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Discontinuous stochastic modelling and discrete numerical approximation for Tuberculosis model with relapse

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Abstract

The objective of this paper is to study a stochastic epidemiological model with infinite Lévy measure and relapse. Using stochastic tools, we prove the existence and uniqueness of global positive solution. Moreover, we also show the extinction and persistence in mean of the disease by the use of Kunita's inequality instead of Burkholder-Davis-Gundy inequality for continuous diffusions. The numerical behaviour of the considered model is analyzed to understand the impact of environmental transmission on the spread of human and zonotic tuberculosis in Morocco.

Keywords: epidemic model, infinite activity, relapse, extinction, persistence in mean, Kunita's inequality.

1. Introduction

Before, epidemiology was only interested in infectious and epidemic diseases, with the appearance of studies on noncommunicable diseases, epidemiology is considered as a scientific discipline that studies the frequency and distribution of diseases in time and space, as will as the role of the factors that determine this frequency and this distribution within human population. Mathematical modelling is one of the most important approach in epidemiology. It has been used to analyze the spread of diseases [1, 2, 3, 4, 5, 6, 7] with the objective of limiting the extent of infection by some form of control like vaccination and media coverage. Moreover, most epidemic models for the transmission of infectious diseases based on the classical model given by Kermack and McKendrick [8]. For some diseases, such as tuberculosis incomplete treatment can lead to relapse, the recovered individual become infectious again [9, 10, 11, 12]. The resulting model called SIRI was formulated by Tudor [13], which consist of a system with three compartments: susceptible, infected and recovered, labelled by S,I and R. The SIRI epidemiological model is suitable for some recurrent diseases feature of animal and human, such as bovine tuberculosis and herpes [14, 15]. The World Organisation for Animal Health (OIE) states the bovine tuberculosis (BTB) is an endemic zoonosis caused by Mycobacterium bovis (M. bovis), which infects many domestic animals and is transmitted to humans through consumption of raw milk or meat from infected animals. In Morocco, bovine tuberculosis remains a major concern impacting both animal and human health. The last national survey based on a skin tuberculin test was conducted in

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2004 and showed an individual prevalence of 18% and a herd prevalence of 33% [16]. A cross-sectional tuberculin study conducted in the Sidi Kacem area in Morocco in 2012 showed an individual prevalence of 20.4% and a herd prevalence of 57.7% [16]. In a review published in 2013, the proportion of zoonotic human tuberculosis (TB) among all TB cases was estimated at 2.8% in Africa [16]. The World Health Organization states a worldwide median prevalence of 3.1% of M. bovis among human TB patients [16]. One previous study found that 17.8% of drug-resistant TB among humans in Morocco was due to M. bovis. The main strategy to control (BTB) in Morocco is based on a test and slaughter scheme. In 1997, Moreira and Wang [17], studied an SIRI model with general saturated nonlinear incidence function. Blower [14] developed a compartmental model for genital herpes, assuming standard incidence for the diseases transmission and constant recruitment rate. In Driessche and Zou [18] considered a more general SIRI model and threshold property were obtained. See also Van den Driessche et al.[19] for an analysis of a related SEIRI model. For other works see [12, 20, 21] and the references therein.

Motivated by the works of Privault and Wang[22] and Caraballo *et al.*[10], in this paper, a deterministic SIRI disease can be modeled as follows:

$$\begin{cases} dS(t) &= [\Lambda - \beta S(t)I(t) - \mu S(t)]dt, \\ dI(t) &= [\beta S(t)I(t) - (\mu+\gamma)I(t) + \delta R(t)]dt, \\ dR(t) &= [\gamma I(t) - (\mu+\delta)R(t)]dt, \end{cases} \tag{1.1}$$

where, $\Lambda, \beta, \mu, \delta, \gamma$ are all positive constants, $S(t)$ is the number of the individuals susceptible to the disease, $I(t)$ are the infected members and $R(t)$ represents who have recovered from the infection at time $t \in \mathbb{R}_+$. In this model, the parameters have the following features: Λ is the total number of the susceptible, β is the disease transmission coefficient, μ represents the natural death rate, γ represents the rate of recovery from infection and δ is the rate of relapse.

The basic reproduction number [23, 24] for this system is given by

$$\mathcal{R}_0 = \frac{\Lambda\beta(\mu+\delta)}{\mu^2(\mu+\gamma+\delta)}.$$

It is defined as the average number of secondary infections produced by an infected case in a population where some individuals are no longer susceptible to infection. The deterministic model (1.1) has been discussed by Vargas-De-León in [25]. He proved that the equilibrium of (1.1) in the deterministic case has been characterized by $\mathcal{R}_0$.

If $\mathcal{R}_0 \leq 1$, the system (1.1) admits a disease free equilibrium

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right),$$

and it is globally asymptotically stable, this means that the endemic disease will not appear.

If $R_0 > 1$, E_0 becomes unstable, and there exists a globally asymptotically stable endemic equilibrium

$$E^* = (S^*, I^*, R^*) = \left(\frac{\Lambda}{\mu\mathcal{R}_0}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1), \frac{\mu\gamma}{\beta(\mu+\delta)}(\mathcal{R}_0 - 1)\right),$$

which means that the disease will be persistent. Considering stochasticity in an SIRI model is important because it allows for the incorporation of random fluctuations in the disease transmission process. This is particularly relevant when dealing with real-world epidemic situations, where factors such as human behavior, environmental conditions, and chance events can lead to unpredictable variations in the spread of the disease. Stochastic SIRI models enable a more realistic representation of the inherent uncertainty in disease dynamics, which can impact the effectiveness of control measures and the persistence of the epidemic. By accounting for stochastic effects, these models provide valuable insights into the potential outcomes of an epidemic and the associated risk factors, aiding in the development of more robust public health strategies. That is way we must use the stochastic differential equation with Lévy jump due to the heavy-tailed nature of Lévy distributions, which allows for the representation of rare but impactful events in the disease transmission process. Unlike other types of jumps, Lévy jumps can capture extreme events that may significantly influence the dynamics of an epidemic, making them more suitable for modeling rare but influential phenomena such as super-spreading events or large-scale behavioral changes. This is particularly relevant in the context of infectious disease dynamics, where rare events can have a disproportionate impact on the spread and persistence of the disease. The heavy-tailed nature of Lévy distributions enables the SIRI model to account for these rare events, providing a more comprehensive understanding of the potential outcomes of an epidemic. Therefore, sudden changes in the form of jumps can impose a more realistic behavior for dynamical processes and the control of the disease. Next, we let the Lévy noise act on the transmission rate as follows :

$$\beta \to \beta + \sigma \dot{B}(t) + \dot{J}(t),$$

where, $B(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. $X(t-)$ is the left limit of $X(t)$, $\sigma > 0$ is the intensity of Brownian motion which is independent of the Lévy jumps $J(t) = \int_0^t \int_{\mathbb{R}^*} C(z)\tilde{N}(dt, dz)$, where $\tilde{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt$ is the compensator of a Poisson counting process with characteristic measure λ on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $C : \mathbb{R}^* \times \Omega \to \mathbb{R}$, is bounded and continuous with respect to λ and is $\mathcal{B}(\mathbb{R}^*) \otimes \mathcal{F}_t$-measurable where $\mathcal{B}(\mathbb{R}^*)$ is a σ-algebra with respect to the set $\mathbb{R}^*$. In this paper, we work in the general setting of infinite Lévy measures λ on $\mathbb{R}^*$. All processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Under these hypotheses, we propose the following system :

$$\begin{cases} dS(t) &= [\Lambda - \beta S(t)I(t) - \mu S(t)]dt - \sigma S(t)I(t)dB(t) - \displaystyle\int_{\mathbb{R}^*} S(t-)I(t-)C(z)\tilde{N}(dt, dz), \\ dI(t) &= [\beta S(t)I(t) - (\mu + \gamma)I(t) + \delta R(t)]dt + \sigma S(t)I(t)dB(t) + \displaystyle\int_{\mathbb{R}^*} S(t-)I(t-)C(z)\tilde{N}(dt, dz), \\ dR(t) &= [\gamma I(t) - (\mu + \delta)R(t)]dt. \end{cases} \tag{1.2}$$

This paper is organized as follows. In the next section, we site some mathematical tools which we use later. In section 3, we proved existence and uniqueness of a global positive solution. In section 4 and 5 we presented the sufficient conditions to the extinction and persistence in mean. In the last section, we present some numerical results to support our theoretical ones.

2. Useful mathematical tools

Now, we will define Kunita's inequality [26], this replaces the Burkholder-Davis-Gundy inequality for continuous martingales [27].

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |M_s|^p\right] \leq C_p\, \mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right], \qquad p > 1, \tag{2.1}$$

where $\langle M, M\rangle_t$ is the quadratic variation of the continuous martingale $(M_t)_{t\in\mathbb{R}_+}$.

2.1. Kunita's inequality

Theorem 2.1. *Let K_t be a jump stochastic integral process defined by*

$$K_t := \int_0^t \int_{\mathbb{R}^3\setminus\{0\}} g_s(z)(N(ds, dz) - \lambda(dz)ds), \qquad t \in \mathbb{R}_+,$$

of the predictable integrand $(g_s(y))_{(s,y)\in\mathbb{R}_+\times\mathbb{R}}$. For $t \in \mathbb{R}_+$ and $p \geq 2$, we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |K_s|^p\right] \leq C_p \left\{\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3\setminus\{0\}} |g_s(z)|^p \lambda(dz)ds\right] + \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^3\setminus\{0\}} |g_s(z)|^2 \lambda(dz)ds\right)^{p/2}\right]\right\}, \tag{2.2}$$

where, $C_p := 2^{2p-2}(\sqrt{e^{\log_2 p}/p} + 8p^{\log_2 p})$.

Proof. The proof is founded in [26] (Theorem 4.4.23).

2.2. Itô's formula for Lévy-type stochastic integrals

Let $b : [0,T]\times\Omega \to \mathbb{R}^d$ and $\sigma : [0,T]\times\Omega \to \mathbb{R}^{d\times d}$ be $\mathcal{F}_t$-adapted processes such that

$$\mathbb{E}\left[\int_0^T \left(|b(s)|^2 + |\sigma(s)|^2\right) ds\right] < \infty,$$

and $H, K : [0,T]\times(\mathbb{R}^d\setminus\{0\})\times\Omega \to \mathbb{R}^d$ predictable processes such that

$$\int_0^T\int_{\mathbb{R}^d\setminus\{0\}} |K(s,z)|N(ds,dz) < \infty \quad \text{and} \quad \int_0^T\int_{\mathbb{R}^d\setminus\{0\}} |H(s,z)|^2\tilde{N}(ds,dz) < \infty, \quad \mathbb{P}\text{-a.s.}$$

We consider the process for $0 \le t \le T$

$$X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dB_s + \int_0^t\int_{|z|\ge1} K(s,z)N(ds,dz) + \int_0^t\int_{|z|<1} H(s,z)\tilde{N}(ds,dz).$$

Theorem 2.2. *Let $X = (X_t)_{t\ge0}$ be a Lévy stochastic integral, then for any function $f \in C^2(\mathbb{R}^d)$, with probability one we have*

$$\begin{aligned} f(X_t) - f(X_0) = \; & \int_0^t \langle \partial_x f(X_{s-}), b(s)\rangle\,ds + \int_0^t \langle \partial_x f(X_{s-}), \sigma(s)\,dB_s\rangle \\ & + \frac{1}{2}\int_0^t Tr(\partial^2_{xx} f(X_{s-})\sigma(s)\sigma(s)^*)\,ds \\ & + \int_0^t\int_{|z|\ge1} \left[f(X_{s-} + K(s,z)) - f(X_{s-})\right] N(ds,dz) \\ & + \int_0^t\int_{|z|<1} \left[f(X_{s-} + H(s,z)) - f(X_{s-})\right] \tilde{N}(ds,dz)) \\ & + \int_0^t\int_{|z|<1} \left[f(X_{s-} + H(s,z)) - f(X_{s-}) - \langle H(s,z), \partial_x f(X_{s-})\rangle\right] ds\,\lambda(dz). \end{aligned}$$

Proof. The proof is stated in [26] (Theorem 4.4.7).

