Hausdorff and fractal dimensions of attractors for functional differential equations in Banach spaces

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Abstract

The main objective of this paper is to obtain estimations of Hausdorff dimension as well as fractal dimension of global attractors and pullback attractors for both autonomous and nonautonomous functional differential equations (FDEs) in Banach spaces. New criterions for the finite Hausdorff dimension and fractal dimension of attractors in Banach spaces are firslty proposed by combining the squeezing property and the covering of finite subspace of Banach spaces, which generalize the method established in Hilbert spaces. In order to surmount the barrier caused by the lack of orthogonal projectors with finite rank, which is the key tool for proving the squeezing property of partial differential equations in Hilbert spaces, we adopt the state decomposition of phase space based on the exponential dichotomy of the studied FDEs to obtain similar squeezing property. The theoretical results are applied to a retarded nonlinear reaction-diffusion equation and a non-autonomous retarded functional differential equation in the natural phase space, for which explicit bounds of dimensions that do not depend on the entropy number but only depend on the spectrum of the linear parts and Lipschitz constants of the nonlinear parts are obtained.

Key words: *global attractors, pullback attractors, Hausdorff dimension, fractal dimension, delay, Banach space, retarded reaction-diffusion equation, functional differential equations*

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1 Introduction

For infinite dimensional dynamical systems generated by partial differential equations or delay differential equations, the phase spaces are generally not locally compact, while the existence of attractors can reduce the essential parts of the flows to compact sets. Furthermore, if the attractors have finite topological dimensions (including Hausdorff dimension and fractal dimension), then the attractors can be described by a finite number of parameters and hence the dynamics of the infinite dynamical systems are likely to be studied by the concepts and methods of finite dimensional dynamical systems. Owing to this, the study of attractors as well as their topological dimensions estimation have received much attention from pure and applied mathematics community during the past decades.

The theory of existence of attractors for deterministic infinite dimensional dynamical systems, especially for a large class of parabolic partial differential equations and delay differential equations, is now well developed. See, for instance, the monographs by Babin and Vishik [4], Hale [23], Ladyzhenskaya [26], Robinson [33] and Temam [36]. With respect to the dimensions estimation of attractors for infinite dimensional dynamical systems, there are several methods. The first one is the squeeze method in Hilbert spaces, which dates back to the pioneering work of Foias and Temam [22], where they showed that under some circumstances a three dimensional flow depends indeed on a finite number of parameters. Then, the idea was adopted in [3] and [25] to show the finite Hausdorff dimensionality of the global attractor for the Navier-Stokes equations, which was then extended to the random case by Debussche in [15]. The idea has also been extended to investigate the exponential attractors of deterministic partial differential equations in [1, 17, 18, 31] and stochastic partial differential equations in [41, 42], which implies the finite fractal dimensions of attractors.

The second method is to compute traces of some linear operators generated by the linearization of the equations, requiring quasi-differentials of the underlying systems, which originates from the early work [16] with an emphasise on the Hausdorff dimension. It was then further extended by Foias and Temam in their later work [12], where they also investigated fractal dimension and the relationship between the Hausdorff dimension of attractors and Lyapunov exponents and Lyapunov numbers. The method has also been adopted to study the dimensions of attractors for a variety of evolution equations. See, for instance, the nonautonomous PDEs in bounded domain [10] and unbounded domain [9], the retarded semilinear partial differential equations in [35] and retarded Navier-Stokes-Voight equation in [32]. More examples can be found in Babin and Vishik [4] and Temam [36] and the references therein.

The above mentioned dimensions estimation methods are mainly obtained in Hilbert spaces, i.e., the phase space endowed with a smooth inner product geometrical structure. Nevertheless, there are many evolution equations arising from real world modelings defined in Banach spaces, such as the delay differential equations [24], delay partial differential equations [38] and the non-autonomous Chafee-Infante equation [5] and so on. Although, in [29, 32, 35], the authors studied dimensions of global attractors for ordinary or partial functional differential equations, they recast the equations into Hilbert

spaces. Therefore, one natural question arises, what can we say about the dimensions of attractors for FDEs in the natural phase space, i.e., the Banach spaces? Málek, Ruzicka and Thäter [28] established a third method for estimating fractal dimension based on a smoothing property of the system, which allows the phase space to be Banach spaces, but requires an auxiliary space that is compactly embedded into the phase space. The method was then extended by Efendiev, Miranville and Zelik [14, 19] to construct pullback and uniform exponential attractors for systems in Banach spaces, which has also been widely used in estimating the fractal dimension and construct exponential attractors of deterministic systems [8, 13, 20, 21] and random systems [6, 34, 40].

Although, the third method is effective for systems in Banach spaces, the estimation of the fractal dimension depends on the choice of another embedding space which may vary from space to space. Furthermore, the dimension estimation depends on the entropy number between two spaces for which is generally quite difficult to obtain an explicit bound. Hence, one naturally wonders whether we can give explicit bounds of topological dimensions of attractors for systems in Banach spaces that only depend on the inner characteristic of the system. The only works tackling topological dimensions estimation of invariant sets for nonlinear maps or attractors for infinite dimensional dynamical systems in Banach spaces that we can find are [30] and [7]. In [30], Mañé showed that negative invariant sets for certain nonlinear maps in Banach spaces have finite fractal dimension, which also imply the finite dimensionality of Hausdorff dimension and was then improved by [7]. Nevertheless, in [30], the results are obtained under the assumption that the derivatives of map are bounded. In this paper, we establish a new method by combining the squeezing property obtained by exponential dichotomy and the covering lemma of the finite dimensional subspace of Banach space established in [30]. We do not need the strict restriction of the boundedness of derivatives and we also provide explicit bounds of the dimensions of the invariant sets in Banach spaces that only depend on the spectrum of the linearized system and state decomposition of the phase space while not relate to the entropy number as [8, 13, 14, 19, 20, 21, 28] did.

We organize the remaining part of this paper as follows. In Section 2, we establish the criteria for both Hausdorff and fractal dimensions estimation for autonomous systems. The corresponding results for nonautonomous systems are obtained in Section 3 followed by applications to a retarded nonlinear reaction-diffusion equations and a non-autonomous retarded functional differential equations which depend only on the spectrum of the linear part and Lipschitz constants of the nonlinear terms are given in Section 4. At last, we summarize the paper and point out some potential directions for future research in Section 5.

2   Dimensions of attractors for autonomous systems

In this section, we study the Hausdorff and fractal dimensions of the attractors for autonomous dynamical systems. Let X be a Banach space with norm $\|\cdot\|_X$ and $S(t): X \to X, t \geq 0$ be a semigroup. A set

$\mathcal{A}$ is said to be invariant, if $S(t)\mathcal{A} = \mathcal{A}$. Moreover, the invariant set $\mathcal{A}$ is said to be a global attractor if $\mathcal{A}$ is a maximal compact invariant set which attracts each bounded set $B \subset X$. The Hausdorff dimension of the compact set $\mathcal{A} \subset X$ is

$$d_H(\mathcal{A}) = \inf \{d : \mu_H(\mathcal{A}, d) = 0\}$$

where, for $d \geq 0$,

$$\mu_H(\mathcal{A}, d) = \lim_{\varepsilon \to 0} \mu_H(\mathcal{A}, d, \varepsilon)$$

denotes the d-dimensional Hausdorff measure of the set $\mathcal{A} \subset X$, where

$$\mu_H(\mathcal{A}, d, \varepsilon) = \inf \sum_i r_i^d$$

and the infimum is taken over all coverings of $\mathcal{A}$ by balls of radius $r_i \leqslant \varepsilon$. It can be shown that there exists $d_H(\mathcal{A}) \in [0, +\infty]$ such that $\mu_H(\mathcal{A}, d) = 0$ for $d > d_H(\mathcal{A})$ and $\mu_H(\mathcal{A}, d) = \infty$ for $d < d_H(\mathcal{A})$. $d_H(\mathcal{A})$ is called the Hausdorff dimension of $\mathcal{A}$.

For a finite dimensional subspace F of a Banach space X, denote by $B_r^F(x)$ the ball in F of center x and radius r, that is $B_r^F(x) = \{y \in F | \|y - x\| \leq r\}$. It is proved in [30] that the following covering lemma of balls in finite dimensional Banach spaces is true.

Lemma 2.1. *For every finite dimensional subspace F of a Banach space X, we have*

$$N\left(r_1, B_{r_2}^F\right) \leq m 2^m \left(1 + \frac{r_1}{r_2}\right)^m, \tag{2.1}$$

for all $r_1 > r_2 > 0$, where $m = \dim F$ and $N\left(r_1, B_{r_2}^F(0)\right)$ is the minimum number of balls needed to cover the ball of radius r_1 by balls $B_{r_2}^F(0)$ of radius r_2 calculated in the metric space X.

To prove the existence of finite Hausdorff dimension of the attractors for a semigroup $\{S(t)\}_{t \geq 0}$ in a Banach space, we impose the following additional assumptions on its attractor $\mathcal{A}$.

Hypothesis A1 There is a finite dimensional projection $P : X \to PX$ with a finite dimension

$$\Lambda = \dim\{PX\}, \tag{2.2}$$

and there exist three positive numbers t_0, M_1, M_2, M_3 and two constants λ_0 and λ_1 such that

$$\|PS(t_0)\varphi - PS(t_0)\psi\| \leq M_1 e^{\lambda_0 t_0} \|\varphi - \psi\| \tag{2.3}$$

and

$$\|(I - P)S(t_0)\varphi - (I - P)S(t_0)\psi\| \leq (M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0}) \|\varphi - \psi\|, \tag{2.4}$$

for any φ, ψ in $\mathcal{A}$.

We can now prove an upper bound for the Hausdorff dimension of the attractors for a semigroup $S(t)$ in a Banach space under **Hypothesis A1**.

