Stochastics and Dynamics

THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS FOR STOCHASTIC EVOLUTION EQUATIONS WITH PANTOGRAPH DELAY

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The polynomial stability problem of stochastic delay differential equations has been studied in recent years. In contrast, there are relatively few works on stochastic partial differential equations with pantograph delay. The present paper is devoted to investigating large-time asymptotic properties of solutions for stochastic pantograph delay evolution equations with nonlinear multiplicative noise. We first show that the mild solutions of stochastic pantograph delay evolution equations with nonlinear multiplicative noise tend to zero with general decay rate (including both polynomial and logarithmic rates) in the pth moment and almost sure senses. The analysis is based on the Banach fixed point theorem and various estimates involving the gamma function. Moreover, by using a generalized version of the factorization formula and exploiting an approximation technique and a convergence analysis, we construct the nontrivial equilibrium solution, defined for $t \in \mathbb{R}$, for stochastic pantograph delay evolution equations with nonlinear multiplicative noise. In particular, the uniqueness, Hölder regularity in time and general stability, in the pth moment and almost sure senses, of the nontrivial equilibrium solution are established.

Keywords: Pantograph delay; Moment general stability; Almost sure general stability; Nonlinear multiplicative noise; Nontrivial equilibrium solution.

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1. Introduction

The stochastic delay differential equations (SDDEs) take into account the perturbations and delays often present in the real world. The stochastic pantograph-delay differential equations are a kind of SDDEs with unbounded delays. Pantograph equations arise in a wide range of applications such as small vertical displacements of a stretched string under gravity [6, 20]. In recent years, stochastic evolution equations have received a great deal of attention. However, there is little work on stochastic evolution equations with pantograph delay.

In this work, Our purpose is to investigate the long time behavior of the following stochastic pantograph delay evolution equations with nonlinear multiplicative noise

$$du(t) = -Au(t)dt + f(t, u(\eta t))dt + g(t, u(\eta t))dB_Q(t), \qquad t \geq 0, \quad \eta \in (0, 1), \quad (1.1)$$

satisfying the initial condition

$$u(0) = u_0. \quad (1.2)$$

Here $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)$, $t \geq 0$ on the space $\mathbb{H}$ and B_Q is a $\mathbb{K}$-valued Brownian motion. In what follows, we assume that the mappings $[0, \infty) \ni t \mapsto f(t, \mu) \in \mathbb{H}$ and $[0, \infty) \ni t \mapsto g(t, \mu) \in \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ are measurable for any $\mu \in L^p(\Omega; \mathbb{H}^\lambda)$, where $\mathbb{H}$, $\mathbb{K}$, $\mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ and $L^p(\Omega; \mathbb{H}^\lambda)$ will be introduced later.

Firstly, we are interested in the existence, uniqueness, pth moment general stability and almost sure general stability of mild solutions for problem (1.1)-(1.2). The analysis is based on the Banach fixed point theorem and various estimates involving the gamma function. Note that many existing works are concerned with the polynomial and exponential stability of SDEs with delay or without delay by using the Razumikhin technique and Lyapunov functions; see for example, [7, 11, 17, 22, 24, 25] and the references therein. For some related works on stability of stochastic differential equations, we mention the interesting papers [2, 10, 12, 16, 18] and references therein. However, there are some systems which are not exponentially stable or polynomially stable, but the solutions do tend to zero asymptotically. Therefore, it is necessary to study general stability. Authors in [8] have considered almost sure stability with general decay rate of the exact solutions for stochastic pantograph differential equations. The moment general stability of exact solutions of the stochastic pantograph differential equation has been investigated in [9].

Moreover, we construct a unique solution u^*, defined for all $t \in \mathbb{R}$, for problem (1.1)-(1.2). In particular, the mean-p Hölder regularity, pth moment general stability and mean-p almost sure general stability of u^* are also established. The existence and uniqueness of u^* follow from constructing a Cauchy convergent sequence of linear versions and using the generalized version of the factorization formula and convergence analysis. Then the Banach fixed theorem allows us to show that the limit u^* has pth moment and almost sure stability with general decay rate. To the best of our knowledge, there are no results on the construction and stability of the nontrivial equilibrium solution for stochastic differential equations with delay. The mean-square exponential stability of the nontrivial equilibrium solution on $t \in \mathbb{R}$ has been established for stochastic reaction-diffusion equations in [19]. It is worth mentioning that, because of the difficulties caused by pantograph delays, we cannot prove the stability with the exponential decay as in [19, Theorem 3]. The presence of pantograph delays also makes the analysis more complicated. Here by using the Banach fixed theorem, we obtain the pth moment stability of the nontrivial equilibrium solution on $t \in \mathbb{R}$ with general decay rate (including the polynomial rate and the logarithmic rate). Furthermore, the almost sure general stability of the nontrivial equilibrium solution is also addressed. In addition, Hölder regularity in time of the nontrivial equilibrium solution is given.

The paper is organized as follows. In Section 2, we present some notations and technical lemmas. The results of pth moment and almost sure α-type stability are considered in Section 3 for stochastic pantograph delay evolution equations with nonlinear multiplicative noise. In Section 4, we shall show the existence and uniqueness of the nontrivial equilibrium solution by constructing the stochastic process u^*. Furthermore, pth moment and almost sure stability with a general decay function $\alpha(t)$ are established for stochastic pantograph delay evolution equations with nonlinear multiplicative noise. We also establish Hölder regularity in time of the nontrivial equilibrium solution. A summary of this work is provided in Section 5. In the end the proof of Theorem 2.1 and a technical proposition are given in the appendix.

Throughout this paper, we denote by C and $\mathbb{C}$ real positive constants which can vary from a line to another and even in the same line. Moreover, let constants C_s and $\mathbb{C}(s)$ denote C and $\mathbb{C}$ depend on some variable s, respectively.

2. Preliminaries

We define the Banach space $\mathbb{H}^\lambda = D(A^\lambda)$, where $D(A^\lambda)$ denotes the domain of the fractional power operator $A^\lambda : \mathbb{H} \to \mathbb{H}$. The norm is given by

$$\|g\|_\lambda := \|A^\lambda g\| \quad \text{for} \;\; g \in \mathbb{H}^\lambda.$$

Denote by $L^p(\Omega; \mathbb{H}^\lambda) = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{H}^\lambda)$ the set of all strongly-measurable, L^p integrable $\mathbb{H}^\lambda$-valued random variable. For any $g \in L^p(\Omega; \mathbb{H}^\lambda)$ define its norm by $\|g(\cdot)\|_{L^p(\Omega;\mathbb{H}^\lambda)} = \big(E\|g(\cdot)\|_\lambda^p\big)^{\frac{1}{p}}$. Let $C\big(c, d; L^p(\Omega; \mathbb{H}^\lambda)\big)$ denote the Banach space of all continuous functions from (c, d) into $L^p(\Omega; \mathbb{H}^\lambda)$ equipped with the sup norm $\big(\sup_{t\in[c,d]} E\|g(t)\|_\lambda^p\big)^{\frac{1}{p}}$.

Let $\mathbb{K}$ be a separable Hilbert space endowed with a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. We denote by $\mathbb{H}$ another Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. Denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. We use the same notation $\|\cdot\|$ for the norms of $\mathbb{K}$ and $\mathcal{L}(\mathbb{K}, \mathbb{H})$, and use $(\cdot, \cdot)$ to denote the inner product of $\mathbb{K}$ for convenience. Let $Q \in \mathcal{L}(\mathbb{K}, \mathbb{K})$ be an operator defined by $Qe_i = \lambda_i e_i$ with finite trace $trQ = \sum_{i=1}^{\infty} \lambda_i < \infty$. Let $\phi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$ and define

$$\|\phi\|_Q^2 := Tr(\phi Q \phi^*) = \sum_{i=1}^{\infty} \left\| \sqrt{\lambda_i} \phi e_i \right\|^2, \tag{2.1}$$

where ϕ^* is the adjoint of the operator ϕ. If $\|\phi\|_Q^2 < \infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Here $\mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space where $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω, $\mathbb{P}$ is the probability measure and $\mathcal{F}_t$ is a right-continuous filtration. Here $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the filtration generated by $B_i(t)$, that is,

$$\mathcal{F}_t := \sigma\{B_i(s) : 0 \leq s \leq t; i \geq 1\}. \tag{2.2}$$

Denote by $B_Q(t)$ the Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume that

$$B_Q(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B_i(t) e_i, \qquad t \geq 0,$$

where $\{B_i(t); t \geq 0\}_{i\geq 1}$ is a sequence of one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, \mathbb{P})$.

The following lemma is needed in this paper.

Lemma 2.1. (*See [3, Theorem 4.36]*) *If* $\phi : [0,T] \times \Omega \to \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ *is a progressively measurable function satisfying* $E\big(\int_0^T \|\phi(s)\|_Q^2 ds\big)^{\frac{p}{2}} < \infty$, *then for any* $t \in [0,T]$,

$$E\Big\| \int_0^t \phi(s) dB_Q(s) \Big\|^p \leq C_p E\Big(\int_0^t \|\phi(s)\|_Q^2 ds \Big)^{\frac{p}{2}}, \tag{2.3}$$

where $C_p > 0$ *is a positive constant depending on* p *and* $p \geq 2$.

We will also need the following theorem which is a corollary of the stochastic Fubini theorem (see, e.g. [4, Theorem 5.2.5]). For the convenience of the reader the proof is given in Appendix A. This result is often referred to as the factorization formula.

Theorem 2.1. *Assume that for some* $\alpha^* \in (0,1)$ *and all* $t \in [t_0, T]$,

$$\int_{t_0}^t (t-s)^{\alpha^*-1} \Big[E\Big(\int_{t_0}^s (s-r)^{-2\alpha^*} \left\| S(t-r)\phi(r) \right\|_Q^2 dr \Big)^{\frac{p}{2}} \Big]^{\frac{1}{p}} ds < +\infty. \tag{2.4}$$

Then

$$B_A(t) = \frac{\sin \alpha^* \pi}{\pi} \int_{t_0}^t (t-s)^{\alpha^*-1} S(t-s) Y_{\alpha^*}(s) ds, \qquad t \in [t_0, T], \tag{2.5}$$

where $t_0 \in \mathbb{R}$, $p \geq 2$ *and*

$$B_A(t) = \int_{t_0}^t S(t-s)\phi(s) dB_Q(s), \qquad Y_{\alpha^*}(s) = \int_{t_0}^s (s-r)^{-\alpha^*} S(s-r)\phi(r) dB_Q(r).$$

3. General stability of mild solutions for PDEs with nonlinear multiplicative noise

This section mainly focuses on pth moment and almost sure stability with general decay rate $\alpha(t)$. First we introduce the following α-type function:

(S_0) (1) $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ is increasing;
(2) $\alpha(0) > 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$;
(3) $\alpha(t)$ satisfies that

$$\limsup_{t\to\infty} e^{-\frac{\delta pt}{4}} \int_0^{\frac{t}{2}} e^{-\delta(\frac{t}{2}-\tau)} (\alpha(\eta\tau))^{-1} d\tau \to 0$$
$$\text{and} \qquad \limsup_{t\to\infty} e^{-\frac{\delta pt}{4}} \alpha(t) \to 0,$$

where η and δ are given in (1.1) and the assumption (S_1) below, respectively.
(4) There exists a positive constant c^* such that

$$\limsup_{t\to\infty} \frac{\alpha(t)}{\alpha(\eta t/2)} = c^*,$$

where $\eta \in (0,1)$ is given in (1.1).

It is clear that $\alpha(t) = 1 + t^{\xi^*} (0 < \xi^* < 1)$ and $\alpha(t) = \log(2 + t)$ satisfy the above requirements.

To study the stability of mild solutions with general decay rate $\alpha(t)$, we need the following assumptions:

(S_1) There exist a real number $\delta > 0$ and positive constants $C_0, C_{\lambda,0} \ge 1$ such that for any $x \in \mathbb{H}$,

$$\big\|A^\lambda S(t)x\big\| \le C_{\lambda,0} e^{-\delta t} t^{-\lambda} \|x\|, \qquad t > 0,$$
$$\big\|S(t)x\big\| \le C_0 e^{-\delta t} \|x\|, \qquad t \ge 0.$$

(S_2) There exist nonnegative functions $L_1, L_2 \in L^\infty(\mathbb{R}^+)$ such that for any $u, v \in L^p(\Omega; \mathbb{H}^\lambda)$ and $t \ge 0$,

$$E\big\|f(t,u) - f(t,v)\big\|^p \le L_1(t) E\|u - v\|_\lambda^p,$$
$$E\big\|g(t,u) - g(t,v)\big\|_Q^p \le L_2(t) E\|u - v\|_\lambda^p.$$

(S_3) There exist nonnegative functions $l_1, l_2 \in L^\infty(\mathbb{R}^+)$ such that for any $t \ge 0$,

$$\|f(t,0)\|^p \le l_1(t), \qquad \|g(t,0)\|_Q^p \le l_2(t),$$

and

$$\Big(\int_0^\infty \big(\alpha(\tau) l_1(\tau)\big)^q d\tau\Big)^{\frac{1}{q}} := \Xi_1 < \infty,$$
$$\Big(\int_0^\infty \big(\alpha(\tau) l_2(\tau)\big)^{q_1} d\tau\Big)^{\frac{1}{q_1}} := \Xi_2 < \infty,$$

where $1/p + 1/q = 1$ and $1/q_1 = 1 - 1/p_1$ with $p_1 \in (1, \frac{1}{2\lambda})$ and $\lambda \in (0, \frac{1}{p})$.

