Stability analysis of stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay

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Abstract

The asymptotic behaviour of stochastic three-dimensional Lagrangian-averaged Navier-Stokes equations with infinite delay and nonlinear hereditary noise is analysed. First, using Galerkin's approximations and the monotonicity method, we prove the existence and uniqueness of solutions when the non-delayed external force is locally integrable and the delay terms are globally Lipschitz continuous with an additional assumption. Next, we show the existence and uniqueness of stationary solutions to the corresponding deterministic equation via the Lax-Milgram and the Schauder theorems. Later, we focus on the stability properties of stationary solutions. To begin with, we discuss the local stability of stationary solutions for general delay terms by using a direct method and then apply the abstract results to two kinds of infinite delays. Besides, the exponential stability of stationary solutions is also established in the case of unbounded distributed delay. Moreover, we investigate the asymptotic stability of stationary solutions in the case of unbounded variable delay by constructing appropriate Lyapunov functionals. Eventually, we establish criteria on the polynomial asymptotic stability of stationary solutions for the special case of proportional delay.

Keywords: `Stochastic three-dimensional Lagrangian-averaged Navier-Stokes equations; Stationary solutions; Exponential convergence; Polynomial asymptotic stability; Infinite delay`
2020 MSC: 35Q30, 35A01, 35B35

1. Introduction

Navier-Stokes equations are known to model the motion of many important fluids which include water, air, oil, etc. Motivated by the first published paper of Leray [29], the authors in references [3, 6, 7, 9–18, 20, 23–25, 27, 31–35, 37, 40] have extensively studied the well-posedness and long-time behavior of solutions to Navier-Stokes equations.

Due to the importance of delay effects in many physical, biological and engineering models [1, 8, 42], delay differential equations have received much attention over the recent years. Besides, the future state of systems may not only depend on their current state, but also on their past history, which plays a nontrivial role in some cases. It is worth mentioning some typical examples, such as the investigation of high-viscosity liquids under the condition of low temperatures, the thermomechanical analysis with respect to polymers, population models, etc (see [22, 36] and the references therein). Thus, we need to take into account some hereditary characteristics such as aftereffect, time lag, memory and time delay in our models.

Many researchers have focused on the relationship between the Navier-Stokes equations and the phenomenon of turbulence for a long time. It is worth stressing that the common assumption relates the onset of turbulence to the randomness of background movement. Moreover, the systems we study are affected by a variety of random factors in real life, so it is necessary to consider some kind of noise in our models. The concept of random dynamical systems was first introduced by Ulam and von Neumann [41] in 1945. Due to the fact that stochastic differential equations originated from random dynamical systems, a growing number of people have studied random dynamical systems since 1980s. Amongst the many notable results, it is remarkable that the importance of the work in [4], Bensoussan and Temam in this article discussed the stochastic Navier-Stokes equations driven by white noise and random forces, providing a more realistic model to solve the problem.

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Based on the previous discussion, we investigate in this paper the stochastic dynamics of the following non-autonomous stochastic three-dimensional Lagrangian-averaged Navier-Stokes (LANS) equation with infinite delay and nonlinear hereditary noise:

$$\begin{cases} \partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u \cdot \nabla)(u - \alpha\Delta u) \\ -\alpha\nabla u^* \cdot \Delta u + \nabla p = f(t) + g_1(t, u_t) + g_2(t, u_t)\dot{W}, \quad \text{in} \quad (\tau, +\infty) \times \mathcal{O}, \\ \nabla \cdot u = 0, \quad \text{in} \quad (\tau, +\infty) \times \mathcal{O}, \\ u = 0, \ Au = 0 \text{ on } (\tau, +\infty) \times \partial\mathcal{O}, \\ u(\tau + s, x) = \phi(s, x), s \in (-\infty, 0], x \in \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subseteq \mathbb{R}^3$ is a bounded open set with sufficiently regular boundary $\partial\mathcal{O}, \tau \in \mathbb{R}$, A is the Stokes operator, the pair (ν, α) of positive coefficients denotes the kinematic viscosity of the fluid and the square of the spatial scale at which fluid motion is filtered respectively, the symbol $*$ denotes the transpose of a matrix, $u = (u_1, u_2, u_3)$ is the averaged (or large-scale) velocity of the fluid, u_t denotes the segment of solutions up to time t, i.e. $u_t(s) = u(t + s)$ for all $s \leq 0$, p is the pressure of the fluid, f is a non-delayed external force field, the terms g_1, g_2 contain some hereditary characteristics, such as memory, unbounded variable or infinite distributed delay, etc, ϕ is an initial velocity field defined in $(-\infty, 0] \times \mathcal{O}$, and $\dot{W}$ denotes the generalized derivative (white noise) of a cylindrical Wiener process, which will be specified later. The LANS model was first used to deal with the turbulence closure problem by using the method of Lagrangian averaging. The study of such models, including the well-posedness of solutions and the existence, uniqueness and asymptotic stability of stationary solutions, has been carried out in some papers, e.g., [12–14, 16] and the references therein. In the special case that $\alpha = 0$, the system (1.1) is reduced to the usual three-dimensional Navier-Stokes equation whose dynamical behavior had been widely investigated in [9, 15, 17, 18, 25, 27, 37]. The deterministic and non-delay version of (1.1), i.e. $g_2 = 0$ and g_1 is independent of u, has been considered in [20, 35]. In the stochastic case without delay, Caraballo et al. in [16] studied the stochastic dynamics of such a system for the first time.

In addition, the delay version of Navier-Stokes equations has also received much attention over the last years. The analysis of Navier-Stokes equations with some hereditary features was first studied by Caraballo and Real in [15] and developed in [9, 12, 14, 23–25, 30–34]. On the one hand, for bounded delay, the authors have discussed several issues including the existence, uniqueness, asymptotic behavior and regularity of solutions, the existence, uniqueness and stability of stationary solutions, the existence, uniqueness of global or pullback (random) attractors. On the other hand, the case of unbounded or infinite delays has been analyzed in [24, 26, 30–34].