2.3. Strong law of large numbers

Theorem 2.3. *Let $M = (M_t)_{t\ge0}$ be a real-valued continuous local martingale vanishing at t = 0 (Theorem1 of [28]). Then,*

$$\lim_{t\to\infty}\langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty}\frac{M_t}{\langle M, M\rangle_t} = 0, \quad a.s.$$

and

$$\limsup_{t\to\infty}\frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty}\frac{M_t}{t} = 0, \quad a.s.$$

More generally, if $A = (A_t)_{t\ge0}$ is a continuous adapted increasing process such that

$$\lim_{t\to\infty} A_t = \infty \quad and \quad \int_0^\infty \frac{d\langle M, M\rangle_t}{(1+A_t)^2} < \infty, \quad a.s.$$

then,

$$\lim_{t\to\infty}\frac{M_t}{A_t} = 0 \quad a.s.$$

Proof. The proof of Theorem 2.3 is founded in [28] (Theorem 1).

3. Global positive solution

Let f be an integrable function on $[0,t]$, we denote

$$\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds, \quad \langle f\rangle^* = \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds, \quad \langle f\rangle_* = \liminf_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds, \quad t>0.$$

We assume that the jump coefficient $C(z)$ in (1.2) satisfy the following assumption :

Assumption $(\mathbf{H}_1)$: $C(z)$ is bounded, $1+C(z)>0$, $\|C\|_\infty < \frac{\mu}{\Lambda}$ and $\int_{\mathbb{R}^*} C^2(z)\lambda(dz) < \infty$.

The next theorem ensures the existence and uniqueness of a global positive solution. Following the same steps of [29] and [30], we can obtain the hereunder results.

Theorem 3.1. *Under the assumption* $(\boldsymbol{H}_1)$*, for any given initial value* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$*, the system* (1.2) *admits a unique positive solution* $(S(t), I(t), R(t))$*,* $t\in\mathbb{R}_+$ *for all* $t\geq 0$ *a.s.*

Proof. By the assumption $(\mathbf{H}_1)$, the drift and the diffusion are locally Lipschitz, then for any given initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there is a unique local solution $(S(t), I(t), R(t))$ for $t\in[0,\tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e=\infty$ a.s. At first, we prove that $(S(t), I(t), R(t))$ do not explode to infinity in a finite time. Let $k_0>0$ be sufficiently large so that $S(0), I(0), R(0) \in [\frac{1}{k_0}, k_0]$. For each integer $k\geq k_0$, we define the stopping time:

$$\tau_k = \inf\left\{t\in[0,\tau_e) : S(t)\notin\left(\frac{1}{k},k\right) \text{ or } I(t)\notin\left(\frac{1}{k},k\right) \text{ or } R(t)\notin\left(\frac{1}{k},k\right)\right\},$$

where τ_k is increasing as $k\uparrow\infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$, whence, $\tau_\infty\leq\tau_e$ a.s. If we can show that $\tau_\infty=\infty$ is true, then $\tau_e=\infty$ and $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ a.s. By absurd, assume that $\tau_\infty<\infty$, then there exist two constants $T>0$ and $0<\varepsilon<1$ such that $\mathbb{P}(\tau_\infty\leq T)\geq\varepsilon$. Thus there is an integer $k_1\geq k_0$ such that

$$\mathbb{P}(\tau_k\leq T)\geq\varepsilon, \quad \forall k\geq k_1. \tag{3.1}$$

Consider the C^2-function $V:\mathbb{R}^3_+\to\mathbb{R}_+$ as follows :

$$V(S,I,R) \;=\; \left(S-a-a\log\frac{S}{a}\right) + (I-1-\log I) + (R-1-\log R), \tag{3.2}$$

where a is a positive constant to be determined bellow and $u-1-\log u\geq 0$, for $u\geq 0$. Applying the Itô formula (see theorem 2.2) to the function $V(S,I,R)$, denote $X(t)=(S(t),I(t),R(t))$, we obtain

$$\begin{aligned} dV(X(t)) = \;& LV(X(t))dt + \frac{\partial V}{\partial S(t)}(-\sigma S(t)I(t))dB(t) + \frac{\partial V}{\partial I(t)}\sigma S(t)I(t)dB(t) \\ & + \int_{\mathbb{R}^*}[V(S(t-) - S(t-)I(t-)C(z); I(t-) + S(t-)I(t-)C(z)) \\ & - V(S(t-), I(t-))]\tilde{N}(dt,dz). \end{aligned}$$

It follows that,

$$\begin{aligned} dV(X(t)) = \;& LV(X(t))dt + \sigma\left[I(t)(S(t)-a) + S(t)(I(t)-1)\right]dB(t) \\ & - \int_{\mathbb{R}^*}[\ln(1+S(t-)C(z)) + a\ln(1-I(t-)C(z))]\tilde{N}(dt,dz). \end{aligned} \tag{3.3}$$

Now, we calculate the infinitesimal operator LV

$$\begin{aligned} LV(X(t)) = \quad & \frac{\partial V}{\partial X} f(X(t)) + \frac{1}{2} tr\left(g^{\top}(X(t)) \frac{\partial^2 V}{\partial X^2} g(X(t)) \right) \\ & + \int_{\mathbb{R}^*} [V(S(t) - S(t)I(t)C(z); I(t) + S(t)I(t)C(z)) \\ & - V(S(t), I(t)) - \left\{ \frac{\partial V}{\partial S(t)}(-S(t)I(t)C(z)) + \frac{\partial V}{\partial I(t)} S(t)I(t)C(z) \right\} \Bigg] \lambda(dz), \end{aligned}$$

where,

$$f(X(t)) = \begin{pmatrix} \Lambda - \beta S(t)I(t) - \mu S(t) \\ (\beta S(t) - (\mu + \gamma))I(t) + \delta R(t) \\ \gamma I(t) - (\mu + \delta)R(t) \end{pmatrix} \quad \text{and} \quad g(X(t)) = \begin{pmatrix} -\sigma S(t)I(t) \\ \sigma S(t)I(t) \\ 0 \end{pmatrix}.$$

Then, we obtain

$$\begin{aligned} LV(X(t)) = \quad & (\Lambda + a\mu + 2\mu + \gamma + \delta) - (\mu + \beta)S(t) + (a\beta - \mu)I(t) \\ & - \frac{a\Lambda}{S(t)} - \delta \frac{R(t)}{I(t)} - \gamma \frac{I(t)}{R(t)} + \frac{1}{2} a\sigma^2 I^2(t) + \frac{1}{2}\sigma^2 S^2(t) \\ & + aM_1(t) + M_2(t), \end{aligned}$$

where,

$$M_1(t) = -\int_{\mathbb{R}^*} [\ln(1 - I(t)C(z)) + I(t)C(z)]\lambda(dz),$$

and

$$M_2(t) = -\int_{\mathbb{R}^*} [\ln(1 + S(t)C(z)) - S(t)C(z)]\lambda(dz).$$

Choosing $a = \frac{\mu}{\beta}$; taking into account that $x - \ln(1 + x) \geq 0$, $\forall x > -1$, one can obtain that

$$LV(X(t)) \leq \Lambda + a\mu + 2\mu + \gamma + \delta + \frac{1}{2} a\sigma^2 K^2 + \frac{1}{2}\sigma^2 K^2 + aM_1(t) + M_2(t).$$

In addition, by Taylor-Lagrange formula, we obtain

$$\ln(1 - I(t)C(z)) = -I(t)C(z) - \frac{I^2(t)C^2(z)}{2(1 - \theta I(t)C(z))^2},$$

where, $\theta \in (0, 1)$ is an arbitrary number. Then, by assumption $(\mathbf{H}_1)$

$$-[\ln(1 - I(t)C(z)) + I(t)C(z)] = \frac{I^2(t)C^2(z)}{2(1 - \theta I(t)C(z))^2} \leq \frac{1}{2(1 - \frac{\Lambda}{\mu}\|C\|_\infty)^2},$$

Similarly, we have

$$-[\ln(1 + S(t)C(z)) - S(t)C(z)] = \frac{S^2(t)C^2(z)}{2(1 + \theta S(t)C(z))^2} \leq \frac{1}{2(1 + \frac{\Lambda}{\mu}\|C\|_\infty)^2} \leq \frac{1}{2(1 - \frac{\Lambda}{\mu}\|C\|_\infty)^2},$$

where, $\theta \in (0, 1)$ is an arbitrary number. Then,

$$LV(X(t)) \leq \Lambda + a\mu + 2\mu + \gamma + \delta + \frac{1}{2}a\sigma^2K^2 + \frac{1}{2}\sigma^2K^2 + \frac{a+1}{2(1-\frac{\Lambda}{\mu}\|C\|_\infty)^2}\lambda(\mathbb{R}^*) := \bar{K}.$$

Integrating both sides of (3.3) over $[0, \tau_k \wedge T]$ and taking the mathematical expectation, we obtain

$$\mathbb{E}[V(X(\tau_k \wedge T))] \quad \leq \quad \bar{K}T + V(X(0)) < \infty. \tag{3.4}$$

Let $\Omega_k = \{\tau_k \leq T\}$. We have $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$, then $\mathbb{P}(\Omega_k) \geq \varepsilon$, for $k \geq k_1$. On the other hand, from (3.2), we have $V(X(\tau_k \wedge T)) \geq 0$, thus

$$\begin{aligned}\mathbb{E}[V(X(\tau_k \wedge T))] &= \mathbb{E}[\mathbf{1}_{\Omega_k}V(X(\tau_k \wedge T))] + \mathbb{E}[\mathbf{1}_{\Omega_k^C}V(X(\tau_k \wedge T))]\\ &\geq \mathbb{E}[\mathbf{1}_{\Omega_k}V(X(\tau_k \wedge T))],\end{aligned}$$

where $\mathbf{1}_{\Omega_k}$ is the indicator function of Ω_k. It follows that,

$$\begin{aligned}V(X(0)) + \bar{K}T &\geq \mathbb{E}\,[\mathbf{1}_{\Omega_k}V(X(\tau_k \wedge T))]\\ &\geq \varepsilon\left[\left(k - a - a\ln\frac{k}{a}\right) \wedge \left(\frac{1}{k} - a - a\ln\frac{1}{ak}\right) \wedge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right)\right].\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\infty > V(X(0)) + \bar{K}T = \infty,$$

which is a contradiction, so we must have $\tau_e = \infty$ a.s. Consequently $S(t)$, $I(t)$ and $R(t)$ are global positive solution to the system (1.2). The proof is complete.