Theorem 2.1. *Assume that $\{S(t)\}_{t\geq 0}$ is a continuous semigroup with global attractor $\mathcal{A}$,* **Hypothesis A1** *holds and there exists $0 < \alpha < 2$ such that*

$$\alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0} < 1. \tag{2.5}$$

Then, the Hausdorff dimension of the global attractor $\mathcal{A}$ satisfies

$$d_H < \frac{-\ln \Lambda - \Lambda \ln(2 + \frac{4}{\alpha})}{\ln(\alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0})}, \tag{2.6}$$

where Λ is the dimension of PX defined by (2.2) and M_1, M_2, M_3, λ_0 and λ_1 are given in **Hypothesis A1**.

Proof. Since $\mathcal{A}$ is a compact subset of X, for any $0 < \varepsilon < 1$, there exist $r_1, \ldots, r_N$ in $(0, \varepsilon]$ and $\tilde{u}_1, \ldots, \tilde{u}_N$ in X such that

$$\mathcal{A} \subset \bigcup_{i=1}^{N} B(\tilde{u}_i, r_i), \tag{2.7}$$

where $B(\tilde{u}_i, r_i)$ represents the ball in X of center $\tilde{u}_i$ and radius r_i. Without loss of generality, we can assume that for any i

$$B(\tilde{u}_i, r_i) \cap \mathcal{A} \neq \emptyset, \tag{2.8}$$

otherwise, it can be deleted from the sequence $\tilde{u}_1, \ldots, \tilde{u}_N$. Therefore, we can choose $u_i, i = 1, 2, \cdots, N$ such that

$$u_i \in B(\tilde{u}_i, r_i) \cap \mathcal{A}, \tag{2.9}$$

and

$$\mathcal{A} \subset \bigcup_{i=1}^{N} (B(u_i, 2r_i) \cap \mathcal{A}). \tag{2.10}$$

It follows from (2.3) and (2.4) that for any $u \in B(u_i, 2r_i) \cap \mathcal{A}$, we have

$$\|PS(t_0)u - PS(t_0)u_i\| \leq 2M_1 e^{\lambda_0 t_0} r_i, \tag{2.11}$$

and

$$\|(I - P)S(t_0)u - (I - P)S(t_0)u_i\| \leq 2(M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0})r_i. \tag{2.12}$$

By Lemma 2.1, for any $\alpha > 0$, we can find $y_i^1, \ldots, y_i^{n_i}$ such that

$$B_{PX}\left(PS(t_0)u_i, 2M_1 e^{\lambda_0 t_0} r_i\right) \subset \bigcup_{j=1}^{n_i} B_{PX}\left(y_i^j, \alpha M_1 e^{\lambda_0 t_0} r_i\right) \tag{2.13}$$

with

$$n_i \leq \Lambda 2^{\Lambda}\left(1 + \frac{2}{\alpha}\right)^{\Lambda}, \tag{2.14}$$

where Λ is the dimension of PX and we have denoted by $B_{PX}(y, r)$ the ball in PX of radius r and center y.

Set
$$u_i^j = y_i^j + (I - P)S(t_0)u_i \tag{2.15}$$

for $i = 1, \ldots, N, j = 1, \ldots, n_i$. Then, for any $u \in B(u_i, 2r_i) \cap \mathcal{A}$, there exists a j such that

$$\left\| S(t_0)u - u_i^j \right\| \leq \left\| PS(t_0)u - y_i^j \right\| + \|(I - P)S(t_0)u - (I - P)S(t_0)u_i\|$$
$$\leq \left(\alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0} \right) r_i \tag{2.16}$$

with

$$n_i \leq \Lambda(2 + \frac{4}{\alpha})^\Lambda \tag{2.17}$$

Denote by $\eta = \alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0}$, then we have

$$S(t_0)(B(u_i, 2r_i) \cap \mathcal{A}) \subset \bigcup_{j=1}^{n_i} B\left(u_i^j, \eta r_i\right). \tag{2.18}$$

Thanks to the invariance of $\mathcal{A}$, i.e., $\mathcal{A} = S(t_0)\mathcal{A}$, we have

$$\mathcal{A} \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{n_i} B\left(u_i^j, \eta r_i\right). \tag{2.19}$$

This implies that, for any $d \geq 0$,

$$\mu_H(\mathcal{A}, d, \eta\varepsilon) \leq \sum_{i=1}^{N} \sum_{j=1}^{n_i} \eta^d r_i^d \leq \Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d \sum_{i=1}^{N} r_i^d, \tag{2.20}$$

we deduce, by taking the infimum over all the coverings of $\mathcal{A}$ by balls of radii less than ε,

$$\mu_H(\mathcal{A}, d, \eta\varepsilon) \leq \Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d \mu_H(\mathcal{A}, d, \varepsilon). \tag{2.21}$$

Applying the formula recursively for k times yields

$$\mu_H\left(\mathcal{A}, d, (\eta\varepsilon)^k\right) \leq [\Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d]^k \mu_H(\mathcal{A}, d, \varepsilon). \tag{2.22}$$

Therefore, if

$$d < \frac{-\ln \Lambda - \Lambda \ln(2 + \frac{4}{\alpha})}{\ln(\alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0})}, \tag{2.23}$$

then

$$\Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d < 1. \tag{2.24}$$

Thus, by taking $k \to \infty$, we have $(\eta\varepsilon)^k \to 0$ and (2.22) leads to

$$\mu_H(\mathcal{A}, d, (\eta\varepsilon)^k) \to 0. \tag{2.25}$$

This completes the proof. $\qquad\square$

Remark 2.1. *By* (2.6), *we can see estimation of* d_H *depends on the parameter* α. *It would be convenient to know what is the optimal bound for this dimension. In other words, what is the value of* α *such that the right hand side of* (2.6) *attains its minimum value. By doing some simulations with some specific values of the parameters involved in* (2.6), *one can check that when* $M_1 e^{\lambda_0 t_0}$ *becomes smaller and smaller the value of* α *at which the minimum is achieved is closer and closer to* 2, *and from one threshold on this minimum is achieved when* $\alpha = 2$. *In conclusion, in these particular situations, if we take* $\alpha \uparrow 2$ *and assume* $2M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0} < 1$, *then for all* $\alpha \in (0,2)$, *we have* $\alpha M_1 e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0} < 1$ *and hence we obtain the estimation*

$$d_H \leq \frac{-\ln \Lambda - \Lambda \ln 4}{\ln(2e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0})}, \tag{2.26}$$

which is independent of α. *But this works only in those mentioned cases in which* $M_1 e^{\lambda_0 t_0}$ *is smaller than some threshold (which depends on all the parameters involved in the problem.*

Next, we study the fractal dimension of attractors for the semigroup $\{S(t)\}_{t\geq 0}$ based on the state space decomposition and squeeze property **Hypothesis A1** following the idea of [15, 22] and [42]. Here, we extend the method to Banach spaces. The fractal dimension (or capacity) of $\mathcal{A}$ is defined as

$$\dim_f \mathcal{A} = \limsup_{\varepsilon \to 0} \frac{\ln N_\varepsilon(\mathcal{A})}{-\ln \varepsilon}, \tag{2.27}$$

where $N_\varepsilon(\mathcal{A})$ is the minimum number of balls of radius less than ε which is necessary to cover $\mathcal{A}$.

Theorem 2.2. *Let* $\mathcal{A}$ *be the global attractor of* $\{S(t)\}_{t\geq 0}$ *with finite diameter* $R_\mathcal{A}$, *that is* $R_\mathcal{A} := \sup_{u \in \mathcal{A}} \|u\|_X < \infty$. *Assume that* **Hypothesis A1** *holds and there exists* $0 < \alpha < M_1$ *such that* $\zeta := \alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0} < 1$. *Then, the fractal dimension of global attractor* $\mathcal{A}$ *has an upper bound*

$$\dim_f \mathcal{A} \leq \frac{\ln \Lambda + \Lambda \ln(2 + \frac{2M_1}{\alpha})}{-\ln \zeta} < \infty, \tag{2.28}$$

where Λ *is the dimension of* PX *defined by* (2.2) *and* M_1, M_2, M_3, λ_0 *and* λ_1 *are given in* **Hypothesis A1**.

Proof. Since $\mathcal{A}$ is a global attractor, then it is compact and hence the number $R_\mathcal{A}$ is well defined. Thus, for any $u_0 \in \mathcal{A}$, we have

$$\mathcal{A} \subseteq B(u_0, R_\mathcal{A}), \tag{2.29}$$

where $B(u_0, R_\mathcal{A})$ is the ball with center u_0 and radius $R_\mathcal{A}$. For any $u \in \mathcal{A} \cap B(u_0, R_\mathcal{A})$, it follows from **Hypothesis A1** that

$$\|PS(t_0)u - PS(t_0)u_0\| \leq M_1 e^{\lambda_0 t_0} R_\mathcal{A}, \tag{2.30}$$

and

$$\|(I-P)S(t_0)u - (I-P)S(t_0)u_0\| \leq M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0} R_\mathcal{A}. \tag{2.31}$$

By Lemma 2.1, we can find $y_0^1, \ldots, y_0^{n_0}$ such that

$$B_{PX}\left(PS\left(t_0\right)u_0, e^{\lambda_0 t_0}M_1 R_{\mathcal{A}}\right) \subset \bigcup_{j=1}^{n_0} B_{PX}\left(y_0^j, \alpha e^{\lambda_0 t_0} R_{\mathcal{A}}\right) \tag{2.32}$$

with

$$n_0 \le \Lambda 2^\Lambda \left(1 + \frac{M_1}{\alpha}\right)^\Lambda, \tag{2.33}$$

where Λ is the dimension of PX.