We may choose several phase spaces for dealing with the infinite delay as given in above references. The first one is the Banach space

$$C_\gamma(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{s \to -\infty} e^{\gamma s}\varphi(s) \text{ exists in } H\}, \text{ where } \gamma > 0, \tag{1.2}$$

where H is the 3D Lebesgue-type Hilbert space. The second one is

$$C_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{s \to -\infty} \varphi(s) \text{ exists in } H\}, \tag{1.3}$$

see [30, 31]. We also use $C_\gamma(V)$ and $C_{-\infty}(V)$, where V is the Sobolev-type subspace instead of H in (1.2) and (1.3).

Our first goal in this paper is to prove the existence and uniqueness of solutions to the stochastic three dimensional LANS Eq. (1.1) in the Banach space $C_\gamma(V)$. As done by Liu and Caraballo [31] for the usual two-dimensional system, we need to assume that the non-delayed external force f is locally integrable (see **Hypothesis F**) and the delay forcing terms $g_i(t, u_t)(i = 1, 2)$ are globally Lipschitz continuous (see **Hypothesis G**). An example is given in the last part of Section 2. The calculation shows that the example (corresponding to infinite distributed delay) satisfies all conditions of **Hypothesis G** in the space $C_\gamma(V)$ for $\gamma > 0$. Besides, we also need an extra assumption on the nonlinear diffusion term g_2 of noise (see **Hypothesis I**).

Under the above assumptions, we use the Galerkin method to construct an approximating sequence. We then give a priori estimates for the approximating sequence ensuring the solutions exist for the whole time interval $[\tau, \tau + T]$ for all $T > 0$. Next, by the monotonicity method established in [39], we obtain the well-posedness of solutions.

Another interesting and challenging topic is to consider the asymptotic behaviour of solutions for Eq. (1.1) towards to the stationary solution. This issue will provide some useful information on future evolution of the system. Thanks to the Lax-Milgram and the Schauder theorems, we first prove the existence and uniqueness of stationary solutions to the corresponding deterministic equation. We then show the local stability of the stationary solution for the general delay term by using a direct method, where the general delay contains the unbounded variable delay and the infinite distributed delay in $C_{-\infty}(V)$. Next, we prove the global stability of the stationary solution. However, to obtain stability results in $C_\gamma(V)$ with $\gamma > 0$, the exponential stability in the case of unbounded variable delay fails to be proved in general (see [31] and [34] for more details). Fortunately, in the case of infinite distributed delay, we are able to prove not only stability of stationary solutions in $C_\gamma(V)$, but also exponential asymptotic stability. Since we can not analyze the exponential stability in the unbounded variable delay case in $C_\gamma(V)$, we will explore, at least, the asymptotic stability in $C_{-\infty}(V)$, by using the Lyapunov functionals construction proposed by Kolmanovskii and Shaikhet [28]. Furthermore, we eventually discuss the polynomial asymptotic stability in the particular case of proportional delay (also known as pantograph delay).

The article is organized as follows. In Section 2, we introduce the cylindrical Wiener process, some notations and linear operators, describe some suitable assumptions about the non-delayed external force f and delay terms $g_i (i = 1, 2)$. In Section 3, we prove the existence and uniqueness of solutions to system (1.1). In Section 4, stationary solutions and their stability results are established. More precisely, on the one hand, we provide some sufficient conditions ensuring the existence and uniqueness of stationary solutions. On the other hand, we further study the convergence of stationary solutions including local stability, exponential stability, asymptotic stability via Lyapunov method and polynomial asymptotic stability.

2. Preliminaries

2.1. The cylindrical Wiener process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ be a complete filtered probability space such that $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub σ-algebras of $\mathcal{F}$, which contains all P-null sets, and further $\mathcal{F}_t = \mathcal{F}_0$ for all $t \le 0$.

Let $\{\beta_t^j, t \ge 0, j = 1, 2, 3, \dots\}$ be a sequence of mutually independent standard real valued $\mathcal{F}_t$-Wiener processes and K a separable Hilbert space with an orthonormal basis $\{e_j;\ j = 1, 2, 3, \dots\}$. Suppose that $\{W(t); t \ge 0\}$ be a K-valued cylindrical Wiener process (with the covariance operator $Q : K \to K$) given by

$$W(t) = \sum_{j=1}^{\infty} \beta^j(t) e_j, \quad t \ge 0. \tag{2.1}$$

Given a separable Hilbert space H_0, we denote by $\mathcal{L}^2(K, H_0)$ the space of Hilbert-Schmidt operators from K into H_0 with following norm

$$\|S\|^2_{\mathcal{L}^2(K,H_0)} = \mathrm{tr}(SQS^*),\ \forall S \in \mathcal{L}^2(K, H_0), \tag{2.2}$$

where tr denotes the trace of an operator and S^* is the adjoint operator of S.

For any separable Banach space X, interval $(a, b) \subset \mathbb{R}$ and $p \ge 1$, we denote by $I^p(a, b; X)$ the Banach space of all processes $\varphi \in L^p(\Omega\times(a, b), \mathcal{F}\otimes\mathcal{B}((a, b)), dP\otimes dt; X)$ such that $\varphi(t)$ is $\mathcal{F}_t$-progressively measurable for a.e. $t \in (a, b)$, where $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra. We also denote by $L^p(\Omega, \mathcal{F}, dP; C(a, b; X))$ with $p \ge 1$ the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes φ such that $\mathbb{E}(\sup_{a\le t\le b} \|\varphi(t)\|^p_X) < \infty$, where $C(a, b; X)$ is the Banach space of all continuous functions from $[a, b]$ into X. For convenience, we write $L^p(\Omega, \mathcal{F}, dP; C(a, b; X))$ as $L^p(\Omega; C(a, b; X))$.

For any positive constant $T > 0$, process $\Phi \in I^2(\tau, \tau + T; \mathcal{L}^2(K, H_0))$ and $t \in [\tau, \tau + T]$, the stochastic integral $\int_\tau^t \Phi(s)dW(s)$ is defined by the unique continuous H_0-valued $\mathcal{F}_t$-martingale such that

$$\Big(\int_\tau^t \Phi(s)dW(s), w\Big)_{H_0} = \sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s)e_j, w\Big)_{H_0} d\beta^j(s),\ \forall w \in H_0, \tag{2.3}$$

where the integral with respect to $\beta^j(s)$ is the Ito integral. By [21], if $\Phi \in I^2(\tau, \tau + T; \mathcal{L}^2(K, H_0))$ and $\phi \in L^2(\Omega, L^\infty(\tau, \tau + T; H_0))$ is $\mathcal{F}_t$-progressively measurable, then

$$\sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s)e_j, \phi(s)\Big)_{H_0} d\beta^j(s) =: \int_\tau^t \Big(\Phi(s)dW(s), \phi(s)\Big),\ \tau \le t \le \tau + T,$$

converges in $L^1(\Omega, C(\tau, \tau + T))$.