□

Theorem 3.2. *All solution of the SIRI system* (1.2) *that initiate in* $\mathbb{R}^3_+$ *are bounded and enter into set*

$$\Gamma = \{(S, I, R) \in \mathbb{R}^3_+ : 0 < S(t) + I(t) + R(t) < \tfrac{\Lambda}{\mu}\}.$$

Proof. Let $N(t) = S(t) + I(t) + R(t)$, (see [31]) where $(S(t), I(t), R(t))$ SIRI system structure (1.2). Differentiating both sides with respect to t, we have

$$\begin{aligned}\frac{dN(t)}{dt} &= \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt}\\ &= \Lambda - \beta S(t)I(t) - \mu S(t) + \beta S(t)I(t) - (\mu + \gamma)I(t)\\ &\quad + \delta R(t) + \gamma I(t) - (\mu + \delta)R(t)\\ &\leq \Lambda - \mu(S(t) + I(t) + R(t)),\end{aligned}$$

Which implies,

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N(t).$$

According to the comparison theorem, we obtain

$$0 < N(t) < N(0)\exp(-\mu t) + \frac{\Lambda}{\mu},$$

as $t \to \infty$, we have $0 < N(t) \leq \frac{\Lambda}{\mu}$. Therefore, all solution of the system structure (1.2) enter into the positivity invariant set $\Gamma = \{(S, I, R) \in \mathbb{R}^3_+ : 0 < N(t) < \frac{\Lambda}{\mu}\}$ as $t \to \infty$. Thus the proof is completed.

□

4. Extinction of the disease

Lemma 4.1. *Let* $(S(t), I(t), R(t))$, $t \in \mathbb{R}_+$ *be the solution of the system* (1.2) *with any initial value* $(S(0), I(0), R(0)) \in \Gamma$. *If the condition* $(\boldsymbol{H}_1)$ *hold for some* $p > 1$*, then*

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{R(t)}{t} = 0, \quad \mathbb{P}\text{-}a.s.$$

Proof. Define $X(t) := S(t) + I(t) + R(t)$. Applying the Itô formula to the function $V(X(t)) = (1 + X(t))^p$, we obtain

$$\begin{aligned}
dV(X(t)) = \quad & LV(X(t))dt + \frac{\partial V}{\partial S(t)}[-\sigma S(t)I(t)]dB(t) + \frac{\partial V}{\partial I(t)}\sigma S(t)I(t)dB(t) \\
& + \int_{\mathbb{R}^*} [V(S(t-) - S(t-)I(t-)C(z); I(t-) + S(t-)I(t-)C(z)) \\
& - V(S(t-), I(t-))]\tilde{N}(dt, dz) \\
= \quad & LV(X(t))dt + p(1 + X(t))^{p-1}[-\sigma S(t)I(t)]dB(t) + p(1 + X(t))^{p-1}\sigma S(t)I(t)dB(t) \\
& + \int_{\mathbb{R}^*} [(1 + S(t-) - S(t-)I(t-)C(z) + I(t-) + S(t-)I(t-)C(z))^p \\
& - (1 + S(t-) + I(t-))^p]\tilde{N}(dt, dz) \\
= \quad & LV(X(t))dt,
\end{aligned}$$

Now, we calculate the infinitesimal operator LV, we obtain

$$\begin{aligned}
LV(X(t)) \quad &= \quad p(1 + X(t))^{p-1}(\Lambda - \mu X(t)) + \frac{p(p-1)}{2}(1 + X(t))^{p-2}\, 2\sigma^2 S^2(t)I^2(t) \\
&\leq \quad p(1 + X(t))^{p-1}(\Lambda - \mu X(t)) + \frac{p(p-1)}{2}(1 + X(t))^{p-2}\, \sigma^2 X^2(t).
\end{aligned}$$

Factoring by $p(1 + X(t))^{p-2}$, we obtain

$$LV(X(t)) \leq p(1 + X(t))^{p-2}\left(\Lambda X(t) - bX^2(t)\right), \tag{4.1}$$

where, $b = \mu - \dfrac{p-1}{2}\sigma^2 > 0$. Hence,

$$dV(X(t)) \leq p(1 + X(t))^{p-2}\left(\Lambda X(t) - bX^2(t)\right). \tag{4.2}$$

For $0 < k < bp$, we have

$$de^{kt}V(X(t)) \quad \leq \quad L\left[e^{kt}V(X(t))\right]dt. \tag{4.3}$$

Integrating both sides of (4.3) from 0 to t yields

$$e^{kt}V(X(t)) - V(X(0)) \quad \leq \quad \int_0^t \left[ke^{ks}V(X(s)) + e^{ks}LV(X(s))\right]ds \tag{4.4}$$

Taking expectation on both sides of (4.4) yields

$$\mathbb{E}\left[e^{kt}V(X(t))\right] \quad \leq \quad V(X(0)) + \mathbb{E}\left[\int_0^t e^{ks}\left[kV(X(s)) + LV(X(s))\right]ds\right]. \tag{4.5}$$

Noting (4.1), we have

$$\begin{aligned} e^{kt}\left[kV(X(t)) + e^{kt}LV(X(t))\right] &\leq e^{kt}\left[k(1+X(t))^p + p(1+X(t))^{p-2}(\Lambda X(t) - bX^2(t))\right] \\ &\leq pe^{kt}(1+X(t))^{p-2}\left[-\left(b - \frac{k}{p}\right)X^2(t) + \Lambda X(t)\right] \\ &\leq pe^{kt}A, \quad t \in \mathbb{R}_+, \end{aligned} \tag{4.6}$$

where,

$$0 < A := 1 + \sup_{x\in\mathbb{R}_+} x^{p-2}\left[-\left(b - \frac{k}{p}\right)x^2 + \Lambda x\right] < \infty.$$

Injecting (4.6) into (4.5), we obtain

$$\begin{aligned} e^{kt}\mathbb{E}\left[(1+X(t))^p\right] &\leq (1+X(0))^p + \mathbb{E}\left[\int_0^t pe^{ks}Ads\right] \\ &\leq (1+X(0))^p + \frac{pA}{k}e^{kt}, \quad t \in \mathbb{R}_+. \end{aligned}$$

Then,

$$\mathbb{E}\left[(1+X(t))^p\right] \leq \frac{(1+X(0))^p}{e^{kt}} + \frac{pA}{k}.$$

For $0 < k < bp$, we have

$$\limsup_{t\to\infty} \mathbb{E}\left[(1+X(t))^p\right] \leq \frac{pA}{k},$$

then there exists $A_0 > 0$, such that

$$\mathbb{E}\left[(1+X(t))^p\right] \leq A_0, \quad t \in \mathbb{R}_+. \tag{4.7}$$

Integrating both sides of (4.2) from τk to t, $t > \tau k$, where $\tau > 0$ and $k = 0, 1, 2, \ldots$, we obtain

$$\int_{\tau k}^t dV(X(s)) \leq p\int_{\tau k}^t (1+X(s))^{p-2}(\Lambda X(s) - bX^2(s))ds,$$

then,

$$(1+X(t))^p \leq (1+X(\tau k))^p + p\int_{\tau k}^t (1+X(s))^{p-2}(\Lambda X(s) - bX^2(s))ds.$$

Afterward,

$$\sup_{\tau k\leq t\leq \tau(k+1)} (1+X(t))^p \leq (1+X(\tau k))^p + p\sup_{\tau k\leq t\leq \tau(k+1)}\left|\int_{\tau k}^t (1+X(s))^{p-2}(\Lambda X(s) - bX^2(s))ds\right|.$$

Applying the expectation on the above inequality, we obtain

$$\mathbb{E}\left[\sup_{\tau k\leq t\leq \tau(k+1)} (1+X(t))^p\right] \leq \mathbb{E}\left[(1+X(\tau k))^p\right] + B.$$

we used the inequality (4.7), where, for some $c > 0$, we have

$$\begin{aligned}
B &:= p\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}\left|\int_{\tau k}^{t}(1+X(s))^{p-2}(\Lambda X(s)-bX^2(s))\,ds\right|\right]\\
&\le p\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}\left|\int_{\tau k}^{t}(1+X(s))^{p-2}\;\Lambda X(s)\,ds\right|\right]\\
&\le p\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}\left|\int_{\tau k}^{t}(1+X(s))^{p-2}\;\Lambda(1+X(s))^2\,ds\right|\right]\\
&\le p\Lambda\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}\left|\int_{\tau k}^{t}(1+X(s))^{p}\,ds\right|\right]\\
&\le c\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}\int_{\tau k}^{t}(1+X(s))^{p}\,ds\right]\\
&\le c\,\mathbb{E}\left[\int_{\tau k}^{\tau(k+1)}(1+X(s))^{p}\,ds\right]\\
&\le c\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\,(\tau(k+1)-\tau k)\right]\\
&\le c\tau\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right],
\end{aligned}$$

where, $c = p\Lambda > 0$. Therefor, we have

$$\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right] \le \mathbb{E}\left[(1+X(\tau k))^{p}\right]+c\,\tau\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right]. \tag{4.8}$$

We choose $\tau > 0$, such that $c\,\tau < \frac{1}{2}$. We combine (4.7) with (4.8), we obtain

$$\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right] \le \mathbb{E}\left[(1+X(\tau k))^{p}\right]+\frac{1}{2}\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right],$$

then,

$$\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right] \le 2A_0. \tag{4.9}$$

Let $\epsilon > 0$ be positive. Applying Chebysheve's inequality, we obtain

$$\mathbb{P}\left(\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p} > (\tau k)^{1+\epsilon}\right) \le \frac{1}{(\tau k)^{1+\epsilon}}\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}(1+X(t))^{p}\right] \le \frac{2A_0}{(\tau k)^{1+\epsilon}} < \infty,$$

for all $k \ge 1$. According to Borel-Cantelli lemma, we obtain that for almost all $\omega \in \Omega$, the bound

$$(1+X(t))^{p} \le (\tau k)^{1+\epsilon},$$

holds for all but finitely many k. Then, there exists a $k_0(\omega)$, such that whenever $k \ge k_0(\omega)$, we have

$$\ln(1+X(t))^{p} \le \ln(\tau k)^{1+\epsilon},$$

we have $\epsilon > 0$ and $\tau k \leq t \leq \tau(k+1)$, then

$$\ln(1+X(t))^p \quad \leq \quad (1+\epsilon)\ln(t).$$

It follows that,

$$\frac{\ln X(t)}{\ln(t)} \quad \leq \quad \frac{1}{p} + \frac{\epsilon}{p}.$$

Let $\epsilon \to 0$, we obtain

$$\limsup_{t\to\infty} \frac{\ln X(t)}{\ln(t)} \quad \leq \quad \frac{1}{p}, \quad \mathbb{P}-a.s, \quad p>1.$$

Set finite random time $T = T(\omega)$, for any small $0 < \zeta < 1 - 1/p$, we have

$$\ln X(t) \quad \leq \quad \left(\zeta + \frac{1}{p}\right)\ln(t), \qquad t \geq T,$$

so,

$$\ln X(t) \quad \leq \quad \ln(t^{\zeta+\frac{1}{p}}), \qquad t \geq T.$$

It follows that,

$$X(t) \quad \leq \quad t^{\zeta+\frac{1}{p}}, \qquad t \geq T.$$

Then,

$$\limsup_{t\to\infty} \frac{X(t)}{t} \quad \leq \quad \limsup_{t\to\infty} \frac{t^{\zeta+\frac{1}{p}}}{t} = 0,$$

It implies,

$$\limsup_{t\to\infty} \frac{S(t)}{t} \leq 0, \quad \limsup_{t\to\infty} \frac{I(t)}{t} \leq 0, \quad \limsup_{t\to\infty} \frac{R(t)}{t} \leq 0, \quad \mathbb{P}\text{-a.s.}$$

By the positivity of the solution, we get

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{R(t)}{t} = 0, \quad \mathbb{P}\text{-a.s.}$$

The proof is completed.

