) = V(Q) - V(Q_{-i}) = 0,$$

which concludes the proof of the Claim.

4 If $|N| = 3$, only case 1 appears.

Let $V \in \mathcal{G}^N$ and $i \in N$ a null player in the global game V. Using the Claim, the linearity of Φ^α, and the decomposition of V in global unanimity games we can write,

$$\Phi_i^\alpha(V) = \sum_{\substack{Q \in \Pi^N \setminus \{\lfloor N \rfloor\} \\ \{i\} \in Q}} \delta_Q(V) \Phi_i^\alpha\left(U_Q\right).$$

Finally, from Definition 4.2 $k(i) = 1$ and the condition imposed on the sharing function α implies that $\Phi_i^\alpha\left(U_Q\right) = 0$ for every $Q \in \Pi^N$ such that $Q \neq \lfloor N \rfloor$ and $\{i\} \in Q$. Therefore, $\Phi_i^\alpha(V) = 0$, which concludes the proof of the existence of a solution satisfying the four properties.

For the uniqueness, let f be a value on $\mathcal{G}^N$ satisfying LIN, EFF, ANO, and NPP. By Theorem 4.1 we already know that there is an $\alpha \in F_n$ such that $f = \Phi^\alpha$. Then, it only remains to check that for every $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \setminus \{(1^n)\}$ with $t_1 = 1$, the sharing function α satisfies

$$\alpha_1\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = 0.$$

Note that the above partition of n is associated with $P \in \Pi^N$ such that $P \neq \lfloor N \rfloor$ and $\{i\} \in P$ for some $i \in N$. Besides, $\{i\} \in P$ implies that $U_P(Q) = U_P\left(Q_{-i}\right)$ for every $Q \in \Pi^N$. Then, i is a null player in U_P and by NPP $\Phi_i^\alpha\left(U_P\right) = 0$. Finally, by Definition 4.2 and the fact that $t_{k(i)} = 1$, the sharing function α satisfies the desired condition. □

An interesting feature of all LEAN values is that when applied to a global game associated with a zero normalized coalitional game by means of Eq. (1) they prescribe the Shapley value of the underlying coalitional game.

Proposition 5.1 *Let $\alpha \in F_n$ be such that $\alpha_1\left(1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = 0$, for every $\left(1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \neq (1^n)$. Then, for every $v \in \mathcal{CG}_0^N$,*

$$\Phi^\alpha\left(V^v\right) = Sh(v).$$

Proof By Lemma 4.1, we can focus on how does the value behave in global unanimity games of partitions of the form $\lceil T \rceil \cup \lfloor N \setminus T \rfloor$, for some $|T| > 1$. By Eq. (6), $\|\lceil T \rceil \cup \lfloor N \setminus T \rfloor\| = \left(1^{n-t}, t^1\right)$, where $t = |T|$. Let $\alpha \in F_n$ satisfy the condition in the statement above. Then, $\alpha_1\left(1^{n-t}, t^1\right) = 0$ and consequently, $\alpha_2\left(1^{n-t}, t^1\right) = 1$. All in all,

$$\Phi_i^\alpha\left(U_{\lceil T \rceil \cup \lfloor N \setminus T \rfloor}\right) = \begin{cases} \frac{1}{t} & \text{if } i \in T \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely the Shapley value of the coalitional unanimity game u_T. Then, the result follows by Lemma 4.1 and LIN. □

To conclude the section we illustrate the family of LEAN values by presenting an instance which is not the Gilboa-Lehrer value.

Example 5.1 In the lattice of partitions $(\Pi^N, \preceq)$, each element $P \in \Pi^N$ covers exactly

$$\sum_{S\in P} 2^{|S|-1} - |P| = \sum_{S\in P} \left(2^{|S|-1} - 1\right)$$

partitions. Consider the global unanimity game U_P, with $P \in \Pi^N \setminus \{\lfloor N \rfloor\}$. The idea is to split 1 equally among the agents in the coalitions whose union gives a coalition in P. That is, let φ be the value on $\mathcal{G}^N$ defined for every $i \in S \in P$ by

$$\varphi_i\left(U_P\right) = \frac{2^{|S|-1} - 1}{|S| \sum_{T\in P}(2^{|T|-1} - 1)}.$$

Note that if $\{i\} \in P$, then $\varphi_i\left(U_P\right) = 0$. Additionally, for every $i, j \in S \in P$, $\varphi_i\left(U_P\right) = \varphi_j\left(U_P\right)$. Moreover, $\varphi_i\left(U_P\right) = \varphi_j\left(U_P\right)$ whenever i and j belong to two different coalitions of P with the same sizes, i.e., $i \in S \in P$, $j \in T \in P$, and $|S| = |T|$.

It can be checked that the linear extension of this value belongs to the LEAN family. Indeed, $\varphi = \Phi^\alpha$ for the sharing function defined for every $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi_n^N \setminus \{(1^n)\}$ by

$$\alpha_{k(i)}\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = \lambda_{k(i)} \frac{2^{t_{k(i)}-1} - 1}{\sum_{r=1}^{p} \lambda_r \left(2^{t_r-1} - 1\right)}.$$

We conclude by illustrating the behavior of this value in the global unanimity game of partition $P = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$. Then,

$$\varphi\left(U_P\right) = \left(0, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Note that, compared to the Gilboa-Lehrer value, this value favors the players who form larger coalitions.

6 Conclusions

We have contributed to the scarce theoretical literature on global cooperative games by studying two families of values in detail. We believe that any sensible Shapley-like value for global games should lie within the family of LEAN or LEA values. We have illustrated these families by providing new values that we plan to study in more detail in the near future.

Our study provides the necessary theoretical framework that eases the application to real problems. Indeed, we have provided a method to identify a value of the family that can better fit a particular situation by only specifying the desired payoffs in global unanimity games. For instance, we believe that it can shed light to the problem of assessing fair transfers or penalties to the countries in international environmental agreements. In the future, we would like to study if the values proposed here could be used to avoid free-riding or at least to minimize the gains from this behavior.

Another future project that we devise is the translation of values and properties that exist in the literature of games with externalities. Indeed, global games can be embedded in the family of games with externalities (Thrall and Lucas 1963) by assigning the same worth to all the coalitions in a partition. Then, any value for games with externalities can be translated to a value for global cooperative games. It is quite simple to check that some properties of values for games with externalities, like the anonymity used for instance in Myerson (1977) and De Clippel and Serrano (2008) translates to the homonymous property used here, but it is not so obvious for other properties.

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