Set

$$u_0^j = y_0^j + (I - P)S\left(t_0\right)u_0 \tag{2.34}$$

for $j = 1, \ldots, n_0$. Then, for any $u \in \mathcal{A} \cap B\left(u_0, R_{\mathcal{A}}\right)$, there exists j such that

$$\begin{aligned}
\left\|S\left(t_0\right)u - u_0^j\right\| &\le \left\|PS\left(t_0\right)u - y_0^j\right\| + \|(I - P)S\left(t_0\right)u - (I - P)S\left(t_0\right)u_0\| \\
&\le \left(\alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0}\right)R_{\mathcal{A}}.
\end{aligned} \tag{2.35}$$

Since $\mathcal{A}$ is invariant, i.e., $\mathcal{A} = S\left(t_0\right)\mathcal{A}$, we have

$$\mathcal{A} = S\left(t_0\right)\left(\mathcal{A} \cap B\left(u_0, R_{\mathcal{A}}\right)\right) \subseteq \bigcup_{j=1}^{n_0} B\left(u_0^j, \left(\alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0}\right)R_{\mathcal{A}}\right). \tag{2.36}$$

Denote by $\zeta = \left(\alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0}\right)$. Applying the formula recursively for k times gives

$$\mathcal{A} = S\left(kt_0\right)\left(\mathcal{A} \cap B\left(u_0, R_{\mathcal{A}}\right)\right) \subseteq \bigcup_{j=1}^{n_0, n_1, \cdots, n_{k-1}} B\left(u_{k-1}^j, \zeta^k R_{\mathcal{A}}\right), \tag{2.37}$$

implying that the minimal number $N_{r_k}\left(\mathcal{A}\right)$ of balls with radius $r_k = \zeta^k R_{\mathcal{A}}$ covering $\mathcal{A}$ in X satisfies

$$N_{r_k}\left(\mathcal{A}\right) \le n_0 \cdot \ldots \cdot n_{k-1} \le [\Lambda 2^\Lambda \left(1 + \frac{M_1}{\alpha}\right)^\Lambda]^k. \tag{2.38}$$

Since we have assumed that $\zeta < 1$, then $r_k \to 0$ as $k \to \infty$. Then it follows from (2.38) that

$$\begin{aligned}
\dim_f \mathcal{A} &= \limsup_{r_k \to 0} \frac{\ln N_{r_k}(\mathcal{A})}{-\ln r_k} \\
&\le \limsup_{k \to \infty} \frac{\ln[\Lambda 2^\Lambda \left(1 + \frac{M_1}{\alpha}\right)^\Lambda]^k}{-\ln(\zeta^k R_{\mathcal{A}})} \\
&= \frac{\ln \Lambda + \Lambda \ln(2 + \frac{2M_1}{\alpha})}{-\ln \zeta} < \infty.
\end{aligned} \tag{2.39}$$

$\square$

Remark 2.2. *As noticed in Remark 2.1, the estimation of $d_f(\mathcal{A})$ depends on the parameter α. The same arguments described there are also valid here and, therefore, in some situations in which $M_1 e^{\lambda_0 t_0}$ is small enough we can deduce a bound for the fractal dimension which is independent of α. Namely, if we*

assume $M_1 e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0} < 1$, then for all $\alpha \in (0, M_1)$, we have $\alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0} < 1$ and hence we deduce, taking $\alpha \uparrow M_1$, that

$$\dim_f \mathcal{A} \leq \frac{\ln \Lambda + \Lambda \ln 4}{-\ln(M_1 e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0})} < \infty, \tag{2.40}$$

which is independent of α.

3 Dimensions of pullback attractors for non-autonomous evolution process

This section is devoted to the investigation of Hausdorff and fractal dimensions of pullback attractors. We first introduce some definitions and preliminaries about evolution processes, pullback attractors as well as their existence criteria.

Definition 3.1. *A family of two-parameter mappings $\{S(t, s) : t, s \in \mathbb{R}, t \geqslant s\}$ acting on X is said to be an evolution process on X if it satisfies*

$$
\begin{aligned}
S(t, \tau)S(\tau, s) &= S(t, s), \quad \forall t, \tau, s \in \mathbb{R}, \quad t \geqslant \tau \geqslant s, \\
S(s, s) &= \mathrm{Id}_X, \quad \forall s \in \mathbb{R},
\end{aligned}
\tag{3.1}
$$

where $\mathrm{Id}_X : X \to X$ represents the identity map on X.

For notation simplicity, we will write $\{S(t, s) : t, s \in \mathbb{R}, t \geqslant s\}$ simply as $\{S(t, s)\}$ in the following. The notion of a pullback attractor is closely related to the following definition of a pullback absorbing set.

Definition 3.2. *The family $\{\mathcal{B}(t)\}_{t \in \mathbb{R}}$ is said to be pullback absorbing with respect to the process $\{S(t, s)\}$ if, for all $t \in \mathbb{R}$ and all $D \subset X$ bounded, there exists $T_D(t) > 0$ such that for all $s \geqslant T_D(t)$*

$$S(t, t - s)D \subset \mathcal{B}(t). \tag{3.2}$$

The absorption is said to be uniform if $T_D(t)$ does not depend on the time variable t.

Definition 3.3. *Let $\{S(t, s)\}$ be a process on a Banach space X. A family of compact sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be a pullback attractor for S if, for all $\tau \in \mathbb{R}$, it satisfies*
(i) $S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geqslant \tau$;
(ii) $\lim_{s \to \infty} \mathrm{dist}(S(t, t - s)D, \mathcal{A}(t)) = 0$ for all bounded subsets D of X.
The pullback attractor is said to be uniform if the attraction property is uniform in time, i.e.

$$\lim_{s \to \infty} \sup_{t \in \mathbb{R}} \mathrm{dist}(S(t, t - s)D, \mathcal{A}(t)) = 0$$

for all bounded subsets $D \subset X$.

In the definition, $\text{dist}(A, B)$ is the Hausdorff semidistance between A and B, defined as

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b), \text{ for } A, B \subseteq X, \tag{3.3}$$

where $d(a, b) = \|b - a\|_X$.

Indeed, just as in the autonomous case, the existence of compact absorbing sets is the crucial property in order to obtain pullback attractors. For the following result, see Babin and Vishik [4] or Temam [36].

Lemma 3.1. *Let $\{S(t, \tau)\}$ be a two-parameter process, and suppose $S(t, \tau) : X \to X$ is continuous for all $t \geqslant \tau$. If there exists a family of compact pullback absorbing sets $\{\mathcal{B}(t)\}_{t \in \mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, and $\mathcal{A}(t) \subset \mathcal{B}(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{D \subset X} \Lambda_D(t)},$$

where

$$\Lambda_D(t) = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{s \geqslant n} S(t, t - s)D}$$

and D is bounded.

We first study the Hausdorff dimension. Similar to **Hypothesis A1**, we impose the following conditions on $\{S(t, s)\}$.

Hypothesis A2 There is a finite dimensional projection $P(t) : X \to PX$ with a finite dimension

$$\Lambda = \dim\{P(t)X\} \tag{3.4}$$

and there are three positive numbers M_1, M_2, M_3 and two constants λ_0 and λ_1 such that

$$\|P(t)S(t, t - s_0)\varphi - P(t)S(t, t - s_0)\psi\| \leq M_1 e^{\lambda_0 s_0} \|\varphi - \psi\| \tag{3.5}$$

and

$$\|(I - P(t))S(t, t - s_0)\varphi - (I - P(t))S(t, t - s_0)\psi\| \leq (M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0}) \|\varphi - \psi\| \tag{3.6}$$

for any $t \in \mathbb{R}$ and some $s_0 \geq 0$ and φ, ψ in $\mathcal{A}(t)$.

Theorem 3.1. *Assume that $\{S(t, s)\}$ is a continuous evolution process with a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, **Hypothesis A2** holds and there exist $0 < \alpha < 2$ such that*

$$\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0} < 1. \tag{3.7}$$

Then, the Hausdorff dimension of the global attractor $\mathcal{A}(t)$ satisfies

$$d_H < \frac{-\ln \Lambda - \Lambda \ln(2 + \frac{4}{\alpha})}{\ln(\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s} + 2M_3 e^{\lambda_0 s_0})}, \tag{3.8}$$

*where Λ is the dimension of $P(t)X$ defined by (3.4) and M_1, M_2, M_3, λ_0 and λ_1 are given in **Hypothesis A1**.*

Proof. Replacing $\mathcal{A}$ and $S(t)$ in the proof of Theorem 2.1 by $\mathcal{A}(t-s_0)$ and $S(t, t-s_0)$ respectively till (2.10). For any $t \in \mathbb{R}$ and the $s_0 \in \mathbb{R}^+$ in **Hypothesis A2**. It follows from (3.5) and (3.6) that for any $u \in B\left(u_i, 2r_i\right) \cap \mathcal{A}(t-s_0)$, we have

$$\|P(t)S(t, t-s_0)u - P(t)S(t, t-s_0)u_i\| \le 2M_1 e^{\lambda_0 s_0} r_i, \tag{3.9}$$

and

$$\|(I-P(t))S(t, t-s_0)u - (I-P(t))S(t, t-s_0)u_i\| \le 2(M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0})r_i. \tag{3.10}$$

By Lemma 2.1, for any $\alpha > 0$, we can find $y_i^1, \ldots, y_i^{n_i}$ such that

$$B_{P(t)X}\left(P(t)S(t, t-s_0)u_i, 2M_1 e^{\lambda_0 s_0} r_i\right) \subset \bigcup_{j=1}^{n_i} B_{P(t)X}\left(y_i^j, \alpha M_1 e^{\lambda_0 s_0} r_i\right) \tag{3.11}$$

with

$$n_i \le \Lambda 2^\Lambda \left(1 + \frac{2}{\alpha}\right)^\Lambda, \tag{3.12}$$

where Λ is the dimension of $P(t)X$.