□

Lemma 4.2. *Let $(S(t), I(t), R(t))$, $t \in \mathbb{R}_+$ be the solution of the system* (1.2) *with any initial value $(S(0), I(0), R(0)) \in \Gamma$. If $(\boldsymbol{H}_1)$ hold for some $p > 2$, Then*
i) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{R}^*} \ln(1+S(s-)C(z))\,\tilde{N}(ds,dz) = 0$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz) = 0$, $\mathbb{P}$*-a.s.*
ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)dB(s) = 0$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)I(s)dB(s) = 0$, $\mathbb{P}$*-a.s.*

Proof. Denote

$$X_1(t) := \int_0^t \int_{\mathbb{R}^*} \ln(1+S(s-)C(z))\,\tilde{N}(ds,dz), \quad X_2(t) := \int_0^t \int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz),$$

and

$$X_3(t) := \int_0^t S(s)dB(s), \quad X_4(t) := \int_0^t S(s)I(s)dB(s), \qquad t \in \mathbb{R}_+.$$

According to the Kunita's inequality (2.2), for any $p \geq 2$, there exists a positive constant c_p, such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X_1(s)|^p\right] \leq \quad c_p\,\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^*} |\ln(1+S(s)C(z))|^2\,\lambda(dz)ds\right)^{p/2}\right] + c_p\,\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^*} |\ln(1+S(s)C(z))|^p\,\lambda(dz)ds\right].$$

Applying Taylor-Lagrange formula, we obtain

$$|\ln(1+S(s)C(z))| \leq \frac{\Lambda}{\mu}\left(1+\frac{\Lambda\|C\|_\infty}{2\mu\left(1-\frac{\Lambda}{\mu}\|C\|_\infty\right)^2}\right)|C(z)|.$$

Therefore, there is a positive constant C_p such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|X_1(s)|^p\right] \leq C_p\, t^{p/2}\left(\int_{\mathbb{R}^*}C^2(z)\lambda(dz)\right)^{p/2} + C_p\, t\int_{\mathbb{R}^*}C^p(z)\lambda(dz), \tag{4.10}$$

where, $C_p = c_p\left[\frac{\Lambda}{\mu}\left(1+\frac{\Lambda\|C\|_\infty}{2\mu\left(1-\frac{\Lambda}{\mu}\|C\|_\infty\right)^2}\right)|C(z)|\right]^p$. Let $\epsilon > 0$. By Doob's martingale inequality, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{\tau k\leq t\leq\tau(k+1)}|X_1(t)|^p > (\tau k)^{1+\epsilon+p/2}\right) \leq&\ \frac{1}{(\tau k)^{1+\epsilon+p/2}}\,\mathbb{E}\left[\sup_{\tau k\leq t\leq\tau(k+1)}|X_1(t)|^p\right]\\ \leq&\ \frac{C_p((\tau(k+1))^{p/2}}{(\tau k)^{1+\epsilon+p/2}}\left(\int_{\mathbb{R}^*}C^2(z)\lambda(dz)\right)^{p/2} + \frac{C_p\tau(k+1)}{(\tau k)^{1+\epsilon+p/2}}\int_{\mathbb{R}^*}C^p(z)\lambda(dz).\end{aligned}$$

According to Borel-Cantelli lemma, we obtain that for almost all $\omega\in\Omega$

$$|X_1(t)|^p \leq (\tau k)^{1+\epsilon+p/2}, \quad \tau k\leq t\leq\tau(k+1),$$

Then, there exists a $k_0(\omega)$, such that whenever $k\geq k_0(\omega)$, we have

$$\ln|X_1(t)|^p \leq \ln(\tau k)^{1+\epsilon+p/2},$$

we have $\epsilon > 0$ and $\tau k\leq t\leq\tau(k+1)$, then

$$\ln|X_1(t)|^p \leq (1+\epsilon+p/2)\ln(t).$$

It implies that,

$$\frac{\ln|X_1(t)|}{\ln(t)} \leq \frac{1+\epsilon+p/2}{p} = \frac{1}{2}+\frac{1+\epsilon}{p}, \quad \epsilon>0,\ \ \tau k\leq t\leq\tau(k+1).$$

Letting $\epsilon\to 0$, yields

$$\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln(t)} \leq \frac{1}{2}+\frac{1}{p}, \quad \mathbb{P}-a.s, \quad p>2.$$

Then, there exists an a.s finite random time $\tilde{T}=\tilde{T}(\omega)$ for any small $0<\tilde{\zeta}<\frac{1}{2}-\frac{1}{p}$, such that

$$\ln|X_1(t)| \leq \left(\frac{1}{2}+\frac{1}{p}+\tilde{\zeta}\right)\ln(t), \quad t\geq\tilde{T},$$

so,

$$\ln|X_1(t)| \leq \ln(t^{\frac{1}{2}+\frac{1}{p}+\tilde{\zeta}}), \quad t\geq\tilde{T}.$$

It follows that,

$$|X_1(t)| \leq t^{\frac{1}{2}+\frac{1}{p}+\tilde{\zeta}}, \quad t\geq\tilde{T}.$$

Then,

$$\limsup_{t\to\infty}\frac{|X_1(t)|}{t} \leq \limsup_{t\to\infty}\frac{t^{\frac{1}{2}+\frac{1}{p}+\tilde{\zeta}}}{t} = 0,$$

which implies,

$$\lim_{t\to\infty}\frac{|X_1(t)|}{t}=0,\quad \mathbb{P}\text{-a.s.}$$

Then,

$$\lim_{t\to\infty}\frac{X_1(t)}{t}=0,\quad \mathbb{P}\text{-a.s.}$$

Following the same steps as above, one can obtain

$$\lim_{t\to\infty}\frac{X_2(t)}{t}=0\quad \mathbb{P}\text{-a.s.}$$

Notice that, $X_3(t) := \int_0^t S(s)dB(s)$. Applying Burkholder-Davis-Gundy inequality (2.1), for any $p \geq 1$, there exists a positive constant C_p, such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|X_3(s)|^p\right] &\leq C_p\,\mathbb{E}\left[\left(\int_0^t |X_3(s)|^2dr\right)^{p/2}\right]\\
&\leq C_p\,\mathbb{E}\left[\left(\sup_{0\le s\le t}|X_3(t)|^2\right)^{p/2}\right]\left(\int_0^t 1\;ds\right)^{p/2}\\
&:= C_p\;t^{p/2}\,\mathbb{E}\left[\sup_{0\le s\le t}|X_3(t)|^p\right].
\end{aligned}$$

Combining the above inequality with (4.9), we obtain

$$\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}|X_3(t)|^p\right] \leq 2A_0C_p(\tau(k+1))^{p/2}.$$

Let $\epsilon > 0$. By Doob's martingale inequality, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{\tau k\le t\le\tau(k+1)}|X_3(t)|^p > (\tau k)^{1+\epsilon+p/2}\right) &\leq \frac{1}{(\tau k)^{1+\epsilon+p/2}}\,\mathbb{E}\left[\sup_{\tau k\le t\le\tau(k+1)}|X_3(t)|^p\right]\\
&\leq \frac{2A_0C_p(\tau(k+1))^{p/2}}{(\tau k)^{1+\epsilon+p/2}}.
\end{aligned}$$

According to Borel-Cantelli lemma, we obtain that for all $\omega \in \Omega$

$$\sup_{\tau k\le t\le\tau(k+1)}|X_3(t)|^p \leq (\tau k)^{1+\epsilon+p/2},$$

then,

$$|X_3(t)|^p \leq (\tau k)^{1+\epsilon+p/2},\quad \tau k \leq t \leq \tau(k+1).$$

Then, there exists a $k_0(\omega)$, such that whenever $k \geq k_0(\omega)$, we have

$$\ln|X_3(t)|^p \leq \ln(\tau k)^{1+\epsilon+p/2},$$

we have $\epsilon > 0$ and $\tau k \leq t \leq \tau(k+1)$, then

$$\ln|X_3(t)|^p \leq (1+\epsilon+p/2)\ln(t).$$

Which implies,

$$\frac{\ln|X_3(t)|}{\ln(t)} \leq \frac{1+\epsilon+p/2}{p}=\frac{1}{2}+\frac{1+\epsilon}{p}.$$

Let $\epsilon \to 0$, we obtain

$$\limsup_{t\to\infty} \frac{\ln|X_3(t)|}{\ln(t)} \leq \frac{1}{2} + \frac{1}{p}, \quad \mathbb{P} - a.s, \quad p > 2.$$

Then, there exists a finite random time $\bar{T} = \bar{T}(\omega)$ for any small $0 < \bar{\zeta} < \frac{1}{2} - \frac{1}{p}$, such that

$$\ln|X_3(t)| \leq \left(\frac{1}{2} + \frac{1}{p} + \bar{\zeta}\right)\ln(t), \qquad t \geq \bar{T},$$

so,

$$\ln|X_3(t)| \leq \ln(t^{\frac{1}{2}+\frac{1}{p}+\bar{\zeta}}), \qquad t \geq \bar{T}.$$

It follows that,

$$|X_3(t)| \leq t^{\frac{1}{2}+\frac{1}{p}+\bar{\zeta}}, \qquad t \geq \bar{T}.$$

Then,

$$\limsup_{t\to\infty} \frac{|X_3(t)|}{t} \leq \limsup_{t\to\infty} \frac{t^{\frac{1}{2}+\frac{1}{p}+\bar{\zeta}}}{t} = 0,$$

this implies,

$$\lim_{t\to\infty} \frac{X_3(t)}{t} = 0, \quad \mathbb{P}\text{-a.s.}$$

By the same way, we have

$$\lim_{t\to\infty} \frac{X_4(t)}{t} = 0, \quad \mathbb{P}\text{-a.s.}$$

The proof is therefore completed.

□

We define the following quantities :

$$\sigma_e = \frac{1}{4}\left\{\left(\frac{\sigma\mu}{\Lambda}\right)^2 + \frac{2}{(1+\frac{\Lambda}{\mu}\|C\|_\infty)^2}\int_{\mathbb{R}^*} C^2(z)\lambda(dz)\right\}, \tag{4.11}$$

and

$$\mathcal{R}_e = \frac{\beta\frac{\Lambda}{\mu}}{\mu + \sigma_e\left(\frac{\Lambda}{\mu}\right)^2}. \tag{4.12}$$

In the next theorem, we establish sufficient conditions for the extinction of system (1.2).

