Atractores pullback:
Existencia y estructura para una ecuación de ondas con amortiguamiento no autónomo

Pullback Attractors:
Existence and structure for a non-autonomous wave equation

Memoria escrita por

Luis Felipe Rivero Garvía

Para optar al grado de doctor en Matemáticas
Programa Oficial de Doctorado Matemáticas
Código MCD2006-00466 Mención de Calidad
Universidad de Sevilla

V^oB^o Los Directores del Trabajo

Tomás Caraballo Garrido   José Antonio Langa Rosado

Dpto. Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas
Universidad de Sevilla

November 16, 2010

Contents

Acknowledgements (Agradecimientos)

Me gustaría dar las gracias a mis padres, porque soy lo que soy gracias a ellos, a mis directores de la tesis Tomás y José Antonio, por ayudarme tanto y hacer que esto haya sido posible, a Patri, mi pareja, por apoyarme en todo este camino y estar siempre a mi lado para los buenos y malos momentos, a Alexandre, por todo el trabajo y la paciencia que ha tenido, a James, por ayudarme tanto cuando estuve en Warwick, a mi hermana Mónica, por ser un ejemplo y una referencia, a Josetxo, por ponerme en contacto con el grupo de Sevilla, al grupo de investigación, por tener una calidad altísima tanto en las matemáticas y como en lo humano, a mis compañeros de la R.I.B., los que están y estuvieron, por todos los buenos momentos, al Departamento de Ecuaciones Diferenciales y Análisis Numérico, a mis vecinos Ana y Manolo, por cuidar siempre de nosotros, a toda mi familia, tanto los que están como los que se fueron, que siempre me ha querido, apoyado, enseñado y animado, a mis amigos de Madrid: Anís (Rosa), Cangrejo (Silvio), Eloy, Rubén, los cubanitos y la gente de la facultad, la gente del barrio y del instituto, a mi gente de capoeira, a los amigos de mis amigos que acabaron siendo mis amigos, al resto de los becarios de la Facultad de Matemáticas de Sevilla, a la gente del sur que me ha ayudando tanto, a los profesores buenos y malos de matemáticas que he tenido, a los referees y editores de las revistas donde he conseguido publicar, al Ministerio de Ciencia e Innovación y a la Universidad de Sevilla, por pagarme cada mes, y a todo el mundo que ha estado en mi vida para lo bueno y lo malo, porque siempre se aprende de cada experiencia.

Chapter 1
Introduction

From the beginning of the civilizations, people have tried to understand and predict all the natural phenomena that occur around them. To this aim, Mathematics has played an important role in all science areas. Since Poincaré initiated the qualitative study of differential equations at the end of the 90th century, the theory of dynamical systems has developed a wide theory to understand many different models. Several dynamical systems can be described using these equations: chemical reactions, population movements and migrations, fluids mechanics and turbulence, planetary dynamics, climate changes, market economic models... This theory combines the classical analytic methods with a geometric point of view, trying to reduce the complexity of the system by the study of the long time behaviour of the solutions.

Two of the oldest and most remarkable types of problems in mathematical physics are: the problems of celestial mechanics, which are finite dimensional problems, and the turbulence of fluids, which is infinite dimensional. The ideas and notions of the first one have penetrated deeply into the theory of infinite dimensional dynamical systems and partial differential equations. Such equations include the Navier-Stokes systems, magneto-hydrodynamics equations or damping wave equations. Within these lines of investigation, scientists have established that there exists a specific object inside the phase space related to the Navier-Stoke equations that attracts all the trajectories ([47, 78, 89]). This result has stimulated investigations of this kind of attractors for other equations appearing in mathematical physics.

The analysis of the global attractor in the autonomous case has been developed within the framework of dynamical systems. A dynamical system is a family of mappings depending on time which represents the evolution of the solution of some physical or theoretical phenomena, although we cannot obtain an explicit expression for it. The classical theory of global attractors is based on the study of the properties of semigroups that ensure the existence of the attractor, and give some information about its structure and continuity. But given a dynamical system starting from a particular initial state, it is not easy to predict if the system will evolve towards rest, towards a simple stationary state or it will explode in a finite time. The mathematical problem here is the study of the long-term behaviour of the system, giving a special interest, in our case, to the dynamics of the different solutions all together.

The study of the global attractor in autonomous problems has been developed extensively over the last few decades and has become now a classical theory with nice works like Chepyzhov & Vishik [33], Hale [47], Ladyzhenskaya [61], Raugel [76], Robinson [78] or Temam [89] among others. The global attractor is the natural mathematical object that describes the stationary state of the system and all the possible dynamics, giving crucial information on the long time behaviour. Indeed, we can track arbitrarily close any trajectory in the phase space (for arbitrarily large time lengths) by trajectories on the global attractor (see Theorem 2.4.1, page 35). This fact shows that the analysis of the geometrical description of the attractor is an important problem. Nowadays, the characterization of global attractors for infinite dimensional dynamical systems is known only for some very specific cases (mostly gradient systems).

Global attractors are closely related to the concept of dissipativeness. A dissipative system experiments a loss of energy due to, for example, damping terms or external forces. This phenomenon causes that the trajectories converge to some sets like fixed points. Results concerning the existence of a global attractors are related to the existence of an absorbing set, that is, a bounded set where all the orbits corresponding to different initial data eventually enter. One of the most important result related to the existence of the global attractor is based on the concept of asymptotic compactness (see Theorem 2.2.6, page 31). Roughly speaking, a dynamical system is asymptotically compact if the dynamics of any bounded sequence is a precompact set (we can obtain convergent subsequences).

In addition to the existence of global attractors and the analyzing of their structure, the continuity of the attractor is another important field to study because it ensures the stability of the original system under small perturbations. Although the upper-semicontinuity is satisfied under only some natural hypotheses, like the convergence of the perturbed semigroup to the limit problem in some sense, the lower-semicontinuity is closely related with the internal structure of the associated attractors. The most celebrated general known result for the characterization of global attractors says that a gradient, asymptotically compact, nonlinear semigroup with a bounded set of equilibria has an attractor which is the union of the unstable manifolds of the set of equilibria (see [47]). A generalization of this concept are the dynamically $\mathcal{E}$-gradient systems (see [2, 26, 23]).

The first attempts to extend the notion of global attractor to the non-autonomous case led to the concepts of uniform attractor and kernel sections, defined by V.V. Chepyzhov and M.I. Vishik in [33] (see also reference therein). The former definition, within the framework of skew-product semiflows, is based on the autonomous definition of global attractor, keeping fixed the more important properties in the autonomous theory. At the same time, but with explicit interest in applications of stochastic differential equations, Crauel and Flandoli in [38] introduced the notion of random pullback attractor in the framework of cocycle maps, which rapidly show its applicability to differential equations with a general enough non-autonomous term (see Caraballo *et al.* [19], Cheban *et al.* [30], Crauel *et al.* [37], Kloeden [56, 59] or Schmalfuß[79, 80, 81] and the references therein).

In a general differential equation we can analyze two kinds of dynamics

- *The forward dynamics*: the behaviour of solutions when the final time goes to infinity.
- *The pullback dynamics*: the behaviour of solutions when, with a fixed final time, the initial time goes to minus infinity.

In the autonomous case, these two dynamics are the same since the dynamics only depends on the elapsed time. But, in general, they are totally unrelated and can produce entirely different qualitative properties. In this way, there is a need to define the concepts in a more general framework and obtain analogous results to those in the autonomous case. This wider context is the framework of evolution processes, that is, families of two-parameter maps (final and initial time) which represent both forward and pullback dynamics of the system. We then can identify a semigroup as an evolution process where only the elapsed time is represented.

In the non-autonomous case, the configuration of the phase space changes depending on the initial time, that is, solutions strongly depend on the initial time. In this way, the pullback attraction is defined for each final time when the initial time goes to minus infinity. Although we can think on natural generalizations of the concepts looking for analogous results in the autonomous case, we found out some problems with these generalizations, and we needed to create new or stronger concepts.

A family of maps $\{S(t,s) : t \geqslant s\}$ from a Banach space X into itself is an evolution process if the following properties are fulfilled:

1) $S(t,t) = I$, for all $t \in \mathbb{R}$,
2) $S(t,s) = S(t,\tau)S(\tau,s)$, for all $t \geqslant \tau \geqslant s$,
3) $\{(t,s) \in \mathbb{R}^2 : t \geqslant s\} \times X \ni (t,s,x) \mapsto S(t,s)x \in X$ is continuous.

If we have the following general non-autonomous system

$$\begin{cases} u_t + A(t)u = f(u,t), \\ u(s) = u_0, \end{cases}$$

we can write the solution of the system as $u(t,s;u_0) = S(t,s)u_0$. A family of compact sets $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is said to be the pullback attractor for $\{S(t,s) : t \geqslant s\}$ if it is invariant ($S(t,s)\mathscr{A}(s) = \mathscr{A}(t)$ for all $t \geqslant s$), attracts all bounded subsets of X 'in the pullback sense' ($\operatorname{dist}(S(t,s)B, \mathscr{A}(t)) \overset{s\to-\infty}{\longrightarrow} 0$ for any bounded set $B \subset X$) and is minimal in the sense that if there exists a family of closed sets $\{C(t) : t \in \mathbb{R}\}$ such that attracts bounded sets of X, then $\mathscr{A}(t) \subset C(t)$, for all $t \in \mathbb{R}$. In the non-autonomous case, the pullback attractor is defined as a not necessary bounded family of sets, which, as in the autonomous case, keeps all its dynamics inside it. As in the autonomous case, pullback dissipativeness and pullback attractors are closely related. The main result on existence of pullback attractors in this memory is Theorem 3.2.4 (page 53), based on the concept of pullback strongly bounded dissipativeness and pullback asymptotically compactness. We need to define a stronger concept of dissipation to ensure the compactness of each set of the attractor, and a very useful bound of $\bigcup_{s\leqslant t} \mathscr{A}(s)$, which ensures that the pullback attractor is the union of all global backwards bounded solutions, providing a first result on the structure of the attractor. The second concept is the natural generalization of the concept of asymptotically compactness for semigroups.

A special case to highlight is the case of pullback point dissipativeness. Within the framework of semigroups, the point dissipative property is very useful to prove existence of the global attractor when dealing with a gradient system (see Theorem 2.2.6 in page 31). But, for the generalization of this result in the non-autonomous case needs to suppose some extra hypotheses, which are satisfied automatically in the autonomous case, like stronger bounds or some equicontinuity properties on the evolution process. Then, we need to define new stronger concepts like strongly point and strongly compact dissipativeness in pullback sense (even more strong than the concept related to bounded subsets before) and a stronger concept of pullback asymptotically compactness (see Theorem 3.2.12, page 59). This shows that point dissipativeness leading to the concept of a pullback attractor is not a simple result as in the autonomous case.

The characterization of non-autonomous attractors in a Banach space has been only developed very recently, and, consequently, results on lower-semicontinuity or characterization of attractors related to non-autonomous perturbations of attractors are relatively new (see [18, 22, 23, 24, 25, 26, 27]). General gradient-like attractors show a concrete structure, the union of the unstable sets of some invariant families os parameterized sets $\Xi(\cdot)$, that is

$$\mathscr{A}(t) = \bigcup_{i=1}^{n} W^u(\Xi_i, t) = \{\, \zeta \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(t) = \zeta \text{ and } \lim_{t\to-\infty} \operatorname{dist}(\xi(t), \Xi_i(t)) = 0\},$$

with $\{\Xi_1, \dots, \Xi_n\}$ a family of invariant isolated sets.

Theorem 3.7.3 shows that any small non-autonomous perturbation of a gradient-like nonlinear semigroup becomes a gradient-like evolution process. Thus, it gives a natural way to construct examples of non-autonomous gradient-like evolution processes as small non-autonomous perturbations of gradient semigroups with all equilibria being hyperbolic, and the isolated global solutions being hyperbolic bounded global solutions (see Definition 3.4.5, page 68).

There are two main strategies to obtain the continuity of attractors: one is to impose detailed assumptions on the structure of the 'unperturbed' attractor of an autonomous systems where the perturbation is a small non-autonomous one of the external force, and the other one is to prove the existence of uniform exponential rate of attraction for the perturbed attractors. In applications, this is the general way to obtain gradient-like pullback attractors and their continuity, that is, for instance, as a small non-autonomous perturbation of a gradient-like semigroup. The upper-semicontinuity is based on the continuity of the processes. But, in the case of the lower-semicontinuity, we need the continuity of the unstable manifolds of the global hyperbolic solutions. Carvalho and Langa in [22] considered the following non-autonomous damped hyperbolic equation in $\mathbb{R}^3$

$$u_{tt} - \Delta u - \beta u_t = g_\eta(t, u),$$

where $g_\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is close to an autonomous function $g_0 : \mathbb{R} \to \mathbb{R}$ in the following sense

$$\sup_{t\in\mathbb{R}} \left(|g_\eta(t,u) - g_0(u)| + \left| \frac{\partial g_\eta}{\partial u}(t,u) - g_0'(u) \right| \right) \xrightarrow{\eta\to 0} 0, \quad \forall u \in \mathbb{R},$$

and they proved that there exists a positive η_0 such that the pullback attractors $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,\eta_0]}$ are gradient-like and upper and lower-semicontinuous when $\eta \to 0$.

In this thesis we present two non-trivial non-autonomous wave equations which necessarily do not come as a perturbation of an autonomous problem: a non-autonomous damping wave equation and a non-autonomous strongly damped wave equation. The non-autonomous nature in our cases comes from a bounded time dependent damping. The first equation is an important second-order linear partial differential equation modeling physical problems such as sound waves, light waves or water waves. It arises in fields such as acoustics, electromagnetic, and fluid dynamics. The damped term works as a dissipative term and makes that the amplitude of oscillation of the wave decreases with time. Historically, the coefficient of this damping is a constant which comes from observations in physical experiments. Two examples of wave equations are the well-known damped pendulum $\ddot{x} + a\dot{x} + \sin x = 0$, where x represent the angle of the rope and the constant a is the friction with the medium, and the sine-Gordon equation with external force, which in the (real) space-time coordinates the equation reads $\phi_{tt} - \phi_{xx} + \sin\phi = f(\phi)$. This last equation was originally considered in the nineteenth century in the study of surfaces of constant negative curvature. This equation has attracted very much attention in the 1970s due to the presence of soliton solutions (see [52, 44, 60, 86]).

The first non-autonomous wave equation which has been studied in the present work is the following,

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f(u) \text{ in } \Omega \\ u(x,t) = 0 \text{ in } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain of $\mathbb{R}^n$ with $n > 1$, the external force satisfies the usual growth conditions and $\beta : \mathbb{R} \to \mathbb{R}$ is a bounded, globally Lipschitz function with $\beta'(t)$ Hölder continuous and such that

$$\beta_0 \leqslant \beta(t) \leqslant \beta_1 \text{ for some } \beta_0, \beta_1 \in (0,\infty).$$

We can observe that the non-autonomous damping $\beta(t)$ may be far away from any constant. We can find many references for similar problems in the autonomous case (see, for example [47] or [89]), because it is a classical example of hyperbolic equation. On the other hand, the non-autonomous case usually comes from the explicit time dependence of the external force as in [53], where the authors show the existence of uniform attractor with exponential rate of convergence. In other works, as [51] or [72], the authors prove the existence of the pullback attractor when the damping is time dependent with a very concrete form, but they do not give any result about its structure. In [45] we can find a non-autonomous wave equation where the damping and the boundary conditions depend on time. The authors show the existence and the well-possedness of the problem, without any reference to attractors.

Identifying $u_t = v$, we can transform the equation above into the following one

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -\beta(t)I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(u) \end{bmatrix},$$

where $A = -\Delta$ with Dirichlet boundary condition. Therefore, we have a problem where the time dependence appears in the operator instead of the external force. In this case, and due to the construction of the operator, the existence of solutions can be obtained from a result in [50]. The existence of the pullback attractor is based on Theorem 3.2.5 (page 54), which is a generalization of a very used result of Hale (see [47]). This result states that, if we can express the process as a sum of one compact process and a process which a decays to zero, then the process is pullback asymptotically compact. All the calculations and estimates are based on energy functionals as

$$V(\varphi, \phi) = \frac{1}{2}\|\varphi\|^2_{H^1_0} + 2b(\varphi, \phi)_{L^2} + \frac{1}{2}\|\phi\|^2_{L^2} - \int_\Omega G(\varphi), \quad (\varphi, \phi) \in H^1_0(\Omega) \times L^2(\Omega),$$

where $G(s) = \int_0^s f(\theta)d\theta$, proving that the linear part of the system has an exponential decay to zero and the non-linear part is compact.

Although the natural phase space for wave equations is $H^1_0(\Omega) \times L^2(\Omega)$, the pullback attractor possesses a higher regularity. Thanks to the variation of constants formula and the decay of the linear part, any globally bounded solution can be written as $\xi(t) = \int_{-\infty}^t L(t,\theta)F(\xi(\theta))d\theta$. Using an iterative procedure, which gives more regularity in each step, we obtain that the pullback attractor is bounded in $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$. This regularity is an important key to obtain the gradient-like structure of the attractor. We also need to assume that there are only finitely many solutions $\{u^*_1, \cdots, u^*_p\}$ of

$$\begin{cases} \Delta u + f(u) = 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases}$$

This is a usual hypothesis in the autonomous framework, especially in gradient systems. Using the convergence of $\{\beta_n(t) = \beta(t_n + t)\}_{n=1}^{\infty}$, for any $\{t_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ and the properties of the energy functional, we can conclude that all globally bounded solutions converge forward and backward to some equilibria. Generally, in non-autonomous systems, there are not necessarily fixed point. Therefore, the concept of hyperbolic bounded global solution generalizes them, which is related to families of projections depending on time which lead to the stable and unstable manifolds (see Definition 4.3.1, page 94).

Although forward and pullback attraction may not be related, there exist some cases when the trajectories converge forward in time to the pullback attractor. The uniform forward attraction gives us trivial examples of pullback attractors that have forward attraction, because a pullback uniform attractor is also a forward uniform attractor and vice versa (see [33, 30]). Other examples are non-autonomous perturbations of gradient-like semigroups, where the forward attraction comes from the autonomous nature of the limit problem (see [23]). As in the autonomous case, results in [62] prove that for each trajectory of the system, another one can be found inside the pullback attractor that tracks the original one, but if we have forward attraction for the pullback attractor, the forward dynamics of the system is also “copied” inside it (Theorem 3.6.1, page 70). In our system there exists an exponential forward attraction (see Theorem 4.3.4, page 96), which is strongly based on the assumption that all equilibrium points are hyperbolic in the sense of Definition 4.3.1, and the fact that there are only a finite number of equilibrium points.

We are also interested in the study of the asymptotic behaviour of the following non-autonomous stronger wave equation (see [14]), where both damping and strongly damping are time-dependent

$$\begin{cases} u_{tt} - \Delta u - \gamma(t)\Delta u_t + \beta_\varepsilon(t)u_t = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

in a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^n$, $\gamma, \beta_\varepsilon : \mathbb{R} \to (0, \infty)$ verifies $0 < \gamma_0 \leqslant \gamma(t) \leqslant \gamma_1 < \infty$, $0 < \beta_{0\varepsilon} \leqslant \beta(t) \leqslant \beta_{1\varepsilon} < \infty$, and $\gamma(t)$ and $\beta_\varepsilon(t)$ are continuously differentiable in $\mathbb{R}$, with bounded derivative uniformly in ε, and $\gamma'(t)$ and $\beta'_\varepsilon(t)$ Hölder continuous uniformly in ε. We also assume the same growth conditions as in the previous case.

There are some relevant physical applications for this kind of equations. In space dimensions $n = 1$ and $n = 2$, if $\gamma(t) \equiv 1$ and $\beta_\varepsilon(t) \equiv 0$, the previous equation models the longitudinal vibrations of a homogeneous bar subjected to viscous effects. The term $-\Delta u_t$ indicates that the stress is proportional to the strain rate as in a linearized Kelvin-Voigt material (see [94]). In dimension three, the model describes the variation from the configuration at rest of a homogeneous and isotropic linear viscoelastic solid with short memory, called rate type (see [42]), in the presence of an external displacement-dependent force. The term βu_t with $\beta > 0$ indicates that the bar is subjected to dynamical friction as well. We also have a perturbed sine-Gordon equation of the form $u_{tt} - \alpha\Delta u_t - \Delta u + \sin u + \beta u_t = f(u)$, describing the evolution of the current u in a Josephson junction (see [35]). One can find some particular cases containing time-dependent damped terms, for example, in [51, 72] (see also the references therein).

This system is parabolic, in contrast to the hyperbolic previous one. The idea in this case is to follow the analogous steps as above, therefore we need that the solutions are in $H_0^1(\Omega) \times L^2(\Omega)$, which is not a trivial task. First of all, we need to set the problem as the following system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -I \\ A & \gamma(t)A + \beta_\varepsilon(t) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}(u) \end{bmatrix}$$

in the larger space $H_0^1(\Omega) \times H^{-1}(\Omega)$. The recent theory of uniformly sectorial operators (see [28]), which is a generalization of the theory of [50], ensures that, under some conditions on the operator and the external force, the solution lives in the same fractional power of the domain of the operator as the initial data. Therefore, taking a suitable fractional power that ensures that the initial data are in $H_0^1(\Omega) \times L^2(\Omega)$, we can ensure that the solution remains in the defined space.

The existence of the attractor, as in the precedent case, is based on estimates obtained by the energy functional and Theorem 3.2.4. The parabolic nature of the problem gives us more regularity of the solution immediately after the initial time, that is, if $U(\cdot, s) = \begin{bmatrix} u(\cdot, s) \\ u_t(\cdot, s) \end{bmatrix}$ is a solution of the problem, then $U \in C((s, t], H_0^1(\Omega) \times H_0^1(\Omega))$. Since for any global solution $\xi(\cdot)$ of the system, $\xi(t) \in C(\mathbb{R}, H_0^1(\Omega) \times H_0^1(\Omega))$, the pullback attractor is embedded in $H_0^1(\Omega) \times H_0^1(\Omega)$ for all $t \in \mathbb{R}$. Using only the energy estimates, we can obtain a bound for the pullback attractor in this space, but, using again iterative methods, the attractor is a bounded subset inside the more regular space $[H^2(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]$.

The continuity of the attractor when $\beta_\varepsilon(\cdot) \to 0$ is based on upper and lower-semicontinuity results. The first one only needs the regularity obtained before and the energy functional. But, in the second case, we need a gradient-like structure for the limit problem. This structure is obtained following the ideas of the previous case and the convergence of $\{\gamma_n(t) = \gamma(t_n + t)\}_{n=1}^{\infty}$. Thanks to the results in [22] about continuity of local stable and unstable manifolds, we need to check two conditions to ensure the lower-semicontinuity (see Theorem 5.4.1 in page 117): all the equilibrium points for the perturbed system are hyperbolic (Theorem 7.6.11 in [50]) and the local unstable manifold of them behaves continuously (Theorem 2.2 in [24]). Therefore, we have a non-trivial example of continuity of pullback attractors when the limit problem is non-autonomous, showing an example far from the dependence of the autonomous case.

This work is divided into two parts. The first part shows results on existence on the existence of attractors, both in the autonomous and the non-autonomous cases. In the second part, we analyze the two non-trivial examples of non-autonomous wave equations where we apply the theoretical results.

Chapter 2 is splitted into six sections. In Section 2.1 we recall some concepts from the framework of semigroups and some relevant results which are necessary in the subsequent sections. In Section 2.2 we give the most relevant existence results for global attractors based on the existence of compact absorbing sets, asymptotically compact semigroups and finite dimensional subspaces. The point dissipative case is also discussed in this section. In Section 2.3 we show some results about continuity of the global attractors. In Section 2.4 we show that there exists a copy of the dynamic of the whole system inside the global attractor, proving the importance of the study of the structure of the attractor in Section 2.5. Finally, some results concerning exponential attractors are included in Section 2.6.

Our aim in Chapter 3 is to establish a generalization of the classical results from Chapter 2 to the non-autonomous framework. In Section 3.1 we state the definitions and concepts related to evolution processes, needed to ensure the existence of the pullback attractor, and an important characterization of pullback asymptotically compact processes. Section 3.2 is dedicated to the generalization of the existence result for global attractors in the autonomous case based on the existence of a family of compact sets which pullback absorbs bounded subsets, the pullback asymptotic compactness property of the process, and the existence of a finite subspace, with a special reference to the pullback point dissipative case. In Section 3.3 we give a view of the most important results concerning the existence of the $\mathcal{D}$-pullback attractor inside the framework of the basis of attraction. The continuity of the attractors under small perturbations is shown in Section 3.4. In Section 3.5 we recall the definitions and results about exponential pullback attractors, which are positive invariant families with pullback attracts exponentially fast. In Section 3.6 we will prove how inside a pullback attractor there exists a "copy" of the forward dynamic of the system. The structure of pullback attractors is showed in Section 3.7, giving the definition of gradient-like process and gradient-like pullback attractors, which is a generalization of the concept of gradient systems in the autonomous case. Finally the relationship between forward and pullback attraction is analyzed in Section 3.8.

The second part of this work is dedicated to the previos non-trivial examples of non-autonomous wave equations, being Chapter 4 devoted to the first one and Chapter 5 to the stronger one.

Finally, in Section 6 we show a set of open problems and researcher lines following the results shown in this work.

Spanish Summary

Este trabajo está dividido en dos partes. Una primera en la que se trata la parte teórica de los sistemas dinámicos autónomos dentro del marco de la teoría de los semigrupos, dando una visión global de esta teoría clásica, así como la teoría más reciente en el caso no autónomo; y una segunda parte en la que se tratan dos ejemplos no autónomos y no triviales de ecuaciones de ondas.

En el Capítulo 2 se ofrece una visión de la teoría clásica sobre atractores globales en el caso autónomo. Esta teoría está dentro del marco de los semigrupos, por lo que la Sección 2.1 está dedicada a establecer las definiciones y conceptos necesarios dentro de este marco, así como ciertos resultados previos. En la Sección 2.2, se muestran los resultados más conocido sobre existencia de atractores globales (véase Teorema 2.2.6), basados en la existencia de conjuntos compactos absorbentes, la propiedad de compacidad asintótica del semigrupo o la existencia de subespacios de dimensión finita. En esta sección también se trata el caso de la disipatividad puntual. En la Sección 2.3 se muestran los conceptos de continuidad de los atractores globales bajo perturbaciones y se desarrollan los conceptos de semicontinuidad superior e inferior. La Sección 2.4 muestra como el atractor global alberga en su interior una copia de toda la dinámica del sistema, por lo que en la aplicaciones es importante hacer un estudio tanto de la dimensión como de la estructura del

mismo. En la Sección 2.5 se muestra los concepts de sistemas gradientes y sistemas $\mathcal{E}$-gradientes, así como la relación que existen entre ambos.

Los objetivos del Capítulo 3 son la generalización de los conceptos previos en el marco de los procesos de evolución. Para ello, es necesario redefinir y generalizar los conceptos ya existentes (ver Sección 3.1). El principal problema a la hora de realizar esta generalización reside en que existen ciertas propiedades que se verifican de manera automática en el caso de los semigrupos, pero que para procesos de evolución en el caso no autónomo no son triviales. Un ejemplo claro es el Teorema 3.2.2, página 52, el cual muestra que con la generalización natural del concepto de disipatividad acotada en sentido pullback no basta para asegurar las propiedades del atractor. Así es necesario definir conceptos más fuertes para obtener resultados análogos al caso autónomo, como el Teorema 3.2.4 (página 53) de la Sección 3.2. Un resultado muy útil en la caracterización de procesos asintóticamente compactos en sentido pullback es el Teorema 3.2.5 (página 54), el cual es una generalización del resultado de Hale que aparece en [47]. Una especial atención requiere el estudio de los procesos disipativos puntuales en sentido pullback, ya que su generalización al caso no autónomos requiere asumir hipótesis extras, como cierto tipo de equicontinuidad o de acotación fuerte, que en el caso autónomo son triviales, o de la definición de atracciones más fuertes en sentido pullback (ver definiciones 3.2.10 o 3.2.11). El Teorema 3.2.12 (página 59) recoge este resultado. En esta sección también se generalizan el resto de resultados de existencia del capítulo anterior. En la Sección 3.3 también se da una visión sobre la teoría de las bases de atracción, así como un resumen de los resultados más relevantes y ejemplos. La continuidad de los atractores pullback se trata en la Sección 3.4, en la cual juega un importante papel cierta estructura específica como son los atractores de tipo gradiente o gradient like. Este tipo de atractores son los que permiten obtener una semicontinuidad inferior bajo pequeas perturbaciones. La continuidad superior se obtiene por medio de técnicas donde sólo la continuidad de los procesos es requerida. La Sección 3.5 trata el concepto de atractor pullback exponencial, familia de conjuntos compactos positivamente invariante que atrae exponencialmente rápido en sentido pullback. En la Sección 3.6 se muestra como dentro del attractor pullback podemos encontrar una copia de la dinámica forward del sistema. Como en el caso autónomo, la estructura del atractor pullback juega un papel muy importante para conocer la dinámica del sistema. En la Sección 3.7 se trata el campo de los procesos de tipo gradiente, generalización de los semigrupos gradientes previamente definidos. La parte final del capítulo (Sección 3.8) está relacionada con la relación entre atracción forward y pullback, resaltando el Teorema 3.6.1, que relaciona la dinámica del sistema con la del propio atractor pullback, tanto hacia adelante como en sentido pullback.

En la segunda parte se estudian dos ecuaciones de ondas no autónomas. De manera habitual, los sistemas no autónomos que conservan un cierto tipo de estructura suelen provenir de ecuaciones autónomas con una pequeña perturbación no autónoma. En nuestro caso las ecuaciones no tienen por qué estar cerca de ningún problema autónomo, aportando en este sentido dos problemas completamente no autónomos sobre existencia, estructura y continuidad de atractores pullback.

En el Capítulo 4 se considera la siguiente ecuación,

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f(u) \text{ en } \Omega \\ u(x,t) = 0 \text{ en } \partial\Omega, \end{cases}$$

siendo Ω un subconjunto de $\mathbb{R}^n$. Se ha trabajado el caso subcrítico ya que se considera como condición de crecimiento sobre el término fuente $|f'(s)| \leqslant c(1 + |s|^{p-1})$ con $p < \frac{n}{n-2}$. En la Sección 4.1 se realiza un estudio sobre la existencia y unicidad de la solución y de la existencia del atractor pullback, usando desigualdades de energía para obtener decaimiento exponencial en la parte lineal y las condiciones de crecimiento para la compacidad de la parte no lineal, y aplicando los teoremas 3.2.5 y 3.2.4. Para obtener la estructura de tipo gradiente, necesitamos previamente que el atractor muestre una mayor regularidad, lo que se muestra en la Sección 4.2.2. Para ello volvemos a usar la ecuación de la energía y el hecho de que, gracias al decaimiento exponencial, podemos eliminar la parte lineal de la fórmula de variación de las constantes de la solución. De esta manera y gracias a un procedimiento iterativo que regulariza en cada paso las soluciones, llegamos a que el atractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ es acotado en $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Con esto, en la misma sección, y suponiendo que

$$\begin{cases} \Delta u + f(u) = 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases}$$

posee un número finito de puntos hiperblicos que denotaremos como $\mathcal{E} = \{e_1^*, \cdots, e_p^*\}$, obtenemos que toda solución global converge hacia delante y hacia atrás a dichos puntos de equilibrio, esto es, que el atractor pullback se puede denotar como

$$\mathscr{A}(t) = \bigcup_{i=1}^{p} W^u(e_i^*)(t), \text{ para todo } t \in \mathbb{R}.$$

En la Sección 4.3 se muestra cómo, bajo la suposición de que los puntos de equilibrio son hiperbólicos en el sentido de la Definición 4.3.1, el atractor pullback muestra también una atracción hacia adelante y de manera exponencial, pudiendo aplicar también el Teorema 3.6.1 y llegando a la conclusión de que existe también una copia de la dinámica forward dentro del atractor pullback.

El Capítulo 5 está dedicado al estudio de la ecuación

$$\begin{cases} u_{tt} - \Delta u - \gamma(t)\Delta u_t + \beta_\varepsilon(t) u_t = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

En este caso, la existencia de la solución necesita de un profundo estudio. En la Sección 5.1 se muestra cómo, usando la teoría de los operadores uniformemente sectoriales, que generaliza la teoría existente para el caso autónomo, tenemos bien definido el problema en el espacio $H^1_0(\Omega) \times H^{-1}(\Omega)$, de manera que si tomamos los datos iniciales en $H^1_0(\Omega) \times L^2(\Omega)$, la solución permanece, como mínimo, en este espacio. Esto nos permite poder usar de nuevo las estimaciones de la energía para probar la existencia del atractor pullback en la Sección 5.2. Usando ideas análogas al caso anterior, en la Sección 5.2.2 me muestra que el atractor es un conjunto acotado de $H^2(\Omega) \cap H^1_0(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega)$. Si sólo usamos las estimaciones de la energía, es decir, sin usar la teoría abstracta de los espacios de potencias fraccionarias, podemos obtener una regularidad para el atractor en $H^1_0(\Omega) \times H^1_0(\Omega)$, como se muestra en la Subsección 5.2.3. Para obtener la continuidad del atractor cuando $\varepsilon \to 0$, necesitamos dotar de estructura al problema límite cuando $\varepsilon = 0$. En la Sección 5.3 y siguiendo las ideas del caso anterior, se prueba que estamos ante un atractor gradiente en el caso límite, lo que nos permitirá en la Sección 5.4 obtener la semicontinuidad inferior de los atractores.

Finalmente, en el Capítulo 6 se muestra un conjunto de problemas abiertos y líneas de investigación a raíz de los resultados mostrados en este trabajo.

Part I

Theory

Chapter 2

The autonomous case

In this chapter we are going to give a global view of the classical theory of global attractors inside the framework of semigroups. Our aim is to give the most important results on existence and characterization that exists in this area avowing proofs, giving only the references, and trying to explain each concept. In Chapter 3 we will see how we can generalizase all these concepts and results. There are a lot of nice works on this subset, as Temam [89], Hale [47], Ladyzhenskaya [61], Babin-Vishik [9], Sell and You [82] or Cholewa and Dlotko [34] for example. Most of the results in this chapter can be found in Robinson [78], indicating other reference when necessary.

2.1 Basic concepts

First we recall some notation about semigroups.

Definition 2.1.1 *A* semigroup *in a metric space X is a family of continuous maps $\{S(t) : t \geqslant 0\}$ from X into itself with the following properties*

1) $S(0) = I$,
2) $S(s+t) = S(t+s) = S(t)S(s)$,
3) $\{t \in \mathbb{R} : t \geqslant 0\} \times X \ni (t,x) \mapsto S(t)x \in X$ *is continuos.*

If we consider an autonomous initial value problem

$$\begin{cases} u_t + Au = f(u), \\ u(0) = u_0, \end{cases}$$

in a Banach space X with a unique solution $u(t; u_0)$, we can construct a semigroup of operators defined as $u(t; u_0) = S(t)u_0$. We are interested in the study of the properties of semigroups to analyze the asymptotic behaviour of the system. The concept of attraction plays an important role.

Definition 2.1.2 *We say that a set* $B \subset X$ attracts *a subset* $A \subset X$ *under* $\{S(t) : t \geqslant 0\}$ *if*

$$\mathrm{dist}(S(t)A, B) \overset{t\to\infty}{\longrightarrow} 0,$$

where dist denote the *Hausdorff semidistance*, defined as

$$dist(A, B) = \sup_{a\in A} \inf_{b\in B} |a - b|,$$

or, equivalently

$$dist = \inf\{\varepsilon > 0 : A \subset N(B, \varepsilon)\}.$$

This function is not a metric since $\mathrm{dist}(A, B) = 0$ does not imply $A = B$, but only that $A \subset \overline{B}$. The *Hausdorff distance* is defined as follows

$$\mathrm{dist}_H(A, B) = \max\{\mathrm{dist}(A, B), \mathrm{dist}(B, A)\}.$$

Some properties of this semidistance are (see [40]):

- $\mathrm{dist}(\emptyset, Y) = 0$, but $\mathrm{dist}(X, \emptyset)$ is not defined,
- $\mathrm{dist}(X, Y) = \mathrm{dist}(\overline{X}, Y)$,
- If $Y_1 \subset Y_2$, then $\mathrm{dist}(X, Y_1) \geqslant \mathrm{dist}(X, Y_2)$,
- If $X_1 \subset X_2$, $\mathrm{dist}(X_1, Y) \leqslant dist(X_2, Y)$.

Invariant sets are important elements in the phase space because their whole dynamics are inside of them, i.e., they can be seen as dynamically independent sets.

Definition 2.1.3 *We say that a subset* A *of* X *is* (positively) invariant under $\{S(t) : t \geqslant 0\}$ *if* $(S(t)A \subseteq A)$ $S(t)A = A$ *for all* $t \geqslant 0$.

Roughly speaking, if we choose an initial data in A, the dynamics of the system always stays in A, that is, no part of A "goes out" as we run the dynamics on the set forward in time.

Now we define the concept of global attractor.

Definition 2.1.4 *A compact set* $\mathscr{A}$ *in* X *is called* the global atractor for $\{S(t) : t \geqslant 0\}$ *if it verifies that for each bounded subset* B *in* X*,*

$$dist(S(t)B, \mathscr{A}) \overset{t\to\infty}{\longrightarrow} 0, \tag{2.1}$$

it is the maximal compact invariant set and it is minimal in the sense of that if there exists a closed set $C \subset X$ *satisfying* (2.1)*, then* $\mathscr{A} \subset C$*.*

The semigroups showing the possibility of the existence of global attractors are called dissipative. This kind of semigroups possesses a bounded subset in the phase space which absorbs all the dynamics of the system in the following sense

Definition 2.1.5 *A semigroup* $\{S(t) : t \geqslant 0\}$ *is called* bounded dissipative *(*compact dissipative, point dissipative*) if there exists a bounded set* $B_0 \subset X$ *which absorbs bounded subset (compact subsets, points) under* $\{S(t) : t \geqslant 0\}$*, that is, if for each bounded set* $B \subset X$ *(compact set* $K \subset X$*, point* $x \in X$*) there exists a time* $t_B = t_B(B)$ *(*$t_K = t_K(K)$*,* $t_x = t_x(x)$*) such that* $S(t)B \subset B_0$ *(*$S(t)K \subset B_0$*,* $S(t)x \in B_0$*) for all* $t \geqslant t_B$ *(*$t \geqslant t_K$*,* $t \geqslant t_x$*). The set* B_0 *is called* the absorbing set.