Recall that given a function $\phi : (-\infty, \tau + T] \to X$, for each $t \in (\tau, \tau + T)$, the segment ϕ_t of ϕ is defined by

$$\phi_t(s) = \phi(t + s),\ \forall\ s \in (-\infty, 0]. \tag{2.4}$$

2.2. Notations and hypotheses

In this subsection, we introduce some notations and linear operators, recall some properties with respect to the nonlinear term $(u\cdot\nabla)(u-\alpha\Delta u)-\alpha\nabla u^*\cdot\Delta u$ in the problem (1.1), and impose some suitable assumptions.

Denote by $\mathbb{L}^2(\mathcal{O}):=(L^2(\mathcal{O}))^3$, $\mathbb{H}_0^1(\mathcal{O}):=(H_0^1(\mathcal{O}))^3$, $\mathbb{C}_0^\infty(\mathcal{O})=(C_0^\infty(\mathcal{O}))^3$ and

$$\mathcal{V}=\{u\in\mathbb{C}_0^\infty(\mathcal{O}):\nabla\cdot u=0\ \text{ in }\ \mathcal{O}\}. \tag{2.5}$$

Let H be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$. Then H is a Hilbert space with the inner product and norm

$$(u,v)=\sum_{j=1}^3\int_{\mathcal{O}}u_j(x)v_j(x)dx,\ |u|^2=(u,u),\ \forall u,v\in H. \tag{2.6}$$

Let V be the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\mathcal{O})$. Then V is a Hilbert space with the inner product

$$((u,v))=(u,v)+\alpha(\nabla u,\nabla v)=(u,v)+\alpha\sum_{i,j=1}^3\int_{\mathcal{O}}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx,\ \forall u,v\in V, \tag{2.7}$$

and the norm $\|u\|^2=((u,u))$. We have $V\subset H\subset V^*$, where V^* is the dual space of V, the injections are dense, continuous and compact.

Denote by $\mathcal{P}$ the Leray projector from $\mathbb{L}^2(\mathcal{O})$ onto H and define the Stokes operator A by

$$Aw=-\mathcal{P}(\Delta w),\ \ \forall\ w\in D(A)=\mathbb{H}^2(\mathcal{O})\cap V, \tag{2.8}$$

where $\mathbb{H}^2(\mathcal{O})=(H^2(\mathcal{O}))^3$. We deduce

$$(Au,v)=(\nabla u,\nabla v),\ \ \|u\|_{\mathbb{H}^2(\mathcal{O})}\le C_1|Au|,\ \ \forall u\in D(A),v\in V, \tag{2.9}$$

where C_1 is a positive constant. In particular, $D(A)$ is a Hilbert space.

Denote by $\langle\cdot,\cdot\rangle$ the duality product between $(D(A))^*$ and $D(A)$, and define a continuous linear operator $\widetilde{A}\in\mathcal{L}(D(A),(D(A))^*)$ by

$$\langle\widetilde{A}u,v\rangle=\nu(Au,v)+\nu\alpha(Au,Av),\ \ \forall u,v\in D(A)=:D(\widetilde{A}). \tag{2.10}$$

It is well-known that the Stokes operator A has a sequence $\{\lambda_k:k\in\mathbb{N}\}$ of eigenvalues satisfying

$$0<\inf_{v\in V\setminus\{0\}}\frac{\|v\|^2}{|v|^2}=\lambda_1\le\lambda_2\le\cdots,\ \ \lambda_k\to\infty \tag{2.11}$$

and a sequence $\{\xi_k\in D(A):k\in\mathbb{N}\}$ of eigenvectors which is orthonormal in H. From (2.10) we have

$$\langle\widetilde{A}\xi_k,v\rangle=\nu\lambda_k((\xi_k,v)) \tag{2.12}$$

and the eigenvalues of the operator $\widetilde{A}$ are given by $\widetilde{\lambda}_k:=\nu\lambda_k$. By (2.10)-(2.12), the operator $\widetilde{A}\in\mathcal{L}(D(A),(D(A))^*)$ satisfies the following conditions:

(A1) $\widetilde{A}$ is self-adjoint;

(A2) For all $u\in D(A)$, $2\langle\widetilde{A}u,u\rangle\ge\widetilde{\alpha}(Au,Au)$, where $\widetilde{\alpha}=2\nu\alpha$;

(A3) $\widetilde{A}\xi_k=\widetilde{\lambda}_k\xi_k$ with $\widetilde{\lambda}_k=\nu\lambda_k$.

As in [14], we associate another inner product on $D(A)=D(\widetilde{A})$, defined by

$$(u,v)_{D(A)}:=\langle\widetilde{A}u,v\rangle,\ \text{and so}\ \ \widetilde{\lambda}_1\|u\|^2\le\|u\|^2_{D(A)},\ \forall u,v\in D(A). \tag{2.13}$$

By (2.9), the above is equivalent to the original inner product $((u,v))+(Au,Av)$ for $u,v\in D(A)$.