Set

$$u_i^j = y_i^j + (I-P(t))S(t, t-s_0)u_i \tag{3.13}$$

for $i = 1, \ldots, N, j = 1, \ldots, n_i$. Then, for any $u \in B\left(u_i, 2r_i\right) \cap \mathcal{A}(t_0 - s)$, there exists j such that

$$\left\|S(t, t-s_0)u - u_i^j\right\| \le \left\|P(t)S(t, t-s_0)u - y_i^j\right\| + \|(I-P(t))S(t, t-s_0)u - (I-P(t))S(t, t-s_0)u_i\|$$
$$\le \left(\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s} + 2M_3 e^{\lambda_0 s_0}\right) r_i \tag{3.14}$$

with

$$n_i \le \Lambda(2 + \frac{4}{\alpha})^\Lambda. \tag{3.15}$$

Denote by $\eta = (\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0})$. Then we have

$$S(t, t-s_0)\left(B\left(u_i, 2r_i\right) \cap \mathcal{A}\right) \subset \bigcup_{j=1}^{n_i} B\left(u_i^j, \eta r_i\right). \tag{3.16}$$

By the invariance property (i) in the definition 3.3, i.e., $\mathcal{A}(t) = S(t, t-s_0)\mathcal{A}(t-s_0)$, we have

$$\mathcal{A}(t) \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{n_i} B\left(u_i^j, \eta r_i\right). \tag{3.17}$$

This gives rise to, for any $d \ge 0$,

$$\mu_H\left(\mathcal{A}(t), d, \eta \varepsilon\right) \le \sum_{i=1}^{N} \sum_{j=1}^{n_i} \eta^d r_i^d \le \Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d \sum_{i=1}^{N} r_i^d, \tag{3.18}$$

we deduce by taking the infimum over all the coverings of $\mathcal{A}(t)$ by balls of radii less than ε:

$$\mu_H\left(\mathcal{A}(t), d, \eta\varepsilon\right) \leq \Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d \mu_H(\mathcal{A}(t - s_0), d, \varepsilon). \tag{3.19}$$

Applying the formula recursively for k times and it follows from the fact that $\mathcal{A}(t) = S(t, t - s_0)S(t, t - s_0) \cdots S(t, t - ks_0)\mathcal{A}(t - ks_0)$ we have

$$\mu_H\left(\mathcal{A}(t), d, \eta\varepsilon\right) \leq [\Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d]^k \mu_H(\mathcal{A}(t - ks_0), d, \varepsilon). \tag{3.20}$$

Therefore, if

$$d_H < \frac{-\ln\Lambda - \Lambda\ln(2 + \frac{4}{\alpha})}{\ln(\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0})}, \tag{3.21}$$

then

$$\Lambda(2 + \frac{4}{\alpha})^\Lambda \eta^d < 1, \tag{3.22}$$

and (3.20) leads to

$$\mu_H(\mathcal{A}(t), d, \varepsilon) \to 0, \tag{3.23}$$

when $k \to \infty$. This completes the proof. $\qquad\square$

Remark 3.1. *Taking into account the content of Remark 2.1 we can obtain some estimation for the dimension which is independent on α. Indeed, if we take $\alpha \uparrow 2$ and assume $2M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0} < 1$, then for all $\alpha \in (0, 2)$, we have $\alpha M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0} < 1$ and hence we deduce*

$$d_H \leq \frac{-\ln\Lambda - \Lambda\ln 4}{\ln(2M_1 e^{\lambda_0 s_0} + 2M_2 e^{\lambda_1 s_0} + 2M_3 e^{\lambda_0 s_0})}, \tag{3.24}$$

which is independent of α, although it may only be optimal for small values of $M_1 e^{\lambda_0 s_0}$.

Subsequently, we study the fractal dimension of pullback attractors for the evolution process $\{S(t, s)\}$.

Theorem 3.2. *Let $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ be the pullback attractor of $\{S(t, s)\}$ with uniform finite diameter $R_{\mathcal{A}} := \sup_{t\in\mathbb{R}} \sup_{u\in\mathcal{A}(t)} \|u\|_X < \infty$. Assume that* **Hypothesis A2** *holds and there exists $0 < \alpha < M_1$ such that $\zeta := \alpha e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0} < 1$. Then, the fractal dimension of global attractor $\mathcal{A}(t)$ has an upper bound*

$$\dim_f \mathcal{A}(t) \leq \frac{\ln\Lambda + \Lambda\ln(2 + \frac{2M_1}{\alpha})}{-\ln\zeta} < \infty, \tag{3.25}$$

where Λ is the dimension of PX defined by (2.2) and M_1, M_2, M_3, λ_0 and λ_1 are given in **Hypothesis A2**.

Proof. Since $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is a pullback attractor, then it is compact and hence the number $R_{\mathcal{A}}$ is well defined. Thus, for any $t \in \mathbb{R}$ and $u_0 \in \mathcal{A}(t)$, we have

$$\mathcal{A}(t) \subseteq B\left(u_0, R_{\mathcal{A}}\right), \tag{3.26}$$

where $B\left(u_0, R_{\mathcal{A}}\right)$ is the ball with center u_0 and radius $R_{\mathcal{A}}$. For any $u \in \mathcal{A}(t-s_0) \cap B\left(u_0, R_{\mathcal{A}}\right)$, it follows from **Hypothesis A2** that

$$\|P(t)S\left(t, t-s_0\right) u - P(t)S(t, t-s_0)u_0\| \leq M_1 e^{\lambda_0 s_0} R_{\mathcal{A}}, \tag{3.27}$$

and

$$\|(I - P(t))S(t, t-s_0)u - (I-P(t))S(t, t-s_0)u_0\| \leq M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0} R_{\mathcal{A}}. \tag{3.28}$$

By Lemma 2.1, we can find $y_0^1, \ldots, y_0^{n_0}$ such that

$$B_{P(t)X}\left(P(t)S(t, t-s_0)u_0, e^{\lambda_0 s_0} M_1 R_{\mathcal{A}}\right) \subset \bigcup_{j=1}^{n_0} B_{P(t)X}\left(y_0^j, \alpha e^{\lambda_0 s_0} R_{\mathcal{A}}\right) \tag{3.29}$$

with

$$n_0 \leq \Lambda 2^{\Lambda}\left(1 + \frac{M_1}{\alpha}\right)^{\Lambda}, \tag{3.30}$$

where Λ is the dimension of $P(t)X$. Set

$$u_0^j = y_0^j + (I - P(t))S(t, t-s_0)u_0 \tag{3.31}$$

for $j = 1, \ldots, n_0$. Then, any $u \in \mathcal{A}(t-s_0) \cap B\left(u_0, R_{\mathcal{A}}\right)$, there exists j such that

$$\left\|S(t, t-s_0)u - u_0^j\right\| \leq \left\|P(t)S(t, t-s_0)u - y_0^j\right\| + \|(I - P(t))S(t, t-s_0)u - (I-P(t))S(t, t-s_0)u_0\|$$
$$\leq \left(\alpha e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0}\right) R_{\mathcal{A}}. \tag{3.32}$$

Since $\mathcal{A}(t)$ is invariant, i.e., $\mathcal{A}(t) = S(t, t-s_0)\mathcal{A}(t-s_0)$, we have

$$\mathcal{A}(t) = S(t, t-s_0)\left(\mathcal{A}(t-s_0) \cap B\left(u_0, R_{\mathcal{A}}\right)\right) \subseteq \bigcup_{j=1}^{n_0} B\left(u_0^j, \left(\alpha e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0}\right) R_{\mathcal{A}}\right). \tag{3.33}$$

Denote by $\zeta = \left(\alpha e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0}\right)$. Applying the formula recursively for k times,

$$\mathcal{A}(t) = S\left(t, t-s_0\right)\cdots S\left(t-(k-1)s_0, t-ks_0\right)\left(\mathcal{A}(t-ks_0) \cap B\left(u_0, R_{\mathcal{A}}\right)\right) \subseteq \bigcup_{j=1}^{n_0, n_1, \cdots, n_{k-1}} B\left(u_{k-1}^j, \zeta^k R_{\mathcal{A}}\right), \tag{3.34}$$

implying that the minimal number $N_{r_k}\left(\mathcal{A}(t)\right)$ of balls with radius $r_k = \zeta^k R_{\mathcal{A}}$ covering $\mathcal{A}(t)$ in X satisfies

$$N_{r_k}\left(\mathcal{A}(t)\right) \leq n_1 \cdot \ldots \cdot n_k \leq \left[\Lambda 2^{\Lambda}\left(1 + \frac{M_1}{\alpha}\right)^{\Lambda}\right]^k. \tag{3.35}$$

Since we have assumed that $\zeta < 1$, then $r_k \to 0$ as $k \to \infty$. Then it follows from (3.35) that

$$\begin{aligned}
\dim_f \mathcal{A}(t) &= \limsup_{r_k \to 0} \frac{\ln N_{r_k}(\mathcal{A}(t))}{-\ln r_k} \\
&\leq \limsup_{k \to \infty} \frac{\ln[\Lambda 2^{\Lambda}\left(1 + \frac{M_1}{\alpha}\right)^{\Lambda}]^k}{-\ln \zeta^k R_{\mathcal{A}}} \\
&= \frac{\ln \Lambda + \Lambda \ln(2 + \frac{2M_1}{\alpha})}{-\ln \zeta} < \infty.
\end{aligned} \tag{3.36}$$

$\square$

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Remark 3.2. *Again, at light of Remark 2.2, if we take $\alpha \uparrow M_1$ and assume $M_1 e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0} < 1$, then for all $\alpha \in (0, M_1)$, we have $\alpha e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0} < 1$ and hence we get an estimation*

$$\dim_f \mathcal{A} \leq \frac{\ln \Lambda + \Lambda \ln 4}{-\ln(M_1 e^{\lambda_0 s_0} + M_2 e^{\lambda_1 s_0} + M_3 e^{\lambda_0 s_0})} < \infty, \tag{3.37}$$

which is independent of α, although it may be only optimal for small values of $M_1 e^{\lambda_0 s_0}$.

4   Applications

In this section, we are concerned about applications of the above established theoretical results to the retarded reaction-diffusion equation and the non-autonomous retarded functional differential equations.

4.1   Retarded reaction-diffusion equation

This subsection is devoted to the Hausdorff and fractal dimensions of global attractors for an autonomous retarded reaction-diffusion equation on the bounded domain $[0, \pi]$ with a Dirichlet boundary condition.