Obviously bounded dissipative implies compact dissipative, which implies point dissipative. In $\mathbb{R}^n$ we have the reciprocals cases since in finite dimensional spaces any closed and bounded set is compact.

Proposition 2.1.6 *Let* $\{S(t) : t \geqslant 0\}$ *be a point dissipative semigroup in* $\mathbb{R}^n$*. Then* $\{S(t) : t \geqslant 0\}$ *is bounded dissipative. Moreover, if* $B \subset \mathbb{R}^n$ *is the point absorbing set, then*

$$\forall \varepsilon > 0, \quad B_\varepsilon = \overline{\bigcup_{t>0} S(t)\overline{N}(B,\varepsilon)},$$

is bounded, positively invariant ($S(t)B_\varepsilon \subset B_\varepsilon$*) and it absorbs any bounded subset of* X *in a fixed time* t_1*, where*

$$\overline{N}(X,\varepsilon) = \{z : z \leqslant y + x, x \in X, y \in \overline{B}(0,\varepsilon)\}.$$

In a general Banach space the previous equivalence does not follow in a straightforward way. We need more assumptions on our semigroup to ensure it. For a special class of semigroups, the so-called asymptotically compact, this equivalence is guaranteed. Moreover, an asymptotically compact semigroup ensures the convergence of subsequences without the necessity of being inside a precompact set.

Definition 2.1.7 *A semigroup is called* asymptotically compact *if for any bounded sequence* $\{x_n\}_{n=1}^{\infty}$ *and* $t_n \overset{n\to\infty}{\longrightarrow} \infty$*, there exists a convergent subsequence of* $\{S(t_n)x_n : n \in \mathbb{N}\}$*.*

The following concepts will play an important role in the existence and structure of global attractors.

Definition 2.1.8 *A solution $\xi(t)$ is a global solution for a semigroup $\{S(t) : t \geqslant 0\}$ if it is defined for all time $t \in \mathbb{R}$ and verifies that $\xi(t+s) = \xi(t)\xi(s)$ for all $s \leqslant t$.*

Definition 2.1.9 *The* ω-limit set *of a set $B \subset X$ under $\{S(t) : t \geqslant 0\}$ is defined as,*

$$\omega(B) = \{y \in X : \exists t_n \to \infty, \{x_n\}_{n=1}^{\infty} \subset B, S(t_n)x_n \overset{n\to\infty}{\longrightarrow} y\}, \tag{2.2}$$

or equivalently

$$\omega(B) = \bigcap_{t\geqslant 0} \overline{\bigcup_{s\geqslant t} S(s)B}. \tag{2.3}$$

Definition 2.1.10 *The* unstable manifold of $z \in X$ *is the set*

$$W^u(z) = \{u_0 \in X : S(t)u_0 \text{ is defined for all } t, S(t)u_0 \overset{t\to-\infty}{\longrightarrow} z\}.$$

In the same way the stable manifold of $z \in X$ *is*

$$W^s(z) = \{u_0 \in X : S(t)u_0 \overset{t\to\infty}{\longrightarrow} z\}.$$

Let be $A \subset X$, the unstable manifold of A *is the set*

$$W^u(A) = \{u_0 \in X : S(t)u_0 \text{ is defined for all } t, dist(S(t)u_0, A) \overset{t\to-\infty}{\longrightarrow} 0\}.$$

The following proposition is a first approach to the existence of the global attractor showing a result related to the properties in Definition 2.1.4.

Proposition 2.1.11 *Let $A \subset X$. If there exists $t_0 > 0$ such that*

$$\overline{\bigcup_{t\geqslant t_0} S(t)A} \tag{2.4}$$

is compact, then $\omega(A)$ is not empty, compact and invariant.

If we are working with an asymptotically compact semigroup, we do not need the compactness condition in (2.4).

Lemma 2.1.12 *Let $\{S(t) : t \geqslant 0\}$ be a semigroup in X. Then,*

1. *For $A, B \subset X$ with A a bounded set attracting B, then $\omega(B) \subset A$.*
2. *If $\{S(t) : t \geqslant 0\}$ is asymptotically compact, then for any bounded subset B of X, $\omega(B)$ is non-empty, compact, invariant and attracts B.*

3. *If $\{S(t) : t \geqslant 0\}$ is asymptotically compact, then for each bounded set B of X, there exists a time $\tau = \tau(B) \geqslant 0$ such that $S(\tau + t)B$ is bounded for all $t \geqslant 0$. Moreover, if B is connected, $\omega(B)$ is also connected.*

The concept of Kuratowski non-compactness measure is, roughly speaking, a measure which shows how close is a set to be a "compact" one.

Definition 2.1.13 *Let X be a metric space and $A \subset X$. The* Kuratowski measure of non-compactness *is defined as:*

$$\alpha(A) = \inf\{\delta > 0 : A \text{ possesses a finite covering by open sets of diameter less than } \delta\}. \tag{2.5}$$

The following properties and results can be found in [40]:

1. $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact;
2. If X is a Banach space and $B_1,\ B_2 \subset X\ \Rightarrow \alpha(B_1+B_2) \leqslant \alpha(B_1)+\alpha(B_2)$;
3. If $B_1 \subset B_2 \Rightarrow \alpha(B_1) \leqslant \alpha(B_2)$;
4. $\alpha(B_1 \cup B_2) \leqslant \max\{\alpha(B_1), \alpha(B_2)\}$;
5. $\alpha(\overline{B}) = \alpha(B)$;
6. $\alpha(kB) = |k|\alpha(B)$ for all $k \in \mathbb{R}$.

Lemma 2.1.14 *Let M be a finite dimensional space. Then, given a ball $B(x, \varepsilon)$ we have that $\alpha(B(x, \varepsilon)) = 2\varepsilon$.*

Lemma 2.1.15 *Let X be a complete metric space and $\{F_n\}$ a decreasing sequence of non-empty, bounded and closed sets such that $\alpha(F_n) \stackrel{n\to\infty}{\longrightarrow} 0$. Then, $\bigcap_{n\in\mathbb{N}} F_n$ is non-empty and compact.*

Definition 2.1.16 *We say that $\{S(t) : t \geqslant 0\}$ in a metric space X is ω-*limit compact *if for all bounded set $B \subset X$ and for all $\varepsilon > 0$ there exists a time $t_0 = t_0(B, \varepsilon)$ such that*

$$\alpha\left(\bigcup_{t\geqslant t_0} S(t)B\right) \leqslant \varepsilon.$$

The previous definition is very close to the concept of asymptotic compactness because the dynamic of each bounded subset is more and more compact according to its Kuratowski measure.

Definition 2.1.17 *A semigroup* $\{S(t) : t \geqslant 0\}$ *is called* set contracting *if there exists a* $\gamma \in [0,1)$ *and a time* t_0 *such that for all bounded set* $B \subset X$ *it holds that* $\alpha(S(t_0)B) \leqslant \gamma\alpha(B)$.

Proposition 2.1.18 *Let* $\{S(t) : t \geqslant 0\}$ *be a set contracting semigroup in a complete metric space* X. *Suppose also that for all bounded subset* $B \in X$, *the set* $\bigcup_{t \geqslant 0} S(t)B$ *is bounded too. Then, the semigroup* $\{S(t) : t \geqslant 0\}$ *is* ω*-limit compact (see [68]).*

2.2 Existence results

In this section we recall some results ensuring the existence of global attractors. We divide this section in three parts, depending on the properties that we need to verify.

2.2.1 Compact absorbing set

The first result of existence is based on the compactness of the absorbing sets, which, based on Lemma 2.1.11, prove that the ω-limit set of the absorbing set is the global attractor.

Theorem 2.2.1 *Suppose that* $\{S(t) : t \geqslant 0\}$ *is bounded dissipative in a Banach space* X *with absorbing set* B. *If* B *is also compact, then there exists the global attractor* $\mathscr{A}$ *and* $\mathscr{A} = \omega(B)$. *Moreover, if* X *is connected, then* $\mathscr{A}$ *is also connected.*

A typical example in analyzing the existence of global attractor in finite dimension is provided by the well-known Lorenz equations. These form a 3D system of ODE's, introduced by Lorenz in 1963 (see [85] for more details). The equations are

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{cases} \tag{2.6}$$

where σ, r and b are positive constants.

Since we are working in $\mathbb{R}^3$, we only need to find a bounded and closed absorbing set because bounded set has compact closure. By Proposition 2.1.6, we just need to prove that there exists a fixed ball $B(0, 0, r + \sigma)$ that is point absorbing.

Let us consider the function $V(x,y,z) = x^2 + y^2 + (z - r - \sigma)^2$, whose time derivative is given by

$$\begin{aligned}\frac{dV}{dt} &= 2x\dot{x} + 2y\dot{y} + 2(z - r - \sigma)\dot{z} \\ &= -2\sigma x^2 - 2y^2 - 2bz^2 + 2b(r+\sigma)z.\end{aligned}$$

Since $-2bz(r+\sigma) = b(z-(r+\sigma))^2 - bz^2 - b(r+\sigma)^2$, we obtain

$$\begin{aligned}\frac{dV}{dt} &= -2\sigma x^2 - 2y^2 - 2bz^2 + 2b(r+\sigma)z \\ &\leqslant -\alpha V + b(r+\sigma)^2,\end{aligned}$$

where $\alpha = \min(2\sigma, b, 2)$. By the Gronwall's Lemma,

$$\begin{aligned}V(t) &\leqslant (V_0 - \frac{b(r+\sigma)^2}{\alpha})e^{-\alpha t} + \frac{b(r+\sigma)^2}{\alpha} \\ &\leqslant \frac{2b(r+\sigma)^2}{\alpha},\end{aligned}$$

for $t \in \mathbb{R}$ large enough. Then, we can apply Theorem 2.2.1.

2.2.2 Asymptotically compact semigroups

In some cases is very difficult to show the compactness of a set. In that case the following existence result is very useful. Moreover, it provides us with an equivalence which is a strong tool in working with gradient systems. This result is one of the most important concerning the existence of a global attractor.

Theorem 2.2.2 *If $\{S(t) : t \geqslant 0\}$ is a semigroup, the following conditions are equivalent:*

i) $\{S(t) : t \geqslant 0\}$ *possesses a global attractor* $\mathscr{A}$*.*

ii) $\{S(t) : t \geqslant 0\}$ *is bounded dissipative and asymptotically compact.*

Moreover, if the bounded set B is the set that absorbs bounded sets under $\{S(t) : t \geqslant 0\}$, then the global attractor is the ω-limit set of B, that is,

$$\mathscr{A} = \omega(B). \tag{2.7}$$

This theorem reveals the important role played by the asymptotic compactness when we are in a general Banach space and how the ω-limit sets describe the global attractor. The decision about which characterization should be used in a given application depends on the properties of the model under study.

In [47], we can find a nice characterization of the asymptotically compact semigroups. Basically, it says that if we can separate our semigroup into two, one of them being compact, and the other one decaying to zero, then the semigroup is asymptotically compact.

Theorem 2.2.3 *Let X be a Banach space and $\{S(t) : t \geqslant 0\}$ a semigroup. Suppose that we can write the semigroup as $S(t) = W(t) + U(t)$ where:*

i) $\{W(t) : t > 0\}$ is a family of maps that for any bounded set $B \subset X$ it verifies

$$\|W(t)B\|_X \leqslant m_1(t)m_2(\|B\|_X)$$

with $m_i : \mathbb{R}^+ \to \mathbb{R}^+$, $\lim_{t\to\infty} m_1(t) = 0$ and $\|B\|_X = \sup\{\|x\|_X : x \in B\}$.

ii) $\{U(t) : t > 0\}$ verifies that for any bounded set $B \subset X$, there exists a time $T(B) > 0$ such that $\bigcup_{t\geqslant T} U(t)B$ is precompact, that is, given a sequence $\{x_n\}_{n=1}^{\infty} \subset B$ and $t_n \overset{n\to\infty}{\longrightarrow} \infty$, there exists a convergent subsequence of $\{S(t_n)x_n\}_{n=1}^{\infty}$.

Then $\{S(t) : t \geqslant 0\}$ is asymptotically compact.

In this work we can find an example of this result for the following one-dimensional damped wave equation

$$u_{tt} + 2\alpha u_t - u_{xx} = f(u),$$

defined in $[0, \pi]$ and the usual growth conditions over f. The author prove that the semigroup $\{S(t) : t \geqslant 0\}$ associated to the solution of the system can be written as $S(t) = L(t) + U(t)$, where $U(t)$ is compact due to the properties of f and $\{L(t) : t \leqslant 0\}$ possess an exponential decay.

2.2.3 ω-limit compact semigroups

A similar result based on the concept of ω-limit compact semigroups is the following one (see [68]).

Theorem 2.2.4 *Let $\{S(t) : t \geqslant 0\}$ be a semigroup in a metric space X. Then, there exists the global attractor if an only if the semigroup is* *i) ω-limit compact, and*

ii) bounded dissipative.

In this theorem the condition of asymptotically compactness is replaced by the ω-limit compactness of the semigroup, but in the following case we change the decomposition in two of the semigroup by the existence of a finite dimensional projection for which the semigroup is bounded and decaying to zero in the complementary space.

Theorem 2.2.5 *Suppose that $\{S(t) : t \geqslant 0\}$ verifies the* flattening property*, that is, for each bounded set $B \in X$ and $\epsilon > 0$ there exists a time $t_0 = t_0(B, \epsilon) > 0$ and a finite dimensional space $X_\varepsilon \subset X$ such that:*

i) $\{\|P_\varepsilon S(t)B\|\}$ is bounded;

ii) $\|(I - P_\varepsilon)S(t)x\| < \epsilon$ for all $t \geqslant t_0$, $x \in B$;

where $P_\varepsilon : H \longrightarrow X_\varepsilon$ is a bounded projection. Then $\{S(t) : t \geqslant 0\}$ is ω-limit compact. Moreover, if X is uniform convex (a normed space such that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that for any $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $\|x + y\|_X \geqslant 2 - \delta$ implies $\|x - y\|_X \leqslant \varepsilon$), the equivalence holds.

In [68], the authors used this results in a 2D Navier-Stokes equation in H_0^1, using also energy estimates, to obtain the existence of the global attractor.

2.2.4 Point dissipative semigroups

The following result shows the existence of the global attractor depending on point dissipative semigroups (see [76]), which are importan in applications because they are closely related with gradient systems.

Theorem 2.2.6 *If $\{S(t) : t \geqslant 0\}$ is a semigroup, the following conditions are equivalent:*

i) $\{S(t) : t \geqslant 0\}$ possesses a global attractor $\mathscr{A}$.

ii) $\{S(t) : t \geqslant 0\}$ is bounded, point dissipative and asymptotically compact.

Moreover, if the bounded set B is the set that absorbs points under $\{S(t) : t \geqslant 0\}$, then the global attractor is the ω-limit set of B, that is,

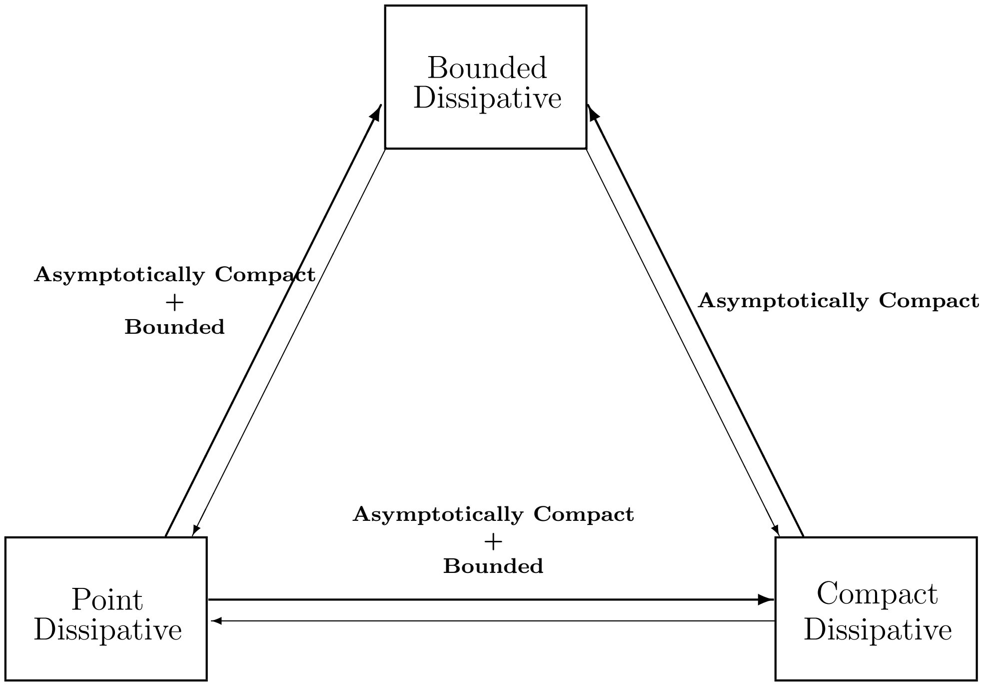
$$\mathscr{A} = \omega(B). \tag{2.8}$$

There exists a relationship between point dissipative and bounded dissipative thanks to the following results:

Lemma 2.2.7 *Let be $\{S(t) : t \geqslant 0\}$ a compact dissipative semigroup. If it is and asymptotically compact, then it is bounded dissipative.*

Lemma 2.2.8 *Let be $\{S(t) : t \geqslant 0\}$ a point dissipative semigroup. If it is bounded and asymptotically compact, then it is compact dissipative.*

We can summarize them in the following picture.



2.3 Upper and lower-semicontinuity

In this section our aim is to know if the attractors possess a continuous behaviour depending on parameters, but first we need to recall the concepts of upper and lower-semicontinuity:

Definition 2.3.1 *Let $\eta \in [0,1]$ be a parameter and X a Banach space. For each $\eta \in [0,1]$ let $A_\eta \subset X$. We say that $\{A_\eta\}_{\eta\in[0,1]}$ is*

- *Upper-semicontinuous at $\eta = 0$ if $\lim_{\eta\to 0} dist(A_\eta, A_0) = 0$.*
- *Lower-semicontinuous at $\eta = 0$ if $\lim_{\eta\to 0} dist(A_0, A_\eta) = 0$.*

- *Continuous at $\eta = 0$ if it is upper and lower semicontinuous.*

The upper-semicontinuity means that all attractors $\mathscr{A}_\eta$ are inside a neighborhood of $\mathscr{A}_0$, that is, $\mathscr{A}_0$ cannot explode (see, for an example, Arrieta *et. al* [7, 8] and reference therein). Obviously, we will need some kind of convergence of the semigroups,

$$\sup_{u \in Y} \|S_\eta(t)u - S_0(t)\|_X \overset{\eta \to 0}{\longrightarrow} 0, \tag{2.9}$$

for any bounded $Y \subset X$ and uniformly for t in compact subsets of $\mathbb{R}$.

Theorem 2.3.2 *Let suppose that there exists a $\eta_0 > 0$ such that the set*

$$\bigcup_{0 \leqslant \eta \leqslant \eta_0} \mathscr{A}_\eta \subset X$$

is bounded and (2.9) *holds. Then dist*$(\mathscr{A}_\eta, \mathscr{A}_0) \overset{\eta \to 0}{\longrightarrow} 0$.

For the lower-semicontinuity we need some structure for the limit attractor $\mathscr{A}_0$, because in this case the $\mathscr{A}_\eta$ wrap the limit attractor more and more while $\eta \to 0$, and we need to know how is the behaviour of the global solutions (see [3, 5, 11]). For this case, the gradient and the gradient-like systems have the most general results based on continuous variation of the local unstable manifolds of the equilibrium points. We can find the following result in [78] or [87].

Theorem 2.3.3 *Under conditions of Theorem 2.3.2 and assuming also that*

1. *for the limit problem there is a finite number of equilibrium points,*

2. *the limit attractor is the union of the closure of the unstable manifolds of the equilibrium points, so that*

$$\mathscr{A}_0 = \bigcup_{e \in \mathcal{E}} \overline{W^u(e)},$$

3. *the local unstable manifolds vary continuously with η near $\eta = 0$.*

Then, the attractor is lower-semicontinuous, that is dist$(\mathscr{A}_0, \mathscr{A}_\eta) \overset{\eta \to 0}{\longrightarrow} 0$. *Consequently, the attractor is continuous in the Hausdorff distance.*

Now we want to show a useful result which provides conditions to ensure the continuous variation of the unstable manifolds. First we need to write the unstable manifolds of any hyperbolic equilibrium point as a graph, and then we will need to prove that these graphs behave continuously when the parameter goes to zero. In [9, 22, 23, 66] we can find more general results about it, but here we only give the autonomous one. This result is based on a perturbation in the nonlinear external force. We consider the following family of problems

$$u_t + Bu = f_\eta(u), \quad u(0) = u_0, \tag{2.10}$$

where $\eta \in [0,1]$, $f_\eta : X \to X$ is Lipschitz continuous, all the problems are well posed and solutions exist for $t \in [0,\infty)$ and any $u_0 \in X$. Each of these problems generates a semigroup that we call $\{L_\eta(t) : t \geqslant 0\}$. We assume also that

$$\lim_{\eta\to 0} \sup_{z\in B_X(0,r)} \|f_\eta(z) - f_0(z)\|_X + \|f'_\eta(z) - f'_0(z)\|_{\mathcal{L}(X)} = 0, \tag{2.11}$$

for all $r > 0$. This condition ensures that we have the convergence of the semigroups as in (2.9) and ensures that given a hyperbolic equilibrium point of the limit problem η_0 for each $\eta \geqslant \eta_0$ there exists an equilibrium point y_η of (2.10) which is also hyperbolic and verifies $\|y_\eta - y_0\|_X \overset{\eta\to 0}{\longrightarrow} 0$.

Now, the idea is to focus on a local neighborhood around one arbitrary equilibrium point. Suppose that y_0 is an equilibrium point of (2.10) for $\eta = 0$. Defining $z = u - y_0$, we can write the limit problem as

$$z_t + Cz = h_0(z), \quad z(0) = z_0, \tag{2.12}$$

where $C = B - f'_0(y_0)$ and $h_0(z) = f_0(y_0 + z) - f(y_0) - f'_0(y_0)z$. Assuming that all equilibrium points are hyperbolic, we can define two projections

$$Q_0 : X \to X^+ = Q_0X \quad \text{and} \quad (I - Q_0) : X \to X^- = (I - Q_0)X.$$

Calling $L_0^+(t) = L_0(t)|_{X^+}$ and $L_0^-(t) = L_0(t)|_{X^-}$ we have that $L_0^\pm(t) \in \mathcal{L}(X)$. Assume also that, for some positive $M_1 \leqslant 1$ and $\beta > 0$

$$\begin{aligned} \|L_0^+(t)\|_{\mathcal{L}(X^+)} &\leqslant M_1 e^{\beta t}, \quad t \leqslant 0 \\ \|L_0^-(t)\|_{\mathcal{L}(X^-)} &\leqslant M_1 e^{-\beta t}, \quad t \geqslant 0. \end{aligned} \tag{2.13}$$

The above assumption can be interpreted as a saddle point in finite dimension, that is, we have an exponential decay when we are moving backwards in time and the same exponential decay when we move forward in time. In the same way we can define $L_\eta^+(t)$ and $L_\eta^-(t)$ with projections Q_η and $(I-Q_\eta)$ for hyperbolic equilibrium points for $L_\eta(t)$ with $\eta \leqslant \eta_0$.

Under the above conditions we have the following result (see [50])

Theorem 2.3.4 *There exists a function $\Sigma_\eta : X^+ \to X^-$ such that the unstable local manifold of the equilibrium point $z \equiv 0$ for (2.12) is given by*

$$W^u(0) = \{z \in X : z = (Q_\eta(z), \Sigma_\eta(Q_\eta(z)))\}.$$

Moreover we have that for any $r > 0$,

$$\sup_{z \in B_X(0,r)} \{\|Q_\eta(z) - Q_0(z)\|_X + \|\Sigma_\eta(Q_\eta(z)) - \Sigma_0(Q_0(z))\|_X\} \stackrel{\eta\to 0}{\longrightarrow} 0.$$

Therefore, if each $\{L_\eta(t) : t \geqslant 0\}$ possesses a global attractor and the limit problem has a gradient-like attractor, we have the upper and lower-semicontinuity of attractors.

2.4 Dynamics inside the attractor

The dynamics inside the attractor helps to determine all possible long-time dynamics of individual trajectories since, for each trajectory of the system, we can find another one inside the attractor which tracks it for a sufficiently large time. Therefore we have a "copy" of all the dynamic inside the attractor, as the following proposition shows (see, for example, [9] or [78]).

Theorem 2.4.1 *Given a trajectory $u(t) = S(t)u_0$, and a sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \stackrel{n\to\infty}{\longrightarrow} 0$ there exists an increasing sequence of times $\{t_n\}_{n=1}^\infty$ with $t_{n+1} - t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ and a sequence of points $\{v_n\}_{n=1}^\infty$ in $\mathscr{A}$ such that $\|u(t) - S(t-t_n)v_n\| \leqslant \varepsilon_n$, for all $t_n \leqslant t \leqslant t_{n+1}$. Furthermore, the "jumps" $\|v_{n+1} - S(t_{n+1}-t_n)v_n\|$ decrease to zero.*

This result is important because the dimension of the global attractor is finite in many cases. We have the following two definitions of dimension: the fractal dimension, based on the number of closed balls of a fixed radius, and the Hausdorff dimension.

Definition 2.4.2 *Let be K a set with compact closure. The* fractal dimension of K, $d_f(K)$, *is given by*

$$d_f(K) = \limsup_{\varepsilon\to 0} \frac{\log_2 n(K,\varepsilon)}{\log_2(\frac{1}{\varepsilon})},$$

where $n(K,\varepsilon)$ denotes the minimum number of balls of radius ε which cover K.

Before the definition of the Hausdorff dimension, we need to define the following measure μ of a set K.

$$\mu(B,d,\varepsilon)=\inf\left\{\sum_i r_i^d : r_i \leqslant \varepsilon \text{ and } B\subseteq \bigcup_i B(x_i,r_i)\right\},$$

where $B(x_i,r_i)$ are balls with radius r_i. This measure is the best approximation of the d-dimensional volume of K.Then, we can define the d-dimensional Hausdorff measure of a set K, $\mathcal{H}^d(K)$, given by

$$\mathcal{H}^d(K)=\lim_{\varepsilon\to 0}\mu(K,d,\varepsilon).$$

Definition 2.4.3 *The* Hausdorff dimension of a compact set K, $d_H(K)$*, is defined by*

$$d_H(K)=\inf_{d>0}\{d:\mathcal{H}^d(K)=0\}.$$

These dimensions verify

i) The fractal dimension is stable under finite unions

$$d_f\left(\bigcup_{i=1}^N K_i\right)\leqslant \max_i d_f(K_i),$$

and the Hausdorff dimension under countable unions

$$d_H\left(\bigcup_{j\in\mathbb{N}} K_j\right)\leqslant \max_j d_H(K_j).$$

ii) If $f: X\to X$ is Hölder continuous with exponent θ, then

$$d_f(f(K))\leqslant \frac{d_f(K)}{\theta}, \qquad d_H(f(K))\leqslant \frac{d_H(K)}{\theta}.$$

iii) $d_f(K_1\times K_2)\leqslant d_f(K_1)+d_f(K_2)$, $d_H(K_1\times K_2)\leqslant d_f(K_1)+d_H(K_2)$.

iv) $d_f(\overline{K})=d_f(K)$.

v) $d_H(K)\leqslant d_f(K)$.

Definition 2.4.4 *We say that* $\{S(t) : t \geqslant 0\}$ *in a Hilbert space* X *is* uniformly differentiable on $\mathscr{A}$ *if for every* $u \in \mathscr{A}$*, there exists a linear operator* $\Lambda(t,u)$ *such that, for all* $t \geqslant 0$*,*

$$\sup_{\substack{u,v \in \mathscr{A} \\ 0 < \|u-v\|_X \leqslant \varepsilon}} \frac{\|S(t)v - S(t)u - \Lambda(t,u)(v-u)\|_X}{\|v-u\|_X} \xrightarrow{\varepsilon \to 0} 0$$

and

$$\sup_{u \in \mathscr{A}} \|\Lambda(t,u)\|_{\mathcal{L}(X)} < \infty, \text{ for each } t \geqslant 0.$$

Consider the following abstract problem

$$\begin{aligned} u_t &= F(u), \\ u(0) &= u_0, \end{aligned}$$

where u_0 is in the Hilbert space X, and assume that it has a unique solution given by $u(t;u_0) = S(t)u_0$ and a global attractor $\mathscr{A}$. Suppose that $\Lambda(t,u)$ is given by the solution of the linearized equation

$$\begin{aligned} w_t &= F'(S(t)u_0)w, \\ u(0) &= u_0. \end{aligned}$$

There exists some results about upper bound for Hausdorff dimension of invariant sets, obtaining a finite dimension for global attractors (see Mañé [70], Mallet-Paret [69] or Robinson [78]). Therefore, inside the attractor there exists a semigroups which “copy” the dynamic of the system in a finite dimensional set.

2.5 Gradient systems

In this section our aim is to give a more detailed structure for global attractors. The attractor is the union of all global and bounded solutions of the system and contains the unstable manifolds of all fixed points and periodic orbits of the semigroup. This gives us a better idea of the dynamics that we can expect. As in the previous sections we denote by $\{S(t) : t \geqslant 0\}$ a semigroup in a Banach (or metric) space X with $\mathscr{A}$ its global attractor. From now on, we denote by $\mathcal{E}$ the set of all equilibrium point for $\{S(t) : t \geqslant 0\}$.

Theorem 2.5.1 *The global attractor* $\mathscr{A}$ *is the union of all global and bounded solutions. Moreover, if* $\{S(t) : t \geqslant 0\}$ *is injective for each point* $x \in \mathscr{A}$*, that is*

$$S(t)u_0 = S(t)v_0 \in \mathscr{A} \quad \text{for some } t > 0 \quad \Rightarrow \quad u_0 = v_0,$$

there exists a unique global solution $\xi : \mathbb{R} \to X$ such that $x = \xi(t)$ for some $t \in \mathbb{R}$.

Remark 1 *If $\{S(t) : t \geqslant 0\}$ is injective in $\mathscr{A}$, then the semigroup is a group inside the attractor, that is $\{S(t)|_{\mathscr{A}} : t \in \mathbb{R}\}$.*

To give a more precise structure of the global attractor, the unstable and stable manifolds of a fixed point play an important role. A specific kind of semigroups, called gradient systems, shows a detailed structure as the union of these unstable manifolds. These systems show a simple asymptotic dynamics and possess a Lyapunov functional, which give a powerful way of "managing" the dynamics.

Definition 2.5.2 *A* Lyapunov function *for $\{S(t) : t \geqslant 0\}$ on a positively invariant set $B \subset X$ is a function $\phi : B \to X$ such that*

1. *for each $x \in B$, the function $t \mapsto \phi(S(t)x)$ is nonincreasing,*
2. *if for $x \in B$ there exists a global solution $\xi(\cdot)$ through $\xi(0) = x$ and there is a $t^* \in \mathbb{R}$ such that $\phi(\xi(t)) = x$ for all $t \geqslant t^*$, then x is a fixed point of the semigroup $\{S(t) : t \geqslant 0\}$.*

Definition 2.5.3 *A semigroup $\{S(t) : t \geqslant 0\}$ with global attractor $\mathscr{A}$ and a finite family of equilibrium points $\mathcal{E} = \{e_1, \dots, e_n\}$ is a* gradient semigroup *if there is a Lyapunov function ϕ on X such that $\phi(S(t)x) = \phi(x)$ if and only if x is an equilibrium point.*

Theorem 2.5.4 *If $K \subset X$ is a compact invariant set, then $W^u(K) \subset \mathscr{A}$. In particular, the unstable manifolds of all invariant sets are contained in the attractor.*

For gradient systems, the attractor can be expressed as the union of the unstable manifolds of the fixed points. Moreover, when we take a point inside the attractor we know that there exists a global solution that converges forward and backward in time to different fixed points of the system, that is

$$\lim_{t\to\infty} S(t)u = e_1, \quad \lim_{t\to-\infty} S(t)u = e_2, \quad e_1, e_2 \in \mathcal{E}, \quad e_1 \neq e_2.$$

When the number of fixed points is finite, there are no homoclinic structure, that is, there do not exist a finite number of fixed points $\{e_{k_1}, ..., e_{k_l}\} \subset \mathcal{E}$ and a set of global solutions $\{\xi_i, 1 \leqslant i \leqslant k\}$ such that, setting $e_{k_{l+1}} = e_{k_1}$,

$$\lim_{t\to-\infty} \xi_i(t) = e_{k_i}, \quad \lim_{t\to\infty} \xi_i(t) = e_{k_{i+1}}, \quad 1 \leqslant i \leqslant k.$$

The following two results show the relation between the equilibrium points and the global attractor for gradient semigroups. The first one shows that the ω-limit set of any point is inside the set of equilibrium points and, in the second one, we use this fact to prove that the attractor is union of the unstable manifolds of the equilibrium points.

Proposition 2.5.5 *Suppose that $\{S(t) : t \geqslant 0\}$ has a Lyapunov function on a positively invariant absorbing set. Then, for every $u_0 \in X$, $\omega(u_0) \subset \mathcal{E}$. Moreover, if X is connected and $\mathcal{E}$ is discrete, then $\omega(u_0) \in \mathcal{E}$.*

Theorem 2.5.6 *Suppose that $\{S(t) : t \geqslant 0\}$ has a Lyapunov function on $\mathscr{A}$. Then $\mathscr{A} = W^u(\mathcal{E})$. Furthermore if X is connected and $\mathcal{E}$ is discrete, then*

$$\mathscr{A} = W^u(\mathcal{E}) = \bigcup_{e\in\mathcal{E}} W^u(e). \tag{2.14}$$

In [26] the authors define the concept of dynamically $\mathcal{J}$-gradient semigroups (previous called 'gradient-like' semigroups in [22, 23, 24, 66]). This kind of systems keeps the dynamical properties of a gradient one, but there may not exist, a priori, reference to the existence of a Lyapunov function.

Definition 2.5.7 *Suppose that $\{S(t) : t \geqslant 0\}$ has a finite number of stationary solutions $\mathcal{E}$. We say that the semigroup is a* dynamically $\mathcal{E}$-gradient semigroup *if the following two conditions are satisfied:*

1. *Given a global solution $\xi : \mathbb{R} \to X$ in $\mathscr{A}$, there exists $e_i, e_j \in \mathcal{E}$ with $e_i \neq e_j$ such that*

$$\lim_{t\to\infty} \xi(t) = e_i, \quad \text{and} \quad \lim_{t\to-\infty} \xi(t) = e_j.$$

2. *$\mathcal{E}$ does not contain any homoclinic structure.*

In this case, and owing to the previous definition, we already know that the global attractor is as in (2.14). The following theorem shows the equivalence between gradient semigroups and dynamically $\mathcal{E}$-gradient semigroups when we have e finite number of equilibrium points (see [2, 26]), so from now on, we will not distinguish them.

Theorem 2.5.8 *A semigroup with a finite collection of equilibrium points is dynamically $\mathcal{E}$-gradient if and only if it is gradient.*

Let us suppose that we have a family of semigroups $\{S_\eta(t) : t \geqslant 0\}_{\eta\in[0,1]}$ with global attractors $\mathscr{A}_\eta$ and $\eta \in [0,1]$. The global attractors $\mathscr{A}_\eta$ keep the structure of the limit problem when it is gradient. The following result, proved in [23], shows this fact under some conditions.

Theorem 2.5.9 *Let X be a Banach space and $\{S_\eta(t) : t \geqslant 0\}_{\eta\in[0,1]}$, $\eta \in [0,1]$ be a family of semigroups in X with global attractors $\mathscr{A}_\eta$ which satisfies*

1. $\overline{\bigcup_{\eta\in[0,1]} \mathscr{A}_\eta}$ *is compact.*
2. *the limit semigroup $\{S_0(t) : t \geqslant 0\}$ is gradient.*
3. *$\{S_\eta(t) : t \geqslant 0\}_{\eta\in[0,1]}$ has a finite number of equilibrium point $\mathcal{E}_\eta = \{y_1^\eta, \ldots, y_n^\eta\}$, and*
$$\sup_{i\in[1,n]} \|y_i^\eta - y_i^0\|_X \stackrel{\eta\to 0}{\longrightarrow} 0.$$
4. *$\|S_\eta(t)u - S_0(t)u\|_X \stackrel{\eta\to 0}{\longrightarrow} 0$ for t in compact subsets of $[0,\infty)$ and u in compact subsets of X.*
5. *There is a $\overline{\eta} > 0$ and neighborhoods V_i^n of y_i^n such that y_i^η is the maximal invariant set for $\{S_\eta(t) : t \geqslant 0\}_{\eta\in[0,1]}$ in V_i^n for each $i = \{1,\ldots,n\}$ and $\eta \leqslant \overline{\eta}$.*

Then, there exists a η_0 such that, for all $\eta \in [0,\eta_0]$,
$$\mathscr{A}_\eta = \bigcup_{i=1}^n W^u(y_i^\eta).$$

Introducing the concept of invariant isolated sets, we can obtain more general concepts and results for dynamically $\mathcal{E}$-semigroups, where we can replace the equilibrium points for the following sets.

Definition 2.5.10 *We say that a family of invariant sets $\mathcal{S} = \{\Gamma_1, \ldots, \Gamma_n\}$ is* a family of invariant isolated sets *if there exists a $\delta > 0$ such that $N(\Gamma_i,\delta) \cap N(\Gamma_j,\delta) = \emptyset$, $1 \leqslant i,j \leqslant n$ and Γ_i is the maximal invariant subset (with respect to $\{S(t) : t \geqslant 0\}$) in the neighborhood $N(\Gamma_i,\delta)$.*

2.6 Exponential attractor

A very interesting part of the study of the global attractor is to find the rate of convergence or attraction. This is very useful in computation because, although we know that the global attractor exists, the computational cost in the approximation of this objet may be very high because the time could be very large. For computer simulations, sometimes it is better to find a set with exponential rate of convergence instead the global attractor (we know that the attractor will be inside that set).