For $u\in D(A)$ and $v\in\mathbb{L}^2(\mathcal{O})$, we regard $(u\cdot\nabla)v$ as the element of $(H^{-1}(\mathcal{O}))^3=:\mathbb{H}^{-1}(\mathcal{O})$ given by

$$\langle(u\cdot\nabla)v,w\rangle_{-1}=\sum_{i,j=1}^3\langle\partial_iv_j,u_iw_j\rangle_{-1},\ \ \forall\ w\in\mathbb{H}_0^1(\mathcal{O}), \tag{2.14}$$

where $\langle \cdot, \cdot \rangle_{-1}$ denotes the duality product between $\mathbb{H}^{-1}(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$ or between $H^{-1}(\mathcal{O})$ and $H_0^1(\mathcal{O})$, and $u_i w_j \in H_0^1(\mathcal{O})$ due to the continuous injections of $H^2(\mathcal{O}) \subset L^\infty(\mathcal{O})$ and $H_0^1(\mathcal{O}) \subset L^6(\mathcal{O})$. Hence, there exists a positive constant $C_2 := C_2(\mathcal{O})$ such that

$$|\langle (u \cdot \nabla)v, w \rangle_{-1}| \leq C_2 |Au||v|\|w\|, \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}). \tag{2.15}$$

If $u \in D(A)$, then $\nabla u^* \in (\mathbb{H}^1(\mathcal{O}))^3 \subset (\mathbb{L}^6(\mathcal{O}))^3$, where $\mathbb{H}^1(\mathcal{O}) = (H^1(\mathcal{O}))^3$ and $\mathbb{L}^6(\mathcal{O}) = (L^6(\mathcal{O}))^3$. For all $v \in \mathbb{L}^2(\mathcal{O})$, we have $\nabla u^* \cdot v \in (L^{3/2}(\mathcal{O}))^3 \subset \mathbb{H}^{-1}(\mathcal{O})$ satisfying

$$\langle \nabla u^* \cdot v, w \rangle_{-1} = \sum_{i,j=1}^{3} \int_{\mathcal{O}} (\partial_j u_i) v_i w_j dx, \quad \forall\ w \in \mathbb{H}_0^1(\mathcal{O}), \tag{2.16}$$

which implies that there exists a positive constant $C_3 := C_3(\mathcal{O})$ such that

$$|\langle \nabla u^* \cdot v, w \rangle_{-1}| \leq C_3 |Au||v|\|w\|, \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}). \tag{2.17}$$

Now, we introduce the trilinear operator as follows:

$$b^{\#}(u, v, w) = \langle (u \cdot \nabla)v, w \rangle_{-1} + \langle \nabla u^* \cdot v, w \rangle_{-1}, \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}). \tag{2.18}$$

By [16, Proposition 2.2], we obtain

$$b^{\#}(u, v, w) = -b^{\#}(w, v, u), \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A), \tag{2.19}$$

which implies that $b^{\#}(u, v, u) = 0, \forall\ (u, v) \in D(A) \times \mathbb{L}^2(\mathcal{O})$. Moreover, there exists a positive constant $c^{\#} := c^{\#}(\mathcal{O})$ such that

$$|b^{\#}(u, v, w)| \leq c^{\#} |Au||v|\|w\|, \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}), \tag{2.20}$$

and

$$|b^{\#}(u, v, w)| \leq c^{\#} \|u\||v||Aw|, \quad \forall\ (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A). \tag{2.21}$$

We then define a bilinear mapping $\widetilde{B} : D(A) \times D(A) \to (D(A))^*$, denoted by

$$\langle \widetilde{B}(u, v), w \rangle = b^{\#}(u, v - \alpha \Delta v, w), \quad \forall\ (u, v, w) \in D(A) \times D(A) \times D(A), \tag{2.22}$$

and $\widetilde{B}(u) := \widetilde{B}(u, u)$ for all $u \in D(A)$. By the definition and properties of $b^{\#}$, we find that there exists a positive constant $\widetilde{c} := \widetilde{c}(\mathcal{O})$ such that

(B1) $\langle \widetilde{B}(u, v), u \rangle = 0$ and $\langle \widetilde{B}(u), v \rangle = -\langle \widetilde{B}(v, u), u \rangle, \quad \forall\ (u, v) \in D(A) \times D(A)$;

(B2) $\|\widetilde{B}(u, v)\|_{(D(A))^*} \leq \widetilde{c} \|u\| \|v\|_{D(A)}, \quad \forall\ (u, v) \in D(A) \times D(A)$;

(B3) $|\langle \widetilde{B}(u, v), w \rangle| \leq \widetilde{c} \|u\|_{D(A)} \|v\|_{D(A)} \|w\|, \quad \forall\ (u, v, w) \in D(A) \times D(A) \times D(A)$.

Recall the phase space

$$C_\gamma(V) = \{\varphi \in C((-\infty, 0]; V) : \lim_{s \to -\infty} e^{\gamma s} \varphi(s) \text{ exists in } V\}, \text{ where } \gamma > 0, \tag{2.23}$$

which is a Banach space with the sup norm

$$\|\varphi\|_{C_\gamma(V)} = \sup_{s \in (-\infty, 0]} e^{\gamma s} \|\varphi(s)\|. \tag{2.24}$$

We now establish some assumptions on the non-delayed external force and delay terms respectively.

Hypothesis F. $f \in I^2(\tau, \tau + T; \mathbb{H}^{-1}(\mathcal{O}))$ for any $\tau \in \mathbb{R}$ and $T > 0$.

Hypothesis G. Let $g_1 : \mathbb{R} \times C_\gamma(V) \to \mathbb{H}^{-1}(\mathcal{O})$ and $g_2 : \mathbb{R} \times C_\gamma(V) \to \mathcal{L}^2(K, \mathbb{L}^2(\mathcal{O}))$ satisfy the following conditions.

(G1) For any $\eta \in C_\gamma(V)$, $g_i(\cdot, \eta)$ are measurable, $i = 1, 2$.

(G2) $g_i(\cdot, 0) = 0, \ i = 1, 2$.