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) - au(x, t) - bu(x, t - r) + f(u(x, t - r)), 0 \leq x \leq \pi, t \geq 0, \\ u(0, t) = u(\pi, t) = 0, t \geq 0, \\ u(x, t) = \phi(t)(x), 0 \leq x \leq \pi, -r \leq t \leq 0, \end{cases} \tag{4.1}$$

where a, b and r are positive constants. Denote by $H = L^2(0, \pi)$ with inner product $(\xi, \eta) = \int_0^\pi \xi(x)\eta(x)dx$, norm $\|\xi\|_H = [\int_0^\pi \xi^2(x)dx]^{1/2}$ for any $\xi, \eta \in H$ and $X = C([-r, 0], H)$ the space of continuous functions from $[-r, 0]$ to H endowed with the supremum norm $\|\phi\| = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\|_H$ for any $\phi \in X$. In order to set the solution in the abstract semigroup framework, we define $A : H \to H$ by

$$Ay = \ddot{y} \tag{4.2}$$

with domain $\text{Dom}(A) = \{y \in C^2([0, \pi]); y(0) = y(\pi) = 0\}$, $L : X \to H$ by

$$L\phi \triangleq -a\phi(0) - b\phi(-r) \tag{4.3}$$

for any $\phi \in C$ and $A_U : X \to X$ by

$$A_U \phi = A\phi(0) + L\phi \tag{4.4}$$

for any $\phi \in X$. It is well known that $A - aI$ generates an analytic compact semigroup $\{T(t)\}_{t \geq 0}$ on H and [38] that A_U generates a semigroup $\{U(t)\}_{t \geq 0}$. Moreover, we assume that f satisfies the following global Lipschitz condition.

Hypothesis A3 $\|f(\phi_1) - f(\phi_2)\|_H \leq L \|\phi_1 - \phi_2\|$ for any $\phi_1, \phi_2 \in X$.

It follows from [38] Theorem 2.6 that (4.1) admits a global solution $u^\phi(\cdot) : [-r, \infty] \to H$ such that $u^\phi(t) = \phi(t)$ for $t \in [-r, 0]$ and

$$u^\phi(t) = T(t)\phi(0) + \int_0^t T(t - s) \left[L\left(u_s^\phi\right) + f\left(u_s^\phi\right) \right] ds, \tag{4.5}$$

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for $t > 0$.

Define $\Phi : \mathbb{R} \times X \to X$ by $\Phi(t)\phi = u_t^\phi(\cdot)$, then it generates an infinite dimensional dynamical system due to the uniqueness of the solution. The existence of attractor for semilinear or nonlinear partial functional differential equations including (4.1) as special case have been reported in much literature. See, for instance, [39] tackles the autonomous case with a nondensely defined linear part. Apparently, (4.1) satisfies other assumptions in [39] and hence it follows from Proposition 3.1 and Theorem 3.1 in [39] that (4.1) admits a global attractor which is stated as follows.

Lemma 4.1. *Assume that* **Hypothesis A3** *holds. Then, for any $\phi \in X$, there exists a constant $\delta > a$ such that the integral solution $u_t^\phi(\cdot)$ of Eq. (4.1) satisfies the following inequality*

$$\|u_t\| \leq \frac{c_1 e^{\delta r}}{a - L_f e^{\delta r}} + e^{\delta r}\left(\|\phi\| - \frac{c_1}{a - L_f e^{\delta r}}\right) e^{(L_f e^{\delta r} - a)t}, \quad t \geq 0, \tag{4.6}$$

where $c_1 = \|f(\mathbf{0})\|$, $a \neq L_f e^{\delta r}$. If $a > L_f e^{\delta r}$, then Eq. (4.1) has a nonempty global attractor $\mathcal{A}$.

We now estimate the dimensions of the global attractor in Lemma 4.1. We first introduce the following state decomposition results of the linear part A_U of (4.4) established in [38]. It follows from [38] that the characteristic values of the linear part A_U are the roots of the following characteristic equation

$$n^2 - \left(\lambda + a + be^{-\lambda r}\right) = 0, n = 1, 2, \cdots. \tag{4.7}$$

Since A_U is compact, it follows from Theorem 1.2 (i) in [38] that the spectrum of A_U are point spectra, which we denote by $\varrho_1 > \varrho_2 > \cdots$ with multiplicity $n_1, n_2, \cdots$, where ϱ_1 is defined as

$$\varrho_1 = \max\left\{\operatorname{Re}\lambda : n^2 - \left(\lambda + a + be^{-\lambda r}\right) = 0\right\}, n = 1, 2, \cdots. \tag{4.8}$$

In the following, we always assume that $b - a < 1$ and it follows from Lemma 1.13 on P73 in [38] that if $a > 0$, $b > 0$ and $b - a < 1$, then $\varrho_1 < 0$. For any given $\varrho_m < 0$, $m \geq 1$, there is a

$$k_m = n_1 + n_2 + \cdots + n_m \tag{4.9}$$

dimensional subspace $X_{k_m}^U$ such that

$$X = X_{k_m}^U \bigoplus X_{k_m}^S$$

is the decomposition of X by ϱ_m. Let P_{k_m} and Q_{k_m} be the projection of X onto $X_{k_m}^U$ and $X_{k_m}^S$ respectively, that is $X_{k_m}^U = P_{k_m}X$, $X_{k_m}^S = (I - P_{k_m})X = Q_{k_m}X$. It follows from the definition of P_{k_m} and Q_{k_m} that

$$\|U(t)Q_{k_m}x\| \leq Ke^{\varrho_m t}\|x\|, \quad t \geq 0, \tag{4.10}$$

where K is a positive constant.

To show the squeezing property, we extend the domain of $U(t)$ to the following space of some discontinuous functions

$$\hat{C} = \left\{\phi : [-r, 0] \to X; \phi\|_{[-r,0)} \quad \text{is continuous and } \lim_{\theta \to 0^-} \phi(\theta) \in X \quad \text{exists}\right\} \tag{4.11}$$

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and introduce the following informal variation of constant formula established in [38]

$$u(t) = U(t)\phi + \int_0^t [U(t-s)X_0 f(u_s)](0)ds, \quad t \geq 0. \tag{4.12}$$

It is proved by Theorem 2.1. in [38] that the function $u : [-r, \infty) \to X$ defined by (4.5) satisfies (4.12) with $u_0 = \phi \in X$, where $X_0 : [-r, 0] \to B(X, X)$ is given by $X_0(\theta) = 0$ if $-r \leq \theta < 0$ and $X_0(0) = Id$.

Remark 4.1. *In general, the solution semigroup defined by (4.12) have no definition at discontinuous functions and the integral in the formula is undefined as an integral in the phase space. However, if interpreted correctly, (4.12) does make sense. Details can be found in [11] Pages 144 and 145.*

We can now prove the squeezing property from which it is clear that the global attractors of (4.1) have finite Hausdorff and fractal dimensions by Theorems 2.1 and 2.2.

Theorem 4.1. *Let P be the finite dimension projection P_{k_m} defined by (4.10), ϱ_1, ϱ_m and K being defined in (4.8) and (4.10) respectively, then we have*

$$\|P\Phi(t)\varphi - P\Phi(t)\psi\| \leq 2e^{(L_f + \varrho_1)t} \|\varphi - \psi\| \tag{4.13}$$

and

$$\|(I-P)\Phi(t)\varphi - (I-P)\Phi(t)\psi\| \leq (Ke^{\varrho_m t} + \frac{KL_f}{\varrho_1 + L_f - \varrho_m} e^{(L_f + \varrho_1)t}) \|\varphi - \psi\| \tag{4.14}$$

for any $t \geq 0$ and $\varphi, \psi \in \mathcal{A}$.

Proof. For any $\varphi, \psi \in X$, denote by $y = \varphi - \psi$ and $w_t = \Phi(t)\varphi - \Phi(t)\psi = u_t^\varphi - u_t^\psi$. Then it follows from (4.12) that

$$w_t = U(t)y + \int_0^t U(t-s)X_0[f(u_s^\varphi) - f(u_s^\psi)]ds, \quad t \geq 0. \tag{4.15}$$

Taking projection $I - P$ on both sides of (4.15) leads to

$$\begin{aligned} \|(I-P)w_t\|_X = &\|(I-P)U(t)y + \int_0^t (I-P)U(t-s)X_0[f(u_s^\varphi) - f(u_s^\psi)]ds\| \\ \leq &Ke^{\varrho_m t}\|y\| + L_f \int_0^t e^{\varrho_1(t-s)}\|(I-P)w_s\|ds. \end{aligned} \tag{4.16}$$

Multiplying both sides of (4.16) by $e^{-\varrho_1 t}$,

$$e^{-\varrho_1 t}\|(I-P)w_t\| \leq Ke^{(\varrho_m - \varrho_1)t}\|y\| + L_f \int_0^t e^{-\varrho_1 s}\|(I-P)w_t\|ds. \tag{4.17}$$

By applying the Gronwall inequality, we have

$$e^{-\varrho_1 t}\|(I-P)w_t\| \leq \|y\|[Ke^{(\varrho_m - \varrho_1)t} + \frac{KL_f}{\varrho_m - \varrho_1 - L_f}(e^{(\varrho_m - \varrho_1)t} - e^{L_f})], \tag{4.18}$$

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indicating that

$$\|(I - P)w_t\| \leq \|y\|[Ke^{\varrho_m t} + \frac{KL_f}{\varrho_m - \varrho_1 - L_f}(e^{\varrho_m t} - e^{(L_f + \varrho_1)t})]$$

$$\leq \|y\|[Ke^{\varrho_m t} + \frac{KL_f}{\varrho_1 + L_f - \varrho_m}e^{(L_f + \varrho_1)t}]. \tag{4.19}$$

Hence, the second part holds with $\lambda_0 = L_f + \varrho_1$, $\lambda_1 = \varrho_m$, $M_2 = K$ and $M_3 = \frac{KL_f}{-\varrho_m + \varrho_1 + L_f}$.