Theorem 4.1. *Let $(S(t), I(t), R(t))$ be the solution of system (1.2) with any initial value $(S(0), I(0), R(0)) \in \Gamma$. If the following conditions $\mathcal{R}_e < 1$ and $2\sigma_e\frac{\Lambda}{\mu} \leq \beta$ hold, then the disease goes out of the population exponentially with probability one. That is to say*

$$\lim_{t\to\infty}\langle S\rangle_t = \frac{\Lambda}{\mu}, \quad \lim_{t\to\infty} I(t) = 0, \quad \textit{and} \quad \lim_{t\to\infty} R(t) = 0, \quad \mathbb{P}\text{-}a.s.$$

Proof. Notice that,

$$\begin{aligned} I(t) + R(t) &= I(0) + R(0) + \beta\int_0^t S(s)I(s)ds - \mu\int_0^t (I(s)+R(s))ds \\ &+ \sigma\int_0^t S(s)I(s)dB(s) + \int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz). \end{aligned}$$

Applying the Itô formula for Lévy-type stochastic integrals to the function $I(t) + R(t) \mapsto \ln(I(t) + R(t))$, we obtain

$$
\begin{aligned}
\ln(I(t)+R(t)) = \quad & \ln(I(0)+R(0)) + \beta \int_0^t \frac{S(s)I(s)}{I(s)+R(s)} ds - \mu t \\
& -\frac{1}{2}\sigma^2 \int_0^t \frac{(S(s)I(s))^2}{(I(s)+R(s))^2} ds + \sigma \int_0^t \frac{S(s)I(s)}{I(s)+R(s)} dB(s) \\
& + \int_0^t \int_{\mathbb{R}^*} \ln\left(\frac{I(s-)+R(s-)+S(s-)I(s-)C(z)}{I(s-)+R(s-)}\right) \tilde{N}(ds,dz) \\
& + \int_0^t \int_{\mathbb{R}^*} \left\{ \ln\left(\frac{I(s)+R(s)+S(s)I(s)C(z)}{I(s)+R(s)}\right) - \frac{S(s)I(s)C(z)}{I(s)+R(s)} \right\} \lambda(dz)ds \qquad (4.13) \\
\leq \quad & \ln(I(0)+R(0)) + \beta \int_0^t S(s)ds - \mu t - \frac{1}{4}\left(\frac{\sigma\mu}{\Lambda}\right)^2 \int_0^t S^2(s)ds \\
& + \sigma \int_0^t S(s)dB(s) + \int_0^t \int_{\mathbb{R}^*} \ln(1+S(s-)C(z))\tilde{N}(ds,dz) \\
& + \int_0^t \int_{\mathbb{R}^*} \left[\ln(1+S(s)C(z)) - S(s)C(z)\right] \lambda(dz)ds,
\end{aligned}
$$

where we used the bound $-(a+b)^{-2} \leq -2^{-1}(a^2b^2)^{-1}$ and $X(t) \in \Gamma,\ t \in \mathbb{R}_+$. Using the Taylor-Lagrange formula, for some $\theta \in (0,1)$ we obtain that

$$
\ln(1+S(t)C(z)) - S(t)C(z) = -\frac{C^2(z)}{2\left(1+\frac{\Lambda}{\mu}\|C\|_\infty\right)^2} S^2(t). \qquad (4.14)
$$

Hence,

$$
\begin{aligned}
\ln(I(t)+R(t)) \leq \quad & \ln(I(0)+R(0)) + \beta \int_0^t S(s)ds - \mu t - \frac{1}{4}\left(\frac{\sigma\mu}{\Lambda}\right)^2 \int_0^t S^2(s)ds \\
& + \sigma \int_0^t S(s)dB(s) + \int_0^t \int_{\mathbb{R}^*} \ln(1+S(s-)C(z))\tilde{N}(ds,dz) \\
& - \int_0^t \int_{\mathbb{R}^*} \frac{C^2(z)}{2\left(1+\frac{\Lambda}{\mu}\|C\|_\infty\right)^2} S^2(s)\lambda(dz)ds.
\end{aligned}
$$

This implies,

$$
\begin{aligned}
\frac{\ln(I(t)+R(t))}{t} \leq \quad & \beta\langle S\rangle_t - \mu - \frac{1}{4}\left\{\left(\frac{\sigma\mu}{\Lambda}\right)^2 + \frac{2}{(1+\frac{\Lambda}{\mu}\|C\|_\infty)^2}\int_{\mathbb{R}^*} C^2(z)\lambda(dz)\right\}\langle S^2\rangle_t \\
& + \frac{1}{t}\left[\ln(I(0)+R(0)) + \sigma\int_0^t S(s)dB(s)\right] \\
& + \frac{1}{t}\int_0^t \int_{\mathbb{R}^*} \ln(1+S(s-)C(z))\tilde{N}(ds,dz).
\end{aligned}
$$

Then,

$$\frac{\ln(I(t)+R(t))}{t} \leq \beta\langle S\rangle_t - \mu - \sigma_e\langle S^2\rangle_t + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \tag{4.15}$$

where,

$$M_1(t) = \ln(I(0)+R(0)) + \sigma\int_0^t S(s)dB(s) \quad \text{and} \quad M_2(t) = \int_0^t\int_{\mathbb{R}^*}\ln(1+S(s-)C(z))\tilde{N}(ds,dz).$$

Integrating from 0 to t on both sides of the system (1.2), we obtain

$$\begin{aligned}\int_0^t dS(s) = \quad & \int_0^t \left[\Lambda - \beta S(s)I(s) - \mu S(s)\right] ds - \sigma\int_0^t S(s)I(s)dB(s) \\ & - \int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz).\end{aligned}$$

It follows that,

$$\begin{aligned}S(t) - S(0) = \quad & \Lambda t - \beta\int_0^t S(s)I(s)ds - \mu\int_0^t S(s)ds - \sigma\int_0^t S(s)I(s)dB(s) \\ & - \int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz),\end{aligned}$$

then,

$$\begin{aligned}\frac{S(t)-S(0)}{t} = \quad & \Lambda - \beta\langle SI\rangle_t - \mu\langle S\rangle_t - \frac{1}{t}\sigma\int_0^t S(s)I(s)dB(s) \\ & -\frac{1}{t}\int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz).\end{aligned} \tag{4.16}$$

We denote that,

$$\begin{aligned}\int_0^t dI(s) = \quad & \beta\int_0^t S(s)I(s)ds - (\mu+\gamma)\int_0^t I(s)ds + \delta\int_0^t R(s)ds + \sigma\int_0^t S(s)I(s)dB(s) \\ & + \int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz),\end{aligned}$$

then,

$$\begin{aligned}\frac{I(t)-I(0)}{t} = \quad & \beta\langle SI\rangle_t - (\mu+\gamma)\langle I\rangle_t + \delta\langle R\rangle_t + \frac{1}{t}\sigma\int_0^t S(s)I(s)dB(s) \\ & +\frac{1}{t}\int_0^t\int_{\mathbb{R}^*} S(s-)I(s-)C(z)\tilde{N}(ds,dz).\end{aligned} \tag{4.17}$$

We have,

$$\int_0^t dR(s) = \quad \gamma\int_0^t I(s)ds - (\mu+\delta)\int_0^t R(s)ds,$$

then,

$$\frac{R(t)-R(0)}{t} = \quad \gamma\langle I\rangle_t - (\mu+\delta)\langle R\rangle_t. \tag{4.18}$$

Combining (4.16) with (4.17) and (4.18), we obtain

$$\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{R(t)-R(0)}{t} \quad = \quad \Lambda - \mu\left(\langle S\rangle_t + \langle I\rangle_t + \langle R\rangle_t\right).$$

which yields

$$\langle S\rangle_t = \frac{\Lambda}{\mu} - (\langle I\rangle_t + \langle R\rangle_t) + \varphi_1(t), \tag{4.19}$$

where, $\varphi_1(t) = \frac{S(0)+I(0)+R(0)}{\mu t} - \frac{S(t)+I(t)+R(t)}{\mu t}$. Using Cauchy-Schwarz inequality and direct computation, we get

$$\begin{aligned}\langle S^2\rangle_t &\geq \langle S\rangle_t^2 = \left[\frac{\Lambda}{\mu} - (\langle I\rangle_t + \langle R\rangle_t) + \varphi_1(t)\right]^2 \\ &\geq \left(\frac{\Lambda}{\mu}\right)^2 - 2\left(\frac{\Lambda}{\mu} + \varphi_1(t)\right)\langle I+R\rangle_t.\end{aligned} \tag{4.20}$$

Injecting (4.19) and (4.20) into (4.15), we obtain

$$\begin{aligned}\frac{\ln(I(t)+R(t))}{t} \leq &\left(\mu + \sigma_e\left(\frac{\Lambda}{\mu}\right)^2\right)(\mathcal{R}_e - 1) - \left(\beta - 2\sigma_e\frac{\Lambda}{\mu}\right)\langle I+R\rangle_t \\ &+ \varphi_2(t) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t},\end{aligned} \tag{4.21}$$

where, $\varphi_2(t) = \beta\varphi_1(t) + 2\varphi_1(t)\langle I+R\rangle_t$. According to the Lemmas 4.1 and 4.2 and the boundedness of the solution to system (1.2), we obtain

$$\lim_{t\to\infty}\frac{M_1(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{M_2(t)}{t} = 0, \quad \lim_{t\to\infty}\varphi_2(t) = 0, \quad \mathbb{P} - a.s. \tag{4.22}$$

Then, for $\mathcal{R}_e < 1$ and $\beta - 2\sigma_e\frac{\Lambda}{\mu} \geq 0$, we have from (4.21)

$$\limsup_{t\to\infty}\frac{\ln(I(t)+R(t))}{t} \leq \left(\mu + \sigma_e\left(\frac{\Lambda}{\mu}\right)^2\right)(\mathcal{R}_e - 1) < 0.$$

This implies,

$$\lim_{t\to\infty}(I(t)+R(t)) = 0, \quad \mathbb{P}\text{-a.s.}$$

By the positivity of $I(t)$ and $R(t)$, we obtain

$$\lim_{t\to\infty} I(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} R(t) = 0, \quad \mathbb{P} - a.s. \tag{4.23}$$

Furthermore, injecting (4.23) into (4.19), we arrive to the equality

$$\lim_{t\to\infty}\langle S\rangle_t = \frac{\Lambda}{\mu}, \qquad \mathbb{P}\text{-a.s.}$$

The proof is therefore completed.

□

5. Persistence in mean of the disease

Next, we will consider stochastic persistence; that is, persistence in the means. Now, we give the definition of persistence in the means.

Definition 5.1. *The system* (1.2) *is said to be persistent in the mean, if*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds>0,\quad \liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)ds>0,\quad \liminf_{t\to\infty}\frac{1}{t}\int_0^t R(s)ds>0,\ \mathbb{P}\text{-}a.s.$$

For convenience, we denote

$$\sigma_p=\frac{1}{2}\left\{\sigma^2+\frac{1}{(1-\frac{\Delta}{\mu}\|C\|_\infty)^2}\int_{\mathbb{R}^*}C^2(z)\lambda(dz)\right\},\tag{5.1}$$

and

$$\mathcal{R}_p=\frac{\beta\frac{\Delta}{\mu}}{\mu+\gamma+\sigma_p\left(\frac{\Delta}{\mu}\right)^2}.\tag{5.2}$$

Theorem 5.1. *If $\mathcal{R}_p>1$, then for any initial value $(S(0),I(0),R(0))\in\Gamma$, the solution $(S(t),I(t),R(t)),t\in\mathbb{R}_+$ of the system* (1.2) *satisfies*

$$\begin{aligned}\liminf_{t\to\infty}\langle S\rangle_t &\geq C_1,\quad \mathbb{P}-a.s,\\ \liminf_{t\to\infty}\langle I\rangle_t &\geq C_2(\mathcal{R}_p-1),\quad \mathbb{P}-a.s,\\ \liminf_{t\to\infty}\langle R\rangle_t &\geq C_3(\mathcal{R}_p-1),\quad \mathbb{P}-a.s,\end{aligned}$$

for some positive constants C_i, $i=1,2,3$.