(G3) There exists $L_{g_i} > 0$ $(i = 1, 2)$ such that for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|g_1(t, \eta) - g_1(t, \zeta)\|_{\mathbb{H}^{-1}(\mathcal{O})} \leq L_{g_1} \|\eta - \zeta\|_{C_\gamma(V)},$$

$$\|g_2(t,\eta)-g_2(t,\zeta)\|_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))}\le L_{g_2}\|\eta-\zeta\|_{C_\gamma(V)};$$

(G4) There exists $C_{g_i}>0$ $(i=1,2)$ such that for all $t\in\mathbb{R}$ and $\eta,\zeta\in C_\gamma(V)$,

$$\int_\tau^t \|g_1(s,u_s)-g_1(s,v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})}ds\le C_{g_1}^2\int_{-\infty}^t\|u(s)-v(s)\|^2ds,$$
$$\int_\tau^t \|g_2(s,u_s)-g_2(s,v_s)\|^2_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))}ds\le C_{g_2}^2\int_{-\infty}^t\|u(s)-v(s)\|^2ds;$$

(G5) There exists $\widetilde{C}_{g_i}>0$ $(i=1,2)$ such that for all $\tau\in\mathbb{R},t\ge\tau$, all decreasing function $\varpi\in C^0([\tau,t])$ and $u,v\in C^0((-\infty,t];V)$

$$\int_\tau^t \varpi(s)\|g_1(s,u_s)-g_1(s,v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})}ds\le \widetilde{C}_{g_1}\int_\tau^t\varpi(s)\|u(s)-v(s)\|^2ds,$$
$$\int_\tau^t \varpi(s)\|g_2(s,u_s)-g_2(s,v_s)\|^2_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))}ds\le \widetilde{C}_{g_2}\int_\tau^t\varpi(s)\|u(s)-v(s)\|^2ds.$$

We infer from (G2)-(G3) that, for all $\eta\in C_\gamma(V)$,

$$\|g_1(t,\eta)\|_{\mathbb{H}^{-1}(\mathcal{O})}\le L_{g_1}\|\eta\|_{C_\gamma(V)},\ \|g_2(t,\eta)\|_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))}\le L_{g_2}\|\eta\|_{C_\gamma(V)}.$$

Next, let us define $\widetilde{f}(t)$ as

$$((\widetilde{f}(t),w))=\langle f(t),w\rangle_{-1},\ \forall\ (t,w)\in\mathbb{R}\times V.$$

By the hypothesis **F**, $\widetilde{f}\in I^2(\tau,\tau+T;(D(A))^*)$ for any $\tau\in\mathbb{R}$ and $T>0$.

In addition, we define $\widetilde{g}_1:\mathbb{R}\times C_\gamma(V)\to V$ such that

$$((\widetilde{g}_1(t,\eta),w))=\langle g_1(t,\eta),w\rangle_{-1},\ \forall\ (t,\eta,w)\in\mathbb{R}\times C_\gamma(V)\times V.$$

Finally, we define $\widetilde{g}_2:\mathbb{R}\times C_\gamma(V)\to\mathcal{L}^2(K,V)$ such that

$$\widetilde{g}_2(t,\eta)=(I+\alpha A)^{-1}\circ\mathcal{P}\circ g_2(t,\eta),\ \forall\ (t,\eta)\in\mathbb{R}\times C_\gamma(V),$$

where I is the identity operator in H and $I+\alpha A:D(A)\to H$ is bijective, moreover,

$$(((I+\alpha A)^{-1}u,w))=(u,w),\ \forall\ u\in H,\ w\in V.$$

Hence, for the orthonormal basis $\{e_j\}$ of K, we have

$$(g_2(t,\eta)e_j,w)=((I+\alpha A)\widetilde{g}_2(t,\eta)e_j,w)=((\widetilde{g}_2(t,\eta)e_j,w)),$$

for all $j\ge1$ and $(t,\eta,w)\in\mathbb{R}\times C_\gamma(V)\times D(A)$, by (2.3), we further obtain that

$$\begin{aligned}\Big(\int_\tau^t g_2(s,\eta)dW(s),w\Big)&=\sum_{j=1}^\infty\int_\tau^t(g_2(s,\eta)e_j,w)d\beta^j(s)\\&=\sum_{j=1}^\infty\int_\tau^t\big((\widetilde{g}_2(s,\eta)e_j,w)\big)d\beta^j(s)\\&=\Big(\Big(\int_\tau^t\widetilde{g}_2(s,\eta)dW(s),w\Big)\Big).\end{aligned}\tag{2.25}$$

By the same method as in [13], one can prove that $\widetilde{g}_1:\mathbb{R}\times C_\gamma(V)\to V$ and $\widetilde{g}_2:\mathbb{R}\times C_\gamma(V)\to\mathcal{L}^2(K,V)$ satisfy the following conditions:

(H1) For any $\eta\in C_\gamma(V)$, $\widetilde{g}_i(\cdot,\eta)$ are measurable, $i=1,2$;

(H2) $\widetilde{g}_i(\cdot,0)=0,\ i=1,2$;

(H3) Taking $L_{\widetilde{g}_1}=L_{g_1},L_{\widetilde{g}_2}=L_{g_2}/\sqrt{1+\alpha\lambda_1}$, we deduce, for all $t\in\mathbb{R}$ and $\eta,\zeta\in C_\gamma(V)$,

$$\|\widetilde{g}_1(t,\eta)-\widetilde{g}_1(t,\zeta)\|\le L_{\widetilde{g}_1}\|\eta-\zeta\|_{C_\gamma(V)},$$

$$\|\widetilde{g}_2(t,\eta)-\widetilde{g}_2(t,\zeta)\|_{\mathcal{L}^2(K,V)}\leq L_{\widetilde{g}_2}\|\eta-\zeta\|_{C_\gamma(V)};$$

(H4) Setting $C_{\widetilde{g}_1}=C_{g_1}, C_{\widetilde{g}_2}=C_{g_2}/\sqrt{1+\alpha\lambda_1}$, we obtain, for all $t\in\mathbb{R}$ and $\eta,\zeta\in C_\gamma(V)$,

$$\int_\tau^t\|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2ds\leq C^2_{\widetilde{g}_1}\int_{-\infty}^t\|u(s)-v(s)\|^2ds,$$
$$\int_\tau^t\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\leq C^2_{\widetilde{g}_2}\int_{-\infty}^t\|u(s)-v(s)\|^2ds;$$

(H5) Letting $\widetilde{C}_{\widetilde{g}_1}=\widetilde{C}_{g_1}, \widetilde{C}_{\widetilde{g}_2}=\widetilde{C}_{g_2}/\sqrt{1+\alpha\lambda_1}$ such that for all $\tau\in\mathbb{R}, t\geq\tau$, and all decreasing function $\varpi\in C^0([\tau,t])$ and $u,v\in C^0((-\infty,t];V)$

$$\int_\tau^t\varpi(s)\|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2ds\leq\widetilde{C}_{\widetilde{g}_1}\int_\tau^t\varpi(s)\|u(s)-v(s)\|^2ds,$$
$$\int_\tau^t\varpi(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\leq\widetilde{C}_{\widetilde{g}_2}\int_\tau^t\varpi(s)\|u(s)-v(s)\|^2ds.$$