Let us state the following definition of mild solution to problem (1.1)-(1.2).

Definition 3.1. Let $T > 0$ and u_0 be an $\mathcal{F}_0$-measurable initial process satisfying $E\|u_0\|_\lambda^p < \infty$. An $\mathcal{F}_t$-measurable stochastic process $u(t)$ is called a mild solution of problem (1.1)-(1.2) on $[0, T]$ if $u \in C(0, T; L^p(\Omega; \mathbb{H}^\lambda))$ and for $t \in [0, T]$,

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau, u(\eta\tau))d\tau + \int_0^t S(t-\tau)g(\tau, u(\eta\tau))dB_Q(\tau), \ \ \mathbb{P}\text{-}a.s. \tag{3.1}$$

Remark 3.1. In fact, by similar arguments in Section 4, the solution $u(t)$ defined by Eq. (3.1) has continuous trajectories with probability 1.

The following theorem shows that mild solutions to Eq. (1.1)-(1.2) are pth moment α-type stable.

Theorem 3.1. *Let $p \geq 2$, $\lambda \in (0, \frac{1}{p})$ and $u_0 \in L^p(\Omega; \mathbb{H}^\lambda)$. Suppose that assumptions (S_0)-(S_3) hold. Let $\|L_1\|_{L^\infty(\mathbb{R}^+)}$ and $\|L_2\|_{L^\infty(\mathbb{R}^+)}$ be sufficiently small such that*

$$\begin{aligned} 2^p C_{\lambda,0}^p \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \|L_1\|_{L^\infty(\mathbb{R}^+)} < 1, \\ 2^p C_p C_{\lambda,0}^p \big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \|L_2\|_{L^\infty(\mathbb{R}^+)} < 1, \\ 2^{2p} c^* C_{\lambda,0}^p \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \|L_1\|_{L^\infty(\mathbb{R}^+)} < 1, \\ 2^{2p} c^* C_p C_{\lambda,0}^p \big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \|L_2\|_{L^\infty(\mathbb{R}^+)} < 1, \end{aligned} \tag{3.2}$$

where c^*, δ, $C_{\lambda,0}$ and C_p are given in the assumptions (S_0)-(S_1) and Lemma 2.1, respectively. Then problem (1.1)-(1.2) has a unique global mild solution u satisfying

$$\sup_{r \in [0,\infty)} \alpha(r) E\|u(r)\|_\lambda^p < \infty. \tag{3.3}$$

Proof. We consider the abstract phase space $C_\vartheta^{p,\lambda} = C_\vartheta\big(0, \infty; L^p(\Omega; \mathbb{H}^\lambda)\big)$ equipped with the norm

$$\|u\|_\vartheta = \sup_{t\in[0,\infty)} \vartheta(t)E\|u(t)\|_\lambda^p, \quad u \in C\big(0, \infty; L^p(\Omega; \mathbb{H}^\lambda)\big),$$

where

$$\vartheta(t) = \begin{cases} \alpha(T), & t \in [0, T], \\ \alpha(t), & t \geq T, \end{cases} \tag{3.4}$$

with $T > 0$ given later. Then $\big(C_\vartheta^{p,\lambda}, \|\cdot\|_\vartheta\big)$ is a Banach space. In order to apply the Banach fixed point theorem, we shall prove that the mapping $\widetilde{\mathcal{T}}$ defined by

$$(\widetilde{\mathcal{T}}u)(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau, u(\eta\tau))d\tau + \int_0^t S(t-\tau)g(\tau, u(\eta\tau))dB_Q(\tau), \tag{3.5}$$

is contractive and bounded on $C_\vartheta^{p,\lambda}$.

Step 1. It follows immediately from (3.5) that

$$\vartheta(t)E\big\|(\widetilde{\mathcal{T}}u)(t) - (\widetilde{\mathcal{T}}v)(t)\big\|_\lambda^p$$

$$
\begin{aligned}
&\leq 2^{p-1}\vartheta(t)E\Big(\int_0^t \big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&+2^{p-1}\vartheta(t)E\Big\|\int_0^t A^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)dB_Q(\tau)\Big\|^p\\
&:=\mathcal{R}_1+\mathcal{R}_2. \qquad (3.6)
\end{aligned}
$$

In view of assumptions (S_1)-(S_2) and Hölder's inequality, we deduce that for $t \in [0,T]$ and any $u,v \in C_\vartheta^{p,\lambda}$,

$$
\begin{aligned}
\mathcal{R}_1 &\leq 2^{p-1}\alpha(T)C^p_{\lambda,0}E\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\leq 2^{p-1}\alpha(T)C^p_{\lambda,0}\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}d\tau\Big)^{p-1}\\
&\quad\times\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau\\
&\leq 2^{p-1}C^p_{\lambda,0}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta. \qquad (3.7)
\end{aligned}
$$

On the other hand, for $t \geq T$ and any $u,v \in C_\vartheta^{p,\lambda}$,

$$
\begin{aligned}
\mathcal{R}_1 &\leq 2^{2p-2}\alpha(t)E\Big(\int_0^{\frac{t}{2}} \big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&\quad+2^{2p-2}\alpha(t)E\Big(\int_{\frac{t}{2}}^t \big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&:=\mathcal{R}_1^1+\mathcal{R}_1^2. \qquad (3.8)
\end{aligned}
$$

Applying Hölder's inequality and assumptions (S_1)-(S_2) results in

$$
\begin{aligned}
\mathcal{R}_1^1 &\leq 2^{2p-2}\alpha(t)C^p_{\lambda,0}E\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\leq 2^{2p-2}\alpha(t)C^p_{\lambda,0}\big(\frac{t}{2}\big)^{-p\lambda}\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}d\tau\Big)^{p-1}\\
&\quad\times\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau \qquad (3.9)\\
&\leq 2^{2p-2}\alpha(t)C^p_{\lambda,0}\|u-v\|_\vartheta\|L_1\|_{L^\infty(\mathbb{R}^+)}\big(\frac{t}{2}\big)^{-p\lambda}\frac{e^{-\delta pt/2}}{\delta^{p-1}}\int_0^{\frac{t}{2}} e^{-\delta(t/2-\tau)}(\alpha(\eta\tau))^{-1}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_1^2 &\leq 2^{2p-2}C^p_{\lambda,0}\alpha(t)E\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\leq 2^{2p-2}C^p_{\lambda,0}\alpha(t)\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}d\tau\Big)^{p-1}\\
&\quad\times\int_{\frac{t}{2}}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau
\end{aligned}
$$

$$\leq 2^{2p-2}C_{\lambda,0}^{p}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^{p}\|u-v\|_{\vartheta}\|L_1\|_{L^\infty(\mathbb{R}^+)}\frac{\alpha(t)}{\alpha(\eta t/2)}. \tag{3.10}$$

Combining (3.7) and (3.8)-(3.10), we can find T large enough such that for all $t \geq 0$,

$$\mathcal{R}_1 < \frac{1}{2}\|u-v\|_{\vartheta}, \tag{3.11}$$

thanks to assumptions (3.2) and (S_0).

Now it remains to estimate the stochastic term. In view of Lemma 2.1, Hölder's inequality and assumptions (S_1)-(S_2), we find that for $t \in [0,T]$,

$$\begin{aligned}
\mathcal{R}_2 &\leq 2^{p-1}\alpha(T)C_pE\Big(\int_0^t \big\|A^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{p-1}\alpha(T)C_pC_{\lambda,0}^pE\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\big\|g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{p-1}\alpha(T)C_pC_{\lambda,0}^p\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}}\\
&\quad\times\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}E\big\|g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big\|_Q^p d\tau\\
&\leq 2^{p-1}C_pC_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_{\vartheta},
\end{aligned} \tag{3.12}$$

and for $t \geq T$,

$$\begin{aligned}
\mathcal{R}_2 &\leq 2^{2p-2}\alpha(t)E\Big\|\int_0^{\frac{t}{2}} S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&\quad+2^{2p-2}\alpha(t)E\Big\|\int_{\frac{t}{2}}^t S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&:= \mathcal{R}_2^1+\mathcal{R}_2^2.
\end{aligned} \tag{3.13}$$

It follows from Lemma 2.1, Hölder's inequality and assumptions (S_1)-(S_2) that

$$\begin{aligned}
\mathcal{R}_2^1 &\leq 2^{2p-2}\alpha(t)C_pE\Big(\int_0^{\frac{t}{2}} \big\|A^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{2p-2}\alpha(t)C_pC_{\lambda,0}^p\Big(\frac{t}{2}\Big)^{-p\lambda}\\
&\quad\times E\Big(\int_0^{\frac{t}{2}} e^{-\frac{2(p-2)}{p}\delta(t-\tau)}e^{-\frac{4}{p}\delta(t-\tau)}\big\|g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{2p-2}\alpha(t)C_pC_{\lambda,0}^p\|u-v\|_{\vartheta}\|L_2\|_{L^\infty(\mathbb{R}^+)}\\
&\quad\times\Big(\frac{t}{2}\Big)^{-p\lambda}\frac{e^{-\delta pt/2}}{(2\delta)^{\frac{p-2}{2}}}\int_0^{\frac{t}{2}} e^{-\delta(t-2\tau)}(\alpha(\eta\tau))^{-1}d\tau,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\mathcal{R}_2^2 &\le 2^{2p-2}\alpha(t)C_pE\Big(\int_{\frac{t}{2}}^{t}\big\|A^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\ &\le 2^{2p-2}\alpha(t)C_pC_{\lambda,0}^p\Big(\int_{\frac{t}{2}}^{t}(t-\tau)^{-2\lambda}e^{-2\delta(t-\tau)}d\tau\Big)^{\frac{p-2}{2}}\\ &\quad\times\int_{\frac{t}{2}}^{t}(t-\tau)^{-2\lambda}e^{-2\delta(t-\tau)}E\big\|g(\tau,u(\eta\tau))-g(\tau,v(\eta\tau))\big\|_Q^p d\tau\\ &\le 2^{2p-2}C_pC_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|u-v\|_\vartheta\|L_2\|_{L^\infty(\mathbb{R}^+)}\frac{\alpha(t)}{\alpha(\eta t/2)}.\end{aligned}\tag{3.15}$$

Inserting (3.14)-(3.15) into (3.13) gives

$$\begin{aligned}\mathcal{R}_2 \le 2^{2p-2}C_pC_{\lambda,0}^p\Big(&(\frac{t}{2})^{-p\lambda}\frac{e^{-\delta pt/2}}{(2\delta)^{\frac{p-2}{2}}}\int_0^{\frac{t}{2}}e^{-\delta(t-2\tau)}(\alpha(\eta\tau))^{-1}d\tau\\ &+\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\frac{\alpha(t)}{\alpha(\eta t/2)}\Big)\|L_2\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta.\end{aligned}\tag{3.16}$$

Then, by assumptions (3.2) and (S_0), in view of (3.12) and (3.16), we can take T sufficiently large such that for any $t\ge 0$,

$$\mathcal{R}_2 < \frac{1}{2}\|u-v\|_\vartheta.\tag{3.17}$$

This together with (3.11) and (3.6) implies that $\widetilde{\mathcal{T}}$ is contractive on the space $C_\vartheta^{p,\lambda}$.
Step 2. By (3.5) we have

$$\begin{aligned}&\vartheta(t)E\big\|(\widetilde{\mathcal{T}}u)(t)\big\|_\lambda^p\\ &\le 3^{p-1}\vartheta(t)C_0^pe^{-\delta pt}E\|u_0\|_\lambda^p+6^{p-1}\vartheta(t)E\Big(\int_0^t\big\|S(t-\tau)f(\tau,0)\big\|_\lambda d\tau\Big)^p\\ &+6^{p-1}\vartheta(t)E\Big\|\int_0^tA^\lambda S(t-\tau)g(\tau,0)dB_Q(\tau)\Big\|^p\\ &+6^{p-1}\vartheta(t)E\Big(\int_0^t\big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,0)\big)\big\|_\lambda d\tau\Big)^p\\ &+6^{p-1}\vartheta(t)E\Big\|\int_0^tA^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,0)\big)dB_Q(\tau)\Big\|^p\\ &:=3^{p-1}\vartheta(t)C_0^pe^{-\delta pt}E\|u_0\|_\lambda^p+\mathcal{R}_3+\mathcal{R}_4+\mathcal{R}_5+\mathcal{R}_6.\end{aligned}\tag{3.18}$$