Subsequently, we prove the first part. Since $U(t)y = PU(t)y + (I - P)U(t)y$, we have

$$\|PU(t)y\| \leq \|U(t)y\| + \|(I - P)U(t)y\|. \tag{4.20}$$

Taking projection of P on both sides of (4.15) gives

$$\|Pw_t\| \leq \|PU(t)y\| + \int_0^t \|PU(t - s)X_0[f(u_s^\varphi) - f(u_s^\psi)]ds\|$$

$$\leq \frac{\|P\|}{\|I - P\|}\|(I - P)U(t)y\|\|U(t)y\| + L_f \int_0^t e^{\varrho_1(t-s)}\|Pw_t\|ds \tag{4.21}$$

$$\leq \frac{|\varrho_m|}{|\varrho_{m+1}|}e^{\varrho_1 t}\|y\| + L_f \int_0^t e^{\varrho_1(t-s)}\|Pw_t\|ds.$$

Multiplying both sides of (4.21) by $e^{-\varrho_1 t}$ implies

$$e^{-\varrho_1 t}\|Pw_t\| \leq \frac{|\varrho_m|}{|\varrho_{m+1}|}\|y\| + L_f \int_0^t e^{-\varrho_1 s}\|Pw_t\|ds. \tag{4.22}$$

By applying the Gronwall inequality, we have

$$e^{-\varrho_1 t}\|Pw_t\| \leq \frac{|\varrho_m|}{|\varrho_{m+1}|}\|y\|e^{L_f t}, \tag{4.23}$$

which means that

$$\|Pw_t\| \leq \frac{|\varrho_m|}{|\varrho_{m+1}|}\|y\|e^{(L_f + \varrho_1)t}. \tag{4.24}$$

Hence, the first part holds by taking $M_1 = \frac{|\varrho_m|}{|\varrho_{m+1}|}$ and $\lambda_0 = L_f + \varrho_1$. $\qquad\square$

It follows from Theorem 2.1 that we have the following results about dimension of attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ for (4.1).

Theorem 4.2. *Let $k_m, \varrho_1, \varrho_m$ and K be defined in (4.8), (4.9) and (4.10) respectively, P be the finite dimensional projection P_{k_m} defined by (4.10). Assume that conditions of Lemma 4.1 are satisfied. Moreover, assume there exist $0 < \alpha < 2$ and $t_0 > 0$ such that*

$$\alpha\frac{|\varrho_m|}{|\varrho_{m+1}|}e^{(L_f + \varrho_1)t_0} + 2Ke^{\varrho_m t_0} + 2\frac{KL_f}{\varrho_1 + L_f - \varrho_m}e^{(L_f + \varrho_1)t_0} < 1. \tag{4.25}$$

Then, the Hausdorff dimension of the global attractor $\mathcal{A}$ satisfies

$$d_H < \frac{-\ln k_m - k_m \ln(2 + \frac{4}{\alpha})}{\ln(\alpha\frac{|\varrho_m|}{|\varrho_{m+1}|}e^{(L_f + \varrho_1)t_0} + 2Ke^{\varrho_m t_0} + 2\frac{KL_f}{\varrho_1 + L_f - \varrho_m}e^{(L_f + \varrho_1)t_0})}. \tag{4.26}$$

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Remark 4.2. *Since ϱ_1 and ϱ_m represent the first and the m-th eigenvalues of the linear part A_U of Eq. (4.1), which depends on the delay effect, we can see the Hausdorff dimension of global attractor $\mathcal{A}$ of Eq. (4.1) depends on the time delay via the distribution of eigenvalues of the linear part A_U of Eq. (4.1). Furthermore, it follows from (4.26) that the Hausdorff dimension depends on the constants of exponential dichotomy, the Lipschitz constant of the nonlinear term and the spectrum gap of the linear part A_U, indicating that the Hausdorff dimension of global attractor $\mathcal{A}$ is very flexible to be tuned by a variety of parameters. If we take $\alpha \uparrow 2$ and impose conditions as Remark 2.1, we can derive an estimation independent of α.*

Particularly, we can see from (4.26) that the Hausdorff dimension of global attractor $\mathcal{A}$ is monotone increasing with respect to m. Thus, in the case $m = 1$, we can obtain the minimum Hausdorff dimension of global attractor $\mathcal{A}$, which is given in the following corollary.

Corollary 4.1. *Let ϱ_1 and K be defined by (4.8) and (4.10) respectively and P be the finite dimensional projection P_{k_1} defined by (4.10). Assume that the conditions of Theorem 4.1 are satisfied a, b, r are appropriately chosen such that $k_1 = 1$. Moreover, assume there exist $0 < \alpha < 2$ and $t_0 > 0$ such that*

$$\alpha \frac{|\varrho_1|}{|\varrho_2|} e^{(L_f + \varrho_1)t_0} + 2K e^{\varrho_1 t_0} + 2K e^{(L_f + \varrho_1)t_0} < 1. \tag{4.27}$$

Then, the Hausdorff dimension of global attractor $\mathcal{A}$ satisfies

$$d < \frac{-\ln(2 + \frac{4}{\alpha})}{\ln(\alpha \frac{|\varrho_1|}{|\varrho_2|} e^{(L_f + \varrho_1)t_0} + 2K e^{\varrho_1 t_0} + 2K e^{(L_f + \varrho_1)t_0})}. \tag{4.28}$$

By Theorem 2.2, we have the following results about the fractal dimension of Eq. (4.1).

Theorem 4.3. *Let $k_m, \varrho_1, \varrho_m, \gamma$ and K be defined in (4.8), (4.9) and (4.10) respectively, P be the finite dimensional projection P_{k_m} defined by (4.10). Assume that the conditions of Lemma 4.1 are satisfied. Moreover, assume there exist $0 < \alpha < \frac{|\varrho_m|}{|\varrho_{m+1}|}$ and $t_0 > 0$ such that $\zeta := \alpha e^{(L_f + \varrho_1)t_0} + K e^{\varrho_m t_0} + \frac{KL_f}{\varrho_1 + L_f - \varrho_m} e^{(L_f + \varrho_1)t_0} < 1$, then, the fractal dimension of global attractor $\mathcal{A}$ has an upper bound*

$$\dim_f \mathcal{A} \leq \frac{\ln k_m + k_m \ln(2 + \frac{2|\varrho_m|}{\alpha |\varrho_{m+1}|})}{-\ln \zeta} < \infty. \tag{4.29}$$

Similar to Remark 4.2 and Corollary 4.4, we have the following corollary about the fractal dimension of global attractor $\mathcal{A}$ in the case $\varrho_m = \varrho_1$.

Corollary 4.2. *Let ϱ_1 and K be defined in (4.8), (4.9) and (4.10) respectively and P be the finite dimensional projection P_{k_1} defined by (4.10). Assume that the conditions of Lemma 4.1 are satisfied and a, b, r are appropriately chosen such that $k_1 = 1$. Moreover, assume there exists $0 < \alpha < t_0$ such that $\alpha e^{(L_f + \varrho_1)t_0} + K e^{\varrho_1 t_0} + K e^{(L_f + \varrho_1)t_0} < 1$, then, the fractal dimension of global attractor $\mathcal{A}$ satisfies*

$$\dim_f \mathcal{A} \leq \frac{\ln(2 + \frac{2|\varrho_1|}{\alpha |\varrho_2|})}{-\ln[(\alpha + K)e^{(L_f + \varrho_1)t_0} + K e^{\varrho_1 t_0}]} < \infty. \tag{4.30}$$

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4.2 Non-autonomous retarded functional differential equation

Consider the following typical non-autonomous RFDE arising in real world applications

$$\dot{u}(t) = \sum_{k=1}^{N} A_k u(t - \omega_k) + \int_{-r}^{0} A(t,\theta) u(t+\theta) d\theta + f(u_t), t \geq \sigma,$$

$$x_\sigma = \phi,$$

(4.31)

where $0 \leq \omega_1 < \omega_2 < \cdots < \omega_N \leq r$, $u(t) \in \mathbb{R}^n$, $A_k \in \mathbb{R}^{n \times n}$, r stands for the delay. The initial condition $\phi \in X \triangleq C([-r,0], \mathbb{R}^n)$, with X being the Banach space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ equipped with the supremum norm $\|\phi\|_X = \sup_{\theta \in [-r,0]} |\phi(\theta)|$ for any $\phi \in X$ and $|\cdot|$ is the usual norm of $\mathbb{R}^n$. $A(t,\theta)$ is integrable in θ for each t and there is a function $a \in \mathcal{L}_1^{loc}((-\infty,\infty), \mathbb{R})$ such that

$$\left| \int_{-r}^{0} A(t,\theta) \phi(\theta) d\theta \right| \leq a(t) |\phi|$$

(4.32)

for all $t \geq \sigma$ and $\phi \in X$. f is a continuous nonlinear mapping from X into $\mathbb{R}^n$

For notation simplicity, define the linear part of (4.31) as a linear mapping $L(t)$ from X into $\mathbb{R}^n$ given by

$$L(t)\varphi = \sum_{k=1}^{N} A_k \varphi(-\omega_k) + \int_{-r}^{0} A(t,\theta) \varphi(\theta) d\theta$$

(4.33)

for any $\varphi \in X$. Following Chapter 6 in [24], assume that there is an $n \times n$ matrix function $\eta(t,\theta)$, measurable in $(t,\theta) \in \mathbb{R} \times \mathbb{R}$, normalized so that

$$\eta(t,\theta) = 0 \quad \text{for} \quad \theta \geq 0, \quad \eta(t,\theta) = \eta(t,-r) \quad \text{for} \quad \theta \leq -r,$$

$\eta(t,\theta)$ is continuous from the left in θ on $(-r,0)$ and has bounded variation in θ on $[-r,0]$ for each t. Further, there is an $m \in \mathcal{L}_1^{loc}((-\infty,\infty), \mathbb{R})$ such that

$$\text{Var}_{[-r,0]} \, \eta(t,\cdot) \leq m(t)$$

and the linear mapping $L(t) : X \to \mathbb{R}^n$ is given by

$$L(t)\varphi = \int_{-r}^{0} d[\eta(t,\theta)]\varphi(\theta)$$

for all $t \in (-\infty,\infty)$ and $\phi \in X$. Obviously, the norm of $L(t)$ satisfies $|L(t)\phi| \leq m(t)|\phi|$. It follows from Theorem 1.1 of Chapter 6 in [24] that under the above assumptions the following non-autonomous linear equation

$$\dot{\tilde{u}}(t) = L(t)\tilde{u}_t,$$

$$\tilde{u}_\sigma = \phi$$

(4.34)

admits a unique global solution $\tilde{u}^\phi(\cdot,\sigma) : [\sigma - r, \infty) \to \mathbb{R}^n$ and hence the two parameters process $S(t,\sigma)$ on X defined by $S(t,\sigma)\phi = \tilde{u}_t^\phi(\cdot,\sigma)$ is a continuous process. We always assume the trivial equilibrium

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$\tilde{u} = 0$ of (4.34) is uniformly asymptotically stable and hence by Lemma 5.3 of Chapter 6 in [24], there exist positive constants γ and K_0 such that

$$\|S(t,\sigma)\phi\| < K_0 e^{-\gamma(t-\sigma)}\|\phi\| \tag{4.35}$$

for all $t \geq \sigma$.