In [78], we can find the following result about the existence of an exponential attracting set

Theorem 2.6.1 *Let be* $\{S(t,s) : t \geqslant s\}$ *a semigroup. Suppose that there exists the global attractor* $\mathscr{A}$ *with finite Hausdorff dimension and the semigroup verifies the Lipschitz property, that is, for* $u, v \in X$ *and* $0 \leqslant t$, $\tau \leqslant T$ *with* $T > 0$, *there exists a* $K = K(T) > 0$ *such that*

$$\|S(t)u - S(\tau)v\|_X \leqslant K(T)(\|u - v\|_X + |t - \tau|).$$

Then there exists a set $\mathcal{K}$ *and* $\sigma > 0$ *which verify*

i) $\mathcal{K}$ *is positive invariant,*

ii) $\mathcal{K}$ *attracts exponentially fast with exponent* σ,

$$dist(S(t)B, \mathcal{K}) \leqslant Ce^{-}\sigma t,$$

iii) $\mathcal{K}$ *has finite Hausdorff dimension* $d_H(\mathcal{K}) \leqslant d_H(\mathscr{A}) + 1$.

The set $\mathcal{K}$ *is called the* exponential attractor.

In [9] we can find some results on uniform exponential attraction under perturbation as the following one,

Theorem 2.6.2 *Suppose that there exists a* η_0 *such that* $\{S_\eta(t) : t \geqslant 0\}$ *is a gradient semigroup for* $0 \leqslant \eta \leqslant \eta_0$ *and there exist* $c = c(B_0)$ *and* $L > 0$ *such that*

$$\|S_\eta(t)u - S_\eta(t)v\|_X \leqslant Le^{ct}\|u - v\|_X \quad \text{for all } u, v \in B_0 \subset X \text{ bounded.}$$

Assuming conditions of Theorem 2.5.9 and also there are $\gamma, M > 0$ *and for each* $1 \leqslant i \leqslant n$, *there exists a neighborhood* V_i *of* y_i^η *such that*

$$dist(S_\eta(t)u, W^u(y_i^\eta)) \leqslant Me^{-\gamma t},$$

for all $u \in V_i$ *and as long as* $S_\eta(s)u \in V_i$. *Then for any bounded set* $B \subset X$, *there are constants* $c(B)$ *and* $\overline{\eta}(B) \in (0, \eta_0]$ *such that*

$$dist(S_\eta(t)u, \mathscr{A}_\eta) \leqslant c(B)e^{-\gamma t}, \quad \text{for all } u \in B, \eta \leqslant \overline{\eta}.$$

Kloeden and Li proved (see [41] and also [9]) that the result above is a necessary and sufficient condition to obtain the lower-semicontinuity of global attractors.

Chapter 3
The non-autonomous case

Our aim in this chapter is to give an analogous theory of the semigroups in the framework of non-autonomous problems. This theory will be a generalization and we will remark the points where the autonomous one can be seen as a special case. However, if the evolution process comes from a non-autonomous differential equation, even though some nice references are already available (cf. Caraballo *et. al.* [19], Carvalho *et. al.* [26], Cheban [29], Chepyzhov and Vishik [33], Kloeden and Rasmussen [58], Sell and You [82]), this is a relatively new field of investigation.

3.1 Evolution processes and pullback attractors

3.1.1 Basic Definitions

Suppose we have a non-autonomous differential equation in a Banach space X

$$\begin{aligned} u_t &= F(u(t), t), \\ u(s) &= u_0, \end{aligned}$$

with a unique solution $u(t, s; u_0)$. We note that the initial time has a very important role because we have an explicit dependence on time of F. This time dependence may appear in the external force, in the operator, in both at the same time or even on the boundary conditions.

In general, a non-autonomous system shows two different important dynamics without relation between them:

- Forward dynamic: the behaviour when final time goes to infinity,

$$\lim_{t\to\infty} u(t, s; u_0).$$

- Pullback dynamic: the behaviour when the initial time goes to minus infinity,
$$\lim_{s\to-\infty} u(t,s;u_0).$$

In the autonomous case, these two dynamics are the same one, but in the general case they can produce entirely different qualitative properties. Consider the following simple examples of non-autonomous equations giving different answers to the relation between pullback and forward dynamics,

$$y_1'(t) = -2ty_1(t) + 2t^2$$

and

$$y_2'(t) = 2ty_2(t) + 2t^2.$$

Both can be solved explicitly with initial value $y_0 \in \mathbb{R}$ at time $s \in \mathbb{R}$ by

$$y_1(t,s) = (y_0 - s)e^{-(t^2-s^2)} + t - e^{-t^2}\int_s^t e^{r^2}dr,$$

$$y_2(t,s) = (y_0 + s)e^{t^2-s^2} + t + e^{t^2}\int_s^t e^{-r^2}dr.$$

In the first case we can observe how the trajectory is closer and closer to $\mathscr{A}_1(t) = t - e^{-t^2}\int_0^t e^{r^2}dr$ when t goes to infinity. In the same way, the trajectories of the second equation are attracted in a pullback sense by the family $\mathscr{A}_2(t) = -t + e^{t^2}\int_{-\infty}^t e^{-r^2}dr$, that is, when initial time $s \to -\infty$. However, $\{\mathscr{A}_1(t) : t \in \mathbb{R}\}$ is forward but not pullback attracting and $\{\mathscr{A}_2(t) : t \in \mathbb{R}\}$ is pullback but not forward.

Then, we need to define a family of two parameters maps, instead of a one parameter family as in the autonomous case, to describe the complete dynamics of the system.

Definition 3.1.1 *An* evolution process *in a metric space X is a family of continuous maps $\{S(t,s) : t \geqslant s\}$ from X into itself with the following properties*

1) $S(t,t) = I$, *for all* $t \in \mathbb{R}$,
2) $S(t,s) = S(t,\tau)S(\tau,s)$, *for all* $t \geqslant \tau \geqslant s$,
3) $\{(t,s) \in \mathbb{R}^2 : t \geqslant s\} \times X \ni (t,s,x) \mapsto S(t,s)x \in X$ *is continuous.*

In this way we can identify the solution of the system with initial data $u_0 \in X$ as $u(t,s;u_0) = S(t,s)u_0$. Inside this definition we can identify a semigroup $\{T(t) : t \geqslant 0\}$ as an autonomous process which verifies $S(t,s) = S(t-s,0) = T(t-s)$. Conversely, given an evolution process which verifies $S(t,s) = S(t-s,0)$, we can define a semigroup $\{T(t) : t \geqslant 0\}$ with $S(t-s,0) = T(t-s)$.

Definition 3.1.2 *A set* $B(t) \subset X$ pullback attracts a set C at time t *under* $\{S(t,s) : t \geqslant s\}$ *if*

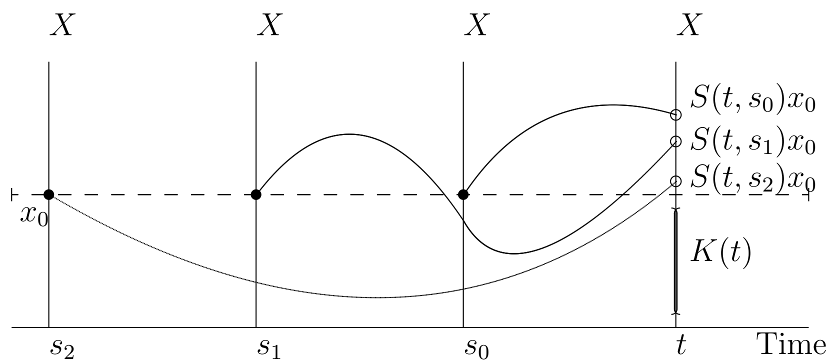
$$\lim_{s \to -\infty} dist(S(t,s)C, B(t)) = 0.$$

A family $\{B(t) : t \in \mathbb{R}\}$ pullback attracts bounded subsets of X *under* $\{S(t,s) : t \geqslant s\}$ *if* $B(t)$ *pullback attracts bounded subsets at time* t *under* $\{S(t,s) : t \geqslant s\}$*, for each* $t \in \mathbb{R}$.

Remark 2 *a) In the autonomous case, forward and pullback dynamics coincide because* $S(t,s) = S(t-s,0)$.

b) The pullback attractor is the sensible concept of attractor in random dynamical systems (see, for example, Caraballo et. al. *in [15], Crauel* et. al. *in [37, 38], Kloeden and Langa in [57] or Schmalfuß in [79, 80, 81])*

As we can see in the previous definition, in the pullback case we have a fixed final time t and we move the initial time $s \to -\infty$, so we have different configurations of the phase space for each final time. The following simple figure tries to illustrate this concept.



In the non-autonomous case we can also define forward attraction. We say that the family $\{C(t) : t \in \mathbb{R}\}$ attracts the bounded set $B \subset X$ if

$$\lim_{t \to \infty} \mathrm{dist}(S(t+s,s)B, C(t+s)) = 0.$$

Since a fixed set A in X will not, in general, remain fixed by a non-autonomous process, the concept of invariance for an evolution process is defined for families of sets.

Definition 3.1.3 *A family of nonempty sets* $\{B(t) : t \in \mathbb{R}\}$ *is* invariant under $\{S(t,s) : t \geqslant s\}$ *if* $S(t,s)B(s) = B(t)$ *for all* $t \geqslant s$ *and* $s \in \mathbb{R}$*. We say that* $\{B(t) : t \in \mathbb{R}\}$ *is* positively invariant *if we only have the inclusion* $S(t,s)B(s) \subset B(t)$.

Now we can define the pullback attractor for an evolution process.

Definition 3.1.4 *A family of compact sets* $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ *is the* pullback attractor for $\{S(t,s) : t \geqslant s\}$ *if it is invariant, attracts all bounded subsets of* X *'in the pullback sense' and is minimal in the sense that if there exists a family of closed sets* $\{C(t) : t \in \mathbb{R}\}$ *such that attracts bounded sets of* X*, then* $\mathscr{A}(t) \subset C(t)$*, for all* $t \in \mathbb{R}$.

Remark 3 *a) An autonomous evolution process* $\{S(t-s,0) : t \geqslant s\}$ *has a pullback attractor* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *if and only if the semigroup* $\{T(t) : t \geqslant 0\}$ *has a global attractor* $\mathscr{A}$ *and in either case* $\mathscr{A}(t) = \mathscr{A}$ *for all* $t \in \mathbb{R}$.

b) The minimality requirement in Definition 3.1.4 is additional relative to the theory of attractors for semigroups. This minimality requirement is essential to ensure uniqueness of pullback attractors. Its inclusion is related to the weakening of the invariance property imposed by the non-autonomous nature of general evolution processes. If $\{T(t) : t \geqslant 0\}$ *is a semigroup and* $\{S(t-s,0) : t \geqslant s\}$ *is the process associated to it, there may exist a family* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *of compact invariant sets that pullback attracts bounded subsets and is not minimal. Indeed, if* $T(t-s) = e^{-(t-s)}x_0$*,* $x_0 \in \mathbb{R}$*,* $t \geqslant s$ *and* $c \geqslant 0$ *the family* $\{[-ce^{-t}, ce^{-t}] : t \in \mathbb{R}\}$ *is invariant,* $[-ce^{-t}, ce^{-t}]$ *is compact and attracts bounded subsets of* $\mathbb{R}$ *at time* t *for each* $t \in \mathbb{R}$.

Definition 3.1.5 *A* global solution for an evolution process $\{S(t,s) : t \geqslant s\}$ *is a function* $\xi : \mathbb{R} \to X$ *such that* $S(t,s)\xi(s) = \xi(t)$ *for all* $t \geqslant s$*. We say that a global solution* $\xi : \mathbb{R} \to X$ *of an evolution process is* backwards bounded *if there is a* $\tau \in \mathbb{R}$ *such that* $\{\xi(t) : t \leqslant \tau\}$ *is a bounded subset of* X.

Definition 3.1.6 *A family* $\{B(t) : t \in \mathbb{R}\}$ pullback absorbs bounded subsets of X *if* $B(t)$ *pullback absorbs bounded sets at time* t*, for each* $t \in \mathbb{R}$*, that is if for each bounded subset* C *of* X*, there exists a time* $s_0 = s_0(C,t)$ *such that for all* $s \leqslant s_0$*,* $S(t,s)C \subset B(t)$*. If there exists a family* $\{B(t) : t \in \mathbb{R}\}$ *of bounded sets which pullback absorbs bounded subsets of* X *then we say that the evolution process* $\{S(t,s) : t \geqslant s\}$ *is* pullback bounded dissipative*. In a completely similar way we define the notions of* pullback point dissipative *or* pullback compact dissipative *processes.*

The notion of a pullback asymptotically compact evolution process is naturally associated with evolution processes which posses a pullback attractor and is the natural generalization of the concept of pullback asymptotic compactness for a semigroup.

Definition 3.1.7 *An evolution process* $\{S(t,s) : t \geqslant s\}$ *in a metric space* X *is said* pullback asymptotically compact *if, for each* $t \in \mathbb{R}$, *sequence* $\{s_k\}_{k\in\mathbb{N}}$ *in* $(-\infty, t]$ *and bounded sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* X *such that*

- $s_k \stackrel{k\to\infty}{\longrightarrow} -\infty$ *and*
- $\{S(t,s_k)x_k : k \in \mathbb{N}\}$ *is bounded,*

the sequence $\{S(t,s_k)x_k\}_{k\in\mathbb{N}}$ *has a convergent subsequence.*

In this way, we can write Theorem 2.2.6 in Section 2.2 as follows

Theorem 3.1.8 *If* $\{T(t) : t \geqslant 0\}$ *is a semigroup and* $\{S(t,s) : t \geqslant s\}$ *is its associated evolution process, then the following conditions are equivalent*

i) $\{S(t,s) : t \geqslant s\}$ *has a pullback attractor* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$.

ii) $\{S(t,s) : t \geqslant s\}$ *is pullback asymptotically compact and pullback bounded dissipative.*

iii) $\{S(t,s) : t \geqslant s\}$ *is pullback bounded, pullback point dissipative and pullback asymptotically compact.*

Definition 3.1.9 *A evolution process is* pullback bounded *if the set*

$$\bigcup_{s\leqslant t} S(t,s)B,$$

is bounded whenever B *is a bounded subset of* X *and for each* $t \in \mathbb{R}$.

Next we give the generalization of the concept of unstable manifold.

Definition 3.1.10 *Let* $\{S(t,s) : t \geqslant s\}$ *be a process and* $\{\Xi(t) : t \in \mathbb{R}\}$ *be an invariant family. If there exists* $\delta > 0$ *such that any global solution* $\xi : \mathbb{R} \to X$ *with* $\xi(t) \in \mathcal{O}_\delta(\cup_{t\in\mathbb{R}}\Xi(t)) := \{z \in X : \mathrm{dist}(z, \cup_{t\in\mathbb{R}}\Xi(t)) < \delta\}$ *for all* $t \in \mathbb{R}$, *must satisfy* $\xi(t) \in \Xi(t)$, *for all* $t \in \mathbb{R}$, *then we say that* $\{\Xi(t) : t \in \mathbb{R}\}$ *is an* isolated invariant family. $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$ *is said a* set of isolated invariant families *if each* Ξ_i^* *is an isolated invariant family and there exists* $\delta > 0$ *such that* $\mathcal{O}_\delta(\cup_{t\in\mathbb{R}}\Xi_i^*(t)) \cap \mathcal{O}_\delta(\cup_{t\in\mathbb{R}}\Xi_j^*(t)) = \emptyset$, $1 \leqslant i < j \leqslant n$.

Definition 3.1.11 *The unstable set of an isolated invariant family* $\Xi^*(\cdot)$ *is defined as*

$$W^u(\Xi^*(\cdot)) = \{(\tau,\zeta) \in \mathbb{R}\times X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(\tau) = \zeta \text{ and } \lim_{t\to-\infty} \mathrm{dist}(\xi(t), \Xi^*(t)) = 0\}.$$

Also, $W^u(\Xi^*(\cdot))(\tau) := \{\, \zeta \in X : (\tau,\zeta) \in W^{\mathrm{u}}(\Xi^*(\cdot))\}$.

For an autonomous evolution process, the above definition of unstable set coincides with the usual definition of an unstable set of an invariant set. That may not be the case for non-autonomous evolution processes. Nonetheless, they coincide if the following condition, which is automatically satisfied for autonomous evolution processes, holds

- If a solution $\xi(t)$ stays inside a suitably small neighborhood of $\Xi_i^*(\cdot)$ for all t in an interval of the form $(-\infty, t_0]$ (respectively, of the form $[t_0, \infty)$), then $\mathrm{dist}(\xi(t), \Xi(t)) \overset{t\to-\infty}{\longrightarrow} 0$ (respectively, $\mathrm{dist}(\xi(t), \Xi(t)) \overset{t\to\infty}{\longrightarrow} 0$).

3.1.2 Preliminary results

To prove (in applications) that a process is asymptotically compact, we will need to assume that the evolution process is pullback strongly bounded as defined next

Definition 3.1.12 *We say that an evolution process $\{S(t,s) : t \geqslant s\}$ in X is* pullback strongly bounded *if, for each $t \in \mathbb{R}$ and bounded subset B of X,*

$$\bigcup_{s\leqslant t}\bigcup_{\tau\leqslant s} S(s,\tau)B$$

is bounded.

Remark 4 *If $\{T(t) : t \geqslant 0\}$ is a semigroup, the associated process $\{S(t-s,0) : t \geqslant s\}$ is pullback strongly bounded if and only if $\{S(t-s,0) : t \geqslant s\}$ is pullback bounded if and only if $\{S(t) : t \geqslant 0\}$ is a bounded semigroup.*

In the pullback case, the definition of ω-limit set becomes a definition of a parameterized family of sets.

Definition 3.1.13 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process in a metric space X and B be a subset of X. The* pullback ω-limit *of B* in t *is defined by*

$$\omega(B,t) := \bigcap_{\sigma\leqslant t} \overline{\bigcup_{s\leqslant\sigma} S(t,s)B}, \tag{3.1}$$

or equivalently

$$\begin{aligned}\omega(B,t) = \{y \in X :\ & \text{there are sequences } \{s_k\}_{k\in\mathbb{N}} \text{ in } (-\infty,t], s_k \overset{k\to\infty}{\longrightarrow} -\infty \\ & \text{and } \{x_k\}_{k\in\mathbb{N}} \text{ in } B, \text{ such that } y = \lim_{k\to\infty} S(t,s_k)x_k\}.\end{aligned} \tag{3.2}$$

Now, we recall some results which will be necessary in the proof of our existence results.

Lemma 3.1.14 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process in a metric space X. If $B \subset X$, then $S(t,s)\omega(B,s) \subset \omega(B,t)$. If B is such that $\omega(B,s)$ is compact and pullback attracts B at time s, then $S(t,s)\omega(B,s) = \omega(B,t)$. Furthermore, if $\omega(B,t)$ pullback attracts C at time t and C is a connected set which contains $\bigcup_{s\leqslant t}\omega(B,s)$, then $\omega(B,t)$ is connected.*

Proof: If $\omega(B,t) = \emptyset$, there is nothing to prove. If $\omega(B,s) \neq \emptyset$, from the continuity of $S(t,s)$ and from (3.2) one immediately sees that $S(t,s)\omega(B,s) \subset \omega(B,t)$.

It remains to show that, if $\omega(B,s)$ is compact and pullback attracts B, then $\omega(B,t) \subset S(t,s)\omega(B,s)$. For $x \in \omega(B,t)$, there are sequences $\sigma_k \to -\infty$, $\sigma_k \leqslant t$ and $x_k \in B$ such that $S(t,\sigma_k)x_k \overset{k\to\infty}{\longrightarrow} x$. Since $\sigma_k \to -\infty$ we have that there exists $k_0 \in \mathbb{N}$ such that $\sigma_k \leqslant s$ for all $k \geqslant k_0$. Hence $S(t,s)S(s,\sigma_k)x_k = S(t,\sigma_k)x_k \to x$ for $k \geqslant k_0$. Since $\omega(B,s)$ is compact and pullback attracts B at time s, we have that $\mathrm{dist}(S(s,\sigma_k)x_k, \omega(B,s)) \overset{k\to\infty}{\longrightarrow} 0$. It is then easy to see that $\{S(s,\sigma_k)x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence (which we again denote by $S(s,\sigma_k)x_k$) for some $y \in \omega(B,s)$. It follows from the continuity of $S(t,s)$ that $S(t,s)y = x$. Hence $\omega(B,t) = S(t,s)\omega(B,s)$.

Now we prove the assertion about the connectedness of $\omega(B,t)$. Suppose that $\omega(B,t)$ is disconnected, then $\omega(B,t)$ is a disjoint union of two compact sets (hence separated by a positive distance), but $\omega(B,t)$ pullback attracts C and this is in contradiction with the fact that $S(t,s)C$ is connected and contains $\omega(B,t)$.

□

Lemma 3.1.15 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process in a metric space X. If B is a nonempty subset of X such that $\overline{\bigcup_{s\leqslant s_0} S(t,s)B}$ is compact, for some $s_0 \in \mathbb{R}$, $s_0 \leqslant t$, then $\omega(B,t)$ is nonempty, compact, invariant and $\omega(B,t)$ pullback attracts B at time t.*

Proof: Since $\overline{\bigcup_{s\leqslant\sigma} S(t,s)B}$ is nonempty and compact for each $\sigma \leqslant s_0$, we have that $\omega(B,t)$ is nonempty and compact.

Let us show that $\omega(B,t)$ pullback attracts B at time t. Suppose not, then there exists $\epsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in B, $\{\sigma_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\sigma_k \leqslant t$, $\sigma_k \overset{k\to\infty}{\longrightarrow} -\infty$, such that $\mathrm{dist}(S(t,\sigma_k)x_k, \omega(B,t)) > \epsilon$ for all $k \in \mathbb{N}$.

Since $\overline{\bigcup_{s\leqslant s_0} S(t,s)B}$ is compact and $\{S(t,\sigma_k)x_k, k \geqslant k_0\} \subset \overline{\bigcup_{s\leqslant s_0} S(t,s)B}$ for some $k_0 \in \mathbb{N}$, $\{S(t,\sigma_k)x_k : k \in \mathbb{N}\}$ has a subsequence which converges to some $y \in \omega(B,t)$. This leads to a contradiction and shows that $\omega(B,t)$ pullback attracts B at time t.

From Lemma 3.1.14, $\omega(B,t)$ is invariant and the proof is complete.

□

In the following lemma we can see how the asymptotic compactness of the process replaces the compactness of $\overline{\bigcup_{s\leqslant s_0} S(t,s)B}$.

Lemma 3.1.16 *If $\{S(t,s) : t \geqslant s\}$ is a pullback asymptotically compact evolution process and B is a nonempty bounded subset of X such that the set $\overline{\bigcup_{\tau\leqslant s_0} S(t,\tau)B}$ is bounded, for some $s_0 \in (-\infty, t]$, then $\omega(B,t)$ is nonempty, compact, invariant and pullback attracts B at time t.*

Proof: First, note that, for any sequences $\{x_k : k \in \mathbb{N}\}$ in B and $\{s_k : k \in \mathbb{N}\}$ in $(-\infty, s_0]$, $s_k \overset{k\to\infty}{\longrightarrow} -\infty$, we have that $\{S(t,s_k)x_k : k \in \mathbb{N}\}$ is bounded. It follows from the fact that $\{S(t,s) : t \geqslant s\}$ is pullback asymptotically compact that there exists $y \in X$ and a subsequence of $\{S(t,s_k)x_k : k \in \mathbb{N}\}$ (which we denote the same) such that $y = \lim_{k\to\infty} S(t,s_k)x_k$. It follows that $y \in \omega(B,t)$ and $\omega(B,t)$ is nonempty.

Now, given a sequence $\{y_k : k \in \mathbb{N}\}$ in $\omega(B,t)$, there are $x_k \in B$ and $s_k \in (-\infty, s_0]$, $s_k \leqslant -k$, such that $\mathrm{dist}(S(t,s_k)x_k, y_k) \leqslant \frac{1}{k}$. Since $\{S(t,s_k)x_k : k \in \mathbb{N}\}$ possesses a convergent subsequence, it follows that $\{y_k : k \in \mathbb{N}\}$ has a convergent subsequence and $\omega(B,t)$ is compact.

We now show that $\omega(B,t)$ pullback attracts B. If not, there exists $\epsilon > 0$ and sequences $x_k \in B$ and $s_k \overset{k\to\infty}{\longrightarrow} -\infty$, such that $\mathrm{dist}(S(t,s_k)x_k, \omega(B,t)) > \epsilon$. Thanks to the pullback asymptotic compactness, there is a $y \in X$ and a subsequence of $\{S(t,s_k)x_k : k \in \mathbb{N}\}$ (which we denote the same) such that $S(t,s_k)x_k \overset{k\to\infty}{\longrightarrow} y$. Clearly $y \in \omega(B,t)$ and that leads to a contradiction. It then follows that $\omega(B,t)$ attracts B.

The invariance of $\omega(B,t)$ follows now from Lemma 3.1.14 and the result is proved.

□

3.2 Existence result

Our aim in this section is to give analogous results to those in Chapter 2 for the existence of pullback attractors, following the structure of the previous chapter.

3.2.1 Family of compact absorbing sets

The first result is based on the existence of a compact family of absorbing sets. This family will ensure that each set of the pullback attractor is compact.

Theorem 3.2.1 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process in a metric space X. Then, the following statements are equivalent*

- $\{S(t,s) : t \geqslant s\}$ *possesses a pullback attractor* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$.
- *There exists a family of compact sets* $\{K(t) : t \in \mathbb{R}\}$ *that pullback attracts bounded subsets of* X *under* $\{S(t,s) : t \geqslant s\}$.

In either case

$$\mathscr{A}(t) = \overline{\bigcup \{\omega(B,t) : \ B \subset X, \ B \ \ \textit{bounded}\ \}} \tag{3.3}$$

and $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *is minimal in the sense that, if there exists another family of closed bounded sets* $\{\hat{\mathscr{A}}(t) : t \in \mathbb{R}\}$ *which pullback attracts bounded subsets of* X *under* $\{S(t,s) : t \geqslant s\}$*, then* $\mathscr{A}(t) \subseteq \hat{\mathscr{A}}(t)$*, for all* $t \in \mathbb{R}$.

Proof: If $\{S(t,s) : t \geqslant s\}$ possesses a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, each $\mathscr{A}(t)$ is compact and pullback attracts bounded subsets of X.

To prove the converse we proceed as follows. First note that, as an immediate consequence of (3.2), we have that $\omega(B,t) \subset K(t)$, for all bounded set $B \subset Z$ and all $t \in \mathbb{R}$. Moreover, we also have that $\omega(B,t)$ attracts B. Indeed, if not there exists $\varepsilon > 0$, a sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers with $s_n \to -\infty$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in B such that $\mathrm{dist}(S(t,s_n)x_n, \omega(B,t)) > \varepsilon$ for all $n \in \mathbb{N}$. As $K(t)$ pullback attracts B, $\mathrm{dist}(S(t,s_n)x_n, K(t)) \overset{n\to 0}{\longrightarrow} 0$. Consequently, $\{S(t,s_n)x_n\}_{n=1}^{\infty}$ has a subsequence which converges to some $x_0 \in K(t)$. Hence $x_0 \in \omega(B,t)$ which leads to a contradiction.

Note that we are now in the hypotheses of Lemma 3.1.14, which imply the invariance of $\omega(B,t)$. Thus, defining $\mathscr{A}(t)$ by (3.3), $\mathscr{A}(t)$ is clearly compact and pullback attracts bounded subsets of X. The invariance of $\mathscr{A}(t)$ holds from the invariance of each set $\omega(B,t)$. Indeed, given $x_0 \in \mathscr{A}(s)$, there exists

$x_n \in \omega(B_n, s)$ with $x_n \to x_0$ as $n \to \infty$. Then, $S(t,s)x_n = y_n \in \omega(B,t)$, and by the continuity of the process $S(t,s)$ we have that $S(t,s)x_n = y_n \to S(t,s)x_0$, which implies that $S(t,s)x_0 \in \mathscr{A}(t)$. Now, take $y_0 \in \mathscr{A}(t)$. Then, there exists $y_n \in \omega(B_n, t)$ with $y_n \to y_0$ as $n \to +\infty$. But then, again by the invariance of the family $\omega(B_n, t)$, there exists $x_n \in \omega(B_n, s)$ with $S(t,s)x_n = y_n$. But each $S(t,s)x_n \in S(t,s)\omega(B_n, s) \subset S(t,s)\mathscr{A}(s)$. As this last set is compact and does not depend on n, we get that $\lim_{n\to+\infty} S(t,s)x_n = y_0 \in S(t,s)\mathscr{A}(s)$.

On the other hand, as $\hat{\mathscr{A}}(t)$ is bounded and pullback attracts bounded sets at t, we have that $\omega(B,t) \subseteq \hat{\mathscr{A}}(t)$, for each bounded subset B of X. Hence $\mathscr{A}(t) \subseteq \hat{\mathscr{A}}(t)$.

□

The application of Theorem 3.2.1 to processes which are not compact may be difficult because one must find a compact set $K(t)$ which pullback attracts bounded subsets of X for each $t \in \mathbb{R}$. Our aim is to provide some alternative results to prove existence of pullback attractors which, in fact, are natural extensions of the ones in the autonomous case, and which help us to have an abstract theory as much complete as possible. It is more, the autonomous results could be understood as corollaries of the following ones (see [26]).

3.2.2 Strongly pullback asymptotically compactness

Not always the natural generalization of the autonomous existence result gives the existence of the pullback attractor. The following result is an example.

Theorem 3.2.2 *If $\{S(t,s) : t \geqslant s\}$ is pullback bounded dissipative and pullback asymptotically compact, then the set $\mathscr{A}(t)$ given by (3.3) is closed, invariant, pullback attracts bounded subsets of X at time t, and the family $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is minimal among the families $\{B(t) : t \in \mathbb{R}\}$ such that $B(t)$ is closed and pullback attracts bounded subsets of X at time t.*

Proof: Observe that we are in the hypotheses of Lemma 3.1.16, so that, given a bounded subset B of X, $\omega(B,t)$ is nonempty, compact, invariant and pullback attracts B at time t. Hence, if $\mathscr{A}(t)$ is defined by (3.3), $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is closed, invariant and pullback attracts bounded subsets of X. If $B(t)$ is closed and pullback attracts bounded sets at time t, it is clear that $\omega(B,t) \subset B(t)$ for each bounded subset B of X and consequently $\mathscr{A}(t) \subset B(t)$. This completes the proof.

□

Observe that the previous result does not ensure any compactness of $\mathscr{A}(t)$. This is not a restriction in the finite dimensional case, as $\mathscr{A}(t)$ is actually bounded and closed. However, the result shows a first difference with respect to analogous results for autonomous evolution processes in the infinite dimensional case. Getting the same kind of results requires to adopt different strategies, by imposing new hypotheses on the dynamics of the processes. In this way we need to define a strongly concept of absorption.

Definition 3.2.3 *We say that an evolution process $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of X which pullback absorbs bounded subsets of X at time τ for each $\tau \leqslant t$; that is, given a bounded subset B of X and $\tau \leqslant t$, there exists $s_0(\tau, D)$ such that $S(\tau, s)B \subset B(t)$, for all $s \leqslant s_0(\tau, D)$.*

Note that the family $\{B(t) : t \in \mathbb{R}\}$ given in the above definition does not need to have a bounded union. Nonetheless, we may choose it in such a way that, for each $t \in \mathbb{R}$, $\bigcup_{s \leqslant t} B(s)$ is bounded. The following theorem gives a sufficient condition for the existence of a compact pullback attractor.

Theorem 3.2.4 *If an evolution process $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative and pullback asymptotically compact, then $\{S(t,s) : t \geqslant s\}$ has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, given by* (3.3), *with the property that $\bigcup_{s \leqslant t} \mathscr{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

Proof: If $\mathscr{A}(t)$ is given by (3.3), it follows from Theorem 3.2.2 that $\mathscr{A}(t)$ is closed, invariant, pullback attracts bounded subsets of X at time t, and $\mathscr{A}(t)$ is minimal among the closed sets that pullback attract bounded subsets of X at time t. From the fact that $\{S(t,s) : t \geqslant s \in \mathbb{R}\}$ is pullback strongly bounded dissipative, there exists a bounded subset $B(t)$ of X that pullback absorbs bounded subsets of X at time τ, for each $\tau \leqslant t$. Since $\omega(B(t), t)$ pullback attracts $B(t)$ at time t (considered as a fixed bounded subset of X), it pullback attracts every bounded subset of X at time t. Indeed, it is enough to prove that, given a bounded subset D of X, $\omega(D,t) \subset \omega(B(t), t)$. If $x_0 \in \omega(D,t)$, there are sequences $\{s_k\}_{k\in\mathbb{N}}$ in $(-\infty, t]$ with $s_k \overset{k\to\infty}{\longrightarrow} -\infty$, and $\{x_k\}_{k\in\mathbb{N}}$ in D such that $S(t, s_k)x_k \overset{k\to\infty}{\longrightarrow} x_0$. Since $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative, given a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \overset{n\to\infty}{\longrightarrow} -\infty$, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\sigma_n \leqslant \tau_n$ such that $S(\tau_n, s)D \subset B(t)$, for all $s \leqslant \sigma_n(\tau_n)$. Given that $s_k \overset{k\to\infty}{\longrightarrow} -\infty$, for each τ_n there exists $k_n \geqslant n$ such that $S(\tau_n, s_{k_n})x_{k_n} \in B(t)$. Thus,

$$S(t, s_{k_n})x_{k_n} = S(t, \tau_n)S(\tau_n, s_{k_n})x_{k_n} \in S(t, \tau_n)B(t),$$

which implies $x_0 \in \omega(B(t),t)$. This proves that $\mathscr{A}(t) \subset \omega(B(t),t)$ and consequently $\mathscr{A}(t)$ is compact. Since clearly $\omega(B(t),t) \subset \mathscr{A}(t)$ we have that $\mathscr{A}(t) = \omega(B(t),t)$.

Finally, since $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative, for each bounded subset D of X, $\omega(D,\tau) \subset \overline{B(t)}$, for all $\tau \leqslant t$. In fact, for any $x_0 \in \omega(D,\tau)$ there is a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $(-\infty,t]$ with $s_n \overset{n\to\infty}{\longrightarrow} -\infty$ and $\{x_n\}_{n\in\mathbb{N}}$ in D such that $\lim_{n\to+\infty} S(\tau,s_n)x_n = x_0$. Hence, $S(\tau,s_n)x_n \in B(t)$ for all suitably large n and so $x_0 \in \overline{B(t)}$. This implies $A(\tau) \subset \overline{B(t)}$ for all $\tau \leqslant t$ and completes the proof.

□

For evolution processes which are pullback strongly bounded, the following result gives sufficient conditions for pullback asymptotic compactness. This result is a generalization of Theorem 2.2.3 (page 30).

Theorem 3.2.5 *Let $\{S(t,s) : t \geqslant s\}$ be a pullback strongly bounded process such that $S(t,s) = T(t,s) + U(t,s)$, where $U(t,s)$ is compact and there exists a non-increasing function*

$$k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

with $k(\sigma,r) \to 0$ when $\sigma \to \infty$, and for all $s \leqslant t$ and $x \in X$ with $\|x\| \leqslant r$, $\|T(t,s)x\| \leqslant k(t-s,r)$. Then, the process $\{S(t,s) : t \geqslant s\}$ is pullback asymptotically compact.

Proof: Let $\{x_n\}_{n=1}^{\infty} \subset B$ with $B \subset X$ bounded and $s_n \in \mathbb{R}$ with $s_n \to -\infty$. We denote

$$B_t = \bigcup_{s\leqslant t} \bigcup_{\tau\leqslant s} S(s,\tau)B, \tag{3.4}$$

where $r > 0$ is such that $\forall x \in B_t$, $\|x\| \leqslant r$ (observe that B_t is a bounded set, so there exists $r > 0$ such that $B_t \subset B(0,r) = B_r$). We define the sets

$$J_j = \{S(t,s_n)x_n : n \geqslant j\}.$$

For each x_n we can write

$$S(t,s_n)x_n = S(t,\tau_n)S(\tau_n,s_n)x_n \subset S(t,\tau_n)B_r,$$

with $\tau_n = \frac{t-s_n}{2}$.

Let s_0 be as in Definition 3.1.6. Then, for all $j \in \mathbb{N}$ such that $\tau_j \leqslant s_0$ and using the Kuratowski measure of non-compactness of Definition 2.1.13, we have

$$\begin{aligned}\alpha(J_j) &\leqslant \alpha(\{T(t,\tau_n)B_r + U(t,\tau_n)B_r : n \geqslant j\})\\ &\leqslant \alpha(\{T(t,\tau_n)B_r : n \geqslant j\})\\ &\leqslant k(t-\tau_j, r) \to 0 \text{ when } j \to \infty.\end{aligned}$$

Since $\alpha(J_1) = \alpha(J_j)$ we conclude that $\{S(t,s_n)x_n\}$ is a precompact set in X.

□

3.2.3 Pullback ω-limit processes

We denote again the Kuratowski measure as $\gamma(\cdot)$. In [57], the authors give a generalization of the ω-limit semigroups in the framework of cocycles and random attractors. We give definitions and results written in our framework, generalizing the results in Section 2.2.

Definition 3.2.6 *A process $\{S(t,s) : t \geqslant s\}$ is* pullback ω-limit compact *if for every time t, bounded subset B and $\varepsilon > 0$ there exists a $s_0 = s_0(t,B,\varepsilon) \leqslant t$ such that*

$$\gamma\left(\bigcup_{s\leqslant s_0} S(t,s)B\right) \leqslant \varepsilon.$$

Definition 3.2.7 *We say that the process $\{S(t,s) : t \geqslant s\}$ in a Hilbert space X verifies the* pullback flattening property *if for any time t, bounded set B and $\varepsilon > 0$ there exists a time $s_1 = s_1(t,B,\varepsilon)$ such that:*

i) The following set

$$\bigcup_{s\leqslant s_1} S(t,s)B,$$

is bounded in X.

ii) There exists a finite dimensional space X_ε and a bounded projection $P_\varepsilon : X \to X_\varepsilon$ such that

$$\|(I-P_\varepsilon)\bigcup_{s\leqslant s_1} S(t,s)B\|_X < \varepsilon. \tag{3.5}$$

The following result is a deterministic version of what was in Kloeden and Langa [57] (see also [58]) and it will give us the analogous result as in Theorem 2.2.5 but for evolution processes.