Proof. Integrating the first equation of the system (1.2) between 0 and t and using the boundedness of $I(t)$, we obtain

$$\begin{aligned}S(t)-S(0)=&\ \Lambda t-\beta\int_0^t S(s)I(s)ds-\mu\int_0^t S(s)ds-\sigma\int_0^t S(s)I(s)dB(s)\\ &-\int_0^t\int_{\mathbb{R}^*}S(s-)I(-s)C(z)\tilde{N}(ds,dz)\\ \geq&\ \Lambda t-\beta\frac{\Lambda}{\mu}\int_0^t S(s)ds-\mu\int_0^t S(s)ds-\sigma\int_0^t S(s)I(s)dB(s)\\ &-\int_0^t\int_{\mathbb{R}^*}S(s-)I(s-)C(z)\tilde{N}(ds,dz)\\ \geq&\ \Lambda t-\left(\mu+\frac{\beta\Lambda}{\mu}\right)\int_0^t S(s)ds-\sigma\int_0^t S(s)I(s)dB(s)\\ &-\int_0^t\int_{\mathbb{R}^*}S(s-)I(s-)C(z)\tilde{N}(ds,dz).\end{aligned}$$

Then,

$$\begin{aligned}\frac{S(t)-S(0)}{t}\geq&\ \Lambda-\left(\mu+\frac{\beta\Lambda}{\mu}\right)\langle S\rangle_t-\sigma\frac{1}{t}\int_0^t S(s)I(s)dB(s)\\ &-\frac{1}{t}\int_0^t\int_{\mathbb{R}^*}S(s-)I(s-)C(z)\tilde{N}(ds,dz).\end{aligned}\tag{5.3}$$

Rearranging the inequality (5.3) and we have

$$\langle S\rangle_t \;\geq\; \left(\mu+\frac{\beta\Lambda}{\mu}\right)^{-1}\left[\Lambda-\frac{S(t)-S(0)}{t}-\sigma\frac{1}{t}\int_0^t S(s)I(s)dB(s)-\frac{1}{t}\int_0^t\int_{\mathbb{R}^*}S(s-)I(s-)C(z)\tilde{N}(ds,dz)\right].$$

Taking inferior limit and using the Lemmas 4.1 and 4.2, we obtain

$$\liminf_{t\to\infty}\langle S\rangle_t \geq \Lambda\left(\mu+\frac{\beta\Lambda}{\mu}\right)^{-1} := C_1 > 0, \quad \mathbb{P}\text{-a.s.}$$

Applying the Itô formula for Lévy-type stochastic integrals to the function $I(t)\mapsto \ln I(t)$, we obtain

$$\begin{aligned}\ln I(t) = \;& \ln I(0)+\beta\langle S\rangle_t t-(\mu+\gamma)\,t+\delta\int_0^t\frac{R(s)}{I(s)}\,ds-\frac{\sigma^2}{2}\int_0^t S^2(s)ds\\ &+\sigma\int_0^t S(s)dB(s)+\int_0^t\int_{\mathbb{R}^*}\left[\ln(1+S(s)C(z))-S(s)C(z)\right]\lambda(dz)ds+M_2(t).\end{aligned}$$

Applying the Taylor-Lagrange formula, we get

$$\ln(1+S(t)C(z))-S(t)C(z)=-\frac{S^2(t)C(z)}{2(1+\theta S(t)C(z))^2},$$

where $\theta\in(0,1)$ is an arbitrary number. By assumption ($\mathbf{H}_1$)

$$\begin{aligned}(1+\theta S(t)C(z))^2 \;&\geq\; (1-\theta S(t)C(z))^2\\ &\geq\; \left(1-\frac{\Lambda}{\mu}\|C\|_\infty\right)^2,\end{aligned}$$

then,

$$-\frac{S^2(t)C^2(z)}{2(1+\theta S(t)C(z))^2}\geq-\frac{S^2(t)C^2(z)}{2(1-\frac{\Lambda}{\mu}\|C\|_\infty)^2}.$$

Which implies,

$$\ln(1+S(t)C(z))-S(t)C(z)\geq-\frac{C^2(z)}{2(1-\frac{\Lambda}{\mu}\|C\|_\infty)^2}S^2(t). \tag{5.4}$$

Using the inequality (5.4), we have

$$\begin{aligned}\frac{\ln I(t)}{t} \geq\;& \frac{\ln I(0)}{t}+\beta\langle S\rangle_t-(\mu+\gamma)-\frac{\sigma^2}{2}\langle S^2\rangle_t+\frac{1}{t}\sigma\int_0^t S(s)dB(s)\\ &-\langle S^2\rangle_t\frac{1}{2(1-\frac{\Lambda}{\mu}\|C\|_\infty)^2}\int_{\mathbb{R}^*}C^2(z)\lambda(dz)+\frac{M_2(t)}{t}\\ \geq\;& \beta\langle S\rangle_t-(\mu+\gamma)-\frac{1}{2}\left\{\sigma^2+\frac{1}{(1-\frac{\Lambda}{\mu}\|C\|_\infty)^2}\int_{\mathbb{R}^*}C^2(z)\lambda(dz)\right\}\langle S^2\rangle_t\\ &+\frac{H_1(t)}{t}+\frac{M_2(t)}{t}\\ \geq\;& \beta\langle S\rangle_t-(\mu+\gamma)-\sigma_p\langle S^2\rangle_t+\frac{H_1(t)}{t}+\frac{M_2(t)}{t},\end{aligned}$$

where, $H_1(t) = \ln I(0) + \sigma \int_0^t S(s)dB(s)$. Then,

$$\frac{\ln I(t)}{t} \geq \beta\langle S\rangle_t - (\mu+\gamma) - \sigma_p\left(\frac{\Lambda}{\mu}\right)^2 + \frac{H_1(t)}{t} + \frac{M_2(t)}{t}, \tag{5.5}$$

where, we used $-S(t) > -\frac{\Lambda}{\mu}$. Combining the first and the second equation of system (1.2) imply that

$$dS(t) + dI(t) = \Lambda - \mu S(t) - (\mu+\gamma)I(t) + \delta R(t).$$

Integrating between 0 and t dividing by t, we obtain

$$\begin{aligned}\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} &= \Lambda - \mu\langle S\rangle_t - (\mu+\gamma)\langle I\rangle_t + \delta\langle R\rangle_t \\ &\geq \Lambda - \mu\langle S\rangle_t - (\mu+\gamma)\langle I\rangle_t.\end{aligned}$$

Hence,

$$\langle S\rangle_t \geq \frac{\Lambda}{\mu} - \frac{\mu+\gamma}{\mu}\langle I\rangle_t + \varphi_3(t), \tag{5.6}$$

where, $\varphi_3(t) = \frac{S(0)+I(0)}{\mu t} - \frac{S(t)+I(t)}{\mu t}$. Injecting (5.6) into (5.5), we obtain

$$\frac{\ln I(t)}{t} \geq \frac{\beta\Lambda}{\mu} - \frac{\beta(\mu+\gamma)}{\mu}\langle I\rangle_t + \beta\varphi_3(t) - \left[\mu+\gamma+\sigma_p\left(\frac{\Lambda}{\mu}\right)^2\right] + \frac{H_1(t)}{t} + \frac{M_2(t)}{t}.$$

Therefore,

$$\frac{\beta(\mu+\gamma)}{\mu}\langle I\rangle_t \geq \frac{\beta\Lambda}{\mu} - \left[\mu+\gamma+\sigma_p\left(\frac{\Lambda}{\mu}\right)^2\right] - \frac{\ln I(t)}{t} + \beta\varphi_3(t) + \frac{H_1(t)}{t} + \frac{M_2(t)}{t}.$$

According to the Lemmas 4.1 and 4.2, we obtain

$$\liminf_{t\to\infty}\langle I\rangle_t \geq C_2\,(\mathcal{R}_p - 1), \quad \mathbb{P}\text{-a.s.}$$

where, $C_2 := \frac{\mu}{\beta}\left(1 + \frac{\sigma_p\left(\frac{\Lambda}{\mu}\right)^2}{\mu+\gamma}\right) > 0$.
From the third equation of system (1.2), we have

$$\frac{R(t)-R(0)}{t} = \gamma\,\langle I\rangle_t - (\mu+\delta)\,\langle R\rangle_t,$$

rearranging, we get

$$(\mu+\delta)\,\langle R\rangle_t = \gamma\,\langle I\rangle_t - \frac{R(t)}{t} + \frac{R(0)}{t}.$$

By the Lemma 4.1, we obtain

$$\liminf_{t\to\infty}\langle R\rangle_t \geq C_3\,(\mathcal{R}_p - 1), \quad \mathbb{P}\text{-a.s.}$$

where, $C_3 := \frac{\gamma}{\mu+\delta}\,C_2 > 0$. The proof is completed.

□

6. Numerical simulations

Numerical simulations of stochastic differential equations are very important in the study of real examples of epidemics diseases. we give some numerical simulations to support our obtained theoretical results of system (1.2). Now, we describe the discretization scheme using the Euler-Maruyama method. According to model (1.2), we consider the stochastic differential equation

$$dX(t) = f(X(t), t)\, dt + g(X(t), t)\, dW(t) + \int_{\xi} h(t, X(t-), z) p_{\varphi}(dz, dt), \tag{6.1}$$

for $t \in [0, T]$, with initial condition $X(0) = x(0)$, where f, g and h are the components of the drift, diffusion and jump coefficients, respectively. Let $W(t)$ be an $\mathcal{F}_t$-adapted Wiener process and $p_{\varphi}(dz, dt)$ be an $\mathcal{F}_t$-adapted Poisson measure with mark space $\xi \subseteq \mathbb{R} \setminus \{0\}$ (see [32]), with intensity measure $\varphi(dz)dt = \lambda F(dz)dt$, where F(·) is a given probability distribution function for the realizations of the marks. Then, the SDEs (6.1) can be written in integral from as:

$$\begin{aligned} X(t) &= X(0) + \int_0^t f(X(s), s)\, ds + \int_0^t g(X(s), s)\, dW(t) + \int_0^t \int_{\xi} h(s, X(s-), z) p_{\varphi}(dz, ds) \\ &= X(0) + \int_0^t f(X(s), s)\, ds + \int_0^t g(X(s), s)\, dW(t) + \sum_{k=1}^{p_{\varphi}(t)} h(\tau_k, X(\tau_k), \varepsilon_k), \end{aligned}$$

where, $\{(\tau_k, \varepsilon_k), k \in \{1, 2, \ldots, p_{\varphi}(t)\}\}$ is the double sequence of pairs of jump times and corresponding marks generated by the Poisson random measure. Next, we wish to solve the SDEs (6.1) on some interval of time $[0, T]$.