Following similar calculations as in (3.8)-(3.10) and (3.13)-(3.15), we conclude that

$$\mathcal{R}_5+\mathcal{R}_6\le 3^{p-1}\|u\|_\vartheta,\quad \text{for}\ \ t\ge T,\tag{3.19}$$

where T is sufficiently large. By using assumptions (S_0), (S_3), and Hölder's inequality, we can find T large enough such that

$$\begin{aligned}\mathcal{R}_3 &\leq \mathbb{C}(\lambda,p)\vartheta(t)\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\|f(\tau,0)\|d\tau\Big)^p \\ &\leq \mathbb{C}(\lambda,p)\|l_1\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\frac{\delta pt}{2}}\Big(\int_0^{\frac{t}{2}} e^{-\delta(t/2-\tau)}(t/2-\tau)^{-\lambda}d\tau\Big)^p \\ &\quad + \mathbb{C}(p,\lambda,\delta)\frac{\alpha(t)}{\alpha(t/2)}\int_{\frac{t}{2}}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}l_1(\tau)\alpha(\tau)d\tau \\ &\leq \mathbb{C}(p,\lambda,\delta)\Big(\frac{\alpha(t)}{\alpha(t/2)}\Xi_1 + \|l_1\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\frac{\delta pt}{2}}\Big) < \infty,\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\mathcal{R}_4 &\leq \mathbb{C}(\lambda,p)\vartheta(t)\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\|g(\tau,0)\|_Q^2d\tau\Big)^{\frac{p}{2}} \\ &\leq \mathbb{C}(\lambda,p)\|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\frac{\delta pt}{2}}\Big(\int_0^{\frac{t}{2}} e^{-2\delta(t/2-\tau)}(t/2-\tau)^{-2\lambda}d\tau\Big)^{\frac{p}{2}} \\ &\quad + \mathbb{C}(p,\lambda,\delta)\frac{\alpha(t)}{\alpha(t/2)}\int_{\frac{t}{2}}^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}l_2(\tau)\alpha(\tau)d\tau \\ &\leq \mathbb{C}(p,\lambda,\delta)\Big(\frac{\alpha(t)}{\alpha(t/2)}\Xi_2 + \|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\frac{\delta pt}{2}}\Big) < \infty.\end{aligned} \tag{3.21}$$

Arguing as in (3.7) and (3.12), we deduce that

$$\mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6 < \mathbb{C}(p,\lambda,\delta,L_1,L_2,l_1,l_2)\big(\alpha(T) + \|u\|_\vartheta\big), \qquad t\in[0,T]. \tag{3.22}$$

The assertion of this theorem follows immediately by applying the Banach fixed point theorem. □

Remark 3.2. If we do not consider the α-type stability, then the global existence and uniqueness of mild solutions to problem (1.1)-(1.2) can be established by using the assumptions (S_1)-(S_2), and the simplified assumption of f and g, i.e.,

$$\|f(t,0)\|^p + \|g(t,0)\|_Q^p \leq \ell(t), \quad t\geq 0, \quad \ell\in L^\infty(\mathbb{R}^+).$$

Remark 3.3. In particular, for the case $\alpha(\tau) = 1+\tau^{\xi^*}(0<\xi^*<1)$, we can find some examples of the function l_1, satisfying the assumption (S_3), such that

$$\int_0^\infty \big(1+\tau^{\xi^*}\big)^q\big(\tau^{\xi^*-1/q}(1+\tau^{\xi^*})^{-3}\big)^q d\tau < \infty, \tag{3.23}$$

or

$$\int_0^\infty \big(1+\tau^{\xi^*}\big)^q e^{-q\tilde{c}\tau}d\tau < \infty, \quad (\tilde{c}>0). \tag{3.24}$$

The assertion for the function l_2 follows similarly.

Based on the result in Theorem 3.1, we further establish almost surely α-type stability of problem (1.1)-(1.2). To this end, let us recall the following Burkholder-Davis-Gundy inequality

$$E\Big(\sup_{t\in[0,T]}\Big\|\int_0^t \Phi(\tau)dB_Q(\tau)\Big\|^{p^*}\Big) \leq c'E\Big|\int_0^T \|\Phi(\tau)\|_Q^2 d\tau\Big|^{\frac{p^*}{2}}, \quad \text{for any } p^*>0, \tag{3.25}$$

where c' is a positive constant depending on p^*.

Theorem 3.2. *Let $\gamma\in(0,1)$. Suppose that all the assumptions of Theorem 3.1 are satisfied. Then the solution of Eq. (1.1)-(1.2) is almost surely α-type stable, that is,*

$$\lim_{t\to\infty}\frac{\log\|u(t)\|_\lambda}{\log\alpha(t)} < -\frac{1-\gamma}{p}, \qquad a.s. \tag{3.26}$$

Proof. Let $s>0$ be given arbitrarily. Note that $N\geq 1$

$$\begin{aligned} u(t) &= S\big(t-\eta^{-(N-1)}s\big)u(\eta^{-(N-1)}s) + \int_{\eta^{-(N-1)}s}^t S(t-\tau)f(\tau,u(\eta\tau))d\tau \\ &\quad + \int_{\eta^{-(N-1)}s}^t S(t-\tau)g(\tau,u(\eta\tau))dB_Q(\tau). \end{aligned} \tag{3.27}$$

In view of Markov's inequality, we have that

$$\begin{aligned} &\mathbb{P}\Big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\|u(t)\|_\lambda^p \geq \big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big) \\ &\leq \big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\|u(t)\|_\lambda^p\big) \\ &\leq 3^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\big\|S\big(t-\eta^{-(N-1)}s\big)u(\eta^{-(N-1)}s)\big\|_\lambda^p\big) \\ &\quad + 6^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\Big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\Big\|\int_{\eta^{-(N-1)}s}^t S(t-\tau)f(\tau,0)d\tau\Big\|_\lambda^p\Big) \\ &\quad + 6^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\Big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\Big\|\int_{\eta^{-(N-1)}s}^t S(t-\tau)g(\tau,0)dB_Q(\tau)\Big\|_\lambda^p\Big) \\ &\quad + 6^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\Big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\Big\|\int_{\eta^{-(N-1)}s}^t S(t-\tau) \\ &\qquad \times \big(f(\tau,u(\eta\tau))-f(\tau,0)\big)d\tau\Big\|_\lambda^p\Big) \\ &\quad + \frac{6^{p-1}}{\big(\alpha(\eta^{-(N-1)}s)\big)^{\gamma-1}}E\Big(\sup_{\eta^{-N}s\leq t\leq \eta^{-(N+1)}s}\Big\|\int_{\eta^{-(N-1)}s}^t S(t-\tau) \\ &\qquad \times \big(g(\tau,u(\eta\tau))-g(\tau,0)\big)dB_Q(\tau)\Big\|_\lambda^p\Big) \\ &:= \mathcal{R}_7+\mathcal{R}_8+\mathcal{R}_9+\mathcal{R}_{10}+\mathcal{R}_{11}. \end{aligned} \tag{3.28}$$

Thanks to assumption (S_1), we obtain

$$\begin{aligned}\mathcal{R}_7 &\le 3^{p-1}C_0^p\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}\\ &\quad\times E\Big(\sup_{\eta^{-N}s\le t\le\eta^{-(N+1)}s} e^{-p\delta(t-\eta^{-(N-1)}s)}\|u(\eta^{-(N-1)}s)\|_\lambda^p\Big)\\ &\le 3^{p-1}C_0^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\big(\alpha(\eta^{-(N-1)}s)\big)E\|u(\eta^{-(N-1)}s)\|_\lambda^p. \end{aligned}\tag{3.29}$$

By Hölder's inequality and assumptions (S_1)-(S_2), we deduce that

$$\begin{aligned}\mathcal{R}_{10} &\le 6^{p-1}C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}\Big(\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}d\tau\Big)^{p-1}\\ &\quad\times\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}E\big\|f(\tau,u(\eta\tau))-f(\tau,0)\big\|^p d\tau\\ &\le 6^{p-1}C_{\lambda,0}^p\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\\ &\quad\times\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\alpha(\tau)L_1(\tau)E\|u(\eta\tau)\|_\lambda^p d\tau\\ &\le 6^{p-1}C_{\lambda,0}^p\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^{p}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\|L_1\|_{L^\infty(\mathbb{R}^+)}\\ &\quad\times\sup_{\tau\in[0,\infty)}\alpha(\tau)/\alpha(\eta\tau)\sup_{\tau\in[0,\infty)}\alpha(\tau)E\|u(\tau)\|_\lambda^p,\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}\mathcal{R}_8 &\le 6^{p-1}C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\Big(\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}d\tau\Big)^{p-1}\\ &\quad\times\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\alpha(\tau)\|f(\tau,0)\|^p d\tau\\ &\le 6^{p-1}C_{\lambda,0}^p\Xi_1\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^{p-1}\big((\delta p)^{p\lambda-1}\Gamma(1-p\lambda)\big)^{\frac{1}{p}}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}.\end{aligned}\tag{3.31}$$

It follows from the Burkholder-Davis-Gundy inequality, Hölder's inequality and assumptions (S_1)-(S_2) that

$$\begin{aligned}\mathcal{R}_{11} &\le 6^{p-1}c'\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}\\ &\quad\times E\Big(\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s}\big\|A^\lambda S(t-\tau)\big(g(\tau,u(\eta\tau))-g(\tau,0)\big)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\\ &\le 6^{p-1}c'C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}\Big(\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}}\\ &\quad\times\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}E\big\|g(\tau,u(\eta\tau))-g(\tau,0)\big\|_Q^p d\tau\\ &\le 6^{p-1}c'C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p-2}{2}}\\ &\quad+\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s} e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}L_2(\tau)\alpha(\tau)E\|u(\eta\tau)\|_\lambda^p d\tau\end{aligned}$$

$$\leq 6^{p-1}c'C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R}^+)}$$
$$\times\sup_{\tau\in[0,\infty)}\alpha(\tau)/\alpha(\eta\tau)\sup_{\tau\in[0,\infty)}\alpha(\tau)E\|u(\tau)\|_\lambda^p. \tag{3.32}$$

Following similar computations as in (3.32), we can derive from assumption (S_3) and Hölder's inequality that

$$\mathcal{R}_9\leq 6^{p-1}c'C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\Big(\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}}$$
$$\times\int_{\eta^{-(N-1)}s}^{\eta^{-(N+1)}s}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\alpha(\tau)l_2(\tau)d\tau \tag{3.33}$$
$$\leq 6^{p-1}c'\Xi_2C_{\lambda,0}^p\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p-2}{2}}$$
$$\times\big((2p_1\delta)^{2p_1\lambda-1}\Gamma(1-2p_1\lambda)\big)^{\frac{1}{p_1}}.$$

Substituting (3.29)-(3.33) into (3.28) yields

$$\begin{aligned}\mathbb{P}\Big(&\sup_{\eta^{-N}s\leq t\leq\eta^{-(N+1)}s}\|u(t)\|_\lambda^p\geq\big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big)\\&\leq\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\mathbb{C}\big(1+\sup_{\tau\in[0,\infty)}\alpha(\tau)E\|u(\tau)\|_\lambda^p\big),\end{aligned}\tag{3.34}$$

where $\mathbb{C}$ is a positive constant independent of η, s. Then it follows immediately from (3.34) that

$$\sum_{N=1}^\infty\mathbb{P}\Big(\sup_{\eta^{-N}s\leq t\leq\eta^{-(N+1)}s}\|u(t)\|_\lambda^p\geq\big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big)<\infty. \tag{3.35}$$

In view of the Borel-Cantelli lemma, we conclude that there exists $\widetilde{\Omega}\subset\Omega$ with $\mathbb{P}(\widetilde{\Omega})=1$ such that for any $\omega\in\widetilde{\Omega}$, there exists an integer $\widetilde{N}=\widetilde{N}(\omega)>0$, such that, for $N\geq\widetilde{N}$ and $\eta^{-N}s\leq t\leq\eta^{-(N+1)}s$,

$$\|u(t)\|_\lambda^p\leq\big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)},$$

which implies the assertion in this theorem. □

4. Nontrivial equilibrium solution for PDEs with nonlinear multiplicative noise

We shall construct a stochastic process u^*, defined for all $t\in\mathbb{R}$, and analyze the α-type stability of u^* for the following stochastic differential equation

$$du(t)=-Au(t)dt+f(t,u(\eta t))dt+g(t,u(\eta t))dB_Q(t),\qquad t\in\mathbb{R},\quad\eta\in(0,1). \tag{4.1}$$

Recall that the cylindrical Brownian motion $B_Q(t)$ in Section 2 is defined only for $t\in\mathbb{R}^+$. In order to consider the mild solution $u^*(t)$ defined for $t\in\mathbb{R}$, we introduce the following infinite-dimensional Brownian motion:

$$B_Q(t)=\sum_{i=1}^\infty\sqrt{\lambda_i}B_i^\star(t)e_i,\quad t\in\mathbb{R}, \tag{4.2}$$

where sequences $\{\lambda_i\}_{i\in\mathbb{N}}, \{e_i\}_{i\in\mathbb{N}}$ have been given in Section 2 and $B_i^\star(t)$ is defined by

$$B_i^\star(t) = \begin{cases} B_i^1(t), & \text{for } t \geq 0, \\ B_i^2(-t), & \text{for } t \leq 0. \end{cases} \tag{4.3}$$

Here B_i^1 and B_i^2 are independent standard one-dimensional Brownian motions. Let

$$\mathcal{F}_t := \sigma\left(\bigcup \left\{B_i^\star(s) - B_i^\star(r) : r \leq s \leq t, i \geq 1\right\}\right), \tag{4.4}$$

be the σ-algebra generated by $\left\{B_i^\star(s) - B_i^\star(r) : r \leq s \leq t, i \geq 1\right\}$.