Theorem 3.2.8 *Let suppose that $\{S(t,s) : t \geqslant s\}$ is a pullback bounded evolution process in a uniform convex Hilbert space X. Then, the following properties are equivalent:*

1. *The process verifies the pullback flattening property.*
2. *The process is pullback ω-limit compact.*
3. *The process is pullback strongly asymptotically compact.*

Proof:

• *1.⇒ 2.* Since we suppose that $\{S(t,s) : t \geqslant s\}$ is pullback bounded, we only need to verify (3.5). Let be B a bounded subset of X. Taking $s_1(t, B, \varepsilon\backslash 2)$ as in Definition 3.2.7, and using the properties of the Kuratowski measure and Lemma 2.1.14

$$\begin{aligned}
\gamma\left(\bigcup_{s\leqslant s_1} S(t,s)B\right) &= \gamma\left((P_\varepsilon + (I-P_\varepsilon))\bigcup_{s\leqslant s_1} S(t,s)B\right)\\
&\leqslant \gamma\left(P_\varepsilon \bigcup_{s\leqslant s_1} S(t,s)B\right) + \gamma\left((I-P_\varepsilon)\bigcup_{s\leqslant s_1} S(t,s)B\right)\\
&\leqslant \gamma(B(0,\varepsilon\backslash 2)) = \varepsilon.
\end{aligned}$$

• *2.⇒ 3.* Let be $\{x_n\}_{n=1}^{\infty} \subset B$ with $B \subset X$ bounded and $t > s_n \overset{n\to\infty}{\longrightarrow} -\infty$. Let us take a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $\varepsilon_k \overset{k\to\infty}{\longrightarrow} 0$. By the pullback ω-limit compactness, for each ε_k there exists a s_{n_k} such that

$$\gamma\left(\bigcup_{s\leqslant s_{n_k}} S(t,s)B\right) \leqslant \varepsilon_k.$$

Denoting

$$C_k(t) = \overline{\bigcup_{s\leqslant s_{n_k}} S(t,s)B},$$

$\gamma(C_k(t)) \overset{k\to\infty}{\longrightarrow} 0$. We can also suppose, without lost of generality, that $s_{n_{k+1}} < s_{n_k}$. Next, we define

$$F_k(t) = \{S(t, s_{n_j})x_{n_j} : j \geqslant k\},$$

and, by construction, $\overline{F_k(t)} \subset C_k(t)$, therefore $\gamma(\overline{F_k(t)}) \leqslant \gamma(C_k(t)) \overset{k\to\infty}{\longrightarrow} 0$, and $\overline{F_{k+1}} \subset \overline{F_k}$. By Lemma 2.1.15

$$\bigcap_{k\in\mathbb{N}} \overline{F_k},$$

is compact and non-empty. Then, there exists a convergent subsequence of $\{S(t,s_n)x_n\}_{n=1}^{\infty}$.

- *3.⇒ 1.* For any bounded set $B \subset X$, let us consider the set

$$C(t) = \bigcap_{k=1}^{\infty} C_k(t) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{s \leqslant s_{n_k}} S(t,s)B}.$$

It is clear that, a point $x \in C(t)$ if and only if there are a sequence $s_k \overset{k\to\infty}{\longrightarrow} -\infty$ and $x_k \in B$ such that $S(t,s_k)x_k \overset{k\to\infty}{\longrightarrow} x$. From the hypothesis of pullback asymptotically compact, $C(t)$ is non-empty. Let prove that is also compact. Now consider a sequence $y_k \in C(t)$, $k = 1, 2, \ldots$ Then, for each y_k, there exist s_k and $z_k \in S(t,s_k)B$ such that $\|y_k - z_k\|_X \leqslant k^{-1}$. By the pullback asymptotical compactness, there is a subsequence (which we denote as the original one) such that $S(t,s_k)z_k \overset{k\to\infty}{\longrightarrow} y$ for some $y \in X$. Therefore, $y_k \overset{k\to\infty}{\longrightarrow} y$ and $C(t)$ is compact.

Suppose that $C(t)$ does not pullback attract B, then there exists a $\delta > 0$ such that $\operatorname{dist}(S(t,s)B, C(t)) \geqslant \delta$. Therefore, we can find a sequence $s_k \overset{k\to\infty}{\longrightarrow} -\infty$ and $x_k \in B$ such that $\operatorname{dist}(S(t,s_k)x_k, C(t)) \geqslant \delta$. By hypothesis, there is a convergent subsequence (which we denote as the original one) such that $S(t,s_k)x_k \overset{k\to\infty}{\longrightarrow} x \in C(t)$. That is impossible, so $C(t)$ pullback attracts B. In particular, for each $\varepsilon > 0$ there exists a time $s_\varepsilon \leqslant t$ such that

$$\operatorname{dist}(S(t,s)B, C(t)) \leqslant \frac{\varepsilon}{4}, \quad \text{for all } s \leqslant s_\varepsilon,$$

in other words, $S(t,s)B \subset N(C(t), \varepsilon/4)$ (with $N(C(t), \varepsilon/4)$ defined as in Proposition 2.1.6, page 25). By compactness, there exists a finite set of points $\{x_1, \ldots, x_{n_\varepsilon}\}$ in X such that

$$C(t) \subset \bigcup_{i=1}^{n_\varepsilon} N\left(x_i, \frac{\varepsilon}{4}\right).$$

Then,

$$\bigcup_{s \leqslant s_\varepsilon} S(t,s)B \subset N(C(t), \varepsilon/4) \subset \bigcup_{i=1}^{n_\varepsilon} N\left(x_i, \frac{\varepsilon}{2}\right).$$

Taking $X_\varepsilon = \mathrm{span}[\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}_\varepsilon}]$, and by the uniform convexity of X, there exits a projection $P_\varepsilon : X \to X_\varepsilon$ such that $\|x - P_\varepsilon x\|_X = \operatorname{dist}(x, X_\varepsilon)$ for each $x \in X$. Then

$$\|(I - P_\varepsilon) \bigcup_{s \leqslant s_\varepsilon} S(t,s)B\|_X \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

□

The proof of the following theorem is a direct consequence of the previous theorem and Theorem 3.2.4.

Theorem 3.2.9 *Let suppose that* $\{S(t,s) : t \geqslant s\}$ *is a pullback bounded evolution process in a uniform convex space* X*, then the following conditions are equivalent*

i) There exists the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$*.*

ii) The process is pullback strongly bounded dissipative and pullback ω*-limit compact.*

iii) The process is pullback strongly bounded dissipative and pullback flattening.

3.2.4 Pullback point dissipativeness

We want to give a result that would enable us to conclude the existence of pullback attractors without having to prove pullback strong bounded dissipativeness but rather pullback strong point dissipativeness (see Hale [47], Raugel [76]). To that end, the notion of pullback asymptotic compactness and pullback dissipativeness associated to the elapsed time presented next are needed.

Definition 3.2.10 *We say that an evolution process* $\{S(t,s) : t \geqslant s\}$ *is* pullback strongly asymptotically compact *if for each* $t \in \mathbb{R}$*, each bounded sequence* $\{x_k : k \in \mathbb{N}\}$ *in* X*, any sequences* $\{s_k : k \in \mathbb{N}\}$*,* $\{\tau_k : k \in \mathbb{N}\}$ *with* $t \geqslant \tau_k \geqslant s_k$ *and* $\tau_k - s_k \stackrel{k\to\infty}{\longrightarrow} \infty$*, then* $\{S(\tau_k, s_k)x_k : k \in \mathbb{N}\}$ *is relatively compact. The process* $\{S(t,s) : t \geqslant s\}$ *is called* strongly compact *if for each time* t *and* $B \subset X$ *bounded there exist a* $T_B \geqslant 0$ *and a compact set* $K \subset X$ *such that* $S(\tau,s)B \subset K$ *for all* $s \leqslant \tau \leqslant t$ *with* $\tau - s \geqslant T_B$*.*

Remark 5 *If* $\{S(t) : t \geqslant 0\}$ *is a semigroup,* $\{S(t-s) : t \geqslant s\}$ *is* pullback strongly asymptotically compact *if and only if* $\{S(t-s) : t \geqslant s\}$ *is* pullback asymptotically compact *if and only if* $\{S(t) : t \geqslant 0\}$ *is* asymptotically compact *in the sense of Definition 2.1.7.*

Definition 3.2.11 *Let* $\{S(t,s) : t \geqslant s\}$ *be an evolution process in a metric space* X*. We say that a bounded set* $B(t)$ *of* X pullback strongly absorbs points (compact subsets) of X at time t *if, for each* $x \in X$ *(compact subset* K *of* X*), there exists* $\sigma_x > 0$ $(\sigma_K > 0)$ *such that* $S(\tau,s)x \in B(t)$ $(S(\tau,s)K \subset$

$B(t)$) for all $s \leqslant \tau \leqslant t$ with $\tau - s \geqslant \sigma_x$ ($\tau - s \geqslant \sigma_K$). We say that $\{S(t,s) : t \geqslant s\}$ is pullback strongly point dissipative (compact dissipative) *if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t) \subset X$ which pullback strongly absorbs points (compact subsets) of X at time t.*

Remark 6 *If a set $B(t)$ pullback strongly absorbs points (compact subsets/-bounded subsets) of X at t, then it pullback strongly absorbs points (compact subsets/bounded subsets) of X at τ for all $\tau \leqslant t$. Also, if $\{T(t) : t \geqslant 0\}$ is a semigroup then $\{S(t-s,0) : t \geqslant s\}$ is pullback strongly point dissipative (compact dissipative) if and only if $\{S(t-s) : t \geqslant s\}$ is pullback point dissipative (compact dissipative) if and only if $\{T(t) : t \geqslant 0\}$ is point dissipative (compact dissipative) in the sense of Definition 2.1.5.*

With these concepts we can prove the following result on existence of pullback attractors. We can see that, in the non-autonomous case, this result is more complicated and we need more hypotheses on the process that are automatically satisfied for semigroups.

Theorem 3.2.12 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process with the property that, for each $t \in \mathbb{R}$ and $\tau > 0$, $\{S(s, s-\tau) : s \leqslant t\}$, is equicontinuous at x for each $x \in X$. If $\{S(t,s) : t \geqslant s\}$ is pullback strongly point dissipative, pullback strongly bounded and pullback strongly asymptotically compact, then $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative. Consequently, $\{S(t,s) : t \geqslant s\}$ has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ with the property that $\bigcup_{s \leqslant t} \mathscr{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

To prove the theorem, we firstly need some auxiliary results which prove that, under certain smoothing properties of the evolution processes, strong pullback point dissipativeness implies strong pullback bounded dissipativeness. The following lemma plays an important role in that procedure.

Lemma 3.2.13 *Let $\{S(t,s) : t \geqslant s\}$ be a pullback strongly point dissipative, pullback asymptotically compact and pullback strongly bounded evolution process. If, for each $t \in \mathbb{R}$ and $\sigma > 0$, the family $\{S(\tau, \tau - \sigma) : \tau \leqslant t\}$ is equicontinuous at each $x \in X$, then $\{S(t,s) : t \geqslant s\}$ is pullback strongly compact dissipative.*

Proof: Fix $t \in \mathbb{R}$ and let $B(t)$ be a bounded subset of X which strongly absorbs points of X at time t.

For $\tau \leqslant t$, let $B^1(t) = \{x \in X : \mathrm{dist}(x,y) < 1 \text{ for some } y \in B(t)\}$ and $C(\tau) = \bigcup_{s \leqslant \tau} S(\tau, s)\overline{B^1(t)}$. Then $C(\tau)$ is a bounded subset of X which strongly absorbs points of X at time τ. Indeed, given $x \in X$, let σ_x be such that $S(r,s)x \in B(t)$, $s+\sigma_x \leqslant r \leqslant \tau \leqslant t$. Then, since $B(t) \subset C(\tau)$, it follows that $C(\tau)$ pullback strongly absorbs points at time τ.

Due to the equicontinuity of the process, if K is a compact subset of X and $x \in K$ there are $\nu_x \in \mathbb{N}$ and $\epsilon_x > 0$ such that $S(r, r-\nu_x)N_{\epsilon_x}(x) \subset B^1(t)$, for all $r \leqslant \tau$. It follows that $S(\tau, r-\nu_x)N_{\epsilon_x}(x) \subset C(\tau)$ for all $r \leqslant \tau$. Since K is compact there is a $p \in \mathbb{N}^*$ and $x_1, \cdots, x_p$ in K such that $K \subset \cup_{i=1}^{p} N_{\epsilon_{x_i}}(x_i)$ and for $\sigma_K = \max\{\sigma_{x_i} : 1 \leqslant i \leqslant p\}$, $S(\tau, r-\sigma_K)K \subset C(\tau)$ for all $r \leqslant \tau$. Then, from the fact that $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded, it follows that $\bigcup_{\tau \leqslant t} C(\tau)$ is bounded and pullback strongly absorbs compact subsets of X at time t.

□

Theorem 3.2.14 *If a process $\{S(t,s) : t \geqslant s\}$ is pullback strongly compact dissipative and pullback strongly asymptotically compact, then $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative.*

Proof: Due to fact that $\{S(t,s) : t \geqslant s\}$ is pullback strongly compact dissipative, there is a closed and bounded set $B(t)$ which pullback strongly absorbs compact subsets of X at time t. First, we prove that, for each bounded subset D of X, $\omega(D,\tau) \subset B(t)$ for each $\tau \leqslant t$. If $y \in \omega(D,\tau)$, there is a sequence $\{s_k : k \in \mathbb{N}\}$ with $s_k \leqslant \tau$ and $s_k \overset{k\to\infty}{\longrightarrow} -\infty$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D$ such that $\mathrm{dist}(S(\tau,s_k)x_k, \omega(D,\tau)) \overset{k\to\infty}{\longrightarrow} 0$. Taking $\{r_k : k \in \mathbb{N}\}$ with $\tau \geqslant r_k \geqslant s_k$ and $\min\{\tau - r_k, r_k - s_k\} \overset{k\to\infty}{\longrightarrow} \infty$ and using the fact that $\{S(t,s) : t \geqslant s\}$ is pullback strongly asymptotically compact (taking subsequences if necessary), there is a $z \in X$ such that $z_k := S(r_k, s_k)x_k \overset{k\to\infty}{\longrightarrow} z$. From the compactness of the set $K = \{z_k : k \in \mathbb{N}\} \cup \{z\}$, there is a $\sigma_K \in \mathbb{N}$ such that $S(\tau, r_k)K \subset B(t)$ whenever $\tau - r_k \geqslant \sigma_K$. Thus, for all suitably large k,

$$S(\tau, s_k)x_k = S(\tau, r_k)S(r_k, s_k)x_k \subset S(\tau, r_k)K \subset B(t).$$

This completes the proof that $\omega(D,\tau) \subset B(t)$ for each $\tau \leqslant t$.

Since $\omega(D,\tau)$ pullback attracts D at time τ, it follows that $B(t)$ pullback attracts bounded subsets of X at time τ for each $\tau \leqslant t$; that is, $\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative.

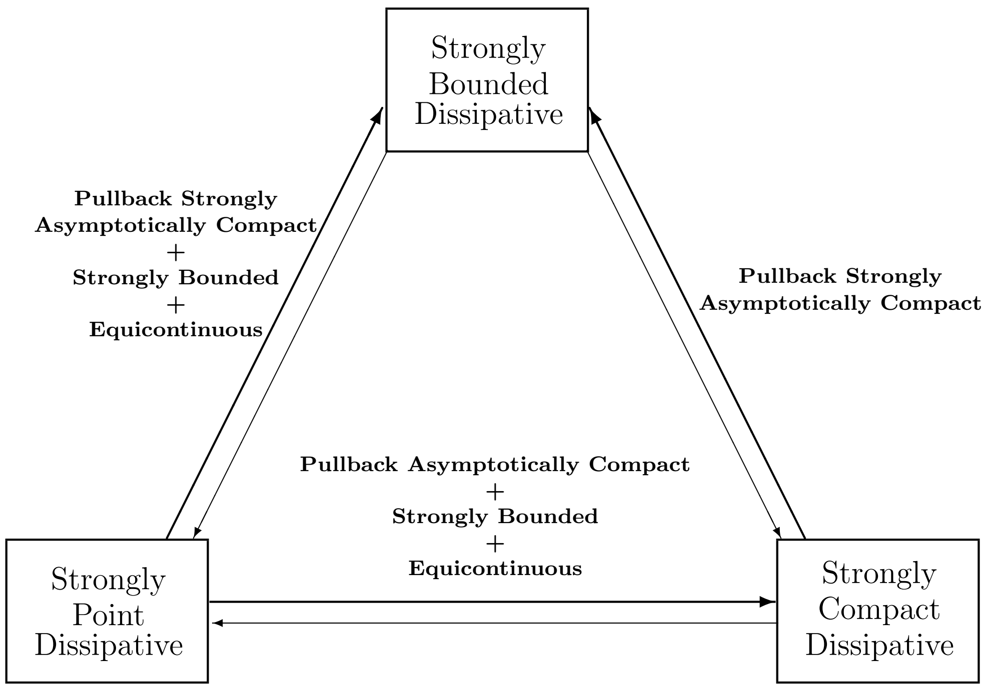
□

As an immediate consequence of Theorem 3.2.4 we have that

Theorem 3.2.15 *If a process $\{S(t,s) : t \geqslant s\}$ is pullback strongly compact dissipative and pullback strongly asymptotically compact, then $\{S(t,s) : t \geqslant s\}$ has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ with the property that $\bigcup_{s \leqslant t} \mathscr{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

The proof of Theorem 3.2.12 is now a direct application of Lemma 3.2.13 and Theorem 3.2.15.

The next diagram, as in the autonomous case, shows an scheme on the relation between the different kinds of pullback dissipation for an evolution process and hypotheses to get one from the others.



In applications, to prove the pullback strongly asymptotic compactness property we use the following result, which is the analogous to Theorem 3.2.5. In this case we need a strongly assumption on the compactness of one of the processes of the sum. The proof is also analogous.

Theorem 3.2.16 *Let $\{S(t,s) : t \geqslant s\}$ be a pullback strongly bounded process such that $S(t,s) = T(t,s) + U(t,s)$, where $U(t,s)$ is strongly compact and there exists a non-increasing function*

$$k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

with $k(\sigma, r) \to 0$ when $\sigma \to \infty$, and for all $s \leqslant t$ and $x \in X$ with $\|x\| \leqslant r$, $\|T(t,s)x\| \leqslant k(t-s,r)$. Then, the process $\{S(t,s) : t \geqslant s\}$ is pullback strongly asymptotically compact.

Proof: Let $\{x_n\} \subset B$ with $B \subset X$ bounded and $t_n, s_n \in \mathbb{R}$ with $t_n - s_n \to \infty$. We denote

$$B_t = \bigcup_{\tau \leqslant t} \bigcup_{s \leqslant \tau} S(\tau, s)B, \tag{3.6}$$

where $r > 0$ is such that $\forall x \in B_t$, $\|x\| \leqslant r$. We define the sets

$$J_j = \{S(t_n, s_n)x_n : n \geqslant j\}.$$

For each x_n we can write

$$S(t_n, s_n)x_n = S(t_n, \tau_n)S(\tau_n, s_n)x_n \subset S(t_n \tau_n)B_r,$$

with $\tau_n = \frac{t_n - s_n}{2}$.

Let T_B be as in Definition 3.2.10. Then, for all $j \in \mathbb{N}$ such that $t_j - \tau_j \geqslant T_B$ we have

$$\begin{aligned}\alpha(J_j) &\leqslant \alpha(\{T(t_n,\tau_n)B_r + U(t_n,\tau_n)B_r : n \geqslant j\}) \\ &\leqslant \alpha(\{T(t_n,\tau_n)B_r : n \geqslant j\}) \\ &\leqslant k(t_j - \tau_j, r) \to 0 \text{ when } j \to \infty.\end{aligned}$$

Since $\alpha(J_1) = \alpha(J_j)$ we conclude that $\{S(t_n, s_n)x_n\}$ is a precompact set in X.

□

3.3 Basis of attraction

All the previous results are based on families of bounded subsets in the phase space X. In our case, the pullback attractor defined in Definition 3.2.10 could be an unbounded subset (see [66]). On the other hand, it is very common in applications that a pullback attractor attracts more general classes of sets, including time-dependent bounded families. Thus, it is natural to obtain pullback attractors in the same class of their associated basis of attraction. In this section we will show general definitions and an existence result for generalized basis of attraction (see, for example, [7, 15, 16, 26, 19]).

In what follows we consider the family $\mathcal{M}$ consisting of all maps from $\mathbb{R}$ to 2^X

$$\tilde{D} : \mathbb{R} \ni t \to \tilde{D}(t) \in 2^X.$$

If $\tilde{D}$ and $\tilde{D}'$ are maps from $\mathbb{R}$ into 2^X, by the inclusion $\tilde{D}' \subset \tilde{D}$ we mean that $\tilde{D}'(t) \subset \tilde{D}(t)$ for all $t \in \mathbb{R}$. A subset $\mathcal{D}$ of $\mathcal{M}$ is said *inclusion closed* if whenever $\tilde{D} \in \mathcal{D}$ and $\tilde{D}' \in \mathcal{M}$ is such that $\tilde{D}' \subset \tilde{D}$, then $\tilde{D}' \in \mathcal{D}$.

Definition 3.3.1 *A subset $\mathcal{D} \subset \mathcal{M}$ which is inclusion closed and such that all maps $\tilde{D}'$ in $\mathcal{D}$ satisfy $\tilde{D}(t) \neq \emptyset$ for all $t \in \mathbb{R}$ is called a* basis of attraction or universe.

The simplest example of this kind of families of maps is given by $D(t) \equiv D$ where D is a bounded subset of X.

Other example very common in the case of random dynamical systems (see [37, 46, 80]) is provided by the *tempered sets* $\mathcal{D}$ consisting of sets such that posses sub-exponential growth with respect to time, that is

$$\sup_{x \in \tilde{D}(t)} \|x\| \leqslant g(t)e^{-t},$$

for some positive function $g(s)$ such that $g(s)e^{-s} \overset{|s|\to\infty}{\longrightarrow} 0$.

Definition 3.3.2 *Let $\mathcal{D}$ be a basis of attraction. Given $t \in \mathbb{R}$, $B(t) \subset X$ is said to be* pullback $\mathcal{D}$-absorbing at time t *if, for every $D \in \mathcal{D}$ there exists $s_0 = s_0(t, D) \leqslant t$ such that*

$$S(t,s)D(s) \subset B(t), \quad \textit{for all } s \leqslant s_0.$$

A family $\{B(t) : t \in \mathbb{R}\}$ is called pullback $\mathcal{D}$-absorbing *if $B(t)$ is pullback $\mathcal{D}$-absorbing at time t for all $t \in \mathbb{R}$.*

The previous definition is a generalization of Definition 3.1.6, so we can as well define a *pullback $\mathcal{D}$-dissipative process* or the concept of *pullback $\mathcal{D}$-attraction.*

Definition 3.3.3 *The pullback $\mathcal{D}$-attractor is a family of compact sets $\{\mathscr{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ satisfying the following properties*

1. *it is invariant, that is $S(t,s)\mathscr{A}_{\mathcal{D}}(s) = \mathscr{A}_{\mathcal{D}}(t)$ for all $s \leqslant t$,*

2. *it pullback $\mathcal{D}$-attracts every element of $\mathcal{D}$, that is*

$$\lim_{s\to-\infty} dist(S(t,s)D(s), \mathscr{A}_{\mathcal{D}}(t)) = 0, \quad \textit{for all } t \in \mathbb{R},$$

3. it is minimal in the sense of Definition 3.2.10.

Before giving the analogous result of Theorem 3.2.4, we need to define the concepts of ω-limit set and pullback strongly asymptotically compact process for this framework.

Definition 3.3.4 *The* pullback ω-limit set *of a family $D \in \mathcal{D}$ is defined by*

$$\omega(D,t) := \bigcap_{\sigma \leqslant t} \overline{\bigcup_{s \leqslant \sigma} S(t,s)D(s)}. \tag{3.7}$$

Definition 3.3.5 *The process $\{S(t,s) : t \geqslant s\}$ is called* $\mathcal{D}$-pullback strongly asymptotically compact *if for each $t \in \mathbb{R}$, sequences $\{s_n\}_{n=1}^{\infty}$ with $s_n \overset{n\to\infty}{\longrightarrow} -\infty$ and $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in D(s_n)$ for some $D \in \mathcal{D}$, then $\{S(t,s_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence.*

We can now establish a result ensuring the existence of pullback $\mathcal{D}$-attractors.

Theorem 3.3.6 *If the process $\{S(t,s) : t \geqslant s\}$ is pullback $\mathcal{D}$-asymptotically compact and pullback $\mathcal{D}$-dissipative with $\tilde{B} = \{B(t) : t \in \mathbb{R}\}$ its pullback $\mathcal{D}$-absorbing family, then there exists the pullback $\mathcal{D}$-attractor and is given by*

$$\{\mathscr{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} = \{\omega(\tilde{B},t) : t \in \mathbb{R}\}.$$

Remark 7 *Anguiano* et. al *in [7], give a nice example of existence of pullback attractor for a reaction-diffusion equation in an unbounded domain. Let us consider the following equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t), \;\; \textit{in } \Omega \times (\tau,+\infty), \\ u = 0, \quad \textit{on} \;\; \partial\Omega \times (\tau,+\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{3.8}$$

where $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and the domain Ω is not necessarily bounded but satisfies the Poincaré inequality, that is there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |\phi(x)|^2 dx \leqslant \lambda_1^{-1} \int_\Omega |\nabla\phi(x)|^2 dx.$$

They consider the set $\mathcal{R}_{\lambda_1}$ of all functions $r : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D}=\{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$, for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$, where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$. In this case the pullback $\mathcal{D}$-attractor belongs to $\mathcal{D}_{\lambda_1}$, which is the basis of attraction.

The natural following question is the relationship between the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ defined in Section 3.1 and the pullback $\mathcal{D}$-attractor $\{\mathscr{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$. If we suppose that the basis of attraction $\mathcal{D}$ includes all bounded subsets of the phase space X, we have that, thanks to the minimality of the pullback attractor, $\mathscr{A}(t) \subset \mathscr{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$. Marín-Rubio and Real give in [71] the following result to obtain the equality.

Theorem 3.3.7 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process which is pullback $\mathcal{D}$-asymptotically compact and pullback strongly $\mathcal{D}$-dissipative, that is, there exists a family $\tilde{B} = \{B(t) : t \in \mathbb{R}\}$ of bounded subsets which is pullback $\mathcal{D}$-absorbing for each $\tau \leqslant t$ (given $\tilde{D} \in \mathcal{D}$ and $\tau \leqslant t$, there exists a time $s_0 = s_0(\tau, \tilde{D})$ such that $S(\tau, s)D(s) \subset B(t)$ for all $s \leqslant s_0$). If $\mathcal{D}$ includes bounded subsets of X, then the pullback $\mathcal{D}$-attractor $\{\mathscr{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ coincides with the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ of bounded subsets of X.*

3.4 Upper and lower-semicontinuity

In this section we analyze the upper and lower-semicontinuity of the pullback attractors. This is an important part of the theory because it ensures that the concept of pullback attractor is a robust concept, that does not explodes under some small perturbations. The lower-semicontinuity results only have sense when the attractor possesses some kind of structure. For this reason we need to study the structure of the pullback attractor in Section 3.7 and improve the concept of uniform attractor of Chepyzhov and Vishik in [33].

First of all, we need to define these two concepts uniformly in time.

Definition 3.4.1 *Let $\{S_\eta(t,s) : t \geqslant s\}_{\eta \in [0,1]}$ be a family of evolution processes in a Banach space X with corresponding pullback attractors $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta \in [0,1]}$. For any bounded interval $I \subset \mathbb{R}$, we say that*

1. *$\mathscr{A}_\eta(\cdot)$ is upper-semicontinuous for $t \in I$ if*

$$\lim_{\eta \to 0} \sup_{t \in I} dist(\mathscr{A}_\eta(t), \mathscr{A}_0(t)) = 0,$$

2. *$\mathscr{A}_\eta(\cdot)$ is lower-semicontinuous for $t \in I$ if*

$$\lim_{\eta \to 0} \sup_{t \in I} dist(\mathscr{A}_0(t), \mathscr{A}_\eta(t)) = 0,$$

3. *$\mathscr{A}_\eta(\cdot)$ is continuous for $t \in I$ if it is upper and lower-semicontinuous for $t \in I$ when $\eta \to 0$.*

As in the purely autonomous setting, it is relatively easy to show that pullback attractors cannot explode under perturbation (upper semicontinuity), and such results are already well known in the non-autonomous setting [17, 18]. It is much harder to show that attractors are lower semicontinuous, i.e. that there is no implosion, but this is true under certain additional conditions.

There are two main strategies to obtain the continuity of pullback attractors. The most common one is to make detailed assumptions on the structure of the 'unperturbed' attractor of an autonomous systems where the perturbation is a non-autonomous small one ([27, 66]). Essentially, if the limit global attractor is given by the union of the unstable manifolds of a finite number of equilibria, and these equilibria and their associated unstable manifolds behave continuously under non-autonomous perturbation, one can prove that the attractors also behave continuously. On the other hand, Babin and Vishik in [9] (see also Kloeden and Li [41] and Kloeden and Rasmussen [58]) proved that a uniform exponential rate of attraction for the perturbed attractors is a sufficient condition for their continuity.

The following theorems in [24] give conditions to ensure the continuity of the pullback attractors. We assume that

- for each $t \in \mathbb{R}$, for each compact subset $K \subset X$ and each $T > 0$,

$$\sup_{s\in[0,T]} \sup_{u\in K} \operatorname{dist}(S_\eta(t,t-s)u, S_0(t,t-s)u) \xrightarrow{\eta\to 0} 0 \tag{3.9}$$

- for a given $\tau \in \mathbb{R}$

$$\bigcup_{\eta\in[0,1]} \bigcup_{t\leqslant\tau} \mathscr{A}_\eta(t) \text{ is bounded} \tag{3.10}$$

- for each $t \in \mathbb{R}$

$$\overline{\bigcup_{\eta\in[0,1]} \mathscr{A}_\eta(t)} \text{ is compact.} \tag{3.11}$$

Then we have the following theorem about the upper-semicontinuity

Theorem 3.4.2 *Under conditions* (3.9), (3.10) *and* (3.11), *the family of attractors* $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,1]}$ *is upper-semicontinuous for each* $t \in \mathbb{R}$ *when* $\eta \to 0$. *Moreover, for each* $I \subset \mathbb{R}$ *bounded*

$$\sup_{t\in I} \operatorname{dist}(\mathscr{A}_\eta(t), \mathscr{A}_0(t)) \xrightarrow{\eta\to 0} 0.$$

As in the autonomous case, the lower-semicontinuity is based on the continuity of the unstable manifolds of the global hyperbolic solutions (see [23, 66]).

Theorem 3.4.3 *In addition to the assumptions in Theorem 3.4.2, assume that*

- *there is a sequence of backward bounded global solutions* $\{\xi_n\}_{n=1}^{\infty}$ *of the limit process* $\{S_0(t,s) : s \leqslant t\}$ *such that*

$$\mathscr{A}_0(t) = \overline{\bigcup_{n=1}^{\infty} W^u(\xi_n)(t)},$$

- *for each* $\eta \in (0,1]$ *there exists a sequence of backward bounded global solutions* $\{\xi_{\eta,n}\}_{n=1}^{\infty}$ *of* $\{S_\eta(t,s) : s \leqslant t\}$ *and* $t_n \in \mathbb{R}$ *such that*

$$\sup_{t \leqslant t_n} \operatorname{dist}(\xi_{\eta,n}(t), \xi_n(t)) \stackrel{\eta \to 0}{\longrightarrow} 0,$$

- *the local unstable manifold of* $\xi_{\eta,n}$ *behaves continuously as* $\eta \to 0$*; that is, for each* $\eta \in (0,1]$ *there are* $t_n \in \mathbb{R}$ *such that*

$$\sup_{t \leqslant t_n} \operatorname{dist_H}(W^u_{loc}(\xi_{\eta,n})(t), W^u_{loc}(\xi_n)(t)) \stackrel{\eta \to 0}{\longrightarrow} 0.$$

Then, the family $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta \in [0,1]}$ *is upper and lower semicontinuous for any bounded interval* $I \subset \mathbb{R}$.

As in the autonomous case (see [23]), we have some useful results about the continuity of the unstable sets of the global hyperbolic solutions. Results in [24] show the case when we have a non-autonomous perturbation of a non-autonomous external force. First of all we want to show how the concept of hyperbolic equilibrium point in the autonomous case turns to the concept of hyperbolic bounded global solution for evolution processes. This concept is related to the existence of two projections in a neighborhood of a global solution where the process shows a behaviour analogous to (2.12) in Section 2.5. The following definition can be found in [50].

Definition 3.4.4 *We say that the linear evolution process* $\{L(t,\tau) : t \geqslant \tau\}$ *in a Banach Space* X *has an* exponential dichotomy with exponent ω and constant M *if there is a family of bounded linear projections* $\{Q(t) : \ t \in \mathbb{R}\}$ *in* X *such that*

1. *$Q(t)L(t,s) = L(t,s)Q(s)$, for all $t \geqslant s$.*
2. *The restriction $L(t,s)|_{R(Q(s))}$, $t \geqslant s$ is an isomorphism from the rank of $Q(s)$ into the rank of $Q(t)$; we denote its inverse by $L(s,t) : R(Q(t)) \to R(Q(s))$.*
3. *There are constants $\omega > 0$ and $M \geqslant 1$ such that*

$$\begin{aligned} \|L(t,s)(I - Q(s))\|_{L(X)} &\leqslant Me^{-\omega(t-s)} \quad t \geqslant s \\ \|L(t,s)Q(s)\|_{L(X)} &\leqslant Me^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{3.12}$$

Definition 3.4.5 *Let $u_t + A(t)u = F(t,u)$ be a general non-autonomous system in a Banach space X with $\{S(t,s) : t \geqslant s\}$ the process associated with its unique solution. We say that a global solution $\xi : \mathbb{R} \to X$ is* hyperbolic *if the process $\{L_\xi(t,\tau) : t \geqslant \tau\}$ corresponding with the linearization around it has an exponential dichotomy.*

Let $A : D(A) \subset X \to X$ be the generator of a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ and $\eta \in [0,1]$. Consider the following family of semilinear problems

$$\begin{cases} u_t = Au + f_\eta(t,u), \\ u(s) = u_0. \end{cases} \tag{3.13}$$

Assume that $f_\eta : \mathbb{R} \times X \to X$ are continuous, Lipschitz continuous with respect to the second variable, uniformly on bounded subsets of X and satisfies

$$\begin{aligned} \lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r)} \{&\|f_\eta(t,z) - f_0(t,z)\|_X + \\ &\|(f_\eta)_z(t,z) - (f_0)_z(t,z)\|_{\mathcal{L}(X)}\} = 0, \end{aligned} \tag{3.14}$$

for all $r > 0$, where $(f_\eta)_z(t,z) \in \mathcal{L}(X)$ denotes the derivative of f_η with respect to the space variable at point (t,z).

If $\xi_\eta : \mathbb{R} \to X$ is a global hyperbolic solution of (3.13) with projections $\{Q_\eta(t) : t \in \mathbb{R}\}$, and $z(t)$ is a solution of

$$\begin{cases} z_t = (A_0(t) + B_\eta(t))\, z + h_\eta(t,z), \\ z(s) = z_0, \end{cases} \tag{3.15}$$

where $A_0(t) = A + f_0'(t,\xi_0(t))$, $B_\eta(t) = f_\eta'(t,\xi_\eta(t)) - f_0'(t,\xi_0(t)) \in \mathcal{L}(X)$ and $h_\eta(t,z) = f_\eta(t,\xi_\eta + z) - f_\eta(t,\xi_\eta) - f_\eta'(t,\xi_\eta)z$, we can write $z = z^+ + z^-$ where $z^+ = Q_\eta(t)z$ and $z^- = z - z^+$, which verify

$$\begin{aligned} z_t^+ &= A_\eta^+(t)z^+ + G^+(t,z^+,z^-) \\ z_t^- &= A_\eta^-(t)z^- + G^-(t,z^+,z^-), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
A_\eta^+(t) &= [A_0(t) + B_\eta(t)]Q_\eta(t),\\
A_\eta^-(t) &= [A_0(t) + B_\eta(t)](I - Q_\eta(t)),\\
G^+(t, z^+, z^-) &= Q_\eta(t)H(z^+ + z^-),\\
G^-(t, z^+, z^-) &= (I - Q_\eta(t))H(z^+ + z^-).
\end{aligned}$$

Then, we have the following result (see Theorem 2.2 in [24]), which guarantees the continuity of local stables and unstable manifolds of hyperbolic global solutions.