It follows from (H2)-(H3) that for all $t\in\mathbb{R}$ and $\eta\in C_\gamma(V)$,

$$\|\widetilde{g}_1(t,\eta)\|\leq L_{\widetilde{g}_1}\|\eta\|_{C_\gamma(V)},\ \|\widetilde{g}_2(t,\eta)\|_{\mathcal{L}^2(K,V)}\leq L_{\widetilde{g}_2}\|\eta\|_{C_\gamma(V)}. \tag{2.26}$$

An example of the delayed terms with (H1)-(H5) is given as follows.
Example 1: For all $t\in\mathbb{R}$ and $\xi\in C_\gamma(V)$, let

$$\widetilde{g}_i(t,\xi)=\int_{-\infty}^0\widetilde{G}_i(t,s,\xi(s))ds,\ i=1,2, \tag{2.27}$$

where $\widetilde{G}_1:\mathbb{R}\times(-\infty,0]\times V\to V$ with $\widetilde{G}_1(t,s,0)=0$, and $\widetilde{G}_2:\mathbb{R}\times(-\infty,0]\times V\to\mathcal{L}^2(K,V)$ with $\widetilde{G}_2(t,s,0)=0$, and both are measurable. Assume that there exist $L_{\widetilde{G}_i}\in L^2(-\infty,0)$ $(i=1,2)$ with $L_{\widetilde{G}_i}(\cdot)e^{-(\gamma+\theta)\cdot}\in L^2(-\infty,0)$ for certain $\theta>0$ such that for all $t\in\mathbb{R}$, $s\in(-\infty,0]$ and $\eta,\zeta\in V$,

$$\|\widetilde{G}_1(t,s,\eta)-\widetilde{G}_1(t,s,\zeta)\|\leq L_{\widetilde{G}_1}(s)\|\eta-\zeta\|,$$
$$\|\widetilde{G}_2(t,s,\eta)-\widetilde{G}_2(t,s,\zeta)\|_{\mathcal{L}^2(K,V)}\leq L_{\widetilde{G}_2}(s)\|\eta-\zeta\|.$$

Thus, we can rewrite the delay terms $\widetilde{g}_i$ $(i=1,2)$ in our problem as $\widetilde{g}_i(t,u_t)=\int_{-\infty}^0\widetilde{G}_i(t,s,u(t+s))ds$ $(i=1,2)$. It follows that the example is within our framework, and $\widetilde{g}_i(i=1,2)$ fulfill conditions (H1)-(H5) (e.g., see [34] for more details).

3. Well-posedness of stochastic 3D LANS equations with infinite delay

In this section, we prove the well-posedness of the stochastic Eq. (1.1), which can be transferred into the following abstract equation:

$$\begin{cases}\dfrac{du}{dt}+\widetilde{A}u(t)+\widetilde{B}(u(t))=\widetilde{f}(t)+\widetilde{g}_1(t,u_t)+\widetilde{g}_2(t,u_t)\dfrac{dW}{dt},\quad\forall\ t>\tau,\\ u(\tau+s)=\phi(s),\quad s\in(-\infty,0].\end{cases} \tag{3.1}$$

Definition 3.1. Suppose that $\phi\in L^2(\Omega,C_\gamma(V))$ (which is a $\mathcal{F}_0$-progressively measurable V-valued processes) and $\tau\in\mathbb{R}$. A stochastic process u defined on $\mathbb{R}$ is called a solution to system (3.1) if

$$u\in I^2(\tau,\tau+T;D(A))\cap L^2(\Omega,L^\infty(\tau,\tau+T;V)),\ \forall\ T>0,$$

$u_\tau=\phi$ and P-almost surely

$$\begin{aligned}&((u(t),w))+\int_\tau^t\langle\widetilde{A}u(s),w\rangle ds+\int_\tau^t\langle\widetilde{B}(u(s)),w\rangle ds\\&=((\phi(0),w))+\int_\tau^t\Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s),w\Big)\Big)ds+\Big(\Big(w,\int_\tau^t\widetilde{g}_2(s,u_s)dW(s)\Big)\Big)\end{aligned} \tag{3.2}$$

for all $t\geq\tau$ and $w\in D(A)$.

Lemma 3.2. *For all $u, v \in D(A)$, we have*

$$\langle -\widetilde{A}\overline{w} - 2(\widetilde{B}(u) - \widetilde{B}(v)), \overline{w}\rangle \leq \sigma \|\overline{w}\|^2 \|v\|^2_{D(A)}, \tag{3.3}$$

where $\overline{w} = u - v$ and $\sigma = \widetilde{c}^2$.

Proof. Note that

$$\langle -\widetilde{A}\overline{w}, \overline{w}\rangle = -\|\overline{w}\|^2_{D(A)}. \tag{3.4}$$

By the property (B1) of the operator $\widetilde{B}$, we have

$$\langle \widetilde{B}(u), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, u), u\rangle = -\langle \widetilde{B}(\overline{w}, u), v\rangle, \tag{3.5}$$

and similarly

$$\langle \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, v), v\rangle. \tag{3.6}$$

Subtracting (3.6) from (3.5),

$$\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}), v\rangle, \tag{3.7}$$

which, together with (B2), implies that

$$\begin{aligned} |\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle| &= |\langle \widetilde{B}(\overline{w}), v\rangle| \\ &\leq \|\widetilde{B}(\overline{w})\|_{(D(A))^*} \|v\|_{D(A)} \\ &\leq \widetilde{c}\|\overline{w}\| \|\overline{w}\|_{D(A)} \|v\|_{D(A)} \\ &\leq \frac{1}{2}\|\overline{w}\|^2_{D(A)} + \frac{\widetilde{c}^2}{2}\|\overline{w}\|^2 \|v\|^2_{D(A)}. \end{aligned} \tag{3.8}$$

Combining (3.4) and (3.8), we obtain (3.3) as desired. □

In the following, we present the well-posedness of problem (3.1). For this end, we further assume **Hypothesis I**. For all $u, v \in L^2(-\infty, \tau + T; D(A))$ and $t \in [\tau, \tau + T]$, Eq. (3.1) satisfies

$$\begin{aligned} &\int_\tau^t \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)\|^2_{\mathcal{L}^2(K,V)} ds + 2\int_\tau^t ((\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), u(s) - v(s)))ds \\ &\leq \sigma \int_\tau^t \|v(s)\|^2_{D(A)} \|u(s) - v(s)\|^2 ds \\ &\quad + 2\int_\tau^t \langle \widetilde{A}(u(s) - v(s)) + \widetilde{B}(u(s)) - \widetilde{B}(v(s)), u(s) - v(s)\rangle ds, \end{aligned} \tag{3.9}$$

where σ is given by (3.3) in Lemma 3.2.