Definition 4.1. A $\mathbb{H}^\lambda$-valued stochastic process $u(t)$ is called a mild solution to problem (4.1) on $\mathbb{R}$ if

1) $u(t)$ is $\mathcal{F}_t$-measurable for each $t \in \mathbb{R}$;
2) $\sup_{t\in\mathbb{R}} \|u(t)\|_{L^p(\Omega;\mathbb{H}^\lambda)} < \infty$;
3) $u(t)$ is continuous almost surely in $t \in \mathbb{R}$ with respect to $\mathbb{H}^\lambda$ norm;
4) it holds that for all $-\infty < t_0 < t < \infty$,

$$\begin{aligned} u(t) = S(t-t_0)u(t_0) &+ \int_{t_0}^t S(t-\tau)f(\tau, u(\eta\tau))d\tau \\ &+ \int_{t_0}^t S(t-\tau)g(\tau, u(\eta\tau))dB_Q(\tau), \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{4.5}$$

4.1. *Linear version*

To construct and analyze solutions of problem (4.1), first we consider the following linear equation:

$$du = -Audt + \zeta(t)dt + \psi(t)dB_Q(t), \qquad t \in \mathbb{R}. \tag{4.6}$$

Theorem 4.1. *Let $p \geq 2$, $\lambda \in (0, \frac{1}{p})$ and the assumption (S_1) be satisfied. Assume that $\zeta(t)$ and $\psi(t)$ in* (4.6) *are $\mathcal{F}_t$-measurable and satisfy*

$$\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p < \infty. \tag{4.7}$$

Then the linear equation (4.6) *has a unique solution $\tilde{u}^*$ in the sense of Definition* 4.1 *which is mean-p Hölder continuous in $t \in \mathbb{R}$, i.e.,*

$$\sup_{t\in\mathbb{R}} \|\tilde{u}^*(t+h) - \tilde{u}^*(t)\|_{L^p(\Omega;\mathbb{H}^\lambda)} \leq \mathbb{C}h^{\frac{1}{2}-\lambda}, \qquad \textit{for each} \;\; h > 0.$$

Furthermore, the solution $\tilde{u}^*$ is exponentially stable, i.e., for any $t_0 \in \mathbb{R}$ and any solution $\tilde{\varrho}(t)$ of Eq. (4.6) *in the sense of Definition* 3.1*, with $\mathcal{F}_{t_0}$-measurable $\tilde{\varrho}(t_0)$ and $E\|\tilde{\varrho}(t_0)\|_\lambda^p < \infty$,*

$$E\big\|\tilde{u}^*(t) - \tilde{\varrho}(t)\big\|_\lambda^p \leq \mathbb{C}e^{-\mathbb{C}(t-t_0)}E\big\|\tilde{u}^*(t_0) - \tilde{\varrho}(t_0)\big\|_\lambda^p.$$

Proof. Let

$$\tilde{u}^*(t) = \int_{-\infty}^{t} S(t-\tau)\zeta(\tau)d\tau + \int_{-\infty}^{t} S(t-\tau)\psi(\tau)dB_Q(\tau). \tag{4.8}$$

Step 1. The process $\tilde{u}^*(t)$ given by (4.8) is well defined.

We define $\Pi_n^1(t)$ and $\Pi_n^2(t)$ by

$$\Pi_n^1(t) := \int_{-n}^{t} S(t-\tau)\zeta(\tau)d\tau, \tag{4.9}$$

and

$$\Pi_n^2(t) := \int_{-n}^{t} S(t-\tau)\psi(\tau)dB_Q(\tau), \tag{4.10}$$

respectively. In view of Lemma 2.1, assumption (S_1) and Hölder's inequality, we find that for $n > m$,

$$\begin{aligned}
&E\big\|\Pi_n^2(t) - \Pi_m^2(t)\big\|_\lambda^p \le C_p E\Big(\int_{-n}^{-m} \big\|A^\lambda S(t-\tau)\psi(\tau)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}} \\
&\le C_p C_{\lambda,0}^p \Big(\int_{-n}^{-m} e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}} \int_{-n}^{-m} e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}E\|\psi(\tau)\|_Q^p d\tau \\
&\le C_p C_{\lambda,0}^p \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p \Big(\int_{-n}^{-m} e^{-\delta(t-\tau)}(t-\tau)^{-2\lambda}e^{-\delta(t-\tau)}d\tau\Big)^{\frac{p}{2}} \qquad (4.11)\\
&\le C_p C_{\lambda,0}^p \big((\delta p_0)^{2p_0\lambda-1}\Gamma(1-2p_0\lambda)\big)^{\frac{p}{2p_0}} \Big(\frac{e^{-\delta q_0 t}(e^{-\delta q_0 m} - e^{-\delta q_0 n})}{\delta q_0}\Big)^{\frac{p}{2q_0}} \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p,
\end{aligned}$$

where we choose $p_0 > 1$ such that $\lambda p_0 < \frac{1}{2}$ and $1/p_0 + 1/q_0 = 1$. Similar to the above arguments, we deduce that for $n > m$,

$$\begin{aligned}
&E\big\|\Pi_n^1(t) - \Pi_m^1(t)\big\|_\lambda^p \\
&\le C_{\lambda,0}^p \Big(\int_{-n}^{-m} e^{-\frac{p\delta}{2(p-1)}(t-\tau)}d\tau\Big)^{p-1} \int_{-n}^{-m} e^{-\frac{p\delta}{2}(t-\tau)}(t-\tau)^{-p\lambda}E\|\zeta(\tau)\|^p d\tau \qquad (4.12)\\
&\le C_{\lambda,0}^p (p\delta/2)^{p\lambda-1}\Gamma(1-p\lambda) \\
&\quad\times \Big(\frac{2(p-1)e^{-\frac{p\delta t}{2(p-1)}}(e^{-\frac{p\delta m}{2(p-1)}} - e^{-\frac{p\delta n}{2(p-1)}})}{p\delta}\Big)^{p-1} \sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p.
\end{aligned}$$

Note that the terms on the right-hand side of (4.11) and (4.12) are as small as possible as $n, m \to \infty$. Then $\{\Pi_n^1(t)\}$ and $\{\Pi_n^2(t)\}$ are Cauchy sequences for each $t \in \mathbb{R}$. Therefore, the process $\tilde{u}^*(t)$ is well defined.

Step 2. The process $\tilde{u}^*$ defined by (4.8) is a solution in the sense of Definition 4.1.

(*I*) The measurability and continuity of $\tilde{u}^*(t)$ in time.
Since $\zeta(t)$ and $\psi(t)$ are $\mathcal{F}_t$-measurable, by (4.4), we obtain that the process $\tilde{u}^*(t)$ is $\mathcal{F}_t$-measurable. Moreover, by using the factorization formula for the stochastic integrals (2.5) and Proposition 6.1, we can derive that the process $\tilde{u}^*(t)$ has continuous trajectories with probability 1.

(II) $\sup_{t\in\mathbb{R}} E\|\tilde{u}^*(t)\|_\lambda^p < \infty$.
On account of assumption (S_1), we have

$$\begin{aligned}
&E\Big\| \int_{-\infty}^t S(t-\tau)\zeta(\tau)d\tau\Big\|_\lambda^p \\
&\le C_{\lambda,0}^p E\Big(\int_{-\infty}^t e^{-\frac{p-1}{p}\delta(t-\tau)}(t-\tau)^{-\frac{p-1}{p}\lambda}\zeta(\tau)e^{-\frac{1}{p}\delta(t-\tau)}(t-\tau)^{-\frac{1}{p}\lambda}d\tau\Big)^p \\
&\le C_{\lambda,0}^p\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p. \qquad (4.13)
\end{aligned}$$

Thanks to Lemma 2.1, by a similar reasoning as in (4.13), we obtain that

$$\begin{aligned}
&E\Big\| \int_{-\infty}^t S(t-\tau)\psi(\tau)dB_Q(\tau)\Big\|_\lambda^p \\
&\le C_p C_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p. \qquad (4.14)
\end{aligned}$$

By (4.13) and (4.14), the assertion follows from (4.8).

(III) The process $\tilde{u}^*(t)$ satisfies (4.5).

$$\begin{aligned}
\tilde{u}^*(t) &= \int_{-\infty}^{t_0} S(t-t_0)S(t_0-\tau)\zeta(\tau)d\tau \\
&\quad + \int_{-\infty}^{t_0} S(t-t_0)S(t_0-\tau)\psi(\tau)dB_Q(\tau) \\
&\quad + \int_{t_0}^t S(t-\tau)\zeta(\tau)d\tau + \int_{t_0}^t S(t-\tau)\psi(\tau)dB_Q(\tau) \qquad (4.15) \\
&= S(t-t_0)\tilde{u}^*(t_0) + \int_{t_0}^t S(t-\tau)\zeta(\tau)d\tau + \int_{t_0}^t S(t-\tau)\psi(\tau)dB_Q(\tau).
\end{aligned}$$

Step 3. The Hölder regularity, exponential stability and uniqueness of $\tilde{u}^*(t)$.

Now we show that $\tilde{u}^*(t)$ is continuous in time. It follows from (4.8) that, for each $h > 0$,

$$\begin{aligned}
&\big\|\tilde{u}^*(t+h) - \tilde{u}^*(t)\big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\le \Big\| \int_{-\infty}^t \big(S(t+h-\tau) - S(t-\tau)\big)\zeta(\tau)d\tau\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{-\infty}^t \big(S(t+h-\tau) - S(t-\tau)\big)\psi(\tau)dB_Q(\tau)\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_t^{t+h} S(t+h-\tau)\zeta(\tau)d\tau\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_t^{t+h} S(t+h-\tau)\psi(\tau)dB_Q(\tau)\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&:= \mathcal{R}_{12} + \mathcal{R}_{13} + \mathcal{R}_{14} + \mathcal{R}_{15}. \qquad (4.16)
\end{aligned}$$

To deal with the term $\mathcal{R}_{13}$, let us consider

$$\int_t^{t+h} \Big\| \int_{-\infty}^t AS(s-\tau)\psi(\tau)dB_Q(\tau)\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} ds.$$

Thanks to Lemma 2.1, in view of assumption (S_1) and Hölder's inequality, we deduce that

$$\begin{aligned}
&\int_t^{t+h} \Big\| \int_{-\infty}^t AS(s-\tau)\psi(\tau)dB_Q(\tau)\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} ds \\
&= \int_t^{t+h} \Big(E\Big\| \int_{-\infty}^t A^{1+\lambda}S(s-\tau)\psi(\tau)dB_Q(\tau)\Big\|^p\Big)^{\frac{1}{p}} ds \\
&\le (C_p)^{\frac{1}{p}} \int_t^{t+h} \Big[E\Big(\int_{-\infty}^t \big\|A^{1+\lambda}S(s-\tau)\psi(\tau)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} ds \\
&\le (C_p)^{\frac{1}{p}} C_{1+\lambda,0} \int_t^{t+h} \Big[E\Big(\int_{-\infty}^t e^{-2\delta(s-\tau)}(s-\tau)^{-2(\lambda+1)}\|\psi(\tau)\|_Q^2 d\tau\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} ds \\
&\le \mathbb{C}(p,\lambda) \int_t^{t+h} \Big(\int_{-\infty}^t e^{-2\delta(s-\tau)}(s-\tau)^{-2(\lambda+1)} d\tau\Big)^{\frac{p-2}{2p}} \\
&\qquad \times \Big(\int_{-\infty}^t e^{-2\delta(s-\tau)}(s-\tau)^{-2(\lambda+1)} E\|\psi(\tau)\|_Q^p d\tau\Big)^{\frac{1}{p}} ds \\
&\le \mathbb{C}(p,\lambda)\big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p\big)^{\frac{1}{p}} \int_t^{t+h} (s-t)^{-\lambda-\frac{1}{2}} ds \\
&= \mathbb{C}(p,\lambda)\big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p\big)^{\frac{1}{p}} h^{\frac{1}{2}-\lambda}.
\end{aligned} \tag{4.17}$$

Then, applying the stochastic Fubini theorem to $\mathcal{R}_{13}$ gives

$$\begin{aligned}
\mathcal{R}_{13} &\le \Big\| \int_{-\infty}^t \int_t^{t+h} AS(s-\tau)\psi(\tau)ds dB_Q(\tau)\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&= \Big\| \int_t^{t+h} \int_{-\infty}^t AS(s-\tau)\psi(\tau)dB_Q(\tau)ds\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\le \mathbb{C}(p,\lambda)\big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p\big)^{\frac{1}{p}} h^{\frac{1}{2}-\lambda}.
\end{aligned} \tag{4.18}$$

By making use of Lemma 2.1, assumption (S_1) and Hölder's inequality, we deduce that

$$\begin{aligned}
\mathcal{R}_{15} &= \Big(E\Big\| \int_t^{t+h} A^\lambda S(t+h-\tau)\psi(\tau)dB_Q(\tau)\Big\|^p\Big)^{\frac{1}{p}} \\
&\le (C_p)^{\frac{1}{p}} C_{\lambda,0}\Big[E\Big(\int_t^{t+h} \big\|A^\lambda S(t+h-\tau)\psi(\tau)\big\|_Q^2 d\tau\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\le \mathbb{C}(p,\lambda)\Big(\int_t^{t+h} e^{-2\delta(t+h-\tau)}(t+h-\tau)^{-2\lambda} d\tau\Big)^{\frac{p-2}{2p}} \\
&\quad \times \Big(\int_t^{t+h} e^{-2\delta(t+h-\tau)}(t+h-\tau)^{-2\lambda} E\|\psi(\tau)\|_Q^p d\tau\Big)^{\frac{1}{p}}
\end{aligned}$$

$$\leq \mathbb{C}(p,\lambda)\big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^p\big)^{\frac{1}{p}} h^{\frac{1}{2}-\lambda}. \tag{4.19}$$