Theorem 3.4.6 *There exists a function $\Sigma_\eta : \mathbb{R} \times X \to X$ such that the unstable manifold of the equilibrium point $(z^+, z^-) \equiv (0, 0)$ for (5.54) is given by*

$$W_\eta^u(0, 0) = \{(\tau, z) \in \mathbb{R} \times X : z = (Q_\eta(\tau)z, \Sigma_\eta(\tau, Q_\eta(\tau)z))\}.$$

Moreover we have that for any $r > 0$,

$$\sup_{t\leqslant\tau} \sup_{z\in B_X(0,r)} \{\|Q_\eta(t)z - Q_0(t)z\|_X + \|\Sigma_\eta(t, Q_\eta(t)z) - \Sigma_0(t, Q_0(t)z)\|_X\} \overset{\eta\to 0}{\longrightarrow} 0.$$

3.5 Exponential pullback attraction

The ratio of attraction, or how fast attracts, of a pullback attractor is a useful element in computational calculus. But the definitions and existence results do not give information about this ratio. Sometimes, in computational simulations, is better to find a family of sets which pullback attracts exponentially fast which contains the pullback attractor. In [63] (following the ideas in [43]), we can find the following definition

Definition 3.5.1 *A family $\{\mathscr{M}(t) : t \in \mathbb{R}\}$ in X is called a* pullback exponential attractor *if it verifies the following properties*

i) *$\mathscr{M}(t)$ is compact for each $t \in \mathbb{R}$.*

ii) *The family is positively invariant, that is $S(t, s)\mathscr{M}(s) \subseteq \mathscr{M}(t)$ for all $s \leqslant t$.*

iii) *The family has a exponential ratio of attraction, i.e., there exist constants $c, \alpha > 0$ such that for any bounded $B \subset X$,*

$$dist(S(t, s)B, \mathscr{M}(t)) \leqslant ce^{-\alpha(t-s)}.$$

In this case we loose the invariant property in order to obtain a better ratio of attraction. However, denoting the pullback attractor as $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, $\mathscr{A}(t) \subset \mathscr{M}(t)$. In [63], the authors show a construction for the exponential pullback attractor, both in discrete and continuos case, giving a finite bound for its dimension. They also apply this the results to the following 2D Navier-Stokes equation

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) & \text{in } (\tau,\infty)\times\Omega, \\ \nabla\cdot u = 0 & \text{in } (\tau,\infty)\times\Omega, \\ u = 0 & \text{on } (\tau,\infty)\times\Gamma, \\ u(\tau) = u_0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded subset with regular boundary Γ and the external force f satisfies that $f \in L^2_{loc}(\mathbb{R}; H)$ (H is the closure of the usual phase space $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \text{div } u = 0\}$ in $(L^2(\Omega))^2$) and $\sup_{r<t}\int_{r-1}^{r} |f(s)|^2 ds < \infty$ for all $t \in \mathbb{R}$. The second condition ensures the finite dimension of the exponential attractor and, due to $\mathscr{A}(t) \subset \mathscr{M}(t)$, the finite dimension of the pullback attractor.

3.6 Dynamic inside the pullback attractor

As in the autonomous case (Theorem 2.4.1 in Section 2.5), results in [62] prove that for each trajectory of $\{S(t,s) : t \geqslant s\}$, another similar one can be found inside the pullback attractor that tracks the original one. The following theorem, published in [77], gives an analogous result based on forward attracting families for evolution processes.

Theorem 3.6.1 *Suppose that the process $\{S(t,s) : t \geqslant s\}$ is Lipschitz in X, that is*

$$\sup_{s\in\mathbb{R}} \|S(t+s,s)u - S(t+s,s)v\|_X \leqslant \kappa(t)\|u-v\|_X, \tag{3.17}$$

with $\kappa : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ bounded in compact subsets and u, v in any bounded subset $B \in X$. Suppose also that there exists a family of compact sets $\{A(t) : t \in \mathbb{R}\}$ that forward attracts bounded sets and is positively invariant under $\{S(t,s) : t \geqslant s\}$. Then, for each trajectory $u(t,s) \in X$ of $\{S(t,s) : t \geqslant s\}$ and positive sequences $\{\varepsilon_n\}_{n=0}^\infty$ and $\{T_n\}_{n=0}^\infty$ with $\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$, $T_n < T_{n+1}$ and $T_n \overset{n\to\infty}{\longrightarrow} \infty$, there exists a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and $v_n \in A(t_n + s)$ such that

$$\sup_{t\in[0,T_n]} \|u(t+t_n+s,s) - S(t+t_n+s, t_n+s)v_n\|_X \leqslant \varepsilon_n. \tag{3.18}$$

Moreover, the 'jumps' $\|v_{n+1} - S(T_n+t_n+s, t_n+s)v_n\|_X$ decrease to zero.

Proof: By the forward attraction and the compactness of each set of the family $\{A(t) : t \in \mathbb{R}\}$, there exists a time $t_0 = t_0(\varepsilon_0, T_0)$ and a $v_0 \in A(t_0 + s)$ such that

$$\|S(t_0 + s, s)u(s) - v_0\|_X \leqslant \frac{\varepsilon_0}{\max_{t\in[0,T_0]} \kappa(t)}.$$

Hence, using (3.17) we have

$$\begin{aligned}
\|S(t+t_0 + s, s)u(s) &- S(t + t_0 + s, t_0 + s)v_0\|_X \\
&= \|S(t + t_0 + s, t_0 + s)S(t_0 + s, s)u(s) - S(t + t_0 + s, t_0 + s)v_0\|_X \\
&\leqslant \max_{t\in[0,T_0]} \kappa(t)\|S(t_0 + s, s)u(s) - v_0\|_X \\
&\leqslant \varepsilon_0 \text{ for all } t \in [0, T_0].
\end{aligned}$$

Now, for ε_1 and T_1 we can find a t_1 and a $v_1 \in A(t_1 + s)$ such that $t_0 < t_1$ and

$$\|S(t_1 + s, s)u(s) - v_1\|_X \leqslant \frac{\varepsilon_1}{\max_{t\in[0,T_1]} \kappa(t)},$$

therefore,

$$\|S(t + t_1 + s, s)u(s) - S(t + t_1 + s, t_1 + s)v_1\|_X \leqslant \varepsilon_1 \text{ for all } t \in [0, T_1].$$

In the same manner, we can see that for any ε_n and T_n there exist a time $t_{n-1} < t_n$ and a $v_n \in A(t_n + s)$ such that

$$\|S(t + t_n + s, s)u(s) - S(t + t_n + s, t_n + s)v_1\|_X \leqslant \varepsilon_n \text{ for all } t \in [0, T_n].$$

Finally, we have

$$\begin{aligned}
\|v_{n+1} - &S(T_n + t_n + s, t_n + s)v_n\|_X \\
&\leqslant \|v_{n+1} - S(T_n + t_n + s, t_n + s)u(t_n + s)\|_X \\
&+ \|S(T_n + t_n + s, t_n + s)u(t_n + s) - S(T_n + t_n + s, t_n + s)v_n\|_X \\
&\leqslant \varepsilon_{n+1} + \varepsilon_n,
\end{aligned}$$

which completes the proof.

□

Remark 8 *As t_n does not depend on the initial time, we can track $u(t, s)$ by trajectories in $\{A(t) : t \in \mathbb{R}\}$ of length T_n from $t_n + s$ to $t_{n+1} + s$ within a distance ε_n.*

3.7 Gradient-like processes

As in the framework of semigroups and global attractors, if an evolution process $\{S(t,s) : t \geqslant s\}$ has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ and $\xi : \mathbb{R} \to X$ is a global backwards bounded solution (the set $\bigcup_{s\leqslant t} \xi(s)$ is bounded for each $t \in \mathbb{R}$), then $\xi(t)$ belongs to $\mathscr{A}(t)$ for all $t \in \mathbb{R}$. If we require, in addition, that the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ has the property that $\cup_{s\leqslant\tau}\mathscr{A}(s)$ is bounded for each $\tau \in \mathbb{R}$, the requirement that the pullback attractor is minimal in Definition 3.1.4 can be dropped and

$$\mathscr{A}(t) = \{\xi(t) : \xi : \mathbb{R} \to X \text{ is a global backwards bounded solution of the evolution process } \{S(t,s) : t \geqslant s\}\}. \tag{3.19}$$

The inclusion $\{\xi(t) : t \in \mathbb{R}\} \subset \{\mathscr{A}(t) : t \in \mathbb{R}\}$ is always true. In fact, given $x \in \mathscr{A}(t)$, we always can construct a global solution such that $\xi(t) = x$ in the following way:

- If $\tau \geqslant t$, we define $\xi(\tau) = S(\tau,t)x$.
- If $\tau \leqslant t$, by (3.3) there exist sequences $\{x_n\}_{n=1}^{\infty}$ bounded and $\{s_n\}_{n=1}^{\infty}$ with $s_n n \to \infty \longrightarrow -\infty$ such that $S(t,s_n)x_n \overset{n\to\infty}{\longrightarrow} x$. Now for each $n \in \mathbb{N}$ we define $y_n = S(s_n, s_{n+1})x_{n+1}$. Then, we can define $\xi(\tau) = S(\tau, s_n)y_n$ with $\tau \in [s_n, s_{n-1}]$.

We need the backwards bound for the pullback attractor because, by (3.3), we need that $\{x_n\}_{n=1}^{\infty}$ be bounded, in this way we have the equivalence. This give us a preliminary structure for the pullback attractor, but it is not enough to ensure the conditions about the lower.semicontinuyty of the attractor. Our aim in this section is to give a generalization of the concept of gradient semigroup.

In [23] we can find an extension of the notion of dynamically $\mathcal{E}$-gradient semigroups (called gradient-like semigroups in that article) to processes (see also [26]). Let $\{S(t,s) : t \geqslant s\}$ be a nonlinear evolution process with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ which contains a finite set of isolated invariant families $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$.

Definition 3.7.1 *A homoclinic structure in* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *is a sequence* $\{\Xi_{\ell_i}^* : 1 \leqslant i \leqslant \mathfrak{p}\}$ *in* $\mathcal{S}$ *and a sequence of global solutions* $\{\xi_i : 1 \leqslant i \leqslant \mathfrak{p}\}$ *such that* $\xi_i(t) \overset{t\to-\infty}{\longrightarrow} \Xi_{\ell_i}^*(t)$, $\xi_i(t) \overset{t\to\infty}{\longrightarrow} \Xi_{\ell_{i+1}}(t)^*$ *and* $\Xi_{\ell_1}^*(\cdot) = \Xi_{\ell_{\mathfrak{p}+1}}^*(\cdot)$.

Definition 3.7.2 *Let $\{S(t,s) : t \geqslant s\}$ be an evolution process in a metric space X with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$. Suppose that there exists a set of isolated invariant families $\mathcal{S} = \{\Xi_i^* : \mathbb{R} \to X : 1 \leqslant i \leqslant n\}$. We say that $\{S(t,s) : t \geqslant s\}$ is a* generalized gradient-like process with respect to $\mathcal{S}$ *if the following two hypotheses are satisfied:*

(H1) For each global solution $\xi : \mathbb{R} \to X$ in $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ there are $1 \leqslant i, j \leqslant n$ such that

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t), \Xi_i^*(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}(\xi(t), \Xi_j^*(t)) = 0.$$

(H2) $\mathcal{S} = \{\Xi_1^*, \cdots, \Xi_n^*\}$ does not contain any homoclinic structure.

The following result is proved in [23]:

Theorem 3.7.3 *Let $\{S_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ be an evolution process in X with pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, $\eta \in [0,1]$. Assume that*

a) $\overline{\cup_{\eta\in[0,1]} \cup_{t\in\mathbb{R}} \mathscr{A}_\eta(t)}$ is compact.

b) $S_0(t,\tau) = T(t-\tau)$, $t \geqslant \tau$ and $\{T(t) : t \geqslant 0\}$ is a gradient semigroup with isolated invariant sets $\{\Xi_{1,0}^*, \cdots \Xi_{n,0}^*\}$.

c) For $\eta \in (0,1]$ there exists a family of sets $\mathcal{S}_\eta = \{\Xi_{1,\eta}^*(t), \ldots, \Xi_{n,\eta}^*(t)\}$, with $\Xi_{i,\eta}^*(\cdot) \subseteq \mathscr{A}(\cdot)$ with $i \in \{1, \ldots, n\}$ such that

$$\lim_{\eta\to 0} \left[\sup_{i\in\{1,\ldots,n\}} \sup_{t\in\mathbb{R}} \operatorname{dist}_{\mathrm{H}}(\Xi_{i,\eta}^*(t), \Xi_{i,0}) \right] = 0.$$

d) $\|S_\eta(t+\tau,\tau)u - S_0(t+\tau,\tau)u\|_X \overset{\eta\to 0}{\longrightarrow} 0$ uniformly for $\tau \in \mathbb{R}$, (t,u) in compact subsets of $[0,\infty) \times X$.

e) There are $\delta > 0$ and $\eta_0 \in (0,1]$ such that, if $\eta < \eta_0$, $\xi_\eta : \mathbb{R} \to X$ is a global solution in $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, $t_0 \in \mathbb{R}$ and $\operatorname{dist}(\xi_\eta(t), \Xi_{i,\eta}^*(t)) < \delta$ for all $t \leqslant t_0$ ($t \geqslant t_0$), then $\operatorname{dist}(\xi_\eta(t), \Xi_{i,\eta}^*(t)) \overset{t\to-\infty}{\longrightarrow} 0$ ($\operatorname{dist}(\xi_\eta(t), \Xi_{i,\eta}^*(t)) \overset{t\to+\infty}{\longrightarrow} 0$).

Then, there exists $\eta_0 > 0$ such that, for all $\eta \leqslant \eta_0$, $\{S_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is a generalized gradient-like nonlinear evolution process. Consequently, there exists $\eta_0 > 0$ such that

$$\mathscr{A}_\eta(t) = \bigcup_{i=1}^{n} W^u(\Xi_{i,\eta}^*)(t), \tag{3.20}$$

for all $t \in \mathbb{R}$ and for all $\eta \leqslant \eta_0$.

This result shows that a small non-autonomous perturbation of a gradient-like nonlinear semigroup becomes a gradient-like evolution process. Thus, it gives a natural way to construct examples of non-autonomous gradient-like evolution processes as small non-autonomous perturbations of gradient-like semigroups with all equilibria being hyperbolic and the perturbed isolated global solutions being hyperbolic bounded global solutions.

Remark 9 *In the Part II of this work we will generalize this previous results, given examples of gradient-like evolutions processes that do not come from small perturbations of gradient semigroups, providing the robustness of this theory and breaking the dependence on the autonomous case.*

3.8 Forward attraction in pullback attractors

Although pullback and forward dynamic may not be related (see [30, 65]), there exist some cases when the trajectories converge forward in time to the pullback attractor. The uniform forward attraction gives us trivial examples of pullback attractors that have forward attraction, because a pullback uniform attractor is also a forward uniform attractor and vice versa. In this case we need a uniform concept of attraction, that is, we say that $B \subset X$ attracts uniformly under the process $\{S(t,s) : t \geqslant s\}$ if for any $C \subset X$

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \operatorname{dist}(S(t+s,s)C, B) = 0. \tag{3.21}$$

We do not distinguish between pullback and forward because if we perform a simple change of variables we obtain

$$\lim_{s\to\infty} \sup_{t\in\mathbb{R}} \operatorname{dist}(S(t,t-s)C, B) = 0. \tag{3.22}$$

The definition of uniform attractor is given by Chepyzhov and Vishik in [33], and is based on the autonomous definition of global attractor, so the authors define it as a not necessarily invariant, in autonomous sense ($S(t,s)A = T(t-s)A = A$), compact subset that is uniformly attracting. However, afterwards the authors introduce the concept of kernel sections of the uniform attractor, which becomes a particular concept of pullback attractor. In [30] we can find the definition of an uniform attractor where the invariant property holds. In both cases, all the results appear in the skew-product framework. A skew-product system consists of a base flow in a base space Σ, which is a metric space with metric ρ, and a flow in the phase space that is, in some sense, driven by the base flow. We also need a group of transformations $\{\theta_t\}_{t\in\mathbb{R}}$ from X to itself such that

- $\theta_0 = Id$.
- $\theta_t \theta_s = \theta_{t+s}$.
- $\theta_t X = X$ for all $t \in \mathbb{R}$.

The dynamics of the phase space X is given by a mapping $\varphi : X \times \Sigma \to X$, calling cocycle, which satisfies

- $\varphi(0, \sigma) = Id_X$ for all $\sigma \in \Sigma$.
- $\varphi(t, \sigma)x$ is continuous in t and x.
- For all $t, s \geqslant 0$ and $\sigma \in \Sigma$, $\varphi(t + s, \sigma) = \varphi(t, \theta_s \sigma)\varphi(s, \sigma)$.

Below we write a general definition and an existence result within the framework of evolution processes. The following definition and result can be found in [30],

Definition 3.8.1 *Let $\{S(t, s) : t \geqslant s\}$ be an evolution process. A family of bounded closed sets $\{\mathscr{A}_u(t) : t \in \mathbb{R}\}$ is called the uniform attractor if the following properties hold:*

1. *There exists a compact set $\hat{A} \subset X$ such that $\bigcup_{t\in\mathbb{R}} \mathscr{A}_u(t) \subset \hat{A}$.*
2. *It is uniformly attracting under $\{S(t, s) : t \geqslant s\}$.*
3. *It is minimal in the sense of Definition 3.1.4.*

Theorem 3.8.2 *If there exists a compact uniformly attracting set, then there exists the uniform attractor.*

Although we need a pullback attracting family, the forward attraction comes from the uniform attractor. Actually, in this case we have a global attractor that is attracting in the pullback sense too.

Other examples are non-autonomous perturbations of gradient semigroups, where the forward attraction comes from the autonomous nature of the limit problem. The following result is Theorem 3.10 of [23] (see also [26]) and show how a pullback attractor possesses a forward attraction too.

Theorem 3.8.3 *Suppose that $\{S_0(t, s) : t \geqslant s\}$ is an autonomous evolution process satisfying that all its stationary points $\{\xi_1, \ldots, \xi_n\}$ are hyperbolic.*

Let be $\{S_\eta(t,s) : t \geqslant s\}_{\eta\in[0,1]}$ *a non-autonomous family of evolutions processes and suppose that there exists a family of hyperbolic bounded solution* $\{\xi_{1,\eta}(\cdot), \ldots, \xi_{1,\eta}(\cdot)\}$ *such that*

$$\sup_{t\in\mathbb{R}} \|\xi_i - \xi_{i,\eta}(t)\|_X \stackrel{\eta\to 0}{\longrightarrow} 0.$$

If we also assume that there is $\gamma > 0$ *and, for each* $1 \leqslant i \leqslant n$*, a neighborhood* $V_{i,\eta}$ *of* $\xi_{i,\eta}$ *such that for any* $u_0 \in V_{i,\eta}$*,* $s \in \mathbb{R}$ *and as long as* $S_\eta(t+s,s)u_0 \in V_{i,\eta}$

$$\sup_{s\in\mathbb{R}} \operatorname{dist}(S_\eta(t+s,s)u_0, W^u(\xi_{i,\eta})(t)) \leqslant Me^{-\gamma t}.$$

Then for any bounded set $B \subset X$*, there is a constant* $c(B) > 0$ *such that*

$$\sup_{s\in\mathbb{R}} \operatorname{dist}(S_\eta(t+s,s)u_0, \mathscr{A}_\eta(t+s)) \leqslant c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \qquad (3.23)$$

In [64, 67] we can find examples of forward attraction in non-autonomous problems which do not come from a perturbation of an autonomous system.

Part II

Wave equations

Chapter 4

Non-autonomous wave equation

In this chapter we are going to work with a more general damped wave equation where the damping depends on time, showing that, under certain assumptions, the solutions of the system are backwards and forward asymptotic to equilibria and homoclinic structures are not present; that is, the associated evolution process is gradient-like in the sense of [22], concluding that the pullback attractor is characterized by the union of the unstable sets of equilibrium points. Although there exists a lot of examples showing existence of pullback attractors (see, for example [7, 9, 16, 19, 27, 56, 80, 83]), the most extended way to obtain a specific structure for the attractor is to considerer a small non-autonomous perturbation of an autonomous system, we will show a non-autonomous system, not necessary close to an autonomous problem, where the pullback attractor is gradient-like. We also show that the pullback attractor is an exponential forward attractor. Note that, in general, there is no relationship between pullback and forward attraction (see [30] or [66]). Most of the results in this chapter can be found in [12] and [13].

4.1 Local well posedness of the problem

In this section we analyze an application of Theorem 3.2.4 of Section 3.2. Let us consider the following non-autonomous wave equation

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f(u) \text{ in } \Omega \\ u(x,t) = 0 \text{ in } \partial\Omega \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain in $\mathbb{R}^n$ with $n > 1$. For $f : \mathbb{R} \to \mathbb{R}$ we assume that:

$$\begin{gathered} f \in C^2(\mathbb{R}), \quad f(0) = 0, \quad |f'(s)| \leqslant c(1 + |s|^{p-1}), \\ |f''(s)| \leqslant c(1 + |s|^{p-2}), \quad \limsup_{|s|\to\infty} \frac{f(s)}{s} \leqslant 0, \end{gathered} \tag{4.2}$$

with $c > 0$ and $p < \frac{n}{n-2}$.

Assume that $\beta : \mathbb{R} \to \mathbb{R}$ is a bounded, globally Lipschitz function with $\beta'(t)$ Hölder continuous and that there are

$$\beta_0 \leqslant \beta(t) \leqslant \beta_1 \text{ for some } \beta_0, \beta_1 \in (0, \infty). \tag{4.3}$$

Remark 10 *We want to remark that assumptions in* (4.2) *imply that* $|f'(s)| \leqslant c(1 + |s|^{p-1})$.

We will prove that the non-autonomous process associated to (4.1) has a pullback attractor by applying Theorem 3.2.4 (page 53). For $u_t = v$ and $V = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, we rewrite (4.1) as

$$V_t = C(t)V + F(V) \tag{4.4}$$

where

$$C(t) = \begin{pmatrix} 0 & I \\ -A & -\beta(t)I \end{pmatrix} \text{ and } F(V) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix} \tag{4.5}$$

$A = -\Delta$ with Dirichlet boundary condition.

Following the theory in [47], we have that A is a sectorial operator in $Y = L^2(\Omega)$ and its domain is $Y^1 = H^2(\Omega) \cap H_0^1(\Omega)$. Denoting by $Z^\beta = Y^\beta \times Y$, with Y^β the fractional power of Y^1 and $0 \leqslant \beta \leqslant 1$, we have that $f : Y^{\frac{1}{2}} = H_0^1(\Omega) \to Y$ is locally Lipschitz (see [47] or [89]). Therefore, for each initial value $V_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Z^{\frac{1}{2}} = H_0^1(\Omega) \times L^2(\Omega)$ and initial time $s \in \mathbb{R}$, system (4.4) possesses a unique solution (see Theorem 7.1.3 in Henry [50]), that is, for each initial time $s \in \mathbb{R}$ there exists a continuous function $V(\cdot, s, w_0) : [s, s+\tau) \to Z^{\frac{1}{2}}$ that is continuous, continuously differentiable in $(s, s+\tau)$, $V(t, s, w_0) \in Y^1$ for $t \in (s, s+\tau)$ and (4.4) is satisfied for all $t \in (s, s+\tau)$ with initial data $V(s, s) = V_0$. This solution can be written as

$$S(t, s)V_0 = L(t, s)V_0 + U(t, s)V_0, \tag{4.6}$$

where $L(t, s)$ is the solution operator for the linear part $V_t = C(t)V$, and

$$U(t, s)V_0 = \int_s^t L(t, \tau)F(S(\tau, s)V_0)d\tau.$$

If $V_0 \in Y^{\frac{1}{2}} \times Y^{\frac{1}{2}}$, then $V(\cdot)$ is twice continuously differentiable in $(0, \tau)$ with values in $Y^{\frac{1}{2}} \times Y^{\frac{1}{2}}$. Indeed, if we apply again Theorem 7.1.3 in [50] to the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \mathcal{A}(t)\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \bar{F}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \ t > t_0 \\ \begin{bmatrix} u(t_0) \\ v(t_0) \\ w(t_0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ -Au_0 - \beta(t_0)v_0 + f(u_0) \end{bmatrix} \end{cases} \tag{4.7}$$

with $\mathcal{Z} = Y^{\frac{1}{2}} \times Y^{\frac{1}{2}} \times Y$, $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset \mathcal{Z} \to \mathcal{Z}$, $D(\mathcal{A}(t)) = Y^1 \times Y^1 \times Y$,

$$\mathcal{A}(t) = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ 0 & A + \beta'(t)I & \beta(t)I \end{bmatrix} \text{ and } \bar{F}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{f}(u,v) \end{bmatrix}$$

with $\mathbf{f}(u,v)(x) := f'(u(x))v(x)$, for each $s \in \mathbb{R}$ there exists a unique solution $W = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in C([s, s+\tau), \mathcal{Z}) \cap C^1((s, s+\tau), \mathcal{Z})$. To that end, we only need to prove that $\mathbf{f} : Y^{\frac{1}{2}} \times Y^{\frac{1}{2}} \to Y$ is locally Lipschitz. Let be $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in Y^{\frac{1}{2}} \times Y^{\frac{1}{2}}$, then

$$\begin{aligned} &\|\mathbf{f}(u_1, v_1) - \mathbf{f}(u_1, v_1)\|_{L^2} \\ &\leqslant \left[\int_\Omega |f'(u_1)|^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \|v_1 - v_2\|_{L^{\frac{2n}{n-2}}} + \left[\int_\Omega |f'(u_1) - f'(u_2)|^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \|v_2\|_{L^{\frac{2n}{n-2}}} \end{aligned}$$

Using (4.2), the Mean Value Theorem, the Hölder inequality, and the Sobolev embeddings we obtain

$$\begin{aligned} &\left[\int_\Omega |f'(u_1) - f'(u_2)|^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \\ &\leqslant \left[\int_\Omega |u_1 - u_2|^{\frac{2n}{n-2}}\right]^{\frac{n-2}{2n}} \left[\int_\Omega |f''(u_1 + \theta(u_2 - u_1))|^{\frac{n}{2}}\right]^{\frac{2}{n}} \\ &\leqslant c\|u_1 - u_2\|_{L^{\frac{2n}{n-2}}} \left[1 + \int_\Omega (|u_1| + |u_2|)^{(\rho-2)\frac{n}{2}}\right]^{\frac{2}{n}} \\ &\leqslant c\|u_1 - u_2\|_{L^{\frac{2n}{n-2}}} \left(1 + \|u_1\|^{\rho-2}_{L^{\frac{n(\rho-2)}{2}}(\Omega)} + \|u_2\|^{\rho-1}_{L^{\frac{n(\rho-2)}{2}}(\Omega)}\right). \end{aligned}$$

Since

$$\frac{n(\rho-2)}{2} \leqslant \frac{2n}{n-2},$$

we have that

$$L^{\frac{n(\rho-1)}{2}}(\Omega) \supset H_0^1(\Omega).$$

With analogous calculations, $\left[\int_\Omega |f'(u_1)|^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \leqslant c\|u_1\|_{H_0^1}^{(\rho-1)}$, which complete the proof.

From now on, we denote $X = Z^{\frac{1}{2}}$.

4.2 The pullback attractor

In this section we will show the existence and the structure of the pullback attractor of system (4.4).

4.2.1 Existence

Note that, if

$$\mathcal{L}(\varphi, \phi) = \frac{1}{2}\|\varphi\|^2_{H_0^1(\Omega)} + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)} - \int_\Omega G(\varphi), \tag{4.8}$$

with $(\varphi, \phi) \in X$, $G(r) = \int_0^r f(\theta)d\theta$, and $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$ a regular solution of (4.4), then

$$\frac{d}{dt}\mathcal{L}(u, u_t) = -\beta(t)\|u_t\|^2_{L^2(\Omega)}.$$

Hence $\mathcal{L} : X \to \mathbb{R}$ is a continuous function which is decreasing along solutions of (4.4). In addition, if $t \mapsto \mathcal{L}(w(t))$ is constant in a non-trivial interval of $\mathbb{R}$, then $w(t)$ is an equilibrium.

This means that $\mathcal{L}$ is a Lyapunov function for (4.4). Nonetheless, we cannot say that the solutions of (4.4) have similar properties to those of gradient autonomous evolution processes (e.g. are backwards and forward asymptotic to equilibria, Lema 3.8.2 in [47], [61], [78], [82] or [89]) since the usual proofs are strongly tied to the properties of the autonomous evolution processes. However, we are going to use this kind of functionals to proof most of our estimations. These functionals are very common in Physics because they represent the energy of the system as addition of the kinetic energy ($\frac{1}{2}\|u_t\|^2_{H_0^1(\Omega)}$) and the potential energy ($\frac{1}{2}\|u\|^2_{L^2(\Omega)}$). In this way they are called *energy functionals*.

Let us prove that $\{S(t, s) : t \geqslant s\}$ is pullback asymptotically compact by using Theorem 3.2.5.

Proposition 4.2.1 *There are positive constants $K > 0$, $\alpha > 0$, such that:*

$$\|L(t,s)\| \leqslant Ke^{-\alpha(t-s)},\ t \geqslant s.$$

Proof: Consider the following function

$$V(\varphi,\phi) = \frac{1}{2}\|\varphi\|^2_{H^1_0} + 2b(\varphi,\phi)_{L^2} + \frac{1}{2}\|\phi\|^2_{L^2},\ (\varphi,\phi) \in X \tag{4.9}$$

If $\frac{b}{2}\lambda_1^{-1} \leqslant \frac{1}{4}$ and $b \leqslant \frac{1}{2}$, we have that

$$\begin{aligned}\frac{1}{4}[\|\varphi\|^2_{H^1_0} + \|\phi\|^2_{L^2}] &\leqslant \frac{1}{2}\|\varphi\|^2_{H^1_0} + 2b(\varphi,\phi) + \frac{1}{2}\|\phi\|^2_{L^2}\\ &\leqslant \frac{3}{4}[\|\varphi\|^2_{H^1_0} + \|\phi\|^2_{L^2}].\end{aligned} \tag{4.10}$$

If $u(t,s)$ is a solution of

$$\begin{cases} u_{tt} + \beta(t)u_t - \Delta u = 0 & \text{in } \Omega\\ u(x,t) = 0 & \text{in } \partial\Omega,\end{cases}$$

then

$$\frac{d}{dt}V(u,u_t) = (u,u_t)_{H^1_0} + 2b(u_t,u_t)_{L^2} + 2b(u,u_{tt})_{L^2} + (u_t,u_{tt})_{L^2}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}V(u(t,s),u_t(t,s)) &= -(\beta(t)-2b)\|u_t\|^2_{L^2} - 2b\|u\|^2_{H^1_0} - 2\beta(t)(u,u_t)_{L^2}\\ &\leqslant -(\beta_0 - 2b - \frac{b\beta_1}{\varepsilon})\|u_t\|^2_{L^2} + (b\beta_1\varepsilon - b\lambda_1)\|u\|^2_{L^2}\\ &\quad - b\|u\|^2_{H^1_0},\end{aligned}$$

with $\varepsilon = \frac{\lambda_1}{\beta_1}$ and $b_0 > 0$ small enough. Then we have that, for all $0 < b \leqslant b_0$ and by (4.10),

$$\begin{aligned}\frac{d}{dt}V(u(t,s),u_t(t,s)) &\leqslant -\frac{\beta_0}{2}\|u_t\|^2_{L^2} - b\|u\|^2_{H^1_0}\\ &\leqslant -\alpha\left(\frac{1}{2}\|u\|^2_{H^1_0} + 2b(u,u_t)_{L^2} + \frac{1}{2}\|u_t\|^2_{L^2}\right)\\ &\leqslant -\alpha V(u(t,s),u_t(t,s)).\end{aligned} \tag{4.11}$$

Hence,

$$V(L(t,s)(\varphi,\phi)) \leqslant V(\varphi,\phi)e^{-\alpha(t-s)}$$

and, consequently, $\|L(t,s)(\varphi,\phi)\|^2_X \leqslant Ke^{-\alpha(t-s)}\|(\varphi,\phi)\|^2_X$ proving the result.

□

Theorem 4.2.2 *$\{S(t,s) : t \geqslant s\}$ is pullback strongly bounded dissipative.*

Proof: Consider the energy functional

$$\mathcal{V}(\varphi,\phi) = \frac{1}{2}\|\varphi\|^2_{H^1_0(\Omega)} + 2b(\varphi,\phi) + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)} - \int_\Omega G(\varphi), \tag{4.12}$$

where $G(s) = \int_0^s f(\theta)d\theta$.

Following [47], it follows from (4.2) that, for each $\delta > 0$ there is a constant $C_\delta > 0$ such that

$$\int_\Omega f(u)u \leqslant \delta\|u\|^2_{L^2(\Omega)} + C_\delta,$$
$$\int_\Omega G(u) \leqslant \delta\|u\|^2_{L^2(\Omega)} + C_\delta$$

for each $u \in L^2(\Omega)$ such that $f(u)u \in L^1(\Omega)$, and $G(u) \in L^1(\Omega)$.

Also, proceeding as in the proof of Proposition 4.1 in [6], there is a constant $c_0 > 0$ and given $r > 0$ there is a constant c_r such that

$$\left.\begin{array}{l} \left|\displaystyle\int_\Omega G(u)\right| \leq c_r\|u\|^2_{H^1_0} \\ \left|\displaystyle\int_\Omega f(u)u\right| \leq c_r\|u\|^2_{H^1_0} \end{array}\right\} \quad \text{for all } u \in H^1_0(\Omega) \text{ with } \|u\|_{H^1_0(\Omega)} \leqslant r. \tag{4.13}$$

From this, if b and δ are sufficiently small, there is a constant $K > 0$ such that

$$\mathcal{V}(\phi,\varphi) \geqslant \frac{1}{8}\left[\|\varphi\|^2_{H^1_0} + \|\phi\|^2_{L^2}\right] - K,$$

and, for each $r > 0$, a constant $K_r > 0$ such that

$$\mathcal{V}(\phi,\varphi) \leqslant K_r\left[\|\varphi\|^2_{H^1_0} + \|\phi\|^2_{L^2}\right]$$

for all $(\phi,\varphi) \in Y^0$ such that $\|(\phi,\varphi)\|_X \leqslant r$.

As in (4.11) we obtain that for some $K > 0$,

$$\frac{d}{dt}\mathcal{V}(u(t,s),u_t(t,s)) \leqslant -\alpha\mathcal{V}(u,u_t) + K. \tag{4.14}$$

Defining $\mathcal{V}_0(\phi,\varphi)=\mathcal{V}(\phi,\varphi)-K$ we obtain that

$$\frac{d}{dt}\mathcal{V}_0(u,u_t)\leqslant -\alpha\mathcal{V}_0(u,u_t)$$

Therefore $\mathcal{V}(u,u_t)\leqslant \mathcal{V}(u_0,v_0)e^{-\alpha(t-s)}+K_1$, and

$$\|S(t,s)(\varphi,\phi)\|_X^2\leqslant C(B)e^{-\alpha(t-s)}+\overline{K}, \tag{4.15}$$

where $C(B)$ only depends on the measure of the set B and $\overline{K}>0$ is a constant.

□

Theorem 4.2.3 $\{U(t,s): t\geqslant s\}$ *is compact.*

Proof: Let $B\subset X$ be bounded and $w_0=\left(\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right)\in B$. Let $t\mapsto S(t,s)w_0$ be the solution of (4.4) which at time s is w_0, and consider

$$U(t,s)w_0=\int_s^t L(t,\theta)F(S(\theta,s)w_0)d\theta.$$

The compactness of U follows easily from the fact that f is bounded from $H_0^1(\Omega)$ into $W^{1,q}(\Omega)$ for some $\frac{2n}{n+2}<q<2$. That is, we have

$$H_0^1\hookrightarrow L^{\frac{2n}{n-2}}\xrightarrow{\ f\ } L^{\frac{2n}{n+2}}\hookrightarrow W^{1,q}\subset\subset L^2,$$

since

$$\begin{aligned}\|f(u)\|_{L^{\frac{2n}{n+2}}}&=\|f(u)-f(0)\|_{L^{\frac{2n}{n+2}}}\\ &\leqslant \||f'(\theta u)||u|\|_{L^{\frac{2n}{n+2}}}\\ &\leqslant \|c(|u|+|u|^p)\|_{L^{\frac{2n}{n+2}}}\\ &\leqslant c\left(\|u\|_{L^{\frac{2n}{n+2}}}+\left[\int_\Omega |u|^{p\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}}\right) \qquad (4.16)\\ &\leqslant c\left(\|u\|_{L^{\frac{2n}{n+2}}}+\|u\|_{L^{\frac{2n}{n-2}}}^{\frac{n+2}{n-2}}\right)\\ &\leqslant c_1\left(\|u\|_{H_0^1}+\|u\|_{H_0^1}^{\frac{n+2}{n-2}}\right),\end{aligned}$$

where we have used the Mean Value Theorem and (4.2). As $H_0^1(\Omega)\times W^{1,q}(\Omega)$ is a compact subset of X, we have the compactness of $U(t,s)$.

□

Hence, applying Theorem 3.2.5 and Theorem 3.2.4, we can conclude that system (4.4) possesses a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ such that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \text{ is bounded in } X. \tag{4.17}$$

Next, our aim is to describe in detail the geometrical structure of the pullback attractor associated to (4.1). In the first section, we prove some auxiliary results on the regularity of the pullback attractor and, in the second one, we show its structure.

4.2.2 Regularity

Now we prove that the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is such that $\cup_{t\in\mathbb{R}}\mathscr{A}(t)$ is a bounded subset of $X^1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. To that end, let $\xi : \mathbb{R} \to X$ be a global bounded solution of (4.1). Then, the set $\{\xi(t) : t \in \mathbb{R}\}$ is a bounded subset of $X = H_0^1(\Omega) \times L^2(\Omega)$.