Remark 3.3. Let

$$C^2_{\widetilde{g}_2} + \frac{2}{\widetilde{\lambda}_1} C^2_{\widetilde{g}_1} \leq \frac{\widetilde{\lambda}_1}{2} \text{ and } u(s + \tau) = v(s + \tau) = \phi(s),\ s \leq 0. \tag{3.10}$$

Then (3.9) in hypothesis I is satisfied. Indeed, by (2.13) and Lemma 3.2, we only need to prove that the following inequality holds:

$$\begin{aligned} &\int_\tau^t \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)\|^2_{\mathcal{L}^2(K,V)} ds + 2\int_\tau^t ((\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), u(s) - v(s)))ds \\ &\leq \widetilde{\lambda}_1 \int_\tau^t \|u(s) - v(s)\|^2 ds. \end{aligned} \tag{3.11}$$

The Young inequality, (H4) and (3.10) imply

$$\int_\tau^t \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)\|^2_{\mathcal{L}^2(K,V)} ds + 2\int_\tau^t ((\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), u(s) - v(s)))ds$$

$$
\begin{aligned}
&\leq C_{\widetilde{g}_2}^2 \int_{-\infty}^t \|u(s)-v(s)\|^2 ds + \frac{2}{\widetilde{\lambda}_1}\int_\tau^t \|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2 ds + \frac{\widetilde{\lambda}_1}{2}\int_\tau^t \|u(s)-v(s)\|^2 ds \\
&\leq \Big(C_{\widetilde{g}_2}^2 + \frac{2}{\widetilde{\lambda}_1} C_{\widetilde{g}_1}^2\Big)\int_{-\infty}^t \|u(s)-v(s)\|^2 ds + \frac{\widetilde{\lambda}_1}{2}\int_\tau^t \|u(s)-v(s)\|^2 ds \\
&= \Big(C_{\widetilde{g}_2}^2 + \frac{2}{\widetilde{\lambda}_1} C_{\widetilde{g}_1}^2 + \frac{\widetilde{\lambda}_1}{2}\Big)\int_\tau^t \|u(s)-v(s)\|^2 ds \\
&\leq \widetilde{\lambda}_1 \int_\tau^t \|u(s)-v(s)\|^2 ds,
\end{aligned}
\tag{3.12}
$$

which implies (3.11) as desired.

Theorem 3.4. *Suppose that hypotheses* **F, G, I** *hold, moreover,* $\phi \in L^4(\Omega, C_\gamma(V))$ *and* $\widetilde{f} \in I^4(\tau, \tau+T; V)$, *then there exists a unique solution* u *to* (3.1)*, which satisfies in addition,*

$$
u \in I^4(\tau,\tau+T;V) \cap L^4(\Omega, L^\infty(\tau,\tau+T;V)). \tag{3.13}
$$

In fact, there exists a positive constant R *depending on* T, $\mathbb{E}\big(\|\phi\|_{C_\gamma(V)}^4\big)$ *and* $\mathbb{E}\big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^4 ds\big)$ *such that*

$$
\mathbb{E}\Big(\sup_{\tau\leq r\leq \tau+T}\|u_r\|_{C_\gamma(V)}^4\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T}\|u(s)\|^4 ds\Big) \leq R. \tag{3.14}
$$

Proof. We split the proof into several steps as follows.

Step 1: *We use the Galerkin method to construct an approximating sequence.* Consider the Hilbert basis $\{w_j; j\in\mathbb{N}\}\subset D(A)$ of V such that $\widetilde{A}w_j = \widetilde{\lambda}_j w_j$, $\forall\ j\geq 1$, denote by V_m the linear space spanned by $\{w_1, w_2, \ldots, w_m\}$ for $m\in\mathbb{N}$, and then put

$$
u^m(t) = \sum_{j=1}^m a_{m,j}(t) w_j, \tag{3.15}
$$

where $a_{m,j}(t)$ $(j=1,\cdots,m)$ will be obtained as the solution of the following finite dimensional system:

$$
\begin{cases}
((u^m(t),w_j)) + \displaystyle\int_\tau^t \langle \widetilde{A}u^m(s), w_j\rangle ds + \int_\tau^t \langle \widetilde{B}(u^m(s)), w_j\rangle ds \\
\quad = ((u^m(\tau),w_j)) + \displaystyle\int_\tau^t \Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m), w_j\Big)\Big) ds \\
\qquad + \Big(\Big(w_j, \displaystyle\int_\tau^t \widetilde{g}_2(s,u_s^m) dW(s)\Big)\Big), \quad \forall\ t\in[\tau,\tau+T],\ j\in[1,m], \quad P\text{-a.s.} \\
u^m(\tau+s) = \mathcal{P}_m\phi(s),\ \forall\ s\in(-\infty,0],
\end{cases}
\tag{3.16}
$$

where $\mathcal{P}_m : V\to V_m$ is the projector.

By the similar argument in [5], for each $m\in\mathbb{N}$, the stochastic ODE (3.16) possesses a (local) solution $\{a_{m,j}(\cdot)\}_{j=1}^m$ in $[\tau, t_m)$ with $\tau < t_m$ (by the initial condition, the value of $a_{m,j}(\cdot)$ in $(-\infty,\tau]$ is well defined). From this, $u^m(\cdot)$ is well-defined in $[\tau, t_m)$ (and thus in $(-\infty, t_m)$). Next, we will give a priori estimate to ensure that the solutions u^m is global, i.e. $t_m = +\infty$.