In view of assumption (S_1) and Hölder's inequality, we have

$$\begin{aligned}
\mathcal{R}_{12} &= \Big\| \int_t^{t+h}\int_{-\infty}^t AS(s-\tau)\zeta(\tau)d\tau ds\Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\leq \int_t^{t+h}\int_{-\infty}^t \Big(E\big\|A^{1+\lambda}S(s-\tau)\zeta(\tau)\big\|^p\Big)^{\frac{1}{p}} d\tau ds \\
&\leq C_{1+\lambda,0}\big(\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p\big)^{\frac{1}{p}} \int_t^{t+h}\int_{-\infty}^t e^{-\delta(s-\tau)}(s-\tau)^{-(\lambda+1)} d\tau ds \\
&\leq \mathbb{C}(\lambda)\big(\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p\big)^{\frac{1}{p}} h^{1-\lambda},
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
\mathcal{R}_{14} &\leq \int_t^{t+h} \big\|S(t+h-\tau)\zeta(\tau)\big\|_{L^p(\Omega;\mathbb{H}^\lambda)} d\tau \\
&\leq C_{\lambda,0}\Big(\int_t^{t+h} e^{-\delta(t+h-\tau)}(t+h-\tau)^{-\lambda} d\tau\Big)^{\frac{p-1}{p}} \\
&\quad \times \Big(\int_t^{t+h} e^{-\delta(t+h-\tau)}(t+h-\tau)^{-\lambda} E\|\zeta(\tau)\|^p d\tau\Big)^{\frac{1}{p}} \\
&\leq \mathbb{C}(\lambda)\big(\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^p\big)^{\frac{1}{p}} h^{1-\lambda}.
\end{aligned} \tag{4.21}$$

Substituting (4.18)-(4.21) into (4.16) yields that $\tilde{u}^*(t)$ is mean-p Hölder continuous.

If $\tilde{\varrho}(t)$ is any solution of (4.6) satisfying $E\|\tilde{\varrho}(t_0)\|_\lambda^p < \infty$, then

$$\tilde{\varrho}(t) = S(t-t_0)\tilde{\varrho}(t_0) + \int_{t_0}^t S(t-\tau)\zeta(\tau)d\tau + \int_{t_0}^t S(t-\tau)\psi(\tau)dB_Q(\tau). \tag{4.22}$$

It follows immediately from (4.15), (4.22) and assumption (S_1) that

$$E\big\|\tilde{u}^*(t) - \tilde{\varrho}(t)\big\|_\lambda^p \leq C_0^p e^{-p\delta(t-t_0)} E\big\|\tilde{u}^*(t_0) - \tilde{\varrho}(t_0)\big\|_\lambda^p, \tag{4.23}$$

which implies that $\tilde{u}^*$ is exponentially stable.

Finally, we show that $\tilde{u}^*(t)$ is unique. Let $v(t)$ be another solution such that $\sup_{t\in\mathbb{R}} E\|v(t)\|_\lambda^p < \infty$. By Definition 4.1 and the assumption (S_1), we obtain that for arbitrary $r \leq t$,

$$E\big\|\tilde{u}^*(t) - v(t)\big\|_\lambda^p \leq C_0^p e^{-p\delta(t-r)} E\big\|\tilde{u}^*(r) - v(r)\big\|_\lambda^p \leq \mathbb{C}e^{-p\delta(t-r)}. \tag{4.24}$$

Letting $r \to -\infty$, we have

$$E\big\|\tilde{u}^*(t) - v(t)\big\|_\lambda^p = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.25}$$

Using Markov's inequality, we deduce that for each $t \in \mathbb{R}$ and any $\varepsilon > 0$,

$$\mathbb{P}\big(\|v(t) - \tilde{u}^*(t)\|_\lambda > \varepsilon\big) \leq \frac{1}{\varepsilon^p} E\|v(t) - \tilde{u}^*(t)\|_\lambda^p, \tag{4.26}$$

and consequently

$$\mathbb{P}\big(\|v(t)-\tilde{u}^*(t)\|_\lambda = 0 \qquad \text{for all} \;\; t\in Q^*\cap\mathbb{R}\big)=1, \tag{4.27}$$

where Q^* denotes the rational numbers. Since the mapping $t\to\|v(t)-\tilde{u}^*(t)\|_\lambda$ is continuous with probability 1, we conclude that

$$\mathbb{P}\big(\|v(t)-\tilde{u}^*(t)\|_\lambda = 0 \qquad \text{for all} \;\; t\in \mathbb{R}\big)=1. \tag{4.28}$$

Therefore, the uniqueness of $\tilde{u}^*(t)$ is confirmed. The proof of this theorem is complete. □

4.2. *Nonlinear version*

The following theorem shows the existence, uniqueness and α-type stability of the solution u^* to problem (4.1).

Theorem 4.2. *Suppose that $p\geq 2$, $\lambda\in(0,\frac{1}{p})$ and assumptions (S_2)-(S_3) hold for $t\in\mathbb{R}$. Let us further assume that assumptions (S_0)-(S_1) hold, and the Lipschitz constants L_1, L_2 in assumption (S_2) are sufficiently small such that*

$$\begin{aligned}
&2^{2p-2}C^p_{\lambda,0}\Big(\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R})} \\
&\qquad\qquad + C_p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R})}\Big) := \mathcal{K}_1<1, \\
&2^{p-1}C^p_{\lambda,0}\Big(C_p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R})} \\
&\qquad\qquad + \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R})}\Big) := \mathcal{K}_2<1,
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
2^pC^p_{\lambda,0}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R})}&<1, \\
2^pC_pC^p_{\lambda,0}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R})}&<1, \\
6^pc^*C^p_{\lambda,0}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R})}&<1, \\
6^pc^*C_pC^p_{\lambda,0}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R})}&<1.
\end{aligned} \tag{4.30}$$

Then, problem (4.1) *has a unique solution $u^*(t)$ in the sense of Definition* 4.1 *which is mean-p Hölder continuous in $t\in\mathbb{R}$, i.e.,*

$$\sup_{t\in\mathbb{R}}\|u^*(t+h)-u^*(t)\|_{L^p(\Omega;\mathbb{H}^\lambda)}\leq \mathbb{C}h^{\frac{1}{2}-\lambda}, \quad \textit{for each} \;\; h>0.$$

Moreover, the solution $u^*(t)$ is α-type stable, that is,

$$\lim_{t\to\infty}\frac{\log E\|u^*(t)-\varrho(t)\|^p_\lambda}{\log\alpha(t)}<0, \tag{4.31}$$

where $\varrho(t)$ is any solution of problem (1.1)-(1.2) *in the sense of Definition* 3.1.

Proof.

Let us construct a sequence of stochastic processes $\{u_n\}$ which converges to the solution u^*. Let $u_0 \equiv 0$. For $n \geq 0$, define $u_{n+1}(t)$ as

$$du_{n+1}(t) = -Au_{n+1}(t)dt + f(t, u_n(\eta t))dt + g(t, u_n(\eta t))dB_Q(t). \tag{4.32}$$

Notice that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} E\big\|f(t, u_n(\eta t))\big\|^p \leq 2^{p-1}\|l_1\|_{L^\infty(\mathbb{R})} + 2^{p-1}\|L_1\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t)\|_\lambda^p, \\ &\sup_{t\in\mathbb{R}} E\big\|g(t, u_n(\eta t))\big\|_Q^p \leq 2^{p-1}\|l_2\|_{L^\infty(\mathbb{R})} + 2^{p-1}\|L_2\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t)\|_\lambda^p. \end{aligned} \tag{4.33}$$

By using Theorem 4.1, we obtain the unique solution $u_{n+1}(t)$ satisfying

$$\sup_{t\in\mathbb{R}} E\|u_{n+1}(t)\|_\lambda^p < \infty, \tag{4.34}$$

and

$$u_{n+1}(t) = \int_{-\infty}^t S(t-\tau)f(\tau, u_n(\eta\tau))d\tau + \int_{-\infty}^t S(t-\tau)g(\tau, u_n(\eta\tau))dB_Q(\tau). \tag{4.35}$$

Step 1. The sequence $\{u_n(t)\}$ converges to the process $u^*(t)$ and the process $u^*(t)$ is a solution in the sense of Definition 4.1.

(1) $\sup_{t\in\mathbb{R}} \|u_n\|_{L^p(\Omega;\mathbb{H}^\lambda)}$ is bounded which is independent of n.
It follows directly from (4.35) that

$$\begin{aligned} E\|u_{n+1}(t)\|_\lambda^p &\leq 2^{p-1}E\Big\|\int_{-\infty}^t S(t-\tau)f(\tau, u_n(\eta\tau))d\tau\Big\|_\lambda^p \\ &\quad + 2^{p-1}E\Big\|\int_{-\infty}^t S(t-\tau)g(\tau, u_n(\eta\tau))dB_Q(\tau)\Big\|_\lambda^p \\ &:= \mathcal{R}_{16} + \mathcal{R}_{17}. \end{aligned} \tag{4.36}$$

By applying Lemma 2.1, assumptions (S_1)-(S_3), Hölder's inequality and (4.33), we obtain

$$\begin{aligned} \mathcal{R}_{17} &\leq 2^{p-1}C_pE\Big(\int_{-\infty}^t \big\|A^\lambda S(t-\tau)g(\tau, u_n(\eta\tau))\big\|_Q^2 d\tau\Big)^{\frac{p}{2}} \\ &\leq 2^{p-1}C_pC_{\lambda,0}^pE\Big(\int_{-\infty}^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\|g(\tau, u_n(\eta\tau))\|_Q^2 d\tau\Big)^{\frac{p}{2}} \\ &\leq 2^{p-1}C_pC_{\lambda,0}^p\Big(\int_{-\infty}^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}} \\ &\quad \times \int_{-\infty}^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}E\|g(\tau, u_n(\eta\tau))\|_Q^p d\tau \\ &\leq 2^{2p-2}C_pC_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \\ &\quad \times \Big(\|l_2\|_{L^\infty(\mathbb{R})} + \|L_2\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t)\|_\lambda^p\Big). \end{aligned} \tag{4.37}$$

Similar to (4.13), we deduce that

$$\mathcal{R}_{16} \le 2^{2p-2} C^p_{\lambda,0}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \times \Big(\|l_1\|_{L^\infty(\mathbb{R})} + \|L_1\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t)\|^p_\lambda\Big). \tag{4.38}$$

Inserting (4.37)-(4.38) into (4.36) gives

$$\sup_{t\in\mathbb{R}} E\|u_{n+1}(t)\|^p_\lambda \le \mathcal{K}_0 + \mathcal{K}_1 \sup_{t\in\mathbb{R}} E\|u_n(t)\|^p_\lambda. \tag{4.39}$$

Then we derive from (4.39), assumption (4.29) and the recursive method that

$$\sup_{t\in\mathbb{R}} E\|u_n(t)\|^p_\lambda \le \frac{\mathcal{K}_0}{1-\mathcal{K}_1}, \tag{4.40}$$

where we have used the notation

$$\mathcal{K}_0 := 2^{2p-2} C^p_{\lambda,0}\Big(\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \|l_1\|_{L^\infty(\mathbb{R})} + C_p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \|l_2\|_{L^\infty(\mathbb{R})}\Big).$$

(2) The sequence $\{u_n\}$ is convergent.
Arguing as in (3.7) and (3.12), it follows from (4.35) that

$$\begin{aligned} &E\|u_{n+1}(t) - u_n(t)\|^p_\lambda \\ &\le 2^{p-1} E\Big\| \int_{-\infty}^t S(t-\tau)\big(f(\tau, u_n(\eta\tau)) - f(\tau, u_{n-1}(\eta\tau))\big)d\tau \Big\|^p_\lambda \\ &\quad + 2^{p-1} E\Big\| \int_{-\infty}^t S(t-\tau)\big(g(\tau, u_n(\eta\tau)) - g(\tau, u_{n-1}(\eta\tau))\big)dB_Q(\tau) \Big\|^p_\lambda \\ &\le 2^{p-1} C_p C^p_{\lambda,0}\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \|L_2\|_{L^\infty(\mathbb{R})} \\ &\quad \times \sup_{t\in\mathbb{R}} E\|u_n(t) - u_{n-1}(t)\|^p_\lambda \\ &\quad + 2^{p-1} C^p_{\lambda,0}\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p \|L_1\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t) - u_{n-1}(t)\|^p_\lambda, \end{aligned} \tag{4.41}$$

which implies that

$$\sup_{t\in\mathbb{R}} E\|u_{n+1}(t) - u_n(t)\|^p_\lambda \le \mathcal{K}_2 \sup_{t\in\mathbb{R}} E\|u_n(t) - u_{n-1}(t)\|^p_\lambda. \tag{4.42}$$