We also assume the nonlinear term f satisfies **Hypothesis A3** and impose the following assumption similar to the exponential dichotomy on the process $S(t,\sigma)$ generated by (4.34).

Hypothesis A4 There exist a positive constants K and a negative constant $\beta < -\gamma$, and an m dimensional projection operator $P(t) : X \to X_m, s \in \mathbb{R}$ and $Q(s) = I - P(t) : X \to X_m^\perp, s \in \mathbb{R}$ such that

$$\|Q(t)S(t,s)\| = \|S(t,s)Q(s)\| \leq Ke^{\beta(t-s)}, \quad t \geq s \tag{4.36}$$

Remark 4.3. *The **Hypothesis A4** is an intermediate and standard assumption in the study of dynamics of non-autonomous RFDEs, such as the boundary problem, the existence of almost periodic solutions [24] and invariant manifolds [37]. In the autonomous case, it degenerates to a fact that the phase space X can be decomposed into a finite dimensional unstable subspace and an infinite dimensional stable subspace, which is guaranteed by imposing some conditions on spectrum distribution of the linear operator. The details can be found in Chapter 7 of [24] and similar results for retarded reaction diffusion equations in Subsection 4.1. For non-autonomous operators, the spectrum condition under which **Hypothesis A4** holds have been studied in the Appendix A of the very recent work [27].*

It follows from Theorem 1.2 of Chapter 6 in [24] and some standard contraction techniques, that under assumption **Hypothesis A3**, the non-autonomous nonlinear equation (4.31) admits a solution $u^\phi(t,\sigma)$ for any $t \in [\sigma - r, \infty)$, which is also continuous with respect to the initial condition. Define the non-autonomous evolution process generated by (2.1) by $\Phi(t,\sigma)\phi = u_t^\phi(\cdot,\sigma)$ for any $\phi \in X$, which is continuous for any $t \geq \sigma$. In the following, we construct exponential attractors for of $\Phi(t,\sigma)$. We first show that $\Phi(t,\sigma)$ admits a family of positive invariant pullback absorbing sets $\mathcal{B}(\sigma)$ for any $\sigma \in \mathbb{R}$.

Theorem 4.4. *Assume that **Hypothesis A4** as well as **Hypothesis A3** hold, $K_0 < 1$ and $K_0 L_f - \gamma < 0$. Then the dynamical system Φ admits an invariant pullback absorbing set $\mathcal{B}$ defined by*

$$\mathcal{B} = \{\phi \in C | \|\phi\| \leq \frac{1}{1-K_0}[\frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}]\}. \tag{4.37}$$

Proof. By the following informal variation of constant formula established in [11]

$$u_t^\phi(\cdot, t-s) = S(t,t-s)\phi + \int_{t-s}^t S(t,t-s-\rho)X_0 f(u_\rho^\phi(\cdot,t-\rho))d\rho, \quad t \geq 0, \tag{4.38}$$

where $X_0 : [-r, 0] \to B(X, X)$ is given by $X_0(\theta) = 0$ if $-r \leq \theta < 0$ and $X_0(0) = Id$, we have

$$\left\| u_t^\phi(\cdot, t - s) \right\| \leq \|(S(t, t - s)\phi)\| + \| \int_{t-s}^t S(t, t - (s - \rho))X_0 f(u_\rho^\phi(\cdot, t - \rho))d\rho\|$$

$$\leq K_0 e^{-\gamma s} \|\phi\| + K_0 L_f \int_{t-s}^t e^{-\gamma(s-\rho)}(\|u_\rho^\phi(\cdot, t - \rho)\| + f(0))\mathrm{d}\rho \tag{4.39}$$

$$\leq K_0 e^{-\gamma s} \|\phi\| + K_0 L_f \int_{t-s}^t e^{-\gamma(s-\rho)}\|u_\rho^\phi(\cdot, t - \rho)\|\mathrm{d}\rho + \frac{K_0 f(0)(1 - e^{-\gamma s})}{\gamma}.$$

Multiplying both sides of (4.39) by $e^{\gamma s}$,

$$e^{\gamma s} \left\| u_t^\phi(\cdot, t - s) \right\| \leq K_0 \|\phi\| + K_0 L_f \int_{t-s}^t e^{\gamma \varrho}\|u_\rho^\phi(\cdot, t - \rho)\|\mathrm{d}\rho + \frac{K_0 f(0)e^{\gamma s}}{\gamma}. \tag{4.40}$$

Applying the Gronwall inequality yields

$$e^{\gamma s} \left\| u_t^\phi(\cdot, t - s) \right\| \leq K_0 \|\phi\| e^{K_0 L_f s} + \frac{K_0 f(0)e^{\gamma s}}{\gamma} + \frac{e^{(\gamma - K_0 L_f)s}}{\gamma - K_0 L_f}, \tag{4.41}$$

and hence

$$\left\| u_t^\phi(\cdot, t - s) \right\| \leq K_0 \|\phi\| e^{(K_0 L_f - \gamma)s} + \frac{K_0 f(0)}{\gamma} + \frac{e^{-K_0 L_f s}}{\gamma - K_0 L_f}$$

$$\leq K_0 \|\phi\| e^{(K_0 L_f - \gamma)s} + \frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}. \tag{4.42}$$

Therefore, in the case $K_0 L_f - \gamma < 0$, for any $\phi \in X$, there exists a $s_{\|\phi\|} > 0$ such that, for all $s \geq s_{\|\phi\|}$,

$$\left\| u_t^\phi(\cdot, t - s) \right\| \leq \frac{1}{1 - K_0}[\frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}]. \tag{4.43}$$

That is, $\mathcal{B}(t)$ is an absorbing set for $\Phi(t, t - s)$. Indeed, for any bounded subset $D \subset X$, denote by $r_D = \sup_{u \in D} \|u\|$, if we take $T_D = \frac{1}{\gamma} \ln \frac{r_D \gamma (1 - K_0)(\gamma - K_0 L_f)}{K_0 f(0)(\gamma - K_0 L_f) + \gamma}$, then we have

$$\Phi(t, t - s)D \subset \mathcal{B}$$

for all $s \geq T_D$.

The invariance property clearly follows since for any $\phi \in \mathcal{B}$, by (4.42) and (4.43), we have

$$\|\Phi(t, t - s)\phi\| = \left\| u_t^\phi(\cdot, t - s) \right\| \leq K_0 \|\phi\| e^{(K_0 L_f - \gamma)s} + \frac{K_0 f(0)}{\gamma} + \frac{e^{-K_0 L_f s}}{\gamma - K_0 L_f}$$

$$\leq (\frac{K_0}{1 - K_0} + 1)[\frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}] \tag{4.44}$$

$$\leq \frac{1}{1 - K_0}[\frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}].$$

This completes the proof. $\qquad\square$

Remark 4.4. *Theorem 4.4 together with continuity and compactness of the evolution process $\Phi(t, \sigma)$ implies it admits a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$. In the sequel, we give upper bounds of Hausdorff and fractal dimensions of the evolution process $\Phi(t, \sigma)$ generated by (4.31).*

Subsequently, we prove the squeezing property of Φ, i.e., $(\mathcal{H}_3)$ holds.

Theorem 4.5. *Let P be the finite dimensional projection P_{k_m} defined by (4.35), K, β, γ and K_0 being defined by (4.35) and (4.36) respectively and assumptions of Theorem 4.4 hold, then we have*

$$\|P(t)\Phi(t,\sigma)\varphi - P(t)\Phi(t,\sigma)\psi\| \leq 2e^{(K_0 L_f - \gamma)(t-\sigma)}\|\varphi - \psi\| \tag{4.45}$$

and

$$\|(I - P(t))\Phi(t,\sigma)\varphi - (I - P(t))\Phi(t,\sigma)\psi\| \leq (Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{-\gamma + L_f - \beta}e^{(L_f - \gamma)(t-\sigma)})\|\varphi - \psi\| \tag{4.46}$$

for any $t \geq 0$ and $\varphi, \psi \in \mathcal{B}$.