If $\xi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} : \mathbb{R} \to X$ is such that $\xi(t) \in \mathscr{A}(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) = L(t,s)\xi(s) + \int_s^t L(t,\theta)F(\xi(\theta))d\theta,$$

and, using Theorem 3.2.5, we have that it can be written as

$$\xi(t) = \int_{-\infty}^t L(t,\theta)F(\xi(\theta))d\theta. \tag{4.18}$$

Consider, for $w_0 = \xi(s)$,

$$W(t,s)w_0 = \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} = \int_s^\tau L(\tau,\theta)F(S(\theta,s)w_0)d\theta$$

and note that,

$$\begin{cases} w_{tt} + \beta(t)w_t = \Delta w + f(u(t,s;w_0)), \\ w(s) = w_t(s) = 0. \end{cases} \tag{4.19}$$

To estimate the solution of (5.25), for w_0 in a bounded subset B of X, we consider, for $b > 0$, the energy functional for $(\varphi,\phi) \in X$

$$V(\varphi,\phi) = \frac{1}{2}\|\varphi\|^2_{H_0^1} + 2b(\varphi,\phi)_{L^2} + \frac{1}{2}\|\phi\|^2_{L^2}, \tag{4.20}$$

to obtain that

$$\begin{aligned}\frac{d}{dt}V(w(t),w_t(t)) &= -(\beta(t)-2b)\|w_t\|^2_{L^2} - 2b\|\nabla w\|^2_{L^2}\\ &\quad + 2b\int_\Omega wf(u) - 2b\beta(t)\int_\Omega ww_t + \int_\Omega w_t f(u)\\ &\leqslant -\frac{\beta_0}{2}\|w_t\|^2_{L^2} - b\|\nabla w\|^2_{L^2} + C,\end{aligned}$$

where we have used (4.17), the fact that f takes bounded subsets of $H^1_0(\Omega)$ into bounded subsets of $L^2(\Omega)$ (from the growth condition in (4.2)) and a chosen value of b small enough. From this we obtain that

$$\bigcup_{s\leqslant\tau\leqslant t} W(\tau,s)B \ \text{ is a bounded subset of } X. \tag{4.21}$$

Hence, if $v = w_t$,

$$\begin{cases} v_{tt} + \beta(t)v_t = \Delta v - \beta'(t)v + f'(u(t,s;u_0))u_t(t,s;u_0)\\ v(s) = 0,\ v_t(s) = f(w_0). \end{cases} \tag{4.22}$$

Proceeding as in [9] we define, for $\epsilon > 0$, $Y^\epsilon = D((-\Delta)^{\frac{\epsilon}{2}})$ with the graph norm and $Y^{-\epsilon} = (Y^\epsilon)'$. Now, to estimate the solution of (4.22) we consider, for $b > 0$, the following energy functional for $(\varphi,\phi) \in Y^{1-\epsilon}\times Y^{-\epsilon}$

$$V_\epsilon(\varphi,\phi) = \frac{1}{2}\|\varphi\|^2_{Y^{1-\epsilon}} + 2b(\varphi,\phi)_{Y^{-\epsilon}} + \frac{1}{2}\|\phi\|^2_{Y^{-\epsilon}}. \tag{4.23}$$

Using (4.17) and (4.21), we have that, for $\epsilon_1 = \frac{(p-1)(n-2)}{2} < 1$ and for some constant $K > 0$,

$$\begin{aligned}\|f'(u)u_t\|_{Y^{-\epsilon_1}} &\leqslant c\|f'(u)u_t\|_{L^{\frac{2n}{n+2\epsilon_1}}}\\ &\leqslant \|u_t\|_{L^2}\|f'(u)\|_{L^{\frac{n}{\epsilon_1}}}\\ &\leqslant c\|u_t\|_{L^2}(1+\|u\|^{p-1}_{L^{\frac{2n}{n-2}}})\\ &\leqslant K.\end{aligned} \tag{4.24}$$

Also

$$\begin{aligned}\frac{d}{dt}V_{\epsilon_1}(v(t),v_t(t)) &= -(\beta(t)-2b)\|v_t\|^2_{Y^{-\epsilon_1}} - 2b\|v\|^2_{Y^{1-\epsilon_1}} \\ &- (2b\beta(t)+\beta'(t))\langle v, v_t\rangle_{Y^{-\epsilon_1}} - 2b\beta'(t)\|v\|^2_{Y^{-\epsilon_1}} \\ &+ 2b\langle v, f'(u)u_t\rangle_{Y^{-\epsilon_1}} + \langle v_t, f'(u)u_t\rangle_{Y^{-\epsilon_1}} \\ &\leqslant -(\beta_0-2b)\|v_t\|^2_{Y^{-\epsilon_1}} - 2b\|v\|^2_{Y^{1-\epsilon_1}} \\ &+ (2b\beta_1+L)\|v\|_{Y^{-\epsilon_1}}\|v_t\|_{Y^{-\epsilon_1}} + 2bL\|v\|^2_{Y^{-\epsilon_1}} \\ &+ 2b\|v\|_{Y^{-\epsilon_1}}\|f'(u)u_t\|_{Y^{-\epsilon_1}} + \|v_t\|_{Y^{-\epsilon_1}}\|f'(u)u_t\|_{Y^{-\epsilon_1}} \\ &\leqslant -\frac{\beta_0}{2}\|v_t\|^2_{Y^{-\epsilon_1}} - b\|v\|^2_{Y^{1-\epsilon_1}} + C,\end{aligned}$$

where we used (4.24), (4.21), (4.17) and chose b sufficiently small. From this, from (5.23) and from the characterization (3.19) in Section 3.7 we obtain by taking norm in (5.25)

$$\|\Delta w\|_{Y^{-\epsilon_1}} \leqslant \|w_{tt}\|_{Y^{-\epsilon_1}} + \lambda_1\beta_1\|w_t\|_{Y^{1-\epsilon_1}} + \|f(u)\|_{L^2} < \infty. \tag{4.25}$$

Then, we can conclude that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \text{ is bounded in } Y^{2-\epsilon_1}\times Y^{1-\epsilon_1}. \tag{4.26}$$

Using (4.26) and restarting from (4.24) with $\epsilon_2 = (p+1)\epsilon_1 - p$ we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \text{ is bounded in } Y^{2-\epsilon_2}\times Y^{1-\epsilon_2}. \tag{4.27}$$

Iterating this procedure a finite number of times, we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \text{ is bounded in } Y^{2}\times Y^{1}. \tag{4.28}$$

Noting that $\xi(t) = \lim_{s\to-\infty} W(t,s)$, since

$$\begin{aligned}\|W(t,s)-\xi(t)\|^2_X &\leqslant K\int_{-\infty}^{s} e^{-\alpha(t-\theta)}\|f(u(\theta)\|_{L^2}d\theta \\ &\leqslant K_2 e^{-\alpha(t-s)}(1+\max_{t\in\mathbb{R}}\|u(t)\|_{H^1_0}) \overset{s\to-\infty}{\longrightarrow} 0,\end{aligned} \tag{4.29}$$

(4.28) implies that

$$\sup_{\xi\in\mathcal{A}}\sup_{t\in\mathbb{R}}\{\|\xi(t)\|_X, \|\xi(t)\|_{X^1}, \|\xi_t(t)\|_X\} < \infty, \tag{4.30}$$

where $\mathcal{A}$ is the set of global bounded solutions for (4.1).

4.2.3 A gradient-like process

Next, we are going to show that, under some assumptions, the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is the union of the unstable manifolds of the equilibrium points of the system. Assume that there are only finitely many solutions $\{u_1^*, \cdots, u_p^*\}$ of

$$\begin{cases} \Delta u + f(u) = 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{4.31}$$

Denote by $\mathcal{E} = \{e_1^*, \cdots, e_p^*\}$ where $e_i^* = \binom{u_i^*}{0}$. Under this assumption, we prove in this section that the evolution process $\{S(t,s) : t \geqslant s\}$ associated to (4.4) is gradient-like; that is, conditions $(H1)$ and $(H2)$ in Definition 3.7.2 are satisfied. As a consequence, we will obtain that

$$\mathscr{A}(t) = \bigcup_{i=1}^{p} W^u(e_i^*)(t), \text{ for all } t \in \mathbb{R}. \tag{4.32}$$

We first observe that the function in (4.12) is such that, given a solution $\xi : [0,\infty) \to X$ of (4.1), then

$$[0,\infty) \ni t \mapsto \mathcal{L}(\xi(t)) \in \mathbb{R}$$

is decreasing. In addition, if $\mathcal{L}(\xi(t))$ is constant in a nontrivial interval of $\mathbb{R}$, then ξ must be an equilibrium. These considerations imply that the functional $\mathcal{L} : X \to \mathbb{R}$ is a Lyapunov function for (4.1) and that, in $\mathcal{E}$, there is no homoclinic structure. The remaining of this section is dedicated to show that all solutions in the pullback attractor of (4.1) are forward and backwards asymptotic to equilibria. These two conditions ensure that $\{S(t,s) : t \geqslant s\}$ is a gradient-like evolution process.

Clearly, if β is a positive constant or, as a consequence of Theorem 3.7.3, if β is uniformly close to a positive constant, $\{S(t,s) : t \geqslant s\}$ is gradient-like. Our goal is to show that even if β is not uniformly close to a constant, the process associated to (4.4) is still gradient-like and, therefore, the pullback attractor is still given by (4.32).

Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For each $n \in \mathbb{N}$, let $\beta_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $\beta_n(t) = \beta(t_n + t)$. Under these assumptions, the family $\{\beta_n\}_{n\in\mathbb{N}}$ is uniformly bounded and uniformly equicontinuous. Consequently, it has a subsequence (which we denote the same) and a globally Lipschitz and bounded function $\gamma : \mathbb{R} \to [0,\infty)$ such that $\beta_n(t) \overset{n\to\infty}{\longrightarrow} \gamma(t)$ uniformly in compact subsets of $\mathbb{R}$.

Now consider the following linear problems

$$\begin{cases} u_{tt} + \beta(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega), \ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{4.33}$$

$$\begin{cases} u_{tt} + \beta_n(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega), \ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{4.34}$$

and

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega), \ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{4.35}$$

Denote by $L(t,s)$, $L_n(t,s)$ and $L_\infty(t,s)$ the processes associated to (4.33), (4.34) and (4.35) in $X = H_0^1(\Omega) \times L^2(\Omega)$, respectively.

Clearly, from Theorem 3.2.5, there are constants $M \geqslant 1$ and $\omega > 0$ such that

$$\begin{aligned} \|L(t,s)\|_{\mathcal{L}(X)} &\leqslant Me^{-\omega(t-s)}, \ t \geqslant s, \\ \|L_n(t,s)\|_{\mathcal{L}(X)} &\leqslant Me^{-\omega(t-s)}, \ t \geqslant s, \\ \|L_\infty(t,s)\|_{\mathcal{L}(X)} &\leqslant Me^{-\omega(t-s)}, \ t \geqslant s. \end{aligned}$$

Also, $L(t_n + t, t_n + s) = L_n(t,s)$. In fact, (4.33) can be rewritten as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ \beta(t)u_t \end{pmatrix}, \\ \begin{pmatrix} u \\ u_t \end{pmatrix}(s) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{cases}$$

and writing

$$L(t,s)U_0 = \begin{pmatrix} \ell_1(t,s)U_0 \\ \ell_2(t,s)U_0 \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad \text{and } C = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

we have, by the variation of constants formula, that

$$\begin{aligned} L(t_n + t, t_n + s)U_0 &= e^{C(t-s)}U_0 - \int_{t_n+s}^{t_n+t} e^{C(t+t_n-\theta)} \begin{pmatrix} 0 \\ \beta(\theta)\ell_2(\theta, t_n + s)U_0 \end{pmatrix} d\theta \\ &= e^{C(t-s)}U_0 - \int_s^t e^{C(t-\theta)} \begin{pmatrix} 0 \\ \beta_n(\theta)\ell_2(t_n + \theta, t_n + s)U_0 \end{pmatrix} d\theta \\ &= L_n(t,s)U_0 \end{aligned}$$

Now,

$$[L_n(t,s)-L_\infty(t,s)]U_0=\int_s^t e^{C(t-\theta)}\beta_n(\theta)\begin{pmatrix}0\\(\ell_2)_n(\theta,s)U_0-(\ell_2)_\infty(\theta,s)U_0\end{pmatrix}d\theta$$
$$+\int_s^t e^{C(t-\theta)}[\beta_n(\theta)-\gamma(\theta)]\begin{pmatrix}0\\(\ell_2)_\infty(\theta,s)U_0\end{pmatrix}d\theta,$$

and a simple application of Gronwall's inequality yields that, for each $T>0$

$$\sup_{t-T\leqslant s\leqslant t}\|L_n(t,s)-L_\infty(t,s)\|_{\mathcal{L}(X)}\overset{n\to\infty}{\longrightarrow}0. \tag{4.36}$$

Now, let $\xi:\mathbb{R}\to X$ be a global bounded solution of (4.4) and recall that, from (5.36),

$$\sup_{t\in\mathbb{R}}\{\|\xi(t)\|_X,\|\xi(t)\|_{X^1},\|\xi_t(t)\|_X\}<\infty.$$

Thus, by the Arzelà-Ascoli Theorem, we have that the sequence ξ_n in $C(\mathbb{R},X)$ defined by $\xi_n(t)=\xi(t_n+t)$ has a subsequence which converges uniformly in compact subsets of $\mathbb{R}$ to a continuous function $\zeta:\mathbb{R}\to X$.

Now, as

$$\xi(t)=\begin{pmatrix}\xi_1(t)\\(\xi_1)_t(t)\end{pmatrix}=L(t,s)\xi(s)+\int_s^t L(t,\theta)\begin{pmatrix}0\\f(\xi_1(\theta))\end{pmatrix}d\theta, \tag{4.37}$$

we also have that

$$\xi(t)=\int_{-\infty}^t L(t,\theta)\begin{pmatrix}0\\f(\xi_1(\theta))\end{pmatrix}d\theta,$$

and, consequently,

$$\begin{aligned}\xi(t+t_n)&=\int_{-\infty}^{t+t_n}L(t+t_n,\theta)\begin{pmatrix}0\\f(\xi_1(\theta))\end{pmatrix}d\theta\\&=\int_{-\infty}^{t}L(t_n+t,t_n+\theta)\begin{pmatrix}0\\f(\xi_1(\theta+t_n))\end{pmatrix}d\theta\\&=\int_{-\infty}^{t}L_n(t,\theta)\begin{pmatrix}0\\f(\xi_1(\theta+t_n))\end{pmatrix}d\theta.\end{aligned}$$

Thanks to this and (4.36), it is not difficult to see that

$$\zeta(t)=\int_{-\infty}^t L_\infty(t,\theta)\begin{pmatrix}0\\f(\zeta(\theta))\end{pmatrix}d\theta$$

and, in particular, $\zeta : \mathbb{R} \to X$ is a global bounded solution of

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = f(u) \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega. \end{cases} \tag{4.38}$$

To that end, we consider the Lyapunov function in (4.8). Then, the mapping $\mathbb{R} \ni t \mapsto \mathcal{L}(\xi(t)) \in \mathbb{R}$ is non-increasing and the only global solution ξ where V is constant are the equilibria in $\mathcal{E}$. Since $\{\xi(t) : t \in \mathbb{R}\}$ lies in a compact set in X, there are real numbers ς_i and ς_j such that

$$\varsigma_i \stackrel{t\to-\infty}{\longleftarrow} \mathcal{L}(\xi(t+r)) \stackrel{t\to\infty}{\longrightarrow} \varsigma_j$$

for all $r \in \mathbb{R}$.

If $t_n \stackrel{n\to\infty}{\longrightarrow} \infty$, taking subsequences, if necessary, $\beta(t_n + r) \stackrel{n\to\infty}{\longrightarrow} \gamma(r)$ uniformly in compact subsets of $\mathbb{R}$, $\xi(t_n + r) \stackrel{n\to\infty}{\longrightarrow} \zeta(r)$ in X, uniformly for r in compact subsets of $\mathbb{R}$, and $(\zeta(t), \zeta_t(t))$ is a global solution of the problem

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \tag{4.39}$$

with the property that $\mathcal{L}(\zeta(t), \zeta_t(t)) = \varsigma_j$, for all $t \in \mathbb{R}$. Hence $\left(\begin{smallmatrix} \zeta(t) \\ \zeta_t(t) \end{smallmatrix}\right) = e_j^*$. Taking $\tilde{t}_n \stackrel{n\to\infty}{\longrightarrow} -\infty$ we obtain an analogous result.

Suppose that there are sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{\bar{t}_n\}_{n\in\mathbb{N}}$ with $t_{n+1} > \bar{t}_n > t_n$, $n \in \mathbb{N}$, such that $\xi(t_n) \stackrel{n\to\infty}{\longrightarrow} e_k^*$ and $\xi(\bar{t}_n) \stackrel{n\to\infty}{\longrightarrow} \bar{e}_k^*$. Now, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $V(\xi(t)) \in (\varsigma_j - \epsilon, \varsigma_j + \epsilon)$ for all $t \in [t_n, \bar{t}_n]$. If $\tau_n \in (t_n, \bar{t}_n)$, $\tau_n \stackrel{n\to\infty}{\longrightarrow} \infty$ and (taking subsequences if necessary), $\beta(\tau_n + r) \stackrel{n\to\infty}{\longrightarrow} \bar{\gamma}(r)$. We have that $\xi(\tau_n + r) \stackrel{n\to\infty}{\longrightarrow} \bar{\zeta}(t)$, which is a solution of

$$\begin{cases} u_{tt} + \bar{\gamma}(t)u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \tag{4.40}$$

with $\mathcal{L}(\bar{\zeta}(t), \bar{\zeta}_t(t)) = \varsigma_j$ for all $t \in \mathbb{R}$, and, consequently, $\bar{\zeta}(t) \equiv e_m^*$ with $\mathcal{L}(e_m^*) = \varsigma_j$. That leads to a contradiction with the fact that there are only finitely many equilibria.

From the fact that the evolution process $\{S(t, s) : t \geqslant s\}$ associated to (4.1) possesses a Lyapunov function (see (4.8)), property $(H2)$ in Definition 3.7.2 is automatically fulfilled. We can summarize all our previous analysis in the following theorem,

Theorem 4.2.4 *Suppose that there are only finitely many solutions* $\{u_1^*, \ldots, u_p^*\}$ *of* (4.31). *Then the evolution process* $\{S(t,s) : t \geqslant s\}$ *associated to* (4.1) *is gradient-like and, as a consequence, we can write the pullback attractor* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *as in* (4.32).

4.2.4 A pullback strongly point dissipative wave equation

Now, we give an example of pullback asymptotically point dissipative system, based on system (4.1). Using the estimations before and Theorem 3.2.12, we will show the existence of the pullback attractor. We consider the damped wave equation in (4.1), assuming the assumptions over $f(s)$ in (4.2), and $\beta(t)$ defined as follow

$$\beta(t) = \begin{cases} \beta_0 & \text{if} \quad t \leqslant 0 \\ b(t) & \text{in} \quad t > 0, \end{cases} \tag{4.41}$$

where $b(t)$ is a bounded and globally Lipschitz function with $b(0) = \beta_0$.

We already know that the process $\{S(t,s) : t \geqslant s\}$ associated to this problem is pullback strongly asymptotically compact and pullback strongly bounded. We want to check that it is also pullback strongly point dissipative and equicontinuous.

When $\beta(t)$ is constant we have an autonomous equation, so the semigroup associated $S(t,s) = \widetilde{S}_0(t-s)$ is point dissipative, i.e. if $\beta(t) \equiv \beta_0$, there exists a set $B_0 \subset X$ such that for each $x \in X$, $\widetilde{S}_0(t)x \subset B_0$ for all $t > \sigma_x$. We now define the following bounded sets

$$B(t) = \bigcup_{\tau \leqslant t} \bigcup_{s \leqslant \tau} S(\tau, s) B_0. \tag{4.42}$$

We take a fixed $t \in \mathbb{R}$, $x \in X$ and consider $S(\tau, s)x$ with $s \leqslant \tau \leqslant t$. If $t \leqslant 0$ we know that there exits $\sigma_x > 0$ such that $S(\tau, s)x = \widetilde{S}_0(\tau - s)x \in B_0 \subset B(t)$ for all $\tau - s \geqslant \sigma_x$. Otherwise, if $t > 0$, we can take a fixed $\widetilde{\tau} < 0$ such that $s \leqslant \widetilde{\tau} < \tau$ and $S(\tau, s)x = S(\tau, \widetilde{\tau})S(\widetilde{\tau}, s)x$. Therefore there exits a $\sigma_x > 0$ such that $S(\widetilde{\tau}, s)x \in B_0$ for all $\widetilde{\tau} - s \geqslant \sigma_x$ and $S(\tau, s)x \subset B(t)$ for all $\tau - s \geqslant \widetilde{\tau} - s \geqslant \sigma_x$. So $\{S(t,s) : t \geqslant s\}$ is pullback strongly point dissipative.

Let $\varepsilon > 0$. By Hale ([47]) we have that $F(\varphi, \phi)$ is locally Lipschitz with constant λ and $(\varphi, \phi) \in X$. Therefore, if we take x, $y \in X$ we have

$$\begin{aligned} \|S(s, s-\tau)x - &S(s, s-\tau)y\| \leqslant \\ &\leqslant K e^{-\alpha\tau}\|x - y\| + \int_s^{s-\tau} K_2 \|S(\theta, s)x - S(\theta, s)y\| e^{-\alpha\theta} d\theta, \end{aligned}$$

and using the Gronwall lemma and taking $\|x-y\| < (Ke^{-\alpha\tau+\frac{K\lambda}{\alpha}})^{-1}$ we can conclude that $\|S(s,s-\tau)x - S(s,s-\tau)y\| < \varepsilon$. Thus, for each $t \in \mathbb{R}$ and $\tau > 0$, $\{S(s,s-\tau) : s \leqslant t\}$, is equicontinuous. Consequently, we can apply Theorem 3.2.12 of Section 3.2.

4.3 Exponential forward attraction

In this section we consider the situation in which (4.1) possesses an exponential pullback attractor in the forward sense. As we will see, the characterization of the pullback attractor and the ideas in Section 4.2.3 play a fundamental role in the proof of the exponential attraction.

Hypothesis 1 *Let $f \in C^2(\mathbb{R})$ and $\beta \in C^1(\mathbb{R})$ be such that conditions* (4.2) *and* (4.3) *are satisfied, and assume that* (4.31) *has a finite number of solutions $\mathcal{E} = \{e_1^*, \cdots, e_p^*\}$ where $e_i^* = \binom{u_i^*}{0}$.*

From the results of Section 4.2.3 we have that the process associated to (4.1) has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ which is characterized by (4.32). Consider the linear process $\{L_i(t,s) : t \geqslant s\}$ associated to

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f'(u_i^*)u, \ t > 0, \ x \in \Omega, \\ u(t,x) = 0, \ t > 0, \ x \in \partial\Omega \\ u(0,\cdot) = u_0 \in H_0^1(\Omega), \ u_t(0,\cdot) = v_0 \in L^2(\Omega). \end{cases} \tag{4.43}$$

Hypothesis 2 *Assume that all equilibria in $\mathcal{E}$ are hyperbolic in the following sense.*

Definition 4.3.1 *We say that the linear evolution process $\{L_i(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has exponential dichotomy with exponent ω and constant M if there is a family of bounded linear projections $\{Q_i(t) : \ t \in \mathbb{R}\}$ in X such that*

1. *$Q_i(t)L_i(t,s) = L_i(t,s)Q_i(s)$, for all $t \geqslant s$.*
2. *The restriction $L_i(t,s)_{|_{R(Q_i(s))}}$, $t \geqslant s$ is an isomorphism from $R(Q_i(s))$ into $R(Q_i(t))$; we denote its inverse by $L_i(s,t) : R(Q_i(t)) \to R(Q_i(s))$ (notice that $R(Q_i(t))$ denotes the range of the operator $Q_i(t)$).*
3. *There are constants $\omega > 0$ and $M \geqslant 1$ such that*

$$\begin{aligned} &\|L_i(t,s)(I - Q_i(s))\|_{L(X)} \leqslant Me^{-\omega(t-s)} \quad t \geqslant s \\ &\|L_i(t,s)Q_i(s)\|_{L(X)} \leqslant Me^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{4.44}$$

When $\{L_i(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ possesses an exponential dichotomy, we say that e_i^* is a hyperbolic equilibrium point.

Remark 11 *We remark that, when β is independent of t, an equilibrium e^* of (4.1) is hyperbolic if and only if zero is not an eigenvalue of A ($A = \Delta + f'(e^*)I$ with homogeneous Dirichlet boundary conditions). Unfortunately, the case when β is time dependent is much more difficult and cannot be easily obtained from the knowledge of the spectrum of A. We conjecture that, under our assumptions, e^* is hyperbolic if and only if $0 \notin \sigma(A)$.*

Under Hypotheses 1 and 2 and Theorem 3.4.6, if $\{Q_i(t) : t \in \mathbb{R}\}$ is the family of projections (given in Definition 3.4.4) associated to e_i^*, for each $1 \leqslant i \leqslant p$, there is a neighborhood $\mathcal{V}_i$ of e_i^* and a function $\Sigma_i : R(Q_i(t)) \to Ker(Q_i(t))$ such that

$$W^u(e_i^*)(t) \cap \mathcal{V}_i = \{e_i^* + Q_i(t)u + \Sigma_i(Q_i(t)u) : u \in X\} \cap \mathcal{V}_i, \tag{4.45}$$

(recall that $\{S(t,s) : t \geqslant s\}$ is a gradient-like evolution process, so the above intersection is the local unstable manifold) and there exists $\gamma > 0$ such that, for any $u_0 \in V_i$, and as long as $S(t+s,s)u_0 \in V_i$,

$$\sup_{s\in\mathbb{R}} \|(I - Q_i(t+s))S(t+s,s)u_0 - \Sigma_i^u((Q_i(t+s)S(t+s,s)u_0))\|_X \leqslant Me^{-\gamma t}.$$

It is easy to see that $\{S(t,s) : t \geqslant 0\}$ is Lipschitz continuous, that is, given a bounded subset B of X, there are constants $c = c(B)$ and $L = L(B) > 0$ such that, for all $u, v \in B$

$$\sup_{s\in\mathbb{R}} \|S(t+s,s)u - S(t+s,s)v\| \leqslant ce^{Lt}\|u - v\|. \tag{4.46}$$

In what follows, based on the results of [9], [23], [26] or [91], we show that the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ of the evolution process associated to (4.1) under Hypothesis 1 and Hypothesis 2 is also an exponential pullback attractor, but first we need the following important results (which extend the corresponding ones in [23], where they are proved for processes which are small perturbations of autonomous evolution processes).

Lemma 4.3.2 *Assume that Hypothesis 1 and Hypothesis 2 are satisfied. If $\{S(t,s) : t \geqslant s\}$ is the evolution process associated to (4.1), given $\delta < \frac{1}{2}\min\{\|e_i^* - e_j^*\|_X : 1 \leqslant i,j \leqslant k,\ i \neq j\}$ and a bounded set $B \subset X$, there is a positive number $T = T(\delta, B)$ such that $\{S(t+s,s)u_0 : 0 \leqslant t \leqslant T\} \cap \cup_{i=1}^n B_\delta(e_i^*) \neq \emptyset$ for all $u_0 \in B$ and for all $s \in \mathbb{R}$.*

Proof: We argue by contradiction. Assume that there is a sequence u_k in B, a sequence of positive numbers $t_k \overset{k\to\infty}{\longrightarrow} \infty$, and a sequence of real numbers s_k such that $\{S(t+s_k, s_k)u_k : 0 \leqslant t \leqslant t_k\} \cap \cup_{i=1}^n B_\delta(e_i^*) = \emptyset$. Extracting subsequences we have that there is a function $\gamma : \mathbb{R} \to [\beta_0, \beta_1]$ and a global solution $\xi : \mathbb{R} \to X$ of (4.38) such that $S(t + \frac{t_k}{2} + s_k, s_k)u_k \to \xi(t)$ uniformly in compact subsets of $\mathbb{R}$. Clearly, by construction, $\xi(t) \notin \cup_{i=1}^n B_\delta(e_i^*)$ for all $t \in \mathbb{R}$ and this contradicts (4.32).

□

Lemma 4.3.3 *Assume that Hypothesis 1 and Hypothesis 2 are satisfied. If* $\{S(t,s) : t \geqslant s\}$ *is the evolution process associated to* (4.1)*, given* $0 < \delta < \frac{1}{2}\min\{\|e_i^* - e_j^*\|_X : 1 \leqslant i, j \leqslant k,\ i \neq j\}$*, there is a* $\delta' > 0$ *such that, if for some* $1 \leqslant i \leqslant n$*,* $\|u_0 - e_i^*\|_X < \delta'$ *and, for some* $t_1 > 0$*,* $\|S(t_1 + s, s)u_0 - e_i^*\|_X \geqslant \delta$*, then* $\|S(t+s,s)u_0 - e_i^*\|_X > \delta'$ *for all* $t \geqslant t_1$ *and for all* $s \in \mathbb{R}$*.*

Proof: Assume that, for some $1 \leqslant i \leqslant n$, there is a sequence u_k in X with $\|u_k - e_i^*\|_X < \frac{1}{k}$ and sequences $s_k \in \mathbb{R}$, and $0 < t_k < \tau_k$ such that $\|S(t_k + s_k, s_k)u_k - e_i^*\|_X \geqslant \delta$ and $\|S(\tau_k + s_k, s_k)u_k - e_i^*\|_X < \frac{1}{k}$. Clearly t_k is bounded from below and that contradicts the fact that $\mathcal{E}$ does not contain any homoclinic structure.

□

Theorem 4.3.4 *There exists* $\gamma > 0$ *and, for each bounded subset* $B \subset X$*, there exists a constant* $c(B) > 0$ *such that, for all* $u_0 \in B$

$$\sup_{s\in\mathbb{R}} \sup_{u_0 \in B} \operatorname{dist}(S(t+s,s)u_0, \mathscr{A}(t+s)) \leqslant c(B)e^{-\gamma t}. \tag{4.47}$$

Proof: To prove (4.47) we first choose $\delta < \delta_0$ such that $B_\delta(e_i^*) \subset V_i$ and V_i is the neighborhood given in (4.45) for e_i^*. From Lemma 4.3.3, for all suitably small δ, there exists $\delta' = \delta'(\delta) < \delta$ such that, if $u_0 \in B_{\delta'}(e_i^*)$ and for some $t_1 > 0$

$$S(t_1 + s, s)u_0 \notin B_\delta(e_i^*),$$

then

$$S(t+s,s)u_0 \notin B_{\delta'}(e_i^*),\ \text{for all } t \geqslant t_1.$$

Now, let B be a bounded subset of X and B_0 be a closed ball centered at $u = 0$ which contains B and $\cup\{B_\delta(u) : u \in A(t),\ t \in \mathbb{R}\}$. From Lemma 4.3.2, there exists $T = T(\delta', B_0)$ such that, for all $u_0 \in B_0$

$$S(t+s,s)u_0 \in \mathcal{O}_{\delta'} = \bigcup_{i=1}^n B_{\delta'}(e_i^*)$$

for some $t \leqslant T$ and $\forall s \in \mathbb{R}$.

Thus, given $u_0 \in B_0$, there are sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M \leqslant n$ and $\{e_i^*\}_{i=1}^M$ such that

$$t_-^0 \leqslant T, \qquad t_-^i - t_+^{i-1} \leqslant T, \quad 1 \leqslant i \leqslant M \qquad t_+^M = +\infty$$

for which $S(t+s,s)u_0 \in \mathcal{O}_\delta(e_i^*)$, for all $t \in [t_-^i, t_+^i]$, $s \in \mathbb{R}$, and $i \in \{1, \ldots, M\}$. Then,

$$\sup_{s\in\mathbb{R}} \operatorname{dist}(S(t+s,s)u_0, \mathscr{A}(t+s)) \leqslant c_0(B_0)e^{-\gamma t},$$

for all $t \in [t_-^i, t_+^i]$.

On the other hand, for $t \in [t_+^{i-1}, t_-^i]$, $t = \sigma + t_+^{i-1}$, for some $\sigma \leqslant T$, and using (4.46) we have that

$$\begin{aligned}
&\operatorname{dist}(S(t+s,s)u_0, \mathscr{A}(t+s)) \\
&\qquad = \operatorname{dist}(S(\sigma + t_+^{i-1} + s, s)u_0, \mathscr{A}(t+s)) \\
&\qquad = \operatorname{dist}(S(\sigma + t_+^{i-1}, t_+^{i-1})S(t_+^{i-1} + s, s)u_0, S(\sigma + t_+^{i-1}, t_+^{i-1})\mathscr{A}(t+s)) \\
&\qquad \leqslant c_1(B_0)e^{kT}\operatorname{dist}(S(t_+^{i-1} + s, s)u_0, \mathscr{A}(t+s)) \\
&\qquad \leqslant c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_+^{i-1}} \\
&\qquad = c(B_0)e^{-\gamma t}.
\end{aligned}$$

□

Remark 12 *Using now Theorem 3.6.1, the attractor has a "copy" of all the dynamic of the system inside it.*

Chapter 5
Non-autonomous strongly damped wave equation

The aim of this chapter is to obtain the existence, regularity, continuity, characterization and continuity of characterization of pullback attractors for a non-autonomous family of strongly damped wave equations in $H_0^1(\Omega)\times L^2(\Omega)$ (see [14]). As in Chapter 4, for the characterization of the pullback attractors we obtain that (when all the equilibria are hyperbolic) all solutions in the pullback attractors are backwards and forward asymptotic to equilibria. We also obtain that the evolution process associated to (5.1) in $H_0^1(\Omega)\times L^2(\Omega)$ is a gradient-like evolution process (homoclinic structures do not exist). This equation is a generalization of an autonomous model which has been investigated in many articles by several authors (see, for example, [3, 21, 10, 31, 32, 54, 73, 75]), with particular emphasis on its asymptotic behaviour. Assuming (critical) growth restrictions on the non-linear term f, the global existence of solutions in the autonomous case has been established in [20], based on the theory of ε-regular solutions. It is remarkable that the model is not written as a small perturbation of its autonomous counterpart.

5.1 Local well posedness of the problem

Let us consider the equation

$$\begin{cases} u_{tt} - \Delta u - \gamma(t)\Delta u_t + \beta_\varepsilon(t) u_t = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

in a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^n$, $\gamma, \beta_\varepsilon : \mathbb{R} \to (0,\infty)$ verify $0 < \gamma_0 \leqslant \gamma(t) \leqslant \gamma_1 < \infty$, $0 < \beta_{0\varepsilon} \leqslant \beta(t) \leqslant \beta_{1\varepsilon} < \infty$, and $\gamma(t)$ and $\beta_\varepsilon(t)$ are continuously differentiable in $\mathbb{R}$ and with bounded derivative uniformly in ε

and $\gamma'(t)$ and $\beta'_\varepsilon(t)$ Hölder continuous uniformly in ε. We also suppose that

$$\begin{aligned} &\beta_{i\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} 0, \text{ for } i=0,1\\ &\frac{\beta_{1\varepsilon}}{\beta_{0\varepsilon}} \overset{\varepsilon\to 0}{\longrightarrow} 1. \end{aligned} \tag{5.2}$$

For the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ we assume the following dissipativeness and growth conditions

$$f\in C^2(\mathbb{R},\mathbb{R}), \quad \limsup_{|s|\to\infty}\frac{f(s)}{s}\leqslant 0, \tag{5.3}$$

$$f(0)=0, \quad |f(s)-f(r)|\leqslant c|s-r|(1+|s|^{\rho-1}+|r|^{\rho-1}), \tag{5.4}$$

$$\tag{5.5}$$

where $1<\rho<\frac{n+2}{n-2}$.

To obtain the local well posedness of the Cauchy problem for (5.1), we will use the results in [28] (see also [84]) which we introduce next

Definition 5.1.1 *Let $\mathcal{Z}$ be a Banach space and $B(t): D\subset\mathcal{Z}\to\mathcal{Z}$ (D fixed) a closed, densely defined, time-dependent operator.*

a) The operator family $B(t)$ is called uniformly sectorial *if there is a constant $C>0$ (independent of $t\in\mathbb{R}$) such that*

$$\|(\lambda I+B(t))^{-1}\|_{\mathcal{L}(\mathcal{Z})}\leqslant\frac{C}{|\lambda|+1}; \forall\lambda\in\mathbb{C} \textit{ with } \mathrm{Re}\lambda\geqslant 0$$

b) $B(t)$ is called uniformly Hölder continuous *if there exist constants $C>0$ and $\varepsilon>0$ such that, for any t, τ and $s\in\mathbb{R}$,*

$$\|\,[B(t)-B(\tau)]\,B^{-1}(s)\|_{\mathcal{L}(\mathcal{Z})}\leqslant C(t-\tau)^{\varepsilon}.$$

Consider the Cauchy problem

$$\begin{aligned} &z_t = B(t)z+F(z), t>s,\\ &z(s)=z_0 \end{aligned} \tag{5.6}$$

Definition 5.1.2 *For a continuous function $F:\mathbb{R}\times\mathcal{Z}^\alpha\to\mathcal{Z}$, $\alpha\in[0,1)$, $z(\cdot,s,z_0):[s,s+\tau)\to\mathcal{Z}^\alpha$ is a solution for (5.6) if it is continuous, continuously differentiable in $(0,\tau)$, $z(t,s,z_0)\in D(B(t))$ for $t\in(s,s+\tau)$ and (5.6) is satisfied for all $t\in(0,\tau)$.*

We can now state the following result (see [28] for a more general version that includes the critical growth case).

Theorem 5.1.3 *If the operator family $B(t)$ is uniformly sectorial and uniformly Hölder continuos, and $F : \mathcal{Z}^\alpha \to \mathcal{Z}$ is Lipschitz continuous in bounded subsets of $\mathcal{Z}^\alpha$ then, given $r > 0$ there exists $\tau > 0$ and for each $z_0 \in \mathcal{Z}^\alpha$ with $\|z_0\|_{\mathcal{Z}^\alpha} \leqslant r$ a function $z(\cdot, s, z_0) \in C([s, s+\tau], \mathcal{Z}^\alpha) \cap C^1((s, s+\tau], \mathcal{Z}^\alpha)$ with the properties that*

$$\{z_0 \in \mathcal{Z}^\alpha : \|z_0\|_{\mathcal{Z}^\alpha} \leqslant r\} \ni z_0 \mapsto z(\cdot, s, z_0) \in C([s, s+\tau], \mathcal{Z}^\alpha)$$

is continuous, $z(\cdot, s, z_0)$ is the unique solution of (5.6).

It is clear from this result (since the time of existence can be chosen uniform in bounded subsets of $\mathcal{Z}^\alpha$) that solutions which do not blow up in $\mathcal{Z}^\alpha$ must exist for all $t \geqslant s$.

Next, we establish the functional analytic framework needed to apply the above results to the problem (5.1). Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ be the positive self-adjoint operator defined by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = -\Delta u$ for all $u \in D(A)$. Let A^α, with $\alpha \in \mathbb{R}$, be the fractional power operators associated to A and, for $\alpha \geqslant 0$, let $X^\alpha = D(A^\alpha)$, with the norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|_X$, be the fractional power spaces associated to A for $\alpha \geqslant 0$. Define $X^{-\alpha}$, $\alpha > 0$, as the completion of X with respect to the norm $\|A^{-\alpha} \cdot\|_X$. With this notation, we will keep denoting by A the closed extension of A to $X^{-\alpha}$, $\alpha \geqslant 0$. Finally, denote by $Y_{(-1)} = X^{\frac{1}{2}} \times X^{-\frac{1}{2}}$

With this, we can view (5.1) as the system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \mathbf{A}(t) \begin{bmatrix} u \\ v \end{bmatrix} = F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \tag{5.7}$$

where $D(\mathbf{A}(\mathbf{t})) = X^{\frac{1}{2}} \times X^{\frac{1}{2}}$,

$$\mathbf{A}(\mathbf{t}) \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A & \gamma(t)A + \beta_\varepsilon(t) \end{bmatrix} \begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} -\varphi \\ A(\phi + \gamma(t)\varphi) + \beta_\epsilon(t)\varphi \end{bmatrix},$$

and

$$F\left(\begin{bmatrix} u \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \mathbf{f}(u) \end{bmatrix},$$

where, for $\phi : \Omega \to \mathbb{R}$, $\mathbf{f}(\phi)$ is defined by $\mathbf{f}(\phi)(x) = f(\phi(x))$, $x \in \Omega$.