Using the recursive method again, in view of (4.40) and the assumption $\mathcal{K}_2 < 1$, we obtain that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \|u_n(t) - u_m(t)\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\ &\le \sum_{j=m}^{n-1} \sup_{t\in\mathbb{R}} \|u_{j+1}(t) - u_j(t)\|_{L^p(\Omega;\mathbb{H}^\lambda)} = \sum_{j=m}^{n-1} \sup_{t\in\mathbb{R}} \big(E\|u_{j+1}(t) - u_j(t)\|^p_\lambda\big)^{\frac{1}{p}} \\ &\le \sum_{j=m}^{n-1} \Big(\sup_{t\in\mathbb{R}} E\|u_{j+1}(t) - u_j(t)\|^p_\lambda\Big)^{\frac{1}{p}} \le \big(\sup_{t\in\mathbb{R}} E\|u_1(t)\|^p_\lambda\big)^{\frac{1}{p}} \sum_{j=m}^{n-1} (\mathcal{K}_2)^{\frac{j}{p}} \end{aligned}$$

$$\leq \big(\frac{\mathcal{K}_0}{1-\mathcal{K}_1}\big)^{\frac{1}{p}} \sum_{j=m}^{n-1} \frac{1}{2^{\frac{j}{p}}} \to 0, \quad \text{as} \;\; n, m \to \infty. \tag{4.43}$$

This means that $u_n(t)$ is a Cauchy sequence, and thus there exists a limiting function $u^*(t)$ such that

$$\sup_{t\in\mathbb{R}} E\|u_n(t) - u^*(t)\|_\lambda^p \to 0, \quad \text{as} \;\; n \to \infty. \tag{4.44}$$

Combining (4.40) and (4.44) results in

$$E\|u^*(t)\|_\lambda^p \leq \frac{\mathcal{K}_0}{1-\mathcal{K}_1}, \qquad \text{for each } t \in \mathbb{R}. \tag{4.45}$$

Since the sequence $\{u_n\}$ is $\mathcal{F}_t$-measurable for each $t \in \mathbb{R}$, the process $u^*(t)$ is $\mathcal{F}_t$-measurable as a limit of $\{u_n\}$.

(3) The process $u^*(t)$ satisfies (4.5) and has continuous trajectories with probability 1.

By similar calculations as in (4.15), it follows from (4.35) that

$$\begin{aligned} u_{n+1}(t) = S(t-t_0)u_{n+1}(t_0) + \int_{t_0}^t S(t-\tau) f(\tau, u_n(\eta\tau))d\tau \\ + \int_{t_0}^t S(t-\tau) g(\tau, u_n(\eta\tau))dB_Q(\tau). \end{aligned} \tag{4.46}$$

To show that $u^*(t)$ satisfies (4.5), we need to pass to the limit in the above identity.

It follows from Markov's inequality and (4.44) that, for each $\varepsilon > 0$,

$$\mathbb{P}\big(\|u_{n+1}(t) - u^*(t)\|_\lambda > \varepsilon\big) \leq \frac{1}{\varepsilon^p} E\|u_{n+1}(t) - u^*(t)\|_\lambda^p \stackrel{n\to\infty}{\longrightarrow} 0, \tag{4.47}$$

which implies that, for each $t \in \mathbb{R}$,

$$u_{n+1}(t) \to u^*(t) \quad \text{in probability}, \quad \text{as} \;\; n \to \infty. \tag{4.48}$$

Due to the fact that $S(t-t_0)$ is a bounded operator, we obtain that

$$S(t-t_0)u_{n+1}(t_0) \longrightarrow S(t-t_0)u^*(t_0) \;\; \text{in probability}, \quad \text{as} \;\; n \to \infty. \tag{4.49}$$

Arguing as in (3.12), in view of Markov's inequality, we deduce that

$$\begin{aligned} &\mathbb{P}\Big(\Big\| \int_{t_0}^t S(t-\tau)\big(g(\tau, u_n(\eta\tau)) - g(\tau, u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda > \varepsilon\Big) \\ &\leq \frac{1}{\varepsilon^p} E\Big\| \int_{t_0}^t S(t-\tau)\big(g(\tau, u_n(\eta\tau)) - g(\tau, u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p \\ &\leq \frac{1}{\varepsilon^p} C_p C_{\lambda,0}^p \big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}} \|L_2\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t) - u^*(t)\|_\lambda^p, \end{aligned} \tag{4.50}$$

which together with (4.44) implies

$$\int_{t_0}^t S(t-\tau) g(\tau, u_n(\eta\tau))dB_Q(\tau)$$

$$\overset{n\to\infty}{\longrightarrow} \int_{t_0}^{t} S(t-\tau)g(\tau, u^*(\eta\tau))dB_Q(\tau), \quad \text{in probability.} \tag{4.51}$$

In a similar way as in (4.50), we find that

$$\begin{aligned}
&\int_{t_0}^{t} S(t-\tau)f(\tau, u_n(\eta\tau))d\tau \\
&\overset{n\to\infty}{\longrightarrow} \int_{t_0}^{t} S(t-\tau)f(\tau, u^*(\eta\tau))d\tau, \quad \text{in probability.}
\end{aligned} \tag{4.52}$$

Finally, by (4.48), (4.49) and (4.51)-(4.52), we can conclude that for all $t \in \mathbb{R}$,

$$\begin{aligned}
u^*(t) = S(t-t_0)u^*(t_0) &+ \int_{t_0}^{t} S(t-\tau)f(\tau, u^*(\eta\tau))d\tau \\
&+ \int_{t_0}^{t} S(t-\tau)g(\tau, u^*(\eta\tau))dB_Q(\tau) \ \text{ a.s.}
\end{aligned} \tag{4.53}$$

i.e. $u^*(t)$ satisfies (4.5). The continuity of the first two terms can be checked straightforwardly, and the continuity of the third term follows from the factorization formula (2.5) and Proposition 6.1. Hence the process $u^*(t)$, defined by (4.2), has continuous trajectories with probability 1.

Step 2. The process u^* is Hölder continuous in $t \in \mathbb{R}$.

By similar arguments as in (4.16)-(4.21) and (4.33), we obtain that for each $h > 0$,

$$\begin{aligned}
&\big\|u^*(t+h) - u^*(t)\big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\le \Big\| \int_{-\infty}^{t} \big(S(t+h-\tau) - S(t-\tau)\big) f(\tau, u^*(\eta\tau))d\tau \Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{-\infty}^{t} \big(S(t+h-\tau) - S(t-\tau)\big) g(\tau, u^*(\eta\tau))dB_Q(\tau) \Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{t}^{t+h} S(t+h-\tau)f(\tau, u^*(\eta\tau))d\tau \Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{t}^{t+h} S(t+h-\tau)g(\tau, u^*(\eta\tau))dB_Q(\tau) \Big\|_{L^p(\Omega;\mathbb{H}^\lambda)} \\
&\le \mathbb{C}(\lambda)\big(\sup_{\tau\in\mathbb{R}} E\|f(\tau, u^*(\eta\tau))\|^p\big)^{\frac{1}{p}} h^{1-\lambda} + \mathbb{C}(p,\lambda)\big(\sup_{\tau\in\mathbb{R}} E\|g(\tau, u^*(\eta\tau))\|_Q^p\big)^{\frac{1}{p}} h^{\frac{1}{2}-\lambda} \\
&\le \mathbb{C}(\lambda)\Big(\|l_1\|_{L^\infty(\mathbb{R})}^{\frac{1}{p}} + \|L_1\|_{L^\infty(\mathbb{R})}^{\frac{1}{p}} \big(\sup_{t\in\mathbb{R}} E\|u^*(t)\|_\lambda^p\big)^{\frac{1}{p}}\Big) h^{1-\lambda} \\
&\quad + \mathbb{C}(\lambda,p)\Big(\|l_2\|_{L^\infty(\mathbb{R})}^{\frac{1}{p}} + \|L_2\|_{L^\infty(\mathbb{R})}^{\frac{1}{p}} \big(\sup_{t\in\mathbb{R}} E\|u^*(t)\|_\lambda^p\big)^{\frac{1}{p}}\Big) h^{\frac{1}{2}-\lambda},
\end{aligned} \tag{4.54}$$

which means that u^* is Hölder continuous in time.

Step 3. The process u^* is α-type stable in the sense of pth moment.

The assertion of this step can be proved by applying the Banach fixed point theorem. Since the proofs of the case $t_0 \geq 0$ is simpler than the case $t_0 < 0$, we assume that $t_0 < 0$. Consider the abstract phase space $C^{p,\lambda}_{\vartheta^*} = C_{\vartheta^*}\big(t_0, \infty; L^p(\Omega; \mathbb{H}^\lambda)\big)$ with the norm

$$\|u\|_{\vartheta^*} = \sup_{t\in[t_0,\infty)} \vartheta^*(t)E\|u(t)\|^p_\lambda, \quad u \in C\big(t_0, \infty; L^p(\Omega; \mathbb{H}^\lambda)\big),$$

where

$$\vartheta^*(t) = \begin{cases} \alpha(T), & t \in [t_0, T], \\ \alpha(t), & t \geq T, \end{cases}$$

with $T > 0$ given later. Then $\big(C^{p,\lambda}_{\vartheta^*}, \|\cdot\|_{\vartheta^*}\big)$ is a Banach space. Put

$$\widehat{\varrho}(t) = \varrho(t) - u^*(t), \tag{4.55}$$

where $\varrho(t)$ is any solution of problem (1.1)-(1.2) in the sense of Definition 3.1. Define the mapping $\mathcal{T}^*$ by

$$\begin{aligned}(\mathcal{T}^*\widehat{\varrho})(t) &= S(t-t_0)\widehat{\varrho}(t_0) \\ &+ \int_{t_0}^t S(t-\tau)\big(f(\tau, \widehat{\varrho}(\eta\tau) + u^*(\eta\tau)) - f(\tau, u^*(\eta\tau))\big)d\tau \\ &+ \int_{t_0}^t S(t-\tau)\big(g(\tau, \widehat{\varrho}(\eta\tau) + u^*(\eta\tau)) - g(\tau, u^*(\eta\tau))\big)dB_Q(\tau).\end{aligned} \tag{4.56}$$

Now we show that $\mathcal{T}^*$ is contractive and bounded on $C^{p,\lambda}_{\vartheta^*}$.

(I) $\mathcal{T}^*$ is a contraction mapping.

Due to (4.56), we have that for any $\widehat{\varrho_1}, \widehat{\varrho_2} \in C^{p,\lambda}_{\vartheta^*}$,

$$\begin{aligned}&\vartheta^*(t)E\big\|(\mathcal{T}^*\widehat{\varrho_1})(t) - (\mathcal{T}^*\widehat{\varrho_2})(t)\big\|^p_\lambda \\ &\leq 2^{p-1}\vartheta^*(t)E\Big\|\int_{t_0}^t S(t-\tau)\big(f(\tau, \widehat{\varrho_1}(\eta\tau) + u^*(\eta\tau)) - f(\tau, \widehat{\varrho_2}(\eta\tau) + u^*(\eta\tau))\big)d\tau\Big\|^p_\lambda \\ &+ 2^{p-1}\vartheta^*(t)E\Big\|\int_{t_0}^t S(t-\tau)\big(g(\tau, \widehat{\varrho_1}(\eta\tau) + u^*(\eta\tau)) \\ &- g(\tau, \widehat{\varrho_2}(\eta\tau) + u^*(\eta\tau))\big)dB_Q(\tau)\Big\|^p_\lambda \\ &:= \mathcal{R}_{28} + \mathcal{R}_{29}.\end{aligned} \tag{4.57}$$

It follows from the assumptions (S_1)-(S_2), Hölder's inequality and (4.55) that for $t \in [t_0, T]$,

$$\begin{aligned}\mathcal{R}_{28} &\leq 2^{p-1}\alpha(T)C^p_{\lambda,0}E\Big(\int_{t_0}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda} \\ &\quad\times \big\|f(\tau, \widehat{\varrho_1}(\eta\tau) + u^*(\eta\tau)) - f(\tau, \widehat{\varrho_2}(\eta\tau) + u^*(\eta\tau))\big\|d\tau\Big)^p \\ &\leq 2^{p-1}\alpha(T)C^p_{\lambda,0}\Big(\int_{t_0}^t e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}d\tau\Big)^{p-1}\end{aligned}$$

$$\times \int_{t_0}^{t} e^{-\delta(t-\tau)}(t-\tau)^{-\lambda} E\big\|f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))-f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big\|^p d\tau$$
$$\leq 2^{p-1}C_{\lambda,0}^p\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|L_1\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho_1}-\widehat{\varrho_2}\|_{\vartheta^*}. \tag{4.58}$$

For $t \geq T$,

$$\begin{aligned}
\mathcal{R}_{28} &\leq 6^{p-1}\alpha(t)E\Big(\int_{t_0}^{0}\big\|S(t-\tau)\big(f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))\\
&\quad - f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&\quad + 6^{p-1}\alpha(t)E\Big(\int_{0}^{\frac{t}{2}}\big\|S(t-\tau)\big(f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))-f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&\quad + 6^{p-1}\alpha(t)E\Big(\int_{\frac{t}{2}}^{t}\big\|S(t-\tau)\big(f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))-f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big)\big\|_\lambda d\tau\Big)^p\\
&:= \mathcal{R}_{28}^1+\mathcal{R}_{28}^2+\mathcal{R}_{28}^3.
\end{aligned} \tag{4.59}$$