1- **Temporal discretization:** we divide the time interval $[0, T]$ into N equal subintervals of width $\Delta t > 0$, we thus obtain a sequence of times

$$0 = t_0 < t_1 < \cdots < t_N = T \quad \text{and} \quad \Delta t = T/N.$$

2- **Euler-Maruyama scheme:** at each time step t_i for $0 < i < N - 1$, we update the approximate variables of the model using the equations:

$$\begin{aligned} S(t_{i+1}) &= S(t_i) + [\Lambda - \beta S(t_i)I(t_i) - \mu S(t_i)]\Delta t - \sigma S(t_i)I(t_i)\Delta W_i - \int_{t_{i+1}}^{t_i} \int_{\xi} C(z)S(t_i)I(t_i)p_{\varphi}(dz, ds) \\ &= S(t_i) + [\Lambda - \beta S(t_i)I(t_i) - \mu S(t_i)]\Delta t - \sigma S(t_i)I(t_i)\Delta W_i - \sum_{i=p_{\varphi}(t_i)+1}^{p_{\varphi}(t_{i+1})} C(\varepsilon_i)S(t_i)I(t_i). \\ I(t_{i+1}) &= I(t_i) + [\beta S(t_i)I(t_i) - (\mu + \gamma)I(t_i) + \delta R(t_i)]\Delta t + \sigma S(t_i)I(t_i)\Delta W_i + \int_{t_{i+1}}^{t_i} \int_{\xi} C(z)S(t_i)I(t_i)p_{\varphi}(dz, ds) \\ &= I(t_i) + [\beta S(t_i)I(t_i) - (\mu + \gamma)I(t_i) + \delta R(t_i)]\Delta t + \sigma S(t_i)I(t_i)\Delta W_i + \sum_{i=p_{\varphi}(t_i)+1}^{p_{\varphi}(t_{i+1})} C(\varepsilon_i)S(t_i)I(t_i). \\ R(t_{i+1}) &= R(t_i) + [\gamma I(t_i) - (\mu + \delta)R(t_i)]\Delta t. \end{aligned}$$

where,

- $S(t_i), I(t_i)$ and $R(t_i)$ are the approximate values of the variables S, I and R resp. at time t_i.
- The random variables $\Delta W_i = W(t_{i+1}) - W(t_i)$ are independent and identically distributed normal random variables with expected value zero and variance Δt .
- The terms $[\Lambda - \beta S(t_i)I(t_i) - \mu S(t_i)]\Delta t$, $[\beta S(t_i)I(t_i) - (\mu + \gamma)I(t_i) + \delta R(t_i)]\Delta t$ and $[\gamma I(t_i) - (\mu + \delta)R(t_i)]\Delta t$ represent the deterministic contributions of the model to the evolution of each variable.
- The terms $-\sigma S(t_i)I(t_i)\Delta W_i$ and $\sigma S(t_i)I(t_i)\Delta W_i$ represent the stochastic contributions (Brownian motion).
- $p_{\varphi}(t) = p_{\varphi}(\xi, [0, t])$ represents the total number of jumps of the Poisson random measure up to time t, which is Poisson distributed with mean λt and $\varepsilon_i \in \xi$ is the ith mark of the Poisson random measure p_{φ}.

This process is repeated for all the time steps of the discretization until the final time t_N is reached, thus producing an approximation of the stochastic spread of the infection over the entire discretized time interval.

6.1. *Zoonotic tuberculosis in Morocco*

in the following, we represent a simulation for transmission dynamic and elimination potential of zoonotic tuberculosis in Morocco [33]. The average lifespan of the Moroccan cattle is 6 years which yields to a death rate of $\mu = 0.167$ per year. The birth rate was estimated to be 0.177 using least squares method based on cattle population data. From the endemic prevalence in cattle the transmission rate of bovine tuberculosis from cattle to cattle was estimated to be $\beta = 0.249$. We choose $\delta = 0.1$, $\gamma = 0.2$, $\sigma = 0.2$, $\sigma_e = 0.01$, and the jump intensity $C(z) = 0.1\dfrac{z}{1+z^2}$ is considered, where $z = 0.5$.

In the following, using time step size $\Delta = 10^{-2}$, with the initial condition $(S(0), I(0), R(0)) = (0.7, 0.17, 0.13)$ and parameters cited above. We note that the deterministic system (1.1) is extinct as $\mathcal{R}_0 = 0.5105 < 1$; on the other hand, for the stochastic system (1.2) we can compute $\mathcal{R}_e = 0.8741 < 1$ and the condition $2\sigma_e \frac{\Lambda}{\mu} = 0.012 < \beta = 0.249$ of Theorem 4.1 satisfied. Thus, the disease is extinct with probability one and the **Figure 1** confirms it.

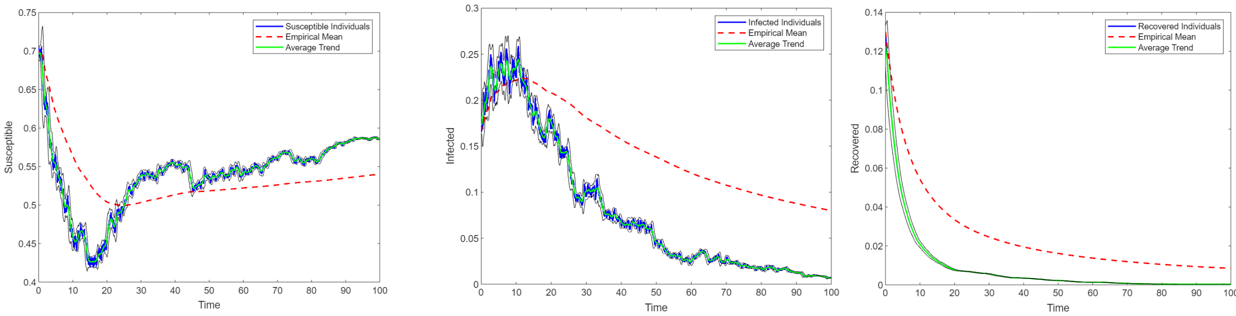

Figure 1: Illustration of extinction of the solution to the deterministic (1.1) and stochastic (1.2) systems respectively, the means, the empirical means and the associated standard deviations using data of Example 6.1.

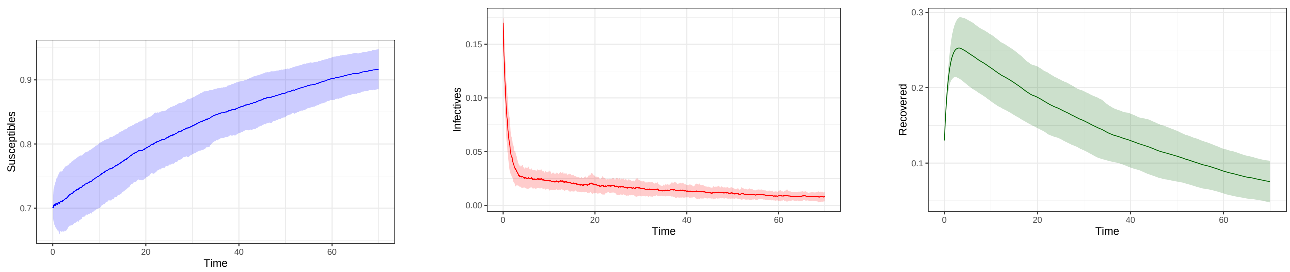

Figure 2: Average over 1000 stochastic simulations of the Lévy jumps model and the associated standard deviations, using data of the Zoonotic tuberculosis in Morocco.

6.2. *Case study of tuberculosis in Morocco*

In Morocco, tuberculosis (TB) is still a serious health risk. Thanks to its National Plan for Tuberculosis and the assistance of both domestic and foreign partners, the kingdom has achieved significant strides in the management and control of tuberculosis. The Tuberculosis (TB) incidence in Morocco was stable around 100 per 100.000 persons[34] . In order to investigate the dynamics of TB in Morocco , we collect the data from the World Bank data for Tuberculosis cases, Morocco population growth, natural death rate from 2000 to 2021.

6.2.1. *Parameter estimation*

We consider the time scale by years. However, since we have small data set on TB in Morocco we discretize the data for each one year into 100 partitions. In this case, if the observations are h = 1/100. In order to illustrated a deterministic or stochastic differential equation, the approximate solution can be simulated using Euler Scheme for Levy Driven Stochastic Differential Equations[35]. We consider a data set of TB cases in Morocco defined as $Y_T \triangleq \{y_0, y_1, \ldots, y_T\}$, where the observation is up to a finite horizon T and $y_t \triangleq [S_t, I_t, R_t]'$ is a column vector in $\mathbb{R}^{3\times 1}$, which represents the daily observed values at time t for the susceptible (S), infected (I), and recovered individuals (R). We need to estimate the unknown vector of parameters θ in the epidemic model. To this end, we consider the predicted epidemic model $\hat{y}_t(\theta)$ defined by

$$\hat{y}_t(\theta) \triangleq \hat{y}_{t-1} + h \begin{bmatrix} \Lambda - \beta \hat{S}_{t-1}\hat{I}_{t-1} - \mu \hat{S}_{t-1} \\ \beta \hat{S}_{t-1}\hat{I}_{t-1} - (\mu+\gamma)\hat{I}_{t-1} + \delta \hat{R}_{t-1} \\ \gamma \hat{I}_{t-1} - (\mu+\delta)\hat{R}_{t-1} \end{bmatrix},$$

where $\hat{y}_0(\theta) = y_0$ as an initial condition at time 0. Using this Euler-Maruyama method to approximate the solution, we calculate the quadratic cost

$$J_T(\theta) \triangleq \sum_{t=0}^{T} \|I_t - \hat{I}_t(\theta)\|^2.$$

Minimizing the quadratic cost J yields the non-linear least square estimator θ_e, where

$$\theta_e \triangleq \arg \min_{\theta \in (0,\infty)^7} J_T(\theta).$$

Shows the estimated parameters derived from fitting the studied models to the provided cumulative case data for Morocco from 2000 to 2021. The predicted $\mathcal{R}_0 = 1.04634$ [9] is as expected, greater than 1, which means the disease will persist. In Fig. 3, we used the estimated parameter β and the parameters deduced from the data to illustrate the patterns of the susceptible (S), infected (I), and recovered individuals (R).

Parameter	Description	Value	Source
Λ	Recruitment rate	675104 year^{-1}	Estimated from World Bank
μ	Natural death rate	0.005 year^{-1}	Estimated from World Bank
β	Transmission rate	7.4059×10^{-9} year^{-1}	Fitted
γ	Recovery rate	0.2 year^{-1}	[36]
δ	Relapse rate	0.00001 year^{-1}	[37]
S_0	Initial Susceptible population	28521577	World Data Bank
I_0	Initial Infected population	32837	World Data Bank
R_0	Initial Recovered population	6567	World Data Bank

Table 1: Table of parameters used in the numerical simulation.

In order to implement a stochastic simulation in 1, we set the stochastic volatility as $\sigma = 4.3016 \times 10^{-9}$.

6.2.2. *Scenario of an extinction of Tuberculosis*

In order to show that the stability equilibrium point for endemic TB Morocco cases can be subject to an extinction of the disease while taking account the perturbation of the environment and since the basic reproduction number is so close to the deterministic threshold. Hence, in Fig. 4, we simulate a prediction for next years. However, we set a volatility of 4.3016×10^{-8}.

6.2.3. *Scenario of a persistence of Tuberculosis*

In this case we set a basic reproduction number of $\mathcal{R}_0 = 2.0926$. Hence, we get the following simulation Fig. 5 for the deterministic and stochastic epidemic models.

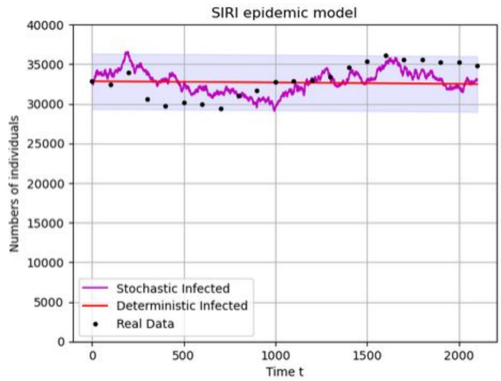

Figure 3: Estimated Infected population for Morocco from 2000 to 2021.