Note that $D(\mathbf{A}(\mathbf{t}))$ (as a vector space) does not depend on t. Denote by $Y^1_{(-1)}$ the space $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ with the norm $\|\mathbf{A}(t) \cdot\|_{Y_{(-1)}}$, where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & -I \\ A & \gamma(t)A + \beta_\epsilon(t)I \end{bmatrix}. \tag{5.8}$$

We have that $Y^1_{(-1)}$ is isomorphic to $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ with its usual norm, uniformly for $t \in \mathbb{R}$. Also note that,

$$\mathbf{A}(t)^{-1} = \begin{bmatrix} \gamma(t)I + \beta_\epsilon(t)A^{-1} & A^{-1} \\ -I & 0 \end{bmatrix}.$$

Denoted by $Y^\alpha_{(-1)}$ the domain of $\mathbf{A}(s)^\alpha$ for some $s \in \mathbb{R}$ and $\alpha \geqslant 0$ with the graph norm; we obtain the fractional power scale spaces associated to $\mathbf{A}^\alpha_{-1}(s)$ (see [50] for more details). As in [20], we can prove that the following embeddings hold (the operator $\mathbf{A}(t)$ is acretive and, as a consequence of that, it will have bounded imaginary powers, and hence the fractional power spaces will coincide with the interpolation spaces):

$$\begin{cases} Y^\alpha_{(-1)} \subset H^1(\Omega) \times H^{2\alpha-1}(\Omega) \subset L^{q_1}(\Omega) \times L^{q_2}(\Omega), \\ \text{for } 1 \leqslant q_1 \leqslant \dfrac{2n}{n-2}, \quad 1 \leqslant q_2 \leqslant \dfrac{2n}{n-2(2\alpha-1)}, \quad \alpha \in [0,1], \quad n \geqslant 3. \end{cases} \tag{5.9}$$

Using this, we establish the following local well posedness result for (5.7).

Theorem 5.1.4 *The operator family $\mathbf{A}(t)$ is uniformly sectorial and uniformly Hölder continuos, and $F : \mathbb{R} \times Y^{\frac{1}{2}}_{(-1)} \to Y_{(-1)}$ defined by $F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \mathbf{f}(u) \end{bmatrix}$ is Lipschitz continuous in bounded subsets of $Y^{\frac{1}{2}}_{(-1)}$ then, given $r > 0$ there exists $\tau > 0$ and for each $U_0 \in Y^{\frac{1}{2}}_{(-1)}$ with $\|U_0\|_{Y^{\frac{1}{2}}_{(-1)}} \leqslant r$ a function $U(\cdot, s, U_0) \in C([s, s+\tau], Y^{\frac{1}{2}}_{(-1)}) \cap C^1((s, s+\tau], Y^{\frac{1}{2}}_{(-1)})$ with the properties that*

$$\{U_0 \in Y^{\frac{1}{2}}_{(-1)} : \|U_0\|_{Y^{\frac{1}{2}}_{(-1)}} \leqslant r\} \ni U_0 \mapsto U(\cdot, s, U_0) \in C([s, s+\tau], Y^{\frac{1}{2}}_{(-1)})$$

is continuous, $U(\cdot, s, U_0)$ is the unique solution of (5.7) (in the sense of Definition 5.1.2) satisfying $U(s, s, U_0) = U_0$. If $U_0 \in X^1 \times X^1$ and assuming

$$f'(0) = 0, \quad |f''(s)| \leqslant c(1 + |u|^{\rho-2}), \tag{5.10}$$

$U(\cdot)$ is twice continuously differentiable in $(0, \tau)$ with values in $X^1 \times X^1$.

The proof of this theorem consists on the verification of the assumptions of Theorem 5.1.3 and the proof of the time regularity in the last statement of it. First we prove the uniform sectoriality and the uniform Hölder continuity of $\mathbf{A(t)}$. Note that

$$(\lambda + \mathbf{A}(t))^{-1} = \begin{pmatrix} ((\lambda + \beta_\epsilon(t))I + \gamma(t)A)R(\lambda) & R(\lambda) \\ -AR(\lambda) & \lambda R(\lambda) \end{pmatrix}$$

where $R(\lambda) = (\lambda(\lambda + \beta_\epsilon(t))I + (1 + \lambda\gamma(t))A)^{-1}$. Therefore, using that the operator A is sectorial in $L^2(\Omega)$ (see [50]) and taking

$$|\lambda| > \max\left\{1, \frac{1}{1-(\gamma_1+\beta_{\varepsilon 1})}, \frac{1}{1-2\gamma_1}, \frac{\beta_{\varepsilon 0}-1}{\gamma_1+1}, \frac{2\beta_{\varepsilon 0}\|A^{-1}\|_{\mathcal{L}(X)}-1}{\gamma_1+2\|A^{-1}\|_{\mathcal{L}(X)}}\right\},$$

we obtain the following estimates

$$\begin{aligned}\|R(\lambda)\|_{\mathcal{L}(X^\alpha)} &= \left|\frac{1}{1+\lambda\gamma(t)}\right| \left\|\left(\frac{\lambda(\lambda+\beta_\varepsilon(t))}{1+\lambda\gamma(t)} + A\right)^{-1}\right\|_{\mathcal{L}(X^\alpha)} \\ &\leqslant \frac{M}{|1+\lambda\gamma(t)| + |\lambda(\lambda+\beta_\varepsilon(t))|},\end{aligned}$$

$$\begin{aligned}\|\lambda R(\lambda)\|_{\mathcal{L}(X^\alpha)} &= \left\|\frac{\lambda}{1+\lambda\gamma(t)}\right\| \left\|\left(\frac{\lambda(\lambda+\beta_\varepsilon(t))}{1+\lambda\gamma(t)} + A\right)^{-1}\right\| \\ &\leqslant \frac{M|\lambda|}{|1+\lambda\gamma(t)| + |\lambda(\lambda+\beta_\varepsilon(t))|}\end{aligned}$$

$$\begin{aligned}\|-AR(\lambda)\|_{\mathcal{L}(X^\alpha)} &= \|\left(\lambda(\lambda+\beta_\varepsilon(t))A^{-1} + (1+\lambda\gamma(t))I\right)^{-1}\|_{\mathcal{L}(X^\alpha)} \\ &= \left|\frac{1}{1+\lambda\gamma(t)}\right| \left\|\left(\frac{\lambda(\lambda+\beta_\varepsilon(t))}{1+\lambda\gamma(t)}A^{-1} + I\right)^{-1}\right\|_{\mathcal{L}(X^\alpha)} \\ &\leqslant \frac{1}{2|1+\lambda\gamma(t)|}.\end{aligned}$$

It is therefore easy to check that

$$\left\|(\lambda + \mathbf{A}(t))^{-1}\right\|_{\mathcal{L}(Y_{(-1)})} \leqslant \frac{C}{|\lambda|+1}.$$

Now,

$$[\mathbf{A}(t) - \mathbf{A}(s)]\mathbf{A}(s)^{-1} = -(\gamma(t)-\gamma(s))\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix} - (\beta_\epsilon(t)-\beta_\epsilon(s))\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}.$$

From this and from the Hölder continuity of γ and β_ϵ uniformly in $\mathbb{R}$ and in ϵ, it is clear that

$$\|[\mathbf{A}(t) - \mathbf{A}(s)]\mathbf{A}(s)^{-1}\|_{\mathcal{L}(Y_{(-1)})} \leqslant M|t-s|^\epsilon$$

which proves the uniform Hölder continuity condition *b*).

Lemma 5.1.5 *If $f:\mathbb{R}\to\mathbb{R}$ satisfies (5.4) then,*

$$\begin{aligned}&\|\mathbf{f}(w_1)-\mathbf{f}(w_2)\|_{L^{\frac{2n}{n+2}}}\\&\quad\leqslant c\|w_1-w_2\|_{H^1}\left(1+\|w_1\|_{H^1}^{\rho-1}+\|w_2\|_{H^1}^{\rho-1}\right),\ w_1,w_2\in H^1(\Omega).\end{aligned}\tag{5.11}$$

Proof: From (5.4), the Hölder inequality, and the Sobolev embeddings we obtain:

$$\begin{aligned}&\|\mathbf{f}(w_1)-\mathbf{f}(w_2)\|_{L^{\frac{2n}{n+2}}(\Omega)}\\
&\leqslant c\left[\int_\Omega\left[|w_1-w_2|(1+|w_1|^{\rho-1}+|w_2|^{\rho-1})\right]^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}}\\
&\leqslant c\left[\int_\Omega|w_1-w_2|^{\frac{2n}{n-2}}\right]^{\frac{n-2}{2n}}\left[\int_\Omega(1+|w_1|^{\rho-1}+|w_2|^{\rho-1})^{\frac{n}{2}}\right]^{\frac{2}{n}}\\
&\leqslant c\|w_1-w_2\|_{L^{\frac{2n}{n-2}}(\Omega)}\|1+|w_1|^{\rho-1}+|w_2|^{\rho-1}\|_{L^{\frac{n}{2}}(\Omega)}\\
&\leqslant \tilde{c}\|w_1-w_2\|_{L^{\frac{2n}{n-2}}(\Omega)}\left(1+\|w_1\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}(\Omega)}+\|w_2\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}(\Omega)}\right).\end{aligned}$$

Since

$$\frac{2n(\rho-1)}{4}\leqslant\frac{2n}{n-2},$$

we have that

$$L^{\frac{n(\rho-1)}{2}}(\Omega)\supset H^1(\Omega).$$

Which completes the proof.

□

Then, the following result is a consequence of the result before.

Lemma 5.1.6 *If $f:\mathbb{R}\to\mathbb{R}$ satisfies the above conditions then,*

$$\begin{aligned}&\|F\begin{pmatrix}\phi_1\\\psi_1\end{pmatrix}-F\begin{pmatrix}\phi_2\\\psi_2\end{pmatrix}\|_{Y_{(-1)}}\leqslant c\|\begin{pmatrix}\phi_1\\\psi_1\end{pmatrix}-\begin{pmatrix}\phi_2\\\psi_2\end{pmatrix}\|_{Y^{\frac12}_{(-1)}}\\
&\quad\leqslant c\left(1+\|\begin{pmatrix}\phi_1\\\psi_1\end{pmatrix}\|^{\rho-1}_{Y^{\frac12}_{(-1)}}+\|\begin{pmatrix}\phi_2\\\psi_2\end{pmatrix}\|^{\rho-1}_{Y_{(-1)}}\right),\ \begin{pmatrix}\phi_1\\\psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\\psi_2\end{pmatrix}\in Y^{\frac12}_{(-1)}.\end{aligned}\tag{5.12}$$

Thanks to the previous result and Theorem 5.1.3, all assertions in Theorem 5.1.4 are fulfilled, except the last one. To see that, if $U_0 \in X^1 \times X^1$, $U(\cdot, U_0) \in C^2((0,\tau), X^{\frac{1}{2}} \times X)$ we apply again Theorem 5.1.3 to the following Cauchy problem

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \mathcal{A}(t)\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \bar{F}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \; t > s \\ &\begin{bmatrix} u(s) \\ v(s) \\ w(s) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ -Au_0 - \gamma'(s)Av_0 - \beta_\epsilon(s)v_0 + \mathbf{f}(u_0) \end{bmatrix} \end{aligned} \tag{5.13}$$

with $\mathcal{Z} = X^1 \times X^1 \times X$, $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset \mathcal{Z} \to \mathcal{Z}$, $D(\mathcal{A}(t)) = X^1 \times X^1 \times X^1$

$$\mathcal{A}(t) = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ 0 & (1+\gamma'(t))A + \beta'_\epsilon(t)I & \gamma(t)A + \beta_\epsilon(t)I \end{bmatrix} \text{ and } \bar{F}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\mathbf{f}}(u,v) \end{bmatrix}$$

with $\mathbf{f}(u,v)(x) := f'(u(x))v(x)$. Firstly, we need to rewrite the problem since $\mathcal{A}(t)$ is singular and is not invertible. Therefore, we consider

$$\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \mathcal{C}(t)\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \bar{H}\begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

with

$$\mathcal{C}(t) = \begin{bmatrix} -I & 0 & 0 \\ 0 & 0 & -I \\ 0 & (N+\gamma'(t))A + \beta'_\epsilon(t)I & \gamma(t)A + \beta_\epsilon(t)I \end{bmatrix}$$

and

$$\bar{H}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u - v \\ 0 \\ \bar{\mathbf{h}}(u,v) \end{bmatrix}$$

with $\mathbf{h}(u,v)(x) := (1+N)Av + f'(u(x))v(x)$ and N large enough. We want to remark that we can write the operator $\mathcal{C}(t)$ as

$$C(t) = \begin{bmatrix} -\frac{1}{N+\gamma'(t)}I & 0 & 0 \\ 0 & 0 & -I \\ 0 & A & \frac{\gamma(t)}{N+\gamma'(t)}A + \frac{\beta_\epsilon(t)}{N+\gamma'(t)}I \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \left(1 + \frac{1}{N+\gamma'(t)}\right)I \\ 0 & \frac{\beta'_\epsilon(t)}{N+\gamma'(t)}I & 0 \end{bmatrix},$$

In this way, it is easy to prove the uniformly sectoriality of the operator based on the calculations for $\mathbf{A}(t)$ and Theorem 1.3.2 in [50]. To show the uniform

Hölder continuity condition, we observe that

$$[\mathcal{C}(t)-\mathcal{C}(s)]\mathcal{C}(\tau)^{-1}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & R_1(t,s,\tau)W(\tau)-R_2(t,s) & R_1(t,s,\tau)\end{bmatrix}$$

where

$$R_1(t,s,\tau)=[(\gamma'(t)-\gamma'(s))A-(\beta'_\varepsilon(t)-\beta'_\varepsilon(s))I][(N+\gamma'(\tau))A+\beta'_\varepsilon(\tau)I]^{-1},$$
$$R_2(t,s)=(\gamma(t)-\gamma(s))A+(\beta_\varepsilon(t)-\beta_\varepsilon(s))I,$$
$$W(\tau)=\gamma(\tau)A+\beta_\epsilon(\tau)I.$$

Due to the assumptions on $\beta_\varepsilon(\cdot)$ and $\gamma(\cdot)$ and the fact that the operator $A:X^1\to X$ is bounded, $|R_2(t,s)|\leqslant M|t-s|$ and $W(\cdot)$ is uniformly bounded. Then we only need to check $R_1(t,s,\tau)$. Taking N large enough such that $\left|\frac{\beta'_\varepsilon(\tau)}{N+\gamma'(\tau)}\right|\|A^{-1}\|_{\mathcal{L}(X)}<\frac{1}{2}$,

$$\begin{aligned}\|(\gamma'(t)-\gamma'(s))A[(N+\gamma'(\tau))A+\beta'_\varepsilon(\tau)I]^{-1}\|_{\mathcal{L}(X)}&\\ &=\frac{|\gamma'(t)-\gamma'(s)|}{|N+\gamma'(\tau))|}\|[I+\frac{\beta'_\varepsilon(\tau)}{N+\gamma'(\tau)}A^{-1}]^{-1}\|_{\mathcal{L}(X)}\\ &\leqslant 2\frac{|\gamma'(t)-\gamma'(s)|}{|N+\gamma'(\tau)|}\\ &\leqslant C|t-s|^\theta.\end{aligned}$$

In a similar way we can prove that

$$\|(\beta'_\varepsilon(t)-\beta'_\varepsilon(s))[(N+\gamma'(\tau))A+\beta'_\varepsilon(\tau)I]^{-1}\|_{\mathcal{L}(X)}\leqslant C|t-s|^\theta.$$

Using techniques similar to those used to study $\mathbf{f}$ in Lemma 5.1.6, $\bar{\mathbf{h}}:\mathcal{Z}\to\mathcal{Z}$ is Lipschitz in bounded subsets of $\mathcal{Z}$, what ensures that $\bar{H}$ also satisfies the hypotheses of Theorem 5.1.3. Therefore there exits a unique solution of system (5.13), which proves the last statement of Theorem 5.1.4.

Remark 13 *Note that, if $u\in X^{\frac{1}{2}}$ and following Lemma 5.1.5, $\mathbf{f}(u)\in L^{\frac{2n}{n+2-2\epsilon}}\hookrightarrow X^{-\frac{1}{2}+\epsilon}$ for some $\epsilon>0$ such that*

$$0<\epsilon=\frac{n(2-\rho)}{2}+\rho. \tag{5.14}$$

Hence, we may obtain that the solutions given in Theorem 5.1.4 with initial data in $X^{\frac{1}{2}+\epsilon}\times X^\epsilon$ are, in fact, in $X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}$ for $t\in(s,s+\tau)$. We can repeat this reasoning, noting that if $u\in X^{\frac{1}{2}+\epsilon}$, then $\mathbf{f}(u)\in X^{-\frac{1}{2}+\rho\epsilon}$ and, after a finite number of steps, we obtain that solutions with initial data in $X^1\times X^{\frac{1}{2}}$ are in $X^1\times X^1$.

5.2 The pullback attractor

From now on, we denote by $Y^0 := H_0^1(\Omega) \times L^2(\Omega)$, which will be the phase space of our process (notice that with the notation used in Section 5.1, $Y^0 = X^{1/2} \times X$ which is isomorphic to $Y^{\frac{1}{2}}_{(-1)}$).

5.2.1 Existence

For $b \in \mathbb{R}^+$, let us define $L_b(\cdot,\cdot) : Y^0 \longrightarrow \mathbb{R}$ as

$$L_b(\phi, \varphi) = \frac{1}{2}\|\phi\|^2_{H_0^1} + \frac{1}{2}\|\varphi\|^2_{L^2} + b(\phi, \varphi)_{L^2} - \int_\Omega G(\phi), \tag{5.15}$$

with $G(\tau) = \int_0^\tau f(\theta)d\theta$. This functional is the same as $\mathcal{V}(\cdot,\cdot)$ that has been defined in (4.12) (page 84).

Thanks to the regularity of the solutions (see Theorem 5.1.4), if (u, u_t) denotes the solution of (5.7) with $(u(s), u_t(s) = (u_0, v_0)$, we can differentiate the expression $L_b(u, u_t)$,

$$\begin{aligned}
\frac{d}{dt}L_b(u, u_t) = {}& (u_{tt} - \Delta u - f(u), u_t)_{L^2} + b\|u_t\|^2_{L^2} \\
&+ b(u, \Delta u + \gamma(t)\Delta u_t - \beta_\varepsilon(t)u_t + f(u))_{L^2} \\
&\leqslant -\gamma_0\|u_t\|^2_{H_0^1} - \beta_{0\varepsilon}\|u_t\|^2_{L^2} - b\gamma(t)(u, u_t)_{H_0^1} - b\|u\|^2_{H_0^1} \\
&- b\beta_\varepsilon(t)(u, u_t)_{L^2} + b(u, f(u))_{L^2} \\
&\leqslant -\frac{\gamma_0}{2}\|u_t\|^2_{H_0^1} - \frac{\lambda_1\gamma_0}{2}\|u_t\|^2_{L^2} - \beta_{0\varepsilon}\|u_t\|^2_{L^2} - b\|u\|^2_{H_0^1} \\
&+b\gamma_1\left(\frac{1}{2a_1}\|u\|^2_{H_0^1} + \frac{a_1}{2}\|u_t\|^2_{H_0^1}\right)+b\beta_{1\varepsilon}\left(\frac{1}{2a_2}\|u\|^2_{L^2} + \frac{a_2}{2}\|u_t\|^2_{L^2}\right) \\
&+ b(\delta\|u\|^2_{H_0^1} + C_\delta).
\end{aligned} \tag{5.16}$$

Hence, for a suitable choice of a_1, a_2, b and δ, there is a $C_0 > 0$ such that

$$\frac{d}{dt}L_b(u, u_t) \leqslant -C_0\left(\|u\|^2_{H_0^1} + \|u_t\|^2_{L^2}\right) + bC_\delta. \tag{5.17}$$

From (5.16) with $b = 0$, given $r > 0$

$$B_r = \sup\{\|u\|^2_{H_0^1(\Omega)} + \|u_t\|^2_{L^2(\Omega)} : t \geqslant s,\ u \in H_0^1(\Omega),\ \text{with}\ \|(u(t), u_t(t))\|_{Y^0} \leqslant r\} < \infty. \tag{5.18}$$

Now, if $\|(u(t),u_t(t))\|^2_{Y^0} > \frac{bC_\delta}{C_0} + \frac{1}{C_0} = r_0^2$ for all $t \geqslant s$, then there exists a time $T_r > 0$ such that $L_b\left(u\left(t\right),u_t\left(t\right)\right) \leqslant 0$ for each $t \geqslant s + T_r$. In other case, there exists t_u such that $\|(u(t_u),u_t(t_u))\|^2_{Y^0} \leqslant \frac{bC_\delta}{C_0} + \frac{1}{C_0} = r_0^2$ (and let t_u be the smallest time with this property), and then $\|(u(t),u_t(t))\|_{Y^0} \leqslant R_1$ for all $t \geqslant t_u$, where R_1 is the radius of a ball containing the set B_{r_0} (defined in (5.18)). If $t_u \leqslant T_r$, then $\|(u(t),u_t(t))\|_{Y^0} \leqslant R_1$ for all $t \geqslant T_r$. If $t_u > T_r$, then $L_b\left(u\left(t\right),u_t\left(t\right)\right) \leqslant 0$ for $t \in [T_r, t_u]$ and $\|(u(t),u_t(t))\|_{Y^0} \leqslant R_1$ for all $t \geqslant t_u$.

This implies that the ball of radius R_0 pullback strongly absorbs bounded subsets of Y^0 where $R_0^2 = \max\{R_1^2, 8K\}$.

Let us now consider the process $\{S(t,s); t \geqslant s\}$ generated by our problem in the phase space Y^0. Recall that $S(t,s)$ is given as follows.

For each initial value $w_0 := (u_0, v_0) \in Y^0$ and each initial time $s \in \mathbb{R}$, system (5.7) possesses a unique solution which can be written as

$$S(t,s)w_0 = T(t,s)w_0 + U(t,s)w_0 = \begin{pmatrix} u(t,s,w_0) \\ u_t(t,s,w_0) \end{pmatrix}, \tag{5.19}$$

where $T(t,s)$ is the evolution process associated to the linear part of (5.7)(i.e. for $f = 0$), and

$$U(t,s)w_0 = \int_s^t T(t,\tau)F(S(\tau,s)w_0)d\tau. \tag{5.20}$$

Proceeding as in (5.16) and using the functional $\tilde{L}_b(u,u_t) = \frac{1}{2}\|u\|^2_{H^1_0} + \frac{1}{2}\|u_t\|_{L^2} + b(u,u_t)_{L^2}$ being now (u,u_t) a solution of the linear part of (5.7), we can prove that this linear part decays exponentially to zero. The compactness of U follows from the following facts.

If we assume that $\rho < \frac{n+2}{n-2}$, then we can choose $s \in (\frac{1}{2}, 1)$ such that $\rho \leqslant \frac{n+2s}{n-2}$, and therefore we have the following chain of inclusions:

$$X^{1/2} \hookrightarrow L^{2n/(n-2)} \xrightarrow{f} L^{2n/(n+2s)} \hookrightarrow X^{-s/2} \subset\subset X^{-1/2},$$

being the last inclusion compact. Thanks to the assumptions on the function f and following the proof of Theorem 4.2.3 (page 85), is easy to see that **f** is compact; this fact implies that F is also compact and, consequently, the operator $U(t,s)$ is compact as well.

Therefore we can apply Theorem 3.2.5 and Theorem 3.2.4 (pages 54 and 53 respectively) to conclude that there exists the pullback attractor $\{\mathscr{A}(t) :$

$t \in \mathbb{R}\}$ in Y^0. Moreover we have that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \quad \text{is bounded in } Y^0, \tag{5.21}$$

and the bound does not depend on ε.

5.2.2 Regularity

5.2.3 Energy estimates in regularity

In this section we will prove that the pullback attractor is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$ using only energy estimates. Taking a solution passing through a point in the pullback attractor, we will denote this solution by $(u(t,s), u_t(t,s))$ for $t > s$. We will omit the arguments in u and u_t when no confusion is possible, and will explicit them otherwise.

Thanks to the analysis carried out in our previous section, we have that the pullback attractor is inside a fixed bounded subset of Y^0. Our aim in this section is to prove more regularity, which will be necessary to obtain a gradient structure for the attractor. We will prove this regularity using the abstract theory of the fractional power of the operator A.

Now we will prove that the attractor is a bounded set of $D(A) \times D(A)$. Thanks to (5.21), we have that the attractor can be written as the set of all global bounded solutions

$$\begin{aligned}\{\mathscr{A}(t) : t \in \mathbb{R}\} = \{&\xi : \mathbb{R} \to Y^0, \text{ such that} \\ &\xi \text{ is a global and bounded solution for (5.1)}\}.\end{aligned} \tag{5.22}$$

Hence, if $\xi(\cdot) = \begin{bmatrix} u(\cdot) \\ u_t(\cdot) \end{bmatrix} : \mathbb{R} \to Y^0$ is such that $\xi(t) \in \mathscr{A}(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) = T(t,s)\xi(s) + \int_s^t T(t,\theta)F(\xi(\theta))d\theta,$$

and we have that it can be written as

$$\xi(t) = \int_{-\infty}^t T(t,\theta)F(\xi(\theta))d\theta. \tag{5.23}$$

Due to Theorem 5.1.4, $\xi(t) \in C(\mathbb{R}, H_0^1(\Omega)\times H_0^1(\Omega))$ and $\xi'(t) \in C(\mathbb{R}, H_0^1(\Omega)\times L^2(\Omega))$. Therefore $\mathscr{A}(t) = S(t,s)\mathscr{A}(s) \subset H_0^1(\Omega) \times H_0^1(\Omega)$ for all $t \in \mathbb{R}$.

Now we will prove a higher regularity following the ideas in [13]. If $\xi = \begin{bmatrix} u \\ u_t \end{bmatrix} : \mathbb{R} \to Y^0$ is a global solution of (5.7) in the pullback attractor and $w_0 = \xi(s)$, consider

$$W(t,s) := \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = \int_s^t T(t,\theta)F(S(\theta,s)w_0)\,d\theta \tag{5.24}$$

and note that,

$$\begin{cases} w_{tt} - \gamma(t)\Delta w_t - \Delta w + \beta_\varepsilon(t)w_t = f(u(t,s;w_0)), \\ w(s) = w_t(s) = 0. \end{cases} \tag{5.25}$$

Proceeding as before, we can prove that

$$\{W(t,s) : t \in \mathbb{R}\} \quad \text{is bounded in } H_0^1 \times L^2. \tag{5.26}$$

Coming back to (5.16), taking $b = 0$ in (5.15) and integrating in $[a,b] \subseteq [t,s]$ we have

$$\int_a^b \frac{d}{dt}L_0 + \frac{\gamma_0}{2}\int_a^b \|u_t\|_{H_0^1}^2 \leqslant 0.$$

Therefore,

$$\int_a^b \|u_t\|_{H_0^1}^2 \leqslant K_1, \tag{5.27}$$

where $K_1 > 0$ is a constant which only depends on the bound of the pullback attractor in Y^0.

Consider the next functional,

$$V(\phi,\varphi,t) = \frac{\gamma(t)}{2}\|\varphi\|_{H_0^1}^2 + (\phi,\varphi)_{L^2} - (f(\phi),\varphi)_{L^2}. \tag{5.28}$$

Multiply (5.1) by u_{tt} and integrate in Ω, obtaining

$$\|u_{tt}\|_{L^2}^2 + \frac{d}{dt}V(u,u_t,t) = (f'(u)u_t,u_t)_{L^2} - \beta_\varepsilon(t)(u_t,u_{tt})_{L^2} - \left(\frac{\gamma'(t)+1}{2}\right)\|u_t\|_{H_0^1}^2.$$

Using the Hölder inequality and the Sobolev inclusions (taking into account that $\rho < \frac{n+2}{n-2}$) we have

$$
\begin{aligned}
(f'(u)u_t, u_t)_{L^2} &\leqslant \int_\Omega |f'(u)||u_t|^2 \\
&\leqslant \left(\int_\Omega |f'(u)|^{n/2}\right)^{2/n} \left(\int_\Omega |u_t|^{2n/(n-2)}\right)^{(n-2)/n} \\
&\leqslant c\left(\int_\Omega (1+|u|^{n/2})^{\rho-1}\right)^{2/n} \|u_t\|^2_{H^1_0} \\
&\leqslant c\left(1+\|u\|^{\rho-1}_{H^1_0}\right)\|u_t\|^2_{H^1_0} \\
&\leqslant c\|u_t\|^2_{H^1_0}
\end{aligned}
\tag{5.29}
$$

Therefore,

$$
\|u_{tt}\|^2_{L^2} + \frac{d}{dt}V(u,u_t,t) \leqslant \left(c + \frac{\beta^2_{1,\epsilon}}{2\lambda_1} + \frac{M_\gamma+1}{2}\right)\|u_t\|^2_{H^1_0} + \frac{1}{2}\|u_{tt}\|^2_{L^2},
$$

where $M_\gamma > 0$ is a bound for $\gamma'(t)$, and we can conclude that

$$
\frac{1}{2}\|u_{tt}(\tau,s)\|^2_{L^2} + \frac{d}{dt}V(u(\tau,s),u_t(\tau,s),\tau) \leqslant C\|u_t(\tau,s)\|^2_{H^1_0}, \quad \text{for all } \tau\in[s,t].
\tag{5.30}
$$

Now, with the help of the previous inequality, we will obtain a bound for $\|u_t(t,s)\|_{H^1_0}$ for $t > s+1$. First, we multiply (5.30) by $(\tau - s)$ and integrate in the interval $[s, s+1]$. Thus,

$$
\int_s^{s+1}(\tau-s)\frac{d}{d\tau}V(u,u_t,\tau)\,d\tau + \frac{1}{2}\int_s^{s+1}(\tau-s)\|u_{tt}\|^2_{L^2}\,d\tau \leqslant C\int_s^{s+1}(\tau-s)\|u_t\|^2_{H^1_0}\,d\tau,
$$

and there is a constant $C_1 > 0$ such that

$$
\begin{aligned}
V(u(s+1,s),u_t(s+1,s),s+1) &+ \frac{1}{2}\int_s^{s+1}(\tau-s)\|u_{tt}\|^2_{L^2}\,d\tau \\
&\leqslant C\int_s^{s+1}(\tau-s)\|u_t\|^2_{H^1_0}\,d\tau + \int_s^{s+1}V(u,u_t,\tau)\,d\tau \leqslant C_1.
\end{aligned}
$$

Here we have used (5.27) and that there exists two constants K_1, $K_2 > 0$ such that

$$
K_1(\|u_t\|^2_{H^1_0} - 1) \leqslant V(u,u_t,t) \leqslant K_2(\|u_t\|^2_{H^1_0} + 1),
\tag{5.31}
$$

for all $(u,u_t) \in \{\mathscr{A}(t) : t\in\mathbb{R}\}$, since

$$
\begin{aligned}
|(u,u_t) - (f(u),u_t)| &\leqslant \frac{1}{2}\|u_t\|^2_{L^2} + \frac{1}{2}\|u\|^2_{L^2} + \|f(u)\|_{L^{\frac{2n}{n+2}}}\|u_t\|_{L^{\frac{2n}{n-2}}} \\
&\leqslant \frac{1}{2}\|u_t\|^2_{L^2} + \frac{1}{2}\|u\|^2_{L^2} + c\left(1+\|u\|^\rho_{H^1_0}\right)\|u_t\|_{H^1_0}.
\end{aligned}
$$

Now, integrating (5.30) in $[s+1,t]$, for $t>s+1$, we obtain

$$\int_{s+1}^{t} \frac{d}{dt} V(u(\tau,s),u_t(\tau,s),\tau)\,d\tau + \frac{1}{2}\int_{s+1}^{t} \|u_{tt}(\tau,s)\|_{L^2}^2\,d\tau \leqslant c\int_{s+1}^{t} \|u_t(\tau,s)\|_{H_0^1}^2\,d\tau,$$

and

$$V(u(t,s),u_t(t,s),t) + \frac{1}{2}\int_{s+1}^{t} \|u_{tt}\|_{L^2}^2\,d\tau \leqslant c\int_{s+1}^{t} \|u_t\|_{H_0^1}^2\,d\tau + V(u(s+1,s),u_t(s+1,s),s+1).$$

By (5.31) and (5.27),

$$K_1(\|u_t(t,s)\|_{H_0^1}^2 - 1) + \frac{1}{2}\int_{s+1}^{t} \|u_{tt}\|_{L^2}^2\,d\tau \leqslant c\int_{s+1}^{t} \|u_t\|_{H_0^1}^2\,d\tau + C_1 \leqslant C.$$

Thus, we have a uniform bound for $\|u_t(t,s)\|_{H_0^1}$ which does not depend on t or s. Choosing s and t accordingly we have that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \quad \text{is bounded in } H_0^1(\Omega)\times H_0^1(\Omega). \tag{5.32}$$

5.2.4 Regularity in $H^2(\Omega)\cap H_0^1(\Omega)\times H^2(\Omega)\cap H_0^1(\Omega)$

In this section we prove that the pullback attractor is more regular, based on the fractional power of the operator and the grow conditions over f.

Using the fact that $\mathbf{f}$ takes bounded subsets of $X^{\frac{1}{2}}$ into bounded subsets of $X^{-\frac{1}{2}+\epsilon_1}$ (with ϵ_1 as in (5.14)) we can state the problem (5.7) in $X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}$ with $\epsilon<\epsilon_1$ (note that $W(0)=\begin{bmatrix}0\\0\end{bmatrix}\in X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}$). We now use (5.24) to obtain the uniform bounds for $W(t,s)$ in $X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}$.

$$\begin{aligned}
&\|W(t,s)\|_{X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}}\\
&\quad\leqslant \int_s^t \|L(t,\theta)\|_{\mathcal{L}\left(X^{\frac{1}{2}+\epsilon_1}\times X^{-\frac{1}{2}+\epsilon_1},X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}\right)} \|F(S(\theta,s)w_0)\|_{X^{\frac{1}{2}+\epsilon_1}\times X^{-\frac{1}{2}+\epsilon_1}}\,d\theta\\
&\quad\leqslant K\int_s^t (t-\theta)^{-1+\epsilon-\epsilon_1} e^{-\alpha(t-\theta)}\,d\theta.
\end{aligned}$$

Therefore, noting that $\xi(t)=\lim_{s\to-\infty} U(t,s)$ as in (4.29) (page 88), we have that

$$\sup_{\xi\in\mathfrak{A}}\sup_{t\in\mathbb{R}}\left\{\|\xi(t)\|_{X^{\frac{1}{2}+\epsilon}\times X^{\frac{1}{2}+\epsilon}}\right\}<\infty, \tag{5.33}$$

where $\mathfrak{A}$ is the set of global bounded solutions for (5.7). We can now repeat this procedure (since $\mathbf{f}$ takes bounded subsets of $X^{\frac{1}{2}+\epsilon}$ into bounded subsets of $X^{-\frac{1}{2}+\rho\epsilon}$) to obtain a bound for the pullback attractor in $X^{\frac{1}{2}+\rho\epsilon} \times X^{\frac{1}{2}+\rho\epsilon}$. In a finite number of steps we arrive at

$$\sup_{\xi\in\mathfrak{A}} \sup_{t\in\mathbb{R}} \{\|\xi(t)\|_{X^1\times X^1}\} < \infty, \tag{5.34}$$

Since $\mathbf{f}$ takes bounded subsets of X^1 into bounded subsets of X, we have a uniform bound for u_{tt} in L^2 inside the attractor, noting that

$$\|u_{tt}\|_{L^2} = \|\Delta u + \gamma(t)\Delta u_t - \beta_\varepsilon(t)u_t + f(u)\|_{L^2} < \infty.$$

From this we deduce that

$$\sup\{\|\xi(t)\|_{X^1\times X^1}+\|\xi'(t)\|_{X^1\times X} : \ \xi \in \mathfrak{A}, \ t \in \mathbb{R}\} < \infty; \tag{5.35}$$

Therefore, noting that $\xi(t) = \lim_{s\to-\infty} W(t,s)$, we have that

$$\sup_{\xi\in\mathfrak{A}} \sup_{t\in\mathbb{R}} \{\|\xi(t)\|_{(H^2\cap H_0^1)\times H_0^1}, \|\xi_t(t)\|_{Y^0}\} < \infty, \tag{5.36}$$

where $\mathfrak{A}$ is the set of global bounded solutions for (5.7). At light of the fact that $\mathbf{f}$ takes bounded subsets of H^2 into bounded subsets of L^2 and using the equation (5.1) we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathscr{A}(t) \quad \text{is bounded in } H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega). \tag{5.37}$$

5.3 Structure of the limit pullback attractor

Let $\{\mathscr{A}_0(t) : t \in \mathbb{R}\}$ be the pullback attractor for (5.7) when the parameter $\varepsilon = 0$. Our aim is to prove the continuity of the attractors showing the upper and lower-semicontinuity. In the second case we will need some particular structure for the limit problem

$$\begin{cases} u_{tt} - \gamma(t)\Delta u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0, \text{ in } \partial\Omega. \end{cases} \tag{5.38}$$

Proceeding as in [13], we need to assume that there exist only finitely many solutions $\{u_1^*, \cdots, u_p^*\}$ of

$$\begin{cases} \Delta u + f(u) = 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{5.39}$$

Denote by $\mathcal{E} = \{e_1^*, \cdots, e_p^*\}$ where $e_i^* = \left(\begin{smallmatrix} u_i^* \\ 0 \end{smallmatrix}\right)$. Under this assumption, we prove in this section that we can write the limit attractor as

$$\mathscr{A}_0(t) = \bigcup_{i=1}^{p} W^u(e_i^*)(t), \text{ for all } t \in \mathbb{R}. \tag{5.40}$$

Due to the fact that $\gamma(\cdot)$ is bounded and Lipschitz, given a sequence $\{t_n\} \subset \mathbb{R}$ we have that $\{\gamma(t_n + t) = \gamma_n(t)\}$ has a convergent subsequence $\gamma_n(t) \to \lambda(t)$ uniformly for t in compact subsets of $\mathbb{R}$, which is also bounded $(\gamma_0 \leqslant \lambda(t) \leqslant \gamma_1)$ and Lipschitz. Let us consider the following problems,

$$\begin{cases} u_{tt} - \gamma_n(t)\Delta u_t - \Delta u = f(u), \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega), \ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{5.41}$$

and

$$\begin{cases} u_{tt} - \lambda(t)\Delta u_t - \Delta u = f(u), \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega), \ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{5.42}$$

with solutions (u, u_t) and (v, v_t) respectively. Our aim is to compare solutions of the above problems with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathscr{A}_n(s)$ where $\{\mathscr{A}_n(t) : t \in \mathbb{R}\}$ is the pullback attractor for (5.41).