Using again the assumptions (S_1)-(S_2), Hölder's inequality and (4.55), we deduce that

$$\begin{aligned}
\mathcal{R}_{28}^1 &\leq 6^{p-1}C_{\lambda,0}^p\alpha(t)\Big(\int_{t_0}^{0}e^{-\delta(t-\tau)}(t-\tau)^{-\lambda}\\
&\qquad \times\big\|f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))-f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big\|d\tau\Big)^p\\
&\leq 6^{p-1}C_{\lambda,0}^p\alpha(t)t^{-p\lambda}\Big(\int_{t_0}^{0}e^{-\delta(t-\tau)}d\tau\Big)^{p-1}\\
&\qquad \times\int_{t_0}^{0}e^{-\delta(t-\tau)}E\big\|f(\tau,\widehat{\varrho_1}(\eta\tau)+u^*(\eta\tau))-f(\tau,\widehat{\varrho_2}(\eta\tau)+u^*(\eta\tau))\big\|^p d\tau\\
&\leq 6^{p-1}C_{\lambda,0}^p\frac{1}{\delta^{p-1}}\|L_1\|_{L^\infty(\mathbb{R})}\alpha(t)t^{-p\lambda}e^{-p\delta t}\int_{t_0}^{0}e^{\delta\tau}E\big\|\widehat{\varrho_1}(\eta\tau)-\widehat{\varrho_2}(\eta\tau)\big\|_\lambda^p d\tau\\
&\leq 6^{p-1}C_{\lambda,0}^p\frac{(\alpha(T))^{-1}}{\delta^p}\|L_1\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho_1}-\widehat{\varrho_2}\|_{\vartheta^*}\alpha(t)t^{-p\lambda}e^{-p\delta t}.
\end{aligned} \tag{4.60}$$

For terms $\mathcal{R}_{28}^2$ and $\mathcal{R}_{28}^3$, by a similar way as in (3.9) and (3.10), we obtain that

$$\begin{aligned}
\mathcal{R}_{28}^2 &\leq 6^{p-1}\alpha(t)C_{\lambda,0}^p\|\widehat{\varrho_1}-\widehat{\varrho_2}\|_{\vartheta^*}\|L_1\|_{L^\infty(\mathbb{R})}\\
&\quad\times\big(\frac{t}{2}\big)^{-p\lambda}\frac{e^{-\delta pt/2}}{\delta^{p-1}}\int_0^{\frac{t}{2}}e^{-\delta(t/2-\tau)}(\alpha(\eta\tau))^{-1}d\tau,
\end{aligned} \tag{4.61}$$

and

$$\mathcal{R}_{28}^3 \leq 6^{p-1}C_{\lambda,0}^p\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^p\|\widehat{\varrho_1}-\widehat{\varrho_2}\|_{\vartheta^*}\|L_1\|_{L^\infty(\mathbb{R})}\frac{\alpha(t)}{\alpha(\eta t/2)}. \tag{4.62}$$

Hence by (4.58) and (4.59)-(4.62), in view of the assumption (4.30), we can take T sufficiently large such that for any $t \geq t_0$,

$$\mathcal{R}_{28} < \frac{1}{2}\|\widehat{\varrho_1}-\widehat{\varrho_2}\|_{\vartheta^*}. \tag{4.63}$$

Thanks to Lemma 2.1, in view of assumptions (S_1)-(S_2), Hölder's inequality and (4.55), we deduce that for $t \in [t_0, T]$,

$$\begin{aligned}
\mathcal{R}_{29} &\leq 2^{p-1}\alpha(T)C_pE\Big(\int_{t_0}^{t}\big\|A^{\lambda}S(t-\tau)\big(g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))\\
&\qquad - g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big)\big\|_Q^2d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{p-1}\alpha(T)C_pC_{\lambda,0}^pE\Big(\int_{t_0}^{t}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\\
&\qquad\times\big\|g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))-g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big\|_Q^2d\tau\Big)^{\frac{p}{2}}\\
&\leq 2^{p-1}\alpha(T)C_pC_{\lambda,0}^p\Big(\int_{t_0}^{t}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}d\tau\Big)^{\frac{p-2}{2}}\\
&\qquad\times\int_{t_0}^{t}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}L_2(\tau)E\big\|\widehat{\varrho}_1(\eta\tau)-\widehat{\varrho}_2(\eta\tau)\big\|_\lambda^pd\tau\\
&\leq 2^{p-1}C_pC_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|L_2\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}.
\end{aligned}\tag{4.64}$$

We see that for $t \geq T$,

$$\begin{aligned}
&\mathcal{R}_{29} \leq 6^{p-1}\alpha(t)E\Big\|\int_{t_0}^{0}S(t-\tau)\big(g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))\\
&- g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&+6^{p-1}\alpha(t)E\Big\|\int_{0}^{\frac{t}{2}}S(t-\tau)\big(g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))-g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&+6^{p-1}\alpha(t)E\Big\|\int_{\frac{t}{2}}^{t}S(t-\tau)\big(g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))-g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&:= \mathcal{R}_{29}^1+\mathcal{R}_{29}^2+\mathcal{R}_{29}^3.
\end{aligned}\tag{4.65}$$

Using Lemma 2.1, assumptions (S_1)-(S_2), Hölder's inequality and (4.55) again, it follows that

$$\begin{aligned}
\mathcal{R}_{29}^1 &\leq 6^{p-1}\alpha(t)C_pC_{\lambda,0}^pE\Big(\int_{t_0}^{0}e^{-2\delta(t-\tau)}(t-\tau)^{-2\lambda}\\
&\qquad\times\big\|g(\tau,\widehat{\varrho}_1(\eta\tau)+u^*(\eta\tau))-g(\tau,\widehat{\varrho}_2(\eta\tau)+u^*(\eta\tau))\big\|_Q^2d\tau\Big)^{\frac{p}{2}}\\
&\leq 6^{p-1}C_pC_{\lambda,0}^p\alpha(t)t^{-p\lambda}\Big(\int_{t_0}^{0}e^{-2\delta(t-\tau)}d\tau\Big)^{\frac{p-2}{2}}(\alpha(T))^{-1}\\
&\qquad\times\int_{t_0}^{0}e^{-2\delta(t-\tau)}L_2(\tau)\alpha(T)E\big\|\widehat{\varrho}_1(\eta\tau)-\widehat{\varrho}_2(\eta\tau)\big\|_\lambda^pd\tau\\
&\leq 6^{p-1}C_pC_{\lambda,0}^p\frac{(\alpha(T))^{-1}}{(2\delta)^{\frac{p}{2}}}\|L_2\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}t^{-p\lambda}\alpha(t)e^{-p\delta t}.
\end{aligned}\tag{4.66}$$

In a similar way as in (3.14) and (3.15), we arrive at

$$\mathcal{R}_{29}^2 \le 6^{p-1}\alpha(t)C_pC_{\lambda,0}^p\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}\|L_2\|_{L^\infty(\mathbb{R})} \times \big(\frac{t}{2}\big)^{-p\lambda}\frac{e^{-\delta pt/2}}{(2\delta)^{\frac{p-2}{2}}}\int_0^{\frac{t}{2}} e^{-\delta(t-2\tau)}(\alpha(\eta\tau))^{-1}d\tau, \tag{4.67}$$

and

$$\mathcal{R}_{29}^3 \le 6^{p-1}C_pC_{\lambda,0}^p\big((2\delta)^{2\lambda-1}\Gamma(1-2\lambda)\big)^{\frac{p}{2}}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}\|L_2\|_{L^\infty(\mathbb{R})}\frac{\alpha(t)}{\alpha(\eta t/2)}. \tag{4.68}$$

Collecting (4.65)-(4.68) and (4.64) together, in view of assumptions (S_0) and (4.30), we choose T large enough such that for all $t \ge t_0$,

$$\mathcal{R}_{29} < \frac{1}{2}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}. \tag{4.69}$$

Consequently, inserting (4.63) and (4.69) into (4.57) yields that the mapping $\mathcal{T}^*$ is contractive on the space $C_{\vartheta^*}^{p,\lambda}$.

(II) $\mathcal{T}^*$ maps $C_{\vartheta^*}^{p,\lambda}$ into itself.

By (4.56) we derive that for any $\widehat{\varrho} \in C_{\vartheta^*}^{p,\lambda}$,

$$\begin{aligned}
&\vartheta^*(t)E\big\|(\mathcal{T}^*\widehat{\varrho})(t)\big\|_\lambda^p\\
&\quad\le 3^{p-1}\vartheta^*(t)E\big\|S(t-t_0)\big(\varrho(t_0)-u^*(t_0)\big)\big\|_\lambda^p\\
&\quad+3^{p-1}\vartheta^*(t)E\Big\|\int_{t_0}^t S(t-\tau)\big(f(\tau,\widehat{\varrho}(\eta\tau)+u^*(\eta\tau))-f(\tau,u^*(\eta\tau))\big)d\tau\Big\|_\lambda^p\\
&\quad+3^{p-1}\vartheta^*(t)E\Big\|\int_{t_0}^t S(t-\tau)\big(g(\tau,\widehat{\varrho}(\eta\tau)+u^*(\eta\tau))-g(\tau,u^*(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\\
&\quad\le 3^{p-1}C_0^p\vartheta^*(t)e^{-\delta p(t-t_0)}E\|\widehat{\varrho}(t_0)\|_\lambda^p+\mathcal{R}_{30}+\mathcal{R}_{31}.
\end{aligned} \tag{4.70}$$

By similar arguments as in (4.59)-(4.62) and (4.65)-(4.68), we deduce that for $t \ge T$,

$$\begin{aligned}
\mathcal{R}_{30}+\mathcal{R}_{31} &\le C\|\widehat{\varrho}\|_{\vartheta^*}(\alpha(T))^{-1}\alpha(t)t^{-p\lambda}e^{-p\delta t}\\
&\quad+C\|\widehat{\varrho}\|_{\vartheta^*}\big(\frac{t}{2}\big)^{-p\lambda}\alpha(t)e^{-\frac{p\delta t}{2}}\int_0^{\frac{t}{2}} e^{-\delta(\frac{t}{2}-\tau)}(\alpha(\eta\tau))^{-1}d\tau\\
&\quad+C\|\widehat{\varrho}\|_{\vartheta^*}\frac{\alpha(t)}{\alpha(\eta t/2)}.
\end{aligned} \tag{4.71}$$

This implies that $\mathcal{R}_{30}+\mathcal{R}_{31} \le \mathbb{C}\|\widehat{\varrho}\|_{\vartheta^*}$ when T is sufficiently large. Following similar computations as in (4.58) and (4.64), we obtain that $\mathcal{R}_{30}+\mathcal{R}_{31} \le \mathbb{C}\|\widehat{\varrho}\|_{\vartheta^*}$ for any $t \in [t_0, T]$. Therefore, the desired assertion follows immediately by the Banach fixed point theorem. □

Remark 4.1. For the case $t_0 \ge 0$ in Step 3, in order to deal with the difficulty caused by pantograph delay, we shall consider the phase space $C_{\vartheta^*_\eta}^{p,\lambda} = C_{\vartheta^*}\big(\eta t_0,\infty; L^p(\Omega;\mathbb{H}^\lambda)\big)$ with the norm

$$\|u\|_{\vartheta^*_\eta} = \sup_{t\in[\eta t_0,\infty)} \vartheta^*_\eta(t)E\|u(t)\|_\lambda^p, \quad u \in C\big(\eta t_0,\infty; L^p(\Omega;\mathbb{H}^\lambda)\big),$$

and define the mapping $\mathcal{T}_\eta^*$ by

$$(\mathcal{T}_\eta^* \widehat{\varrho})(t) = \begin{cases} S(t-t_0)\widehat{\varrho}(t_0) + \int_{t_0}^t S(t-\tau)\big(f(\tau, \widehat{\varrho}(\eta\tau) + u^*(\eta\tau)) \\ \quad - f(\tau, u^*(\eta\tau))\big)d\tau + \int_{t_0}^t S(t-\tau)\big(g(\tau, \widehat{\varrho}(\eta\tau) + u^*(\eta\tau)) \\ \quad - g(\tau, u^*(\eta\tau))\big)dB_Q(\tau), \quad t \geq t_0, \\ \widehat{\varrho}(t), \quad t \in [\eta t_0, t_0], \end{cases} \tag{4.72}$$

where $\widehat{\varrho}(t)$ and $\varrho(t)$ are given in Theorem 4.2, and

$$\vartheta_\eta^*(t) = \begin{cases} \alpha(T), & t \in [\eta t_0, T], \\ \alpha(t), & t \geq T. \end{cases}$$

The following result gives almost surely α-type stability of u^* to problem (4.1).

Theorem 4.3. *Let $\gamma \in (0,1)$. Suppose that all the assumptions of Theorem 4.2 are satisfied. Then the solution u^* of Eq. (4.1) defined on $\mathbb{R}$ is almost surely α-type stable, that is,*

$$\lim_{t\to\infty} \frac{\log \|u^*(t) - \varrho(t)\|_\lambda}{\log \alpha(t)} < -\frac{1-\gamma}{p}, \qquad a.s. \tag{4.73}$$

where $\varrho(t)$ is any solution of problem (1.1)-(1.2) in the sense of Definition 3.1.