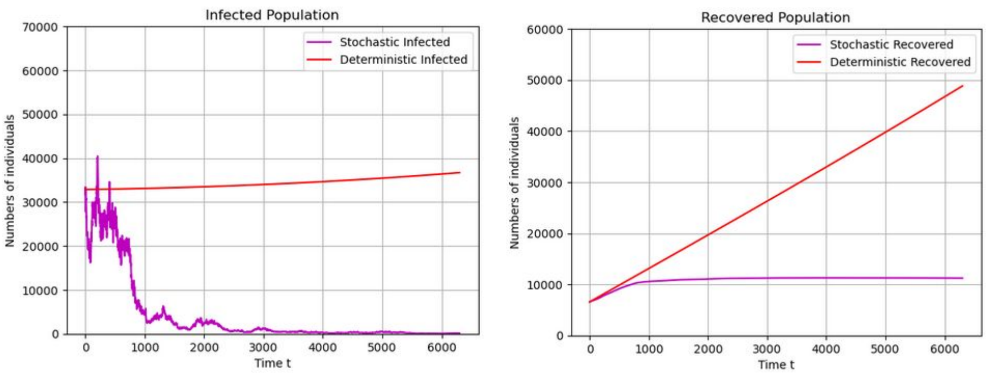

Figure 4: Stochastic and deterministic scenario for infected and recovered compartments in the next 40 Years.

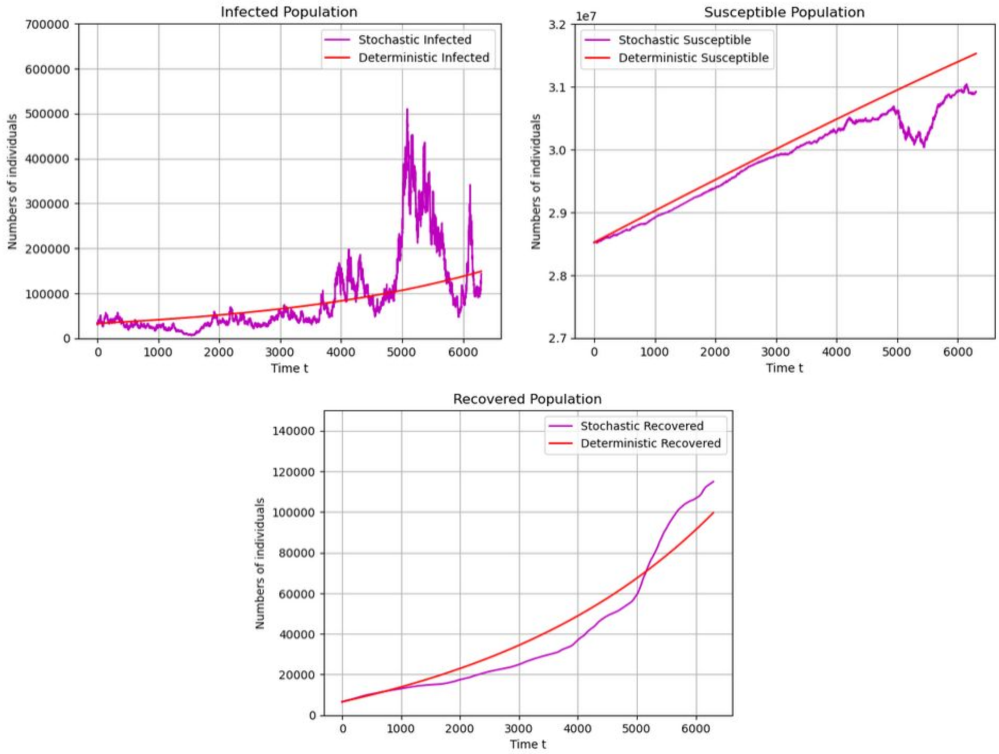

Figure 5: Stochastic and deterministic scenario for infected and recovered compartments in the next 40 Years.

6.3. *Herpes simplex virus type 2*

Herpes simplex virus type 2 (HSV-2) is a sexually transmitted infection that causes genital herpes. The virus usually infects the genital tract or oral mucosa. It is a lifelong condition that can cause painful blisters or ulcers that can recur over time. Most people have no symptoms or only mild symptoms, but the infection can be more severe in people with suppressed immune systems, such as HIV-infected persons. In this example, we are interested in presenting a simulation to illustrate the case of (HSV-2). For this we consider the same parameters presented in [14]. Let $\Lambda = 0.1$, $\beta = 0.2$, $\mu = 0.05$, $\sigma = 0.2$ and $\gamma = \delta = 0.2857$. Then, the deterministic system is persistent as $\mathcal{R}_0 = 4.3219 > 1$. Furthermore, for the stochastic system we have $\mathcal{R}_p = 1.0647 > 1$, then according to Theorem 5.1 the system (1.2) is persistent and the disease becomes endemic, as Fig. 6 clearly support this result.

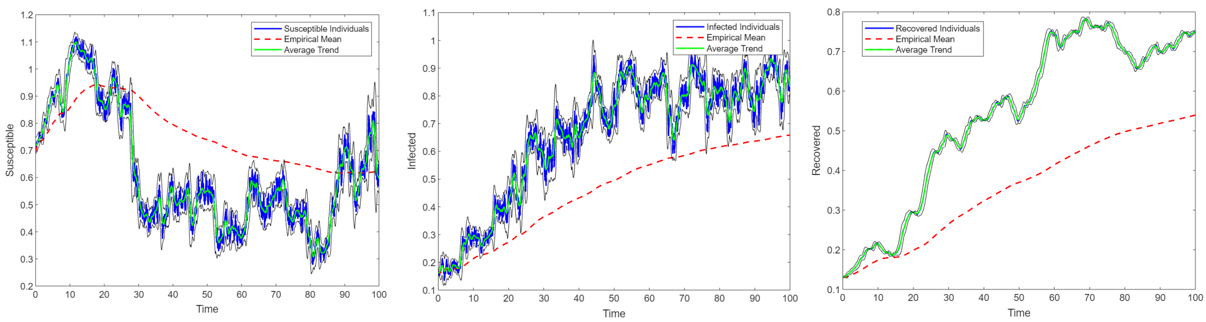

Figure 6: Illustration of persistence in mean of the solution to the deterministic (1.1) and stochastic (1.2) systems respectively, the means, the empirical means and the associated standard deviations using data of Example 6.2.

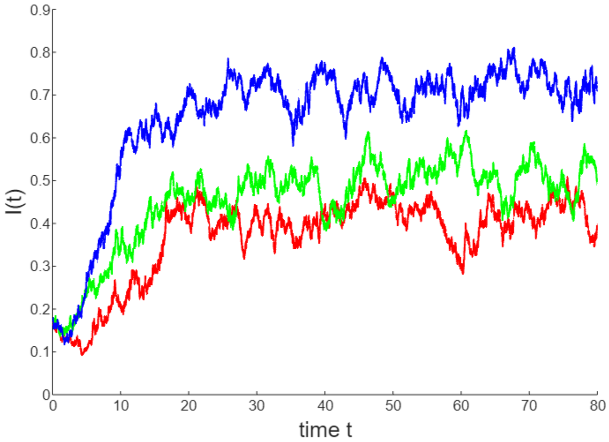

Figure 7: Paths of the solution to the stochastic system (1.2) with the parameter values as in Example 6.2 and $\delta = 0.14, 0.2$ and 0.4.

6.4. *Sensitivity index of the basic reproduction number $\mathcal{R}_0$*

To gain a better understanding of the most effective strategies for preventing and controlling the spread of a disease, we will now examine the impact of various parameters on the dynamics of the stochastic epidemic system through sensitivity analysis. This analysis will provide valuable insights into which factors have the greatest influence on the spread of the disease, enabling us to develop appropriate intervention strategies. In the following, we choose disease parametrers as follows:

Table 2:

Λ	β	μ	γ	σ	δ	λ	σ_p	$C(z)$	$\lambda(\mathbb{R}^*)$
0.177	1.2	0.167	0.2	0.2	0.1	0.5	0.044	0.06	1

6.4.1. *Sensitivity index of $\mathcal{R}_0$ with respect to relapse parameter*

In order to assess the impact of changes in the relapse rate on the spread of the disease, we will analyze the sensitivity index of the basic reproduction number $\mathcal{R}_0$, in relation to the relapse parameter δ. This analysis will help us understand how variations in the relapse rate affect the overall spread of the disease. The normalized forward sensitivity index of $\mathcal{R}_0$, which depends differentiably on a parameter δ, is defined as:

$$\frac{\partial \mathcal{R}_0}{\partial \delta}\frac{\delta}{\mathcal{R}_0} = \frac{\gamma\delta}{(\mu+\delta)(\mu+\gamma+\delta)}.$$

The positive sign of the sensitivity index indicates that as the relapse parameter increases, the basic reproduction number $\mathcal{R}_0$ also increases. This, in turn leads to the persistence of the disease within the population over time. The following figure represents a variation of the relapse parameter $\delta = 0.14, 0.2$ and 0.4. By observing Fig. 7, we can conclude that as the relapse rate increases, there is an increase in the number of infected individuals.

6.4.2. *Sensitivity index of $\mathcal{R}_0$ with respect to recovery parameter*

Now, we examine the sensitivity index of the basic reproduction number $\mathcal{R}_0$ with respect to the recovery parameter γ. Then, the recovery sensitivity parameter is given by:

$$\frac{\partial \mathcal{R}_0}{\partial \gamma}\frac{\gamma}{\mathcal{R}_0} = -\frac{\gamma}{\mu+\gamma+\delta}.$$

The negative sensitivity index indicates that an increase in the recovery parameter will result in a decrease in the basic reproduction number $\mathcal{R}_0$, which ultimately leads to the extinction of the disease in the population. The next figure illustrates a variation of the recovery rate $\gamma = 0.09, 0.18$ and 0.22 with parameters values as in example 6.2. From Fig. 8, it is noticeable that increasing the recovery rate is associated with a decline in the number of infected individuals.

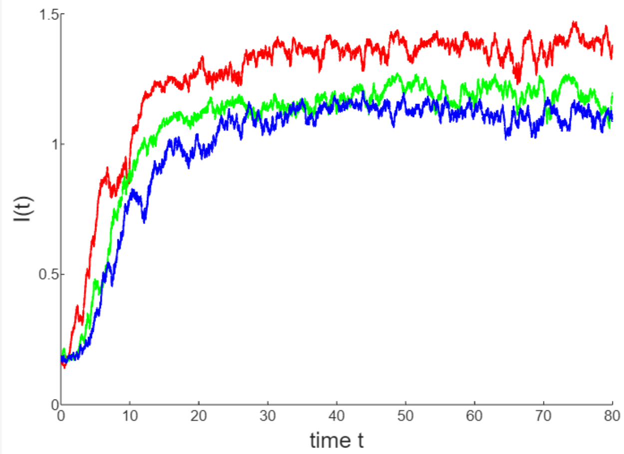

Figure 8: Paths of the solution to the stochastic system (1.2) with the parameter values as in Example 6.2 and $\gamma = 0.09, 0.18$ and 0.22.

Conclusion. In this work, we have formulated an SIRI epidemic model with disease relapse, driven by correlated Brownian motion and Lévy jump components with infinite characteristic measure $\lambda(\mathbb{R}^*) = \infty$. The analysis of the stochastic system (1.2) shows the existence and uniqueness of global positive solution. We present new solution estimates using Kunita's inequality for jump processes rather than Burtholder-Davis-Gundy inequality for continuous diffusions in the key Lemmas 4.1 and 4.2. In theorem 4.1 we proved that the disease goes to zero exponentially with probability one wherever $\mathcal{R}_e < 1$, this means that the endemic disease will disappear in the population. Moreover, as long as $\mathcal{R}_p > 1$, the epidemic disease will be persistent.

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