We note that, proceeding exactly as in the previous section we obtain that

$$\bigcup \{\mathscr{A}_n(t) \cup \mathscr{A}_0(t) : n \in \mathbb{N}, \ t \in \mathbb{R}\} \tag{5.43}$$

is bounded in $H_0^1 \times H_0^1$. For $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathscr{A}_n(s)$, let $\xi_n(t)$ and $\xi_\infty(t)$ be the solutions of (5.41) and (5.42), respectively.

Defining $z = \xi_n - \xi_\infty$, we have

$$\begin{cases} z_{tt} = \Delta z + \lambda(t)\Delta z_t + (\gamma_n(t) - \lambda(t))\Delta(\xi_n)_t + f(\xi_n) - f(\xi_\infty) \quad \text{in } \Omega \\ z = 0 \quad \text{in } \partial\Omega. \end{cases} \tag{5.44}$$

To prove the convergence of these solutions we consider the following,

$$\begin{aligned}
\frac{d}{dt}\Big[\frac{1}{2}\|z\|^2_{H^1_0}+\frac{1}{2}\|z_t\|^2_{L^2}+b(z,z_t)_{L^2}\Big] &= (-\Delta z+z_{tt},z_t)_{L^2}+b\|z_t\|^2_{L^2}+b(z_{tt},z)_{L^2}\\
&\leqslant -\gamma_0\|z_t\|^2_{H^1_0}-b\|z\|^2_{H^1_0}+|\gamma_n(t)-\lambda(t)||((\xi_n)_t,z_t)_{H^1_0}|+b\lambda(t)|(z_t,z)_{H^1_0}|\\
&+b|\gamma_n(t)-\lambda(t)||((\xi_n)_t,z)_{H^1_0}|+|(f(\xi_n)-f(\xi_\infty),z_t)_{L^2}|\\
&+b|(f(\xi_n)-f(\xi_\infty),z)_{L^2}|\\
&\leqslant -\frac{\gamma_0}{2}\|z_t\|^2_{H^1_0}-\frac{b}{2}\|z\|^2_{H^1_0}+K|\gamma_n(t)-\lambda(t)|,
\end{aligned}$$

where we have used the bounds in (5.43) and similar estimates to those in (5.16). Therefore,

$$\begin{aligned}
\|z\|^2_{H^1_0}+\|z_t\|^2_{L^2} &\leqslant K\int_s^t|\gamma_n(\theta)-\lambda(\theta)|d\tau\\
&\leqslant \max_{\theta\in[t,s]}|\gamma_n(\theta)-\lambda(\theta)|K(t-s)\overset{n\to 0}{\longrightarrow}0,
\end{aligned}$$

and we can conclude that, for t in compact subsets of $\mathbb{R}$,

$$\|\xi_n(t)-\xi_\infty(t)\|_{Y^0}\overset{n\to\infty}{\longrightarrow}0. \tag{5.45}$$

We consider the Lyapunov function

$$\mathcal{L}(\phi,\varphi)=\frac{1}{2}(\|\phi\|^2_{H^1_0}+\|\varphi\|^2_{L^2})-\int_\Omega\int_0^{\phi(x)}f(s)\,dsdx.$$

Then $\mathbb{R}\ni t\mapsto\mathcal{L}(\xi(t),\xi_t(t))\in\mathbb{R}$ is non-increasing and the only global solution ξ where $\mathcal{L}$ is constant are the equilibria in $\mathcal{E}$. Since $\{\xi(t):t\in\mathbb{R}\}$ lies in a compact set, there are real numbers ℓ_i and ℓ_j such that

$$\ell_i\overset{t\to-\infty}{\longleftarrow}\mathcal{L}(\xi(t+r))\overset{t\to\infty}{\longrightarrow}\ell_j$$

for all $r\in\mathbb{R}$.

If $t_n\overset{n\to\infty}{\longrightarrow}\infty$, taking subsequences, if necessary, $\gamma(t_n+r)\overset{n\to\infty}{\longrightarrow}\lambda(r)$ uniformly in compact subsets of $\mathbb{R}$, $\xi(t_n+r)\overset{n\to\infty}{\longrightarrow}\zeta(r)$ in Y^0, uniformly for r in compact subsets of $\mathbb{R}$, and $(\zeta(t),\zeta_t(t))$ is a global solution of the problem

$$\begin{cases} u_{tt}-\lambda(t)\Delta u_t-\Delta u=f(u), \text{ in } \Omega,\\ u=0 \text{ in } \partial\Omega,\end{cases} \tag{5.46}$$

with the property that $\mathcal{L}(\zeta(t), \zeta_t(t)) = \ell_j$, for all $t \in \mathbb{R}$. Hence $\begin{pmatrix} \zeta(t) \\ \zeta_t(t) \end{pmatrix} = e_j^*$. Taking $\tilde{t}_n \overset{n\to\infty}{\longrightarrow} -\infty$, we obtain an analogous result.

Suppose that there are sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{\bar{t}_n\}_{n\in\mathbb{N}}$ with $t_{n+1} > \bar{t}_n > t_n$, $n \in \mathbb{N}$, such that $\xi(t_n) \overset{n\to\infty}{\longrightarrow} e_k^*$ and $\xi(\bar{t}_n) \overset{n\to\infty}{\longrightarrow} \bar{e}_k^*$. Now, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\mathcal{L}(\xi(t)) \in (\ell_j - \epsilon, \ell_j + \epsilon)$ for all $t \in [t_n, \bar{t}_n]$. If $\tau_n \in (t_n, \bar{t}_n)$, $\tau_n \overset{n\to\infty}{\longrightarrow} \infty$ and (taking subsequences if necessary), $\gamma(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{\lambda}(r)$. We have that $\xi(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{\zeta}(t)$, which is a solution of

$$\begin{cases} u_{tt} - \bar{\lambda}(t)\Delta u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \tag{5.47}$$

with $\mathcal{L}(\bar{\zeta}(t), \bar{\zeta}_t(t)) = \ell_j$ for all $t \in \mathbb{R}$, and, consequently, $\bar{\zeta}(t) \equiv e_m^*$ with $\mathcal{L}(e_m^*) = \ell_j$. That leads to a contradiction with the fact that there are only finitely many equilibria. Therefore, we can write the pullback attractor as in (5.40).

5.4 Continuity of the attractors

In this section we prove the continuity of the pullback attractors for (5.1) when $\beta_\varepsilon(t) \overset{\varepsilon\to 0}{\longrightarrow} 0$. From now on, we will denote by $\{S_\varepsilon(t, s) : t \geqslant s\}$ the process associated to (5.7) and by $\{\mathscr{A}_\varepsilon(t) : t \in \mathbb{R}\}$ the pullback attractor of $S_\varepsilon(t, s)$ for each ε.

5.4.1 Upper-semicontinuity

Let $U_0 \in H_0^1 \times L^2$, $v = S_0(t+s, s)U_0$, $u = S_\varepsilon(t+s, s)U_0$ and $w = u - v$, where $\{S_0(t, s) : t \geqslant s\}$ is the evolution process associated to the limit problem (5.38). We have that

$$\begin{cases} w_{tt} = \Delta w + \gamma(t)\Delta w_t - \beta_\varepsilon(t)u_t + f(u) - f(v), \\ w(s) = w_t(s) = 0. \end{cases} \tag{5.48}$$

Let us consider the functional $H(w, w_t) = \frac{1}{2}(\|w\|_{H_0^1}^2 + \|w_t\|_{L^2}^2)$. Then, there is a constant $K > 0$ such that

$$\begin{aligned} \frac{d}{dt}H(w, w_t) &= (w_{tt} - \Delta w, w_t)_{L^2} \\ &\leqslant -\gamma_0\|w_t\|_{H_0^1}^2 + \beta_\varepsilon(t)(u_t, w_t)_{L^2} + (f(u) - f(v), w_t)_{L^2} \\ &\leqslant K\, H(w, w_t) + K\beta_{1\varepsilon}, \end{aligned}$$

where we used that $f : H_0^1 \to L^{2n/n+2}$ is Lipschitz, Hölder's inequality and (5.43). By the Gronwall Lemma,

$$\|w\|_{H_0^1}^2 + \|w_t\|_{L^2}^2 \leqslant C\beta_{1\varepsilon} \int_s^t e^{K(t-\tau)} d\tau \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{5.49}$$

in compact subsets of $\mathbb{R}$, uniformly for U_0 in bounded subsets of $H_0^1 \times L^2$.

Let $\tau \in \mathbb{R}$ be such that $\operatorname{dist}(S_0(t,\tau)\mathcal{B}, \mathscr{A}_0(t)) < \frac{\delta}{2}$ where $\bigcup_{s\in\mathbb{R}} \mathscr{A}_\varepsilon(s) \subset \mathcal{B}$ for all $\delta > 0$. Therefore, using (5.49), there exists $\epsilon_0 > 0$ such that $\sup_{a_\epsilon \in \mathscr{A}_\epsilon} \|S_\varepsilon(t,\tau)a_\varepsilon(\tau) - S_0(t,\tau)a_\varepsilon(\tau)\| < \frac{\delta}{2}$ for all $\epsilon \leqslant \epsilon_0$. Then,

$$\begin{aligned}\operatorname{dist}(\mathscr{A}_\varepsilon(t), \mathscr{A}_0(t)) &\leqslant \sup_{a_\epsilon \in \mathscr{A}_\epsilon(\tau)} d(S_\varepsilon(t,\tau)a_\varepsilon, S_0(t,\tau)a_\varepsilon) + \operatorname{dist}(S_0(t,\tau)\mathscr{A}_\varepsilon(\tau), \mathscr{A}_0(t)) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta,\end{aligned}$$

and the upper-semicontinuity is proved.

5.4.2 Lower-semicontinuity

First of all, we need to suppose that all equilibrium points in $\mathcal{E}$ are hyperbolic for the limit problem in the sense of Definition 4.3.1 in Section 4.3 (page 94).

The key of the proof of the lower-semicontinuity is based on the proof of the local continuity of the sets $W_0^u(e_i^*)$ and $W_\varepsilon^u(e_i^*)$ defined as in (3.1.11) (page 47), writing them first as a graph and showing later the continuity. Our aim is to parallel the analysis carried out in Section 2 from [24], using the following theorem, which is a particular case of Theorem 3.4.3 (page 67).

Theorem 5.4.1 *Let X be a Banach space and consider the family $\{S_\varepsilon(t,s) : t \geqslant s\}$, $\varepsilon \in [0,1]$, of nonlinear processes in X. Assume that for any x in a compact subset of X, $\|S_\varepsilon(t,s)x - S_0(t,s)x\| \overset{\varepsilon\to 0}{\longrightarrow} 0$ for $[s,t] \subset \mathbb{R}$ and suppose that for each $\varepsilon \in [0,1]$ there exits a pullback attractor $\{\mathscr{A}_\varepsilon(t) : t \in \mathbb{R}\}$, such that $\overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon(t)}$ is compact and we can write $\{\mathscr{A}_0(t) : t \in \mathbb{R}\}$ as in (5.40). Furthermore, assume that for each $e_j^* \in \mathcal{E}$:*

1. *given $\delta > 0$, there exists $\varepsilon_{j,\delta}$, such that for all $0 < \varepsilon < \varepsilon_{j,\delta}$ there is a global hyperbolic solution $\xi_{j,\varepsilon}$ of (5.7) that satisfies*

$$\sup_{t\in\mathbb{R}} \|\xi_{j,\varepsilon}(t) - e_j^*\|_X < \delta,$$

2. *the local unstable manifold of $\xi_{j,\varepsilon}$ behaves continuously as $\varepsilon \to 0$; that is,*

$$dist_H \left(W^u_{0,loc}(e^*_j), W^u_{\varepsilon,loc}(\xi_{j,\varepsilon})\right) \to 0,$$

where $dist_H(A,B) = \max\{dist(A,B), dist(B,A)\}$ is the symmetric Hausdorff distance and $W^u_{loc}(\cdot) = W^u(\cdot) \cap B_X(\cdot,\rho)$, with $\rho > 0$.

Then the family $\{\mathscr{A}_\varepsilon(t) : t \in \mathbb{R}, 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ is lower-semicontinuous at $\varepsilon = 0$, i.e.

$$dist_H(\mathscr{A}_\varepsilon(t), \mathscr{A}_0(t)) \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.50}$$

The compactness of the union of all pullback attractors is obtained in (5.37) and, in Section 5.3, we have proved that the pullback attractor of the limit problem is gradient-like. Therefore, we need to prove the two conditions in Theorem 5.4.1. These are just consequences of the stability of the hyperbolic equilibria under perturbation (see [22]).

We then need to prove that all the equilibrium points in $\mathcal{E}$ are hyperbolic for all $S_\varepsilon(t,s)$. If $u(t)$ is a solution of (5.38) and defining $z(t) = u(t) - e^*_j$ for any $e^*_j \in \mathcal{E}$, then $z(\cdot)$ satisfies

$$\begin{cases} z_{tt} - \Delta z - \gamma(t)\Delta z_t - f'(e^*_j)z = h(z) \\ z(s) = z_0; z_t(s) = z_1, \end{cases} \tag{5.51}$$

where $h(z) = f(z+e^*_j) - f(e^*_j) - f'(e^*_j)z$. Then, we can construct the following system

$$\begin{pmatrix} z \\ x \end{pmatrix}_t + \bar{\mathbf{A}}_0(t)\begin{pmatrix} z \\ x \end{pmatrix} = H\begin{pmatrix} z \\ x \end{pmatrix} \tag{5.52}$$

where

$$\bar{\mathbf{A}}_0(t) = \begin{pmatrix} 0 & -I \\ -\Delta - f'(e^*_j) & -\gamma(t)\Delta \end{pmatrix} \text{ and } H\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ h(z) \end{pmatrix}.$$

In the same way, taking $v(t)$ solution of (5.1) and defining $z(t) = v(t) - e^*_j$ we have

$$\begin{pmatrix} z \\ x \end{pmatrix}_t + \left[\bar{\mathbf{A}}_0(t) + B_\varepsilon(t)\right]\begin{pmatrix} z \\ x \end{pmatrix} = H\begin{pmatrix} z \\ x \end{pmatrix} \tag{5.53}$$

where

$$B_\varepsilon(t) = \begin{pmatrix} 0 & 0 \\ 0 & \beta_\varepsilon(t) \end{pmatrix},$$

with $H(0) \equiv 0$ and the Jacobian matrix $J_H(0) \equiv 0 \in \mathcal{L}(Y^0)$.

Let $\{Z_0(t,s) : t \geqslant s\}$ and $\{Z_\varepsilon(t,s) : t \geqslant s\}$ denote the processes associated to (5.52) and (5.53) respectively. Note that

$$\sup_{t\in\mathbb{R}} \|B_\varepsilon(t)\|_{\mathcal{L}(Y^0)} = \sup_{t\in\mathbb{R}} |\beta_\varepsilon(t)| \leqslant \beta_{1\varepsilon} \xrightarrow{\varepsilon\to 0} 0,$$

then, by Theorem 7.6.11 in [50], it follows that for $\varepsilon_0 > 0$ sufficiently small, $\{Z_\varepsilon(t,s) : t \geqslant s\}$ has an exponential dichotomy and, consequently, every point in $\mathcal{E}$ is also hyperbolic for $\{S_\varepsilon(t,s) : t \geqslant s\}$, for all $\varepsilon < \varepsilon_0$.

The way in which we obtain (5.53) allows us to concentrate on the existence of invariant manifolds of equilibrium points around zero stationary solutions. Then, if $V = (v, v_t)$ is a solution of (5.53), and we write $\{Q_\varepsilon(t) : t \in \mathbb{R}\}$ for the projections defined in Definition 3.4.4, we define $V^+(t) = Q_\varepsilon(t)V(t)$ and $V^-(t) = V(t) - V^+(t)$, which satisfies

$$\begin{aligned} V_t^+ &= A_\varepsilon^+(t)V^+ + G^+(t, z^+, z^-) \\ V_t^- &= A_\varepsilon^-(t)V^- + G^-(t, z^+, z^-), \end{aligned} \tag{5.54}$$

where

$$\begin{aligned} A_\varepsilon^+(t) &= [A_0(t) + B_\varepsilon(t)]Q_\varepsilon(t), \\ A_\varepsilon^-(t) &= [A_0(t) + B_\varepsilon(t)](I - Q_\varepsilon(t)), \\ G^+(t, z^+, z^-) &= Q_\varepsilon(t)H(V^+ + V^-), \\ G^-(t, z^+, z^-) &= (I - Q_\varepsilon(t))H(V^+ + V^-). \end{aligned}$$

Therefore, we can apply Theorem 3.4.6 in Section 3.4, obtaining the lower-semicontinuity of the pullback attractors.

Remark 14 *Observe that, in particular, we have proved upper and lower semicontinuity of pullback attractors with respect to the autonomous strongly damped wave equation, i.e., the one with* $\gamma(t) = \gamma > 0$ *and* $\beta_\varepsilon(t) = \beta_\varepsilon > 0$.

Chapter 6

Conclusions and open problems

In this work we have showed a complete theory about pullback attractors. We have described analogous results on existence, structure and continuity as in the autonomous case. These results are generalizations of the classical theory of semigroups, indeed we can include the autonomous results in this framework as particular cases. In this way, we have written part of the theory of global attractors inside the framework of the evolution processes, making a robust and self-contained theory. Our contributions on this theory has been focused in the study of the pullback asymptotically compact and pullback point dissipative evolution processes.

In this work we also have shown two non-autonomous problems which are not necessary close to any autonomous one. Therefore, we give more consistence to the theory of evolution processes and continuity of pullback attractors, avoiding the relationship between the structure of a pullback attractor with a limit gradient global attractor. These two non-autonomous wave equations are not trivial examples, so we needed to make a whole study of them, by showing the existence and uniqueness of the solution (using some new techniques and theory like uniform sectoriality, specially in the strongly damped case), the existence of the pullback attractor, its structure and the continuity when the limit problem is also a non-autonomous system, applying the theory and results previously obtained. We also give a pullback attractor which possesses an exponential ratio of forward attraction. But not only we give new results in the non-autonomous case, es we could consider constant damping in both problems. Therefore we give also non-trivial examples of structure and convergence of global attractors.

But there are some open problems that born from this study. In both equations we have studied the subcritical case, that is, when the growth conditions ensure the compactness of the non linear part of the solution and ensure the higher regularity. Both critical and supercritical cases are still open problems. The continuity of the pullback attractor in the strongly damped case has been studied when $\beta_{0\varepsilon} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$. A different and more difficult problem is when we define the strongly damped as $\gamma_\varepsilon(t)\Delta u_t$ with $\gamma_\varepsilon(t) \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ in some sense. In this case the continuity of the processes depends directly of the continuity of the operators, instead on a small perturbation of them. Therefore we need a generalization of the Trotter-Kato theorem (see [55, 90]), which is an open problem nowadays. This limit problem will be a hyperbolic problem due to the strongly damping, which ensures the higher regularity of the time derivative of the solution, becomes zero. Then, we will need to deal also with a loose of regularity in the limit.

Other interesting problem is a generalization of the equations, using a non-autonomous external force $f(u,t)$. In this case the hyperbolic global solutions will not be the equilibrium points due to they will not be constant. This give us an open field in the structure and continuity of the pullback attractor. There also remain a study about the dimension of the pullback attractors for this two equations.

Sun *et. al* in [88], Wang *et. al* in [92] or Xiao in [93], studied some non-classical parabolic equations (non-classical diffusion equations) where the strongly damping Δu_t appear in the equation

$$u_t - \Delta u_t - \Delta u = f(u) + g(x),$$

proving the existence of the global attractor in the autonomous case. On the other hand, Cung and Tang in [39] made a study of the upper-semicontinuity for this class of non-autonomous nonclassical diffusion equations

$$u_t - \varepsilon\Delta u_t - \Delta u + f(u) = g(t),$$

where the non-autonomous nature comes from the external force. The ideas of these problems are very close to the ideas in this work, therefore a non-autonomous study with a time dependent coefficient in the strongly damping will be an interesting problem, also a study of the lower-semicontinuity in the second equation.

Hale and Raugel in [49] studied the lower-semicontinuity of the global attractor of the following hyperbolic equation

$$\varepsilon u_{tt} + u_t - \Delta u = -f(u) - g,$$

when $\varepsilon \to 0$, changing the lower regularity of the problem to a higher one in the limit, because when $\varepsilon = 0$ we have a parabolic system, i.e., they compared the global attractor of a wave equation with the attractor of a diffusion equation. The generalization of this problem to the non-autonomous case still remains open (see Arrieta *et al.* [4]).

The theory showed in this work is focused on the non-autonomous case, but it could have applications in stochastic problems, therefore there are a wide range of problems where this theory could been applied, specially when the stochastic nature of the equation comes as a random perturbation of a deterministic autonomous problem. In the same vein, we could extrapolate this theory to problems on unbounded domains.

Bibliography

[1] M. Anguiano, T. Caraballo, J. Real, Existence of pullback attractor for a reaction-diffusion equation in some unbounded domanis with non-autonomous forcing term in H^{-1}, *Inter. Journal of Bifurcation and Chaos*, to appear.

[2] E.R. Aragão-Costa, T. Caraballo, A.N. Carvalho, J.A. Langa, Stabilitiy of gradient semigroups under perturbation, *preprint*.

[3] J.M. Arrieta, A.N. Carvalho, Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain, *J. Differential Equations* **199**, (2004) 143–178.

[4] J.M. Arrieta, A. Carvalho, J.A. Langa, A. Rodrguez-Bernal, Continuity of dynamical structures for non-autonomous evolution equations under singular perturbations, *preprint*.

[5] J.M. Arrieta, A.N. Carvalho, G. Lozada-Cruz, Dynamics in dumbbell domains. III. Continuity of attractors, *J. Differential Equations*, vol. 247, **1**, (2009) 225-259.

[6] J. Arrieta, A.N. Carvalho, J.K. Hale, A damped hyperbolic equation with critical exponent, *Communications in Partial Differential Equations*, vol. 17, **5-6**, (1992) 841–866.

[7] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, Perturbation of the diffusion and upper semicontinuity of attractors, *Appl. Math. Lett.*, vol. 12, **5**, (1999) 37-42.

[8] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds, *Comm. Partial Differential Equations*, vol. 25, **1-2**, (2000) 1-37.

[9] A.B. Babin, M.I. Vishik, *Attractors of evolution equations.* North Holland (1992).

[10] S.M. Bruschi, A.N. Carvalho, J.W. Cholewa, T. Dlotko, Uniform Exponential Dychotomy and Continuity of Attractors for Singular Perturbed Damped Equations, *J. of Dynamics and Differential Eq.* vol. 18, **3**, (2006) 767–814.

[11] S.M. Bruschi, A.N. Carvalho, J.W. Cholewa, T. Dlotko, Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations, *J. Dynam. Differential Equations*, vol. 18, **3**, (2006) 767–814.

[12] T. Caraballo, A.N. Carvalho, J.A. Langa, F. Rivero, Existence of pullback attractors for pullback asymptotically compact processes, *Nonlinear Analysis*, **72**, (2010) 1967–1976.

[13] T. Caraballo, A.N. Carvalho, J.A. Langa, F. Rivero, Some gradient-like non-autonomous evolution processes, *Int. Journal of Bifurcation and Chaos*, vol. 20, **9**, (2010) DOI: 10.1142/S0218127410027337.

[14] T. Caraballo, A.N. Carvalho, J.A. Langa, F. Rivero, A non-autonomous strongly damped wave equation: Existence and continuity of the pullback attractor, *Nonlinear Analysis*, to appear.

[15] T. Caraballo, M.J. Garrido-Atienza, B. Schmalfuß, J. Valero, Nonautonomous and random attractors for delay random semilinear equations without uniqueness. *Discrete Contin. Dyn. Syst.*, vol. 21, **2**, (2008) 415-443.

[16] T. Caraballo, P.E. Kloeden, J.A. Langa, Atractores globales para sistemas diferenciales no autónomos. *CUBO: Matemática Educacional*, vol. 5, **2**, (2003) 305–329.

[17] T. Caraballo, J.A. Langa, On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, vol. 10, **4**, (2003) 491–513.

[18] T. Caraballo, J.A. Langa, J.C. Robinson, Upper semicontinuity of attractors for small random perturbations of dynamical systems, *Comm. Partial Differential Equations*, vol. 23, **9-10**, (1998) 1557–1581.

[19] T. Caraballo, G. Lukaszewicz, J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems. *Nonlinear Analysis* **64**, (2006) 484–498.

[20] A.N. Carvalho, J.W. Cholewa, Local well-posedness for strongly damped wave equation with critical nonlinearities, *Bull. Austral. Math. Soc*, **66**, (2002) 443–463.

[21] A.N. Carvalho, J.W. Cholewa, T. Dlotko, Strongly damped wave problems: Bootstrapping and regularity of solutions, *J. Differential Equations*, **244**, (2008) 2310–2333.

[22] A.N. Carvalho, J.A. Langa, Non-autonomous perturbation of autonomous semilinear differential equations: Continuity of local stable and unstable manifolds, *J. Differential Equations,* **233**, (2007) 622–653.

[23] A.N. Carvalho, J.A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, *J. Differential Equations*, vol. 246, **7**, (2009) 2646–2668.

[24] A.N. Carvalho, J.A. Langa, J.C. Robinson, Lower semicontinuity of attractors for non-autonomous dynamical systems, *Ergodic Theory Dynam. Systems*, vol. 29, **6**, (2009) 1765–1780.

[25] A.N. Carvalho, J.A. Langa, J.C. Robinson, On the continuity of pullback attractors for evolution processes, *Nonlinear Analysis*, **71** (2009) 1812–1824.

[26] A.N. Carvalho, J.A. Langa, J.C. Robinson, *Attractors for infinite dimensional dynamical systems*, Springer, to appear.

[27] A.N. Carvalho, J.A. Langa, J.C. Robinson, A.Suárez, Characterization of non-autonomous attractors of a perturbed gradient system, *J. Differential Equations,***236**, (2007) 570–603.

[28] A.N. Carvalho, M.J. Nascimento, Singularly non-autonomous semilinear parabolic problems with critical exponents, *Discrete Contin. Dyn. Syst. Ser. S*, vol. 2, **3**, (2009) 449–471.

[29] D. Cheban, *Global Attractors of Non-Autonomous Dissipative Dynamical Systems.*World Scientific Publishing Co. Pte. Ltd (2004).

[30] D.N. Cheban, P.E. Kloeden, B. Schmalfuß, The relationship between pullback, forward and global attractors of nonautonomous dynamical systems, *Nonlinear Dyn. Syst. Theory* vol. 2, **2**, (2002) 125–144.

[31] S. Chen, R. Triggiani, Proof of extensions of two conjetures on structural damping for elastic systems, *Pacific J. of Mathematics*, vol. 135, **1**, (1989) 15–55.

[32] S. Chen, R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, *J. Differential Equations*, vol. 88, **2** (1990) 279–293.

[33] V.V. Chepyzhov, M.I. Vishik, *Attractors for Equations of Mathematical Physics*. Colloquium Publications, **49**. American Mathematical Society (2002).

[34] J.W. Cholewa, T. Dlotko, *Global attractors in abstract parabolic problems*. London Mathematical Society Lecture Note Series, **278**. Cambridge University Press, (2000).

[35] P.L. Christiansen, P.S. Lomdhal, O.H. Soerensen, Soliton excitations in Josephson tunnel junctions, *Phys. Rev.* **25**, (1982) 5737–5748.

[36] C. Conley, *Isolated invariant sets and the Morse index.* CBMS Regional Conference Series in Mathematics, **38**. American Mathematical Society, Providence, R.I (1978).

[37] H. Crauel, A. Debussche, F. Flandoli, Random attractors, *J. Dynam. Differential Equations* vol. 9, **2**, (1997) 307–341.

[38] H. Crauel, F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Relat Fields*, **100**, (1994) 365–393.

[39] Cung The Anh, Tang Quoc Bao, Pullback attractors for a non-autonomous semi-linear degenerate parabolic equation, *Glasg. Math. J.*, vol. 52, **3**, (2010) 537-554.

[40] K. Deimling, *Nonlinear Functional Analysis*. Springer-Verlag (1985).

[41] L. Desheng, P.E. Kloeden, Equi-attraction and the continuous dependence of pullback attractors on parameters, *Stoch. Dyn.*, vol. 4, **3**, (2004) 373-384.

[42] G. Duvaut, J-L. Lions, *Inequalities in Mechanics and Physics*, Springer (1976).

[43] M. Efendiev, A. Miranville, S. Zelik, Infinite dimensional exponential attractors for a nonautonomous reaction-diffusion system, *Math. Nachr.*, **248/249**, (2003) 72–96.

[44] J.C. Eilbeck, G.R. McGuire, Numerical study of the regularized long-wave equation. II. Interaction of solitary waves. *J. Computational Phys.*, vol. 23, **1**, (1977) 63-73.

[45] A. Favini, C. Gal, G. Ruiz, J. Goldstein, S. Romanelli, The non-autonomous wave equation with general Wentzell boundary conditions, *Proceedings of the Royal Society of Edinburgh*, **135A**, (2005) 317-329.

[46] F. Flandoli, B. Schmalfuß, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise. *Stochastics Stochastics Rep.* vol. 59, **1–2**, (1996) 21-45.

[47] J.K. Hale, *Asymptotic Behavior of Dissipative System*, American Mathematical Society (1989).

[48] J.K. Hale, G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.*, vol. 154, **4**, (1989) 281–326.

[49] J.K. Hale, G. Raugel, Lower semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *J. Dynam. Differential Equations* vol. 2, **1**, (1990) 19-67.

[50] D. Henry, *Geometry Theory of Semilinear Parabolic Equations*, Springer-Verlag (1981).

[51] F. Hirosawa, J. Wirth, Generalised energy conservation law for wave equations with variable propagation speed, *J. Math. Anal. Appl.* **358**, (2009) 56-74.

[52] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation. *J. Mathematical Phys.*, **14**, (1973) 805-809.

[53] L. Hongyan, Z. Shengfan, On non-autonomous strongly damped wave equations with a uniform attractor and some averaging, *J. Math. Anal. Appl.* **341**, (2008) 791-802.

[54] V. Kalantarov, S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, *J. Differential Equations*, vol. 247, **247**, (2009) 1120–1155.

[55] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, (1976)

[56] P.E. Kloeden, Pullback attractors of nonautonomous semidynamical system, *Stochastics and Dynamics*, vol. 3, **1**, (2003) 101–112.

[57] P.E. Kloeden, J.A. Langa, Flattening, squeezing and the existence of random attractors, *Proc. Royal Society A*, **463**, (2007) 163–181.

[58] P.E. Kloeden, M. Rasmussen, Nonautonomous Dynamical System, *AMS monographs*, to appear.

[59] P.E. Kloeden, B. Schmalfuß, Nonautonomous system, cocycle attractors and variable time-step discretization, *Numerical Algorithms*, **3**, (1997) 141–152.

[60] M. Kruskal, *Nonlinear wave equations. Dynamical systems, theory and applications*, Lecture Notes in Phys., vol. 38, Springer, Berlin (1975).

[61] O. Ladyzhenskaya, *Attractors for semigroups and evolution equations.* Lezioni Lincee. [Lincei Lectures] Cambridge University Press, Cambridge, (1991).

[62] J.A. Langa, Asymptotically finite dimensional pullback behaviour of nonautonomous PDEs, *Arch. Math*, **80**, (2003) 525–535.

[63] J.A. Langa, A. Miranville, J. Real, Pullback exponential attractors, *Discrete Contin. Dyn. Syst.*, vol. 26, **4**, (2010) 1329-1357.

[64] J.A. Langa, J.C. Robinson, A. Rodríguez-Bernal, A. Suárez, Permanence and asymptotically stable complete trajectories for nonautonomous Lotka-Volterra models with diffusion, *SIAM J. Math. Anal.*, vol. 40, **6**, (2009) 2179-2216.

[65] J.A. Langa, J.C. Robinson, A. Rodríguez-Bernal, A. Suárez, V. Vidal-López, Existence and nonexistence of unbounded forwards attractor for a class of non-autonomous reaction diffusion equations, *Discrete Contin. Dyn. Syst.*, vol. 18, **2-3**, (2007) 483-497.

[66] J.A. Langa, J.C. Robinson, A. Suárez, A. Vidal-López The stability of attractors for non-autonomous perturbations of gradient-like systems, *J. Differential Equations,* **234**, (2007) 607-625.

[67] J.A. Langa, A. Rodríguez-Bernal, A. Suárez, On the long time behavior of non-autonomous Lotka-Volterra models with diffusion via the sub-supertrajectory method, *J. Differential Equations*, vol. 249, **2** (2010) 414-445.

[68] Q. Ma, S. Wang, Ch. Zhong, Necessary and sufficient Conditions for the Existence of Global Attractors for semigroups and Applications. *Indiana University Mathematics Journal*, vol. 51, **6**, (2002) 1541–1559.

[69] J. Mallet-Paret, Negatively invariant sets of compact maps and an extension of a theorem of Cartwright, *J. Differential Equations*, vol. 22, **2**, (1976) 331-348.

[70] R. Mañé, On the dimension of the compact invariant sets of certain nonlinear maps, *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980), Lecture Notes in Math., 898*, Springer, Berlin-New York, (1981) 230-242.

[71] P. Marín-Rubio, J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Analysis*, **71**, (2009) 3956–3963.

[72] K. Nishihara a, J. Zhai, Asymptotic behaviors of solutions for time dependent damped wave equations, *J. Math. Anal. Appl.*, **360**, (2009) 412-421.

[73] V. Pata, M. Squassina, On the Strongly Damped Wave equation, *Commum. Math. Phys*, **253**, (2005) 511–533.

[74] A. Parker, On exact solutions of the regularized long-wave equation: a direct approach to partially integrable equations. I. Solitary wave and solitons, *J. Math. Phys.*, vol. 36, **7**, (1995) 3498-3505.

[75] V. Pata, S. Zelik, Smooth attractors for strongly damped wave equation, *Nonlinearity*, **19**, (2006) 1495–1506.

[76] G. Raugel, Global Attractor in Partial Differential Equations, *Handbook of dynamical systems*, **2**, (2002) 885–982, North-Holland, Amsterdam.

[77] F. Rivero, Forward and pullback attraction on pullback attractors, *Bol. Sociedad Española Matemática Aplicada*, **52**, (2010) 155–162.

[78] J.C. Robinson, *Infinite-Dimensional Dynamical System. An introduction to dissipative parabolic PDEs and the theory of global attractors.* Cambridge Text in Applied Mathematics (2001).

[79] B. Schmalfuß, Endlichdimensionale Approximation der Lsung der stochastischen Navier-Stokes-Gleichung, *Statistics*, vol. 21, **1**, (1990) 149-157.

[80] B. Schmalfuß, Attractors for the nonautonomous dynamical systems. *International Conference of Diffential Equations*, vol. 1, **2**, (1999).

[81] B. Schmalfuß, Measure attractors and random attractors for stochastic partial differential equations. *Stochastic Anal. Appl.*, vol. 17, **6**, (1999) 1075-1101.

[82] G.R. Sell, Y. You, *Dynamics of evolutionary equations.* Applied Mathematical Sciences, 143. Springer-Verlag (2002).

[83] G.R. Sell, Nonautonomous differential equations and dynamical systems. *Trans. Amer. Math. Soc.*, **127**, (1967) 241–283.

[84] P. E. Sobolevskiĭ, *Equations of parabolic type in a Banach space*, Amer. Math. Soc. Transl. Series 2, vol. 49: Ten Papers on Functional Analysis and Measure Theory,ó American Mathematical Society, Providence, R.I., (1966), 1–62.

[85] C.T. Sparrow, *The Lorentz Equations: Bifurcations, Chaos and Strange Attractors.* Springer-Verlag (1982).

[86] O. Steinmann, Two-soliton solutions of relativistic field equations. *Comm. Math. Phys.*, vol. 60, **3**, (1978) 269-276.

[87] A.M. Stuart, A.R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambeidge University Press (1996).

[88] C.Y. Sun, S. Y. Wang, C. Zhong, Global attractors for a nonclassical diffusion equation. *Acta Math. Sin.*, vol. 23, **7**, (2007) 1271-1280.

[89] R. Temam, *Infinite-Dimensional Dynamical System in Mechanics and Physics.* Springer-Verlag (1988).

[90] H.F. Trotter, Approximation of semi-groups of operators, *Pacific J. Math.*, **8**, (1958), 887–919.

[91] M.I. Vishik, S. V. Zelik, Regular attractors and their nonautonomous perturbations, *preprint.*

[92] S. Wang, L. Desheng; C. Zhong, On the dynamics of a class of nonclassical parabolic equations. *J. Math. Anal. Appl.*, vol. 317, **2**, (2006) 565-582.

[93] Y. Xiao, Attractors for a nonclassical diffusion equation. *Acta Math. Appl. Sin. Engl. Ser.*, vol. 18, **2**, (2002) 273-276.

[94] V.G. Zvyagin, M.V. Turbin, Investigation of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids, *Sovrem. Mat. Fundam. Napravl.*, **31**, (2009) 3-144.