Proof. The proofs are still concerned with the case any $t_0 < 0$. We find that for $N \geq 1$ and any given $s > 0 > t_0$,

$$\begin{aligned} u^*(t) - \varrho(t) = {} & S\big(t - \eta^{-(N-1)}s\big)\big(u^*(\eta^{-(N-1)}s) - \varrho(\eta^{-(N-1)}s)\big) \\ & + \int_{\eta^{-(N-1)}s}^t S(t-\tau)\big(f(\tau, u^*(\eta\tau)) - f(\tau, \varrho(\eta\tau))\big)d\tau \\ & + \int_{\eta^{-(N-1)}s}^t S(t-\tau)\big(g(\tau, u^*(\eta\tau)) - g(\tau, \varrho(\eta\tau))\big)dB_Q(\tau). \end{aligned} \tag{4.74}$$

Thanks to Markov's inequality, by a similar way as in (3.29), (3.30) and (3.32), we derive that

$$\begin{aligned} & \mathbb{P}\Big(\sup_{\eta^{-N}s \leq t \leq \eta^{-(N+1)}s} \|u^*(t) - \varrho(t)\|_\lambda^p \geq \big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big) \\ & \leq \big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma} E\big(\sup_{\eta^{-N}s \leq t \leq \eta^{-(N+1)}s} \|u^*(t) - \varrho(t)\|_\lambda^p\big) \\ & \leq 3^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma} E\big(\sup_{\eta^{-N}s \leq t \leq \eta^{-(N+1)}s} \big\|S\big(t - \eta^{-(N-1)}s\big)\big(u^*(\eta^{-(N-1)}s) \\ & \qquad - \varrho(\eta^{-(N-1)}s)\big)\big\|_\lambda^p\big) \\ & \quad + 3^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma} E\Big(\sup_{\eta^{-N}s \leq t \leq \eta^{-(N+1)}s} \Big\| \int_{\eta^{-(N-1)}s}^t S(t-\tau)\big(f(\tau, u^*(\eta\tau)) \end{aligned}$$

$$
\begin{aligned}
&\quad - f(\tau, \varrho(\eta\tau))\big)d\tau\Big\|_\lambda^p\Big) \\
&\quad + 3^{p-1}\big(\alpha(\eta^{-(N-1)}s)\big)^{1-\gamma}E\Big(\sup_{\eta^{-N}s\le t\le\eta^{-(N+1)}s}\Big\|\int_{\eta^{-(N-1)}s}^t S(t-\tau)\big(g(\tau,u^*(\eta\tau)) \\
&\quad - g(\tau,\varrho(\eta\tau))\big)dB_Q(\tau)\Big\|_\lambda^p\Big) \\
&\le \mathbb{C}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\big(\alpha(\eta^{-(N-1)}s)\big)E\|u^*(\eta^{-(N-1)}s)-\varrho(\eta^{-(N-1)}s)\|_\lambda^p \\
&\quad + \mathbb{C}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}\sup_{\tau\in[0,\infty)}\alpha(\tau)/\alpha(\eta\tau)\sup_{\tau\in[t_0,\infty)}\alpha(\tau)E\|u^*(\tau)-\varrho(\tau)\|_\lambda^p. \quad (4.75)
\end{aligned}
$$

This, together with (4.31) given in Theorem 4.2 and assumption (S_0), yields

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{\eta^{-N}s\le t\le\eta^{-(N+1)}s}\|u^*(t)-\varrho(t)\|_\lambda^p \ge \big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big) \\
&\quad \le \mathbb{C}\big(\alpha(\eta^{-(N-1)}s)\big)^{-\gamma}, \quad (4.76)
\end{aligned}
$$

where $\mathbb{C}$ is a positive constant independent of η, s. Hence,

$$
\sum_{N=1}^{\infty}\mathbb{P}\Big(\sup_{\eta^{-N}s\le t\le\eta^{-(N+1)}s}\|u^*(t)-\varrho(t)\|_\lambda^p \ge \big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)}\Big) < \infty. \quad (4.77)
$$

Thanks to the Borel-Cantelli lemma, we conclude that there exists $\widetilde{\Omega} \subset \Omega$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that for any $\omega \in \widetilde{\Omega}$, there exists an integer $\widetilde{N} = \widetilde{N}(\omega) > 0$, when $N \ge \widetilde{N}$ and $\eta^{-N}s \le t \le \eta^{-(N+1)}s$,

$$
\|u^*(t)-\varrho(t)\|_\lambda^p \le \big(\alpha(\eta^{-(N-1)}s)\big)^{-(1-\gamma)},
$$

which implies that assertion (4.73) holds. □

5. Conclusions

In this work we studied stochastic evolution equations with pantograph delay and nonlinear multiplicative noise. First we established the pth moment general decay stability and almost sure general decay stability (including both polynomial and logarithmic rates) of mild solutions. We then constructed the nontrivial equilibrium solution defined for $t \in \mathbb{R}$ by using the generalized version of the factorization formula and an approximation technique. Moreover, we established the Hölder regularity and general stability of the nontrivial equilibrium solution in the pth moment and almost sure senses. It is worth mentioning that the stability analysis here is based on the Banach fixed point theorem and various estimates involving the gamma function, which are quite different from the Razumikhin technique and Lyapunov functions usually used in stochastic differential equations with pantograph delay. One technical challenge is that the presence of pantograph delay makes the analysis more complicated. Another highlight of the work is the construction and the general stability of the nontrivial equilibrium solution, which can be used to study stochastic evolution equations with discrete and distributed delays in the bounded and unbounded cases.

6. Appendix

Proof. [**Proof of Theorem 2.1**] Thanks to condition (2.4), it follows that

$$\begin{aligned}
&\frac{\sin\alpha^*\pi}{\pi}\int_{t_0}^t (t-s)^{\alpha^*-1}S(t-s)Y_{\alpha^*}(s)ds \\
&= \frac{\sin\alpha^*\pi}{\pi}\int_{t_0}^t (t-s)^{\alpha^*-1}S(t-s)\int_{t_0}^s (s-r)^{-\alpha^*}S(s-r)\phi(r)dB_Q(r)ds \\
&= \frac{\sin\alpha^*\pi}{\pi}\int_{t_0}^t \Big[\int_r^t (t-s)^{\alpha^*-1}(s-r)^{-\alpha^*}ds\Big]S(t-r)\phi(r)dB_Q(r).
\end{aligned} \tag{6.1}$$

Since

$$\int_r^t (t-s)^{\alpha^*-1}(s-r)^{-\alpha^*}ds = \frac{\pi}{\sin\alpha^*\pi}, \qquad t_0 \le r \le t, \quad \alpha^* \in (0,1), \tag{6.2}$$

the result follows. □

Proposition 6.1. *Assume that* $\jmath_0 > 1$, $\jmath_1 \ge 0$, $\delta > 0$, $\alpha^* > \frac{1}{\jmath_0} + \jmath_1$, $1-\alpha^* < \varsigma < 1 - (1/\jmath_0 + \jmath_1)$ *and that* E_1, E_2 *are Banach spaces such that for* $t \in [0,T]$ *and* $x \in E_2$,

$$\|S(t)x\|_{E_1} \le Ct^{-\jmath_1}e^{-\delta t}\|x\|_{E_2}, \qquad \|S(t)x - x\|_{E_2} \le Ct^{\varsigma}\|A^{\varsigma}x\|_{E_2}. \tag{6.3}$$

Then G_{α^*} *defined by*

$$G_{\alpha^*}f(t) = \int_{-\infty}^t (t-s)^{\alpha^*-1}S(t-s)f(s)ds, \qquad t \in [-\infty, T], \tag{6.4}$$

is a bounded linear operator from $L^{\jmath_0}(-\infty, T; E_2) =: L^{\jmath_0}$ *into* $C([-\infty, T]; E_1)$.

Proof. Following the similar arguments of [3, Proposition 5.9], in view of the conditions (6.3) and $\alpha^* > \frac{1}{\jmath_0} + \jmath_1$, we deduce that

$$\begin{aligned}
&\Big\|\int_{-\infty}^t (t-s)^{\alpha^*-1}S(t-s)f(s)ds\Big\|_{E_1} \le C\int_{-\infty}^t (t-s)^{\alpha^*-1-\jmath_1}e^{-\delta(t-s)}\|f\|_{E_2}ds \\
&\le C\Big(\int_{-\infty}^t (t-s)^{(\alpha^*-1-\jmath_1)\frac{\jmath_0}{\jmath_0-1}}e^{-\frac{\jmath_0\delta(t-s)}{\jmath_0-1}}ds\Big)^{\frac{\jmath_0-1}{\jmath_0}}\|f\|_{L^{\jmath_0}},
\end{aligned} \tag{6.5}$$

which means

$$\sup_{-\infty \le t \le T}\|G_{\alpha^*}f(t)\|_{E_1} \le C\|f\|_{L^{\jmath_0}}. \tag{6.6}$$

By (6.4), we have

$$\begin{aligned}
&\|G_{\alpha^*}f(t) - G_{\alpha^*}f(s)\|_{E_1} \\
&\le \Big\|\int_{-\infty}^s (t-\sigma)^{\alpha^*-1}S(t-\sigma)f(\sigma)d\sigma - \int_{-\infty}^s (s-\sigma)^{\alpha^*-1}S(s-\sigma)f(\sigma)d\sigma\Big\|_{E_1}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Big\| \int_s^t (t-\sigma)^{\alpha^*-1} S(t-\sigma) f(\sigma) d\sigma \Big\|_{E_1} \\
&\le \int_{-\infty}^s (t-\sigma)^{\alpha^*-1} \big\| \big(S(t-s)-I\big) S(s-\sigma) f(\sigma) \big\|_{E_1} d\sigma \\
&\quad + \int_{-\infty}^s \big((t-\sigma)^{\alpha^*-1} - (s-\sigma)^{\alpha^*-1}\big) \big\| S(s-\sigma) f(\sigma) \big\|_{E_1} d\sigma \\
&\quad + \int_s^t (t-\sigma)^{\alpha^*-1} \big\| S(t-\sigma) f(\sigma) \big\|_{E_1} d\sigma \\
&:= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \qquad (6.7)
\end{aligned}
$$

It follows from (6.3) and Hölder's inequality that

$$
\begin{aligned}
\mathcal{A}_1 &\le C \int_{-\infty}^s (t-\sigma)^{\alpha^*-1} (t-s)^{\varsigma} \Big\| A^{\varsigma} S\big(\frac{s-\sigma}{2}\big) S\big(\frac{s-\sigma}{2}\big) f(\sigma) \Big\|_{E_1} d\sigma \\
&\le C(t-s)^{\varsigma} \int_{-\infty}^s (t-\sigma)^{\alpha^*-1} e^{-\frac{\delta(s-\sigma)}{2}} \big(\frac{s-\sigma}{2}\big)^{-\varsigma} e^{-\frac{\delta(s-\sigma)}{2}} \big(\frac{s-\sigma}{2}\big)^{-\jmath_1} \|f(\sigma)\|_{E_2} d\sigma \\
&\le C(t-s)^{\varsigma+\alpha^*-1} \Big(\int_{-\infty}^s (s-\sigma)^{-\frac{\jmath_0(\jmath_1+\varsigma)}{\jmath_0-1}} e^{-\frac{\jmath_0 \delta(s-\sigma)}{\jmath_0-1}} d\sigma \Big)^{\frac{\jmath_0-1}{\jmath_0}} \Big(\int_{-\infty}^s \|f(\sigma)\|_{E_2}^{\jmath_0} d\sigma \Big)^{\frac{1}{\jmath_0}} \\
&\le C(t-s)^{\varsigma+\alpha^*-1} \|f\|_{L^{\jmath_0}(-\infty,T;E_2)}. \qquad (6.8)
\end{aligned}
$$

Noting that $1-\alpha^* < \varsigma < 1-(1/\jmath_0+\jmath_1)$, we have $\mathcal{A}_1 \to 0$ as $t \to s$.
For $\mathcal{A}_2$, in view of (6.3), we arrive at

$$
\mathcal{A}_2 \le C \int_{-\infty}^s \big((t-\sigma)^{\alpha^*-1} - (s-\sigma)^{\alpha^*-1}\big)(s-\sigma)^{-\jmath_1} \|f(\sigma)\|_{E_2} d\sigma. \qquad (6.9)
$$

Applying the dominated convergence theorem to (6.9) results in $\mathcal{A}_2 \to 0$ as $t \to s$.
For $\mathcal{A}_3$, by (6.3) and Hölder's inequality we obtain

$$
\begin{aligned}
\mathcal{A}_3 &\le C \int_s^t (t-\sigma)^{\alpha^*-1-\jmath_1} \|f(\sigma)\|_{E_2} d\sigma \\
&\le C(t-s)^{1+\frac{\jmath_0(\alpha^*-1-\jmath_1)}{\jmath_0-1}} \|f\|_{L^{\jmath_0}(-\infty,T;E_2)} \to 0 \quad \text{as} \quad t \to s.
\end{aligned} \qquad (6.10)
$$

□

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