Proof. For any $\varphi, \psi \in X$, denote by $y = \varphi - \psi$ and $w_t(\cdot, \sigma) = \Phi(t,\sigma)\varphi - \Phi(t,\sigma)\psi = u_t^\varphi(\cdot, \sigma) - u_t^\psi(\cdot, \sigma)$. Then it follows from (4.38) that

$$w_t(\cdot, \sigma) = S(t,\sigma)y + \int_{t-s}^t S(t,s)X_0[f(u_s^\varphi) - f(u_s^\psi)]ds, \quad t \geq 0. \tag{4.47}$$

Taking projection $I - P(t)$ on both sides of (4.47) leads to

$$
\begin{aligned}
\|(I - P(t))w_t(\cdot, \sigma)\| &= \|(I - P(t))S(t, t-s)y + \int_{t-s}^t (I - P(t))S(t,s)X_0[f(u_s^\varphi) - f(u_s^\psi)]ds\| \\
&\leq Ke^{\beta(t-\sigma)}\|y\| + L_f K_0 \int_{t-s}^t e^{-\gamma(t-s)}\|(I - P(t))w_s\|ds.
\end{aligned}
\tag{4.48}
$$

Multiplying both sides of (4.48) by $e^{\gamma(t-\sigma)}$,

$$e^{\gamma(t-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\| \leq Ke^{(\beta+\gamma)(t-\sigma)}\|y\| + L_f K_0 \int_{t-s}^t e^{\gamma(s-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\|ds. \tag{4.49}$$

By applying the Gronwall inequality, we have

$$e^{\gamma(t-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\| \leq \|y\|[Ke^{(\beta+\gamma)(t-\sigma)} + \frac{KL_f K_0}{\beta + \gamma - L_f K_0}(e^{(\beta+\gamma)(t-\sigma)} - e^{L_f K_0})], \tag{4.50}$$

indicating that

$$
\begin{aligned}
\|(I - P(t))w_t(\cdot, \sigma)\| &\leq \|y\|[Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{\beta + \gamma - L_f K_0}(e^{\beta(t-\sigma)} - e^{(K_0 L_f - \gamma)(t-\sigma)})] \\
&\leq \|y\|[Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{-\gamma + L_f K_0 - \beta}e^{(K_0 L_f - \gamma)(t-\sigma)}].
\end{aligned}
\tag{4.51}
$$

Hence, the second part holds with $\lambda_0 = L_f K_0 - \gamma$, $\lambda_1 = \beta$, $M_2 = K$ and $M_3 = \frac{KL_f K_0}{-\beta - \gamma + L_f K_0}$.

Subsequently, we prove the first part. Since $S(t,\sigma)y = P(t)S(t,\sigma)y + (I - P(t))S(t,\sigma)y$, we have

$$\|P(t)S(t,\sigma)y\| \leq \|S(t,\sigma)y\| + \|(I - P(t))S(t,\sigma)y\|. \tag{4.52}$$

Taking projection of $P(t)$ on both sides of (4.38) and on account of (4.52),

$$\|P(t)w_t(\cdot,\sigma)\| = \|S(t,\sigma)y\| + \|(I - P(t))S(t,\sigma)y\| + \int_{t-s}^{t} \|P(t)S(t,s)X_0[f(u_s^{\varphi}) - f(u_s^{\psi})]ds\|$$

$$\leq (K_0 e^{-\gamma s} + K e^{\beta(t-\sigma)})\|y\| + L_f K_0 \int_{t-s}^{t} e^{-\gamma(t-s)} \|Pw_t(\cdot,\sigma)\| ds \qquad (4.53)$$

$$\leq (K_0 + K)e^{-\gamma s}\|y\| + L_f K_0 \int_{t-s}^{t} e^{-\gamma(t-s)} \|Pw_t(\cdot,\sigma)\| ds.$$

Multiplying both sides of (4.54) by $e^{\gamma(t-\sigma)}$,

$$e^{\gamma(t-\sigma)}\|Pw_t(\cdot,\sigma)\| \leq (K_0 + K)\|y\| + L_f K_0 \int_{t-s}^{t} e^{\gamma(s-\sigma)} \|Pw_t(\cdot,\sigma)\| ds. \qquad (4.54)$$

By applying the Gronwall inequality, we have

$$e^{\gamma(t-\sigma)}\|Pw_t(\cdot,\sigma)\| \leq (K_0 + K)\|y\|e^{L_f K_0(t-\sigma)}, \qquad (4.55)$$

indicating that

$$\|Pw_t(\cdot,\sigma)\| \leq (K_0 + K)\|y\|e^{(L_f K_0 - \gamma)(t-\sigma)}. \qquad (4.56)$$

Hence, the first part holds by taking $M_1 = (K_0 + K)$ and $\lambda_0 = L_f K_0 - \gamma$. $\qquad \square$

Apparently, $\lambda_1 = \beta < -\gamma < L_f K_0 - \gamma = \lambda_0$ in Theorem 2.1. Hence, it follows from Theorem 2.1 that we have the following results about the dimension of pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ of (4.1).

Theorem 4.6. *Let K_0, γ, β, K and $P(t)$ be defined in (4.35) and (4.36) respectively. Assume that the conditions of Theorems 4.4 and 4.5 are satisfied. Moreover, assume there exist $0 < \alpha < 2$ and $t_0 > 0$ such that*

$$\alpha(K_0 + K)e^{(L_f K_0 - \gamma)t_0} + 2Ke^{\beta t_0} + 2\frac{L_f K_0 - \gamma}{(L_f K_0 - \gamma + \beta)}e^{(L_f K_0 - \gamma)t_0} < 1. \qquad (4.57)$$

Then, the Hausdorff dimension of the global attractor $\mathcal{A}$ satisfies

$$d_H < \frac{-\ln k_m - k_m \ln(2 + \frac{4}{\alpha})}{\ln(\alpha(K_0 + K)e^{(L_f K_0 - \gamma)t_0} + 2Ke^{\beta t_0} + 2\frac{L_f K_0 - \gamma}{(L_f K_0 - \gamma + \beta)}e^{(L_f K_0 - \gamma)t_0})}. \qquad (4.58)$$

Particularly, we can see from (4.58) the Hausdorff dimension of pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is monotone increasing with respect to m. Thus, in the case $m = 1$, we can obtain the minimum Hausdorff dimension of pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, which is given in the following corollary.

Corollary 4.3. *Let K_0, γ, β, K and $P(t)$ be defined in (4.35) and (4.36) respectively. Assume that the conditions of Theorems 4.4 and 4.5 are satisfied. Moreover, assume there exist $0 < \alpha < 2$ and $t_0 > 0$ such that*

$$\alpha(K_0 + K)e^{(L_f K_0 - \gamma)t_0} + 2Ke^{\beta t_0} + 2\frac{L_f K_0 - \gamma}{(L_f K_0 - \gamma + \beta)}e^{(L_f K_0 - \gamma)t_0} < 1. \qquad (4.59)$$

Then, the Hausdorff dimension of the global attractor $\mathcal{A}$ satisfies

$$d_H < \frac{-\ln(2 + \frac{4}{\alpha})}{\ln(\alpha(K_0 + K)e^{(L_f K_0 - \gamma)t_0} + 2Ke^{\beta t_0} + 2\frac{L_f K_0 - \gamma}{(L_f K_0 - \gamma + \beta)}e^{(L_f K_0 - \gamma)t_0})}. \qquad (4.60)$$

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By Theorem 2.2, we have the following results about the fractal dimension of Eq. (4.1).

Theorem 4.7. *Let K_0, γ, β, K and $P(t)$ be defined in (4.35) and (4.36) respectively. Assume that the conditions of Theorems 4.4 and 4.5 are satisfied. Moreover, assume there exist $0 < \alpha < K_0 + K$ such that $\zeta := \alpha e^{(L_f K_0 - \gamma)t_0} + K e^{\beta t_0} + \frac{K L_f}{L_f K_0 - \gamma + \beta} e^{(L_f K_0 - \gamma)t_0} < 1$. Then, the fractal dimension of pullback attractor $\mathcal{A}(t)$ has an upper bound*

$$\dim_f \mathcal{A}(t) \leq \frac{\ln m + m \ln(2 + \frac{2K_0 + K}{\alpha})}{-\ln \zeta} < \infty. \tag{4.61}$$

Similar to Remark 4.2 and Corollary 4.4, we have the following corollary about the fractal dimension of pullback attractor $\mathcal{A}(t)$ in the case $m = 1$.

Corollary 4.4. *Let K_0, γ, β, K and $P(t)$ be defined in (4.35) and (4.36) respectively. Assume that the conditions of Theorems 4.4 and 4.5 are satisfied. Moreover, assume there exist $0 < \alpha < K_0 + K$ such that $\zeta := \alpha e^{(L_f K_0 - \gamma)t_0} + K e^{\beta t_0} + \frac{K L_f}{L_f K_0 - \gamma + \beta} e^{(L_f K_0 - \gamma)t_0} < 1$. Then, the fractal dimension of pullback attractor $\mathcal{A}(t)$ satisfies*

$$\dim_f \mathcal{A}(t) \leq \frac{\ln(2 + \frac{2K_0 + K}{\alpha})}{-\ln \zeta} < \infty. \tag{4.62}$$

Remark 4.5. *In [29], the author proved that the negative invariant sets of an autonomous retarded functional differential equation have finite fractal dimension by requiring differentiability of the solution semiflow. Furthermore, they recast the equation into a Hilbert space and did not provide explicit bounds of the dimension. Here, we directly investigate the problem in the natural phase space, i.e., a Banach space and do not require the smooth condition. Moreover, we consider the non-autonomous case and establish an explicit estimation that only depends on the inner characteristic of the studied equation. Apparently, if we take $\alpha \uparrow K_0 + K$ and impose conditions as Remark 3.2, we can deduce an estimation independent of α.*

5 Conclusions

In this paper, we have established some new criterions for the upper bounds of Hausdorff and fractal dimensions of global attractors and pullback attractors for both autonomous and nonautonomous dynamical systems in Banach spaces. The methods show a wide applicability to infinite dimensional dynamical systems generated by functional differential equations, especially the ones of which the linear parts admit exponential dichotomies with decomposition of state spaces. They may be also available to investigate topological dimensions of attractors for neutral partial functional differential equations, the infinite delay case as well as some other evolution equations with certain squeeze properties in Banach spaces.

In the applications, we only consider partial functional differential equations on bounded domain. Actually, there are many real world process evolution on infinite domain, such as the mature population

of species living in an infinite habitat. In such a scenario, the Laplace operator has a continuous spectrum, $H^1(\mathbb{R}^n)$ is not compactly embedded in $L^2(\mathbb{R}^n)$ and the solution semiflow do not have absorbing sets that are compact in the original topology, causing the method developed here no longer effective and new techniques should be established. This will be studied in an upcoming paper. Generally, random effects are omnipresent in mathematical modelings. Therefore, anther question is, what can we say about the topological dimensions of random attractors for partial functional differential equations perturbed by random effects, i.e. the stochastic functional differential equations(SFDEs). Indeed, even under what conditions do SFDEs generate random dynamical systems have not been perfectly tackled. This problem also deserves much efforts in the future.

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