Step 2: *We give a-priori estimates for the approximating sequence.* We first claim that, for any $T>0$, the following inequality holds:

$$
\mathbb{E}\Big(\sup_{\tau\leq r\leq\tau+T}\|u_r^m\|_{C_\gamma(V)}^2\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T}\|u^m(s)\|_{D(A)}^2 ds\Big) \leq R_1, \tag{3.17}
$$

where R_1 is a positive constant depending on T, $\mathbb{E}\big(\|\phi\|_{C_\gamma(V)}^2\big)$ and $\mathbb{E}\big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2 ds\big)$.

Indeed, multiplying (3.16) by $a_{m,j}$, summing those relations for $j=1,\cdots,m$ and applying Ito's formula to $\|u^m(t)\|^2$, we obtain that

$$
\|u^m(t)\|^2 + 2\int_\tau^t \|u^m(s)\|_{D(A)}^2 ds = \|u^m(\tau)\|^2 + 2\int_\tau^t \Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m), u^m(s)\Big)\Big) ds \tag{3.18}
$$

$$+\int_\tau^t \|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds + 2\int_\tau^t \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big).$$

By (2.26), we can rewrite (3.18) as

$$\begin{aligned}
\|u^m(t)\|^2 + 2\int_\tau^t \|u^m(s)\|^2_{D(A)}ds &\le \|\phi(0)\|^2 + 2\int_\tau^t \Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m), u^m(s)\Big)\Big)ds \\
&\quad + \int_\tau^t \|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds + 2\Big(\Big(\int_\tau^t u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \\
&\le \|\phi(0)\|^2 + 2\int_\tau^t \|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|\|u^m(s)\|ds \\
&\quad + L^2_{\widetilde{g}_2}\int_\tau^t \|u_s^m\|^2_{C_\gamma(V)}ds + 2\Big|\Big(\Big(\int_\tau^t u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|. \qquad (3.19)
\end{aligned}$$

By (2.13), the Young inequality and (2.26), we find

$$\begin{aligned}
&2\int_\tau^t \|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|\|u^m(s)\|ds \\
&\le 2\widetilde{\lambda}_1^{-\frac{1}{2}}\int_\tau^t \|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|\|u^m(s)\|_{D(A)}ds \\
&\le 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^2ds + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}\int_\tau^t \|u_s^m\|^2_{C_\gamma(V)}ds + \int_\tau^t \|u^m(s)\|^2_{D(A)}ds. \qquad (3.20)
\end{aligned}$$

Substituting (3.20) into (3.19), we obtain

$$\begin{aligned}
\|u^m(t)\|^2 + \int_\tau^t \|u^m(s)\|^2_{D(A)}ds &\le \|\phi(0)\|^2 + 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^2ds + c_1\int_\tau^t \|u_s^m\|^2_{C_\gamma(V)}ds \\
&\quad + 2\Big|\int_\tau^t \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|, \qquad (3.21)
\end{aligned}$$

where $c_1 = L^2_{\widetilde{g}_2} + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}$. The above inequality implies

$$\begin{aligned}
\|u_t^m\|^2_{C_\gamma(V)} &\le \max\Big\{\sup_{\theta\le\tau-t} e^{2\gamma\theta}\|u^m(t+\theta)\|^2, \sup_{\tau-t\le\theta\le 0} e^{2\gamma\theta}\|u^m(t+\theta)\|^2\Big\} \\
&\le \max\Big\{\sup_{\theta\le\tau-t} e^{2\gamma\theta}\|\phi(t+\theta-\tau)\|^2, \sup_{\tau-t\le\theta\le 0} e^{2\gamma\theta}\|u^m(t+\theta)\|^2\Big\} \\
&\le \max\Bigg\{\sup_{\theta\le 0} e^{2\gamma(\theta-t+\tau)}\|\phi(\theta)\|^2, \sup_{\tau-t\le\theta\le 0} e^{2\gamma\theta}\Bigg(\|\phi(0)\|^2 + 2\widetilde{\lambda}_1^{-1}\int_\tau^{t+\theta}\|\widetilde{f}(s)\|^2ds \\
&\quad + c_1\int_\tau^{t+\theta}\|u_s^m\|^2_{C_\gamma(V)}ds + 2\Big|\int_\tau^{t+\theta}\Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Bigg)\Bigg\} \\
&\le e^{-2\gamma(t-\tau)}\|\phi\|^2_{C_\gamma(V)} + \|\phi\|^2_{C_\gamma(V)} + 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^2ds + c_1\int_\tau^t \|u_s^m\|^2_{C_\gamma(V)}ds \\
&\quad + 2\sup_{\tau-t\le\theta\le 0} e^{2\gamma\theta}\Big|\int_\tau^{t+\theta}\Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|. \qquad (3.22)
\end{aligned}$$

Taking supremum and expectation of (3.22), we find

$$\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u_r^m\|^2_{C_\gamma(V)}\Big) &\le 2\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^2ds\Big) + c_1\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds \\
&\quad + 2\mathbb{E}\Big(\sup_{\tau\le r\le t}\sup_{\tau-r\le\theta\le 0} e^{2\gamma\theta}\Big|\int_\tau^{r+\theta}\Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big). \qquad (3.23)
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and (2.26), the last term of (3.23) is bounded by

$$2\mathbb{E}\Big(\sup_{\tau\le r\le t}\sup_{\tau-r\le\theta\le 0} e^{2\gamma\theta}\Big|\int_\tau^{r+\theta}\Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big)$$

$$
\begin{aligned}
&\leq 2\mathbb{E}\Big(\sup_{\tau\leq r+\theta\leq t}\Big|\int_\tau^{r+\theta}\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big)\\
&\leq 2c_1\mathbb{E}\Bigg(\Big(\int_\tau^t\|u^m(s)\|^2\|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac{1}{2}}\Bigg)\\
&\leq 2c_1\mathbb{E}\Bigg(\sup_{\tau\leq r\leq t}\|u_r^m\|_{C_\gamma(V)}\Big(\int_\tau^t\|\widetilde{g}_2(s,u_s^m)\|^2ds\Big)^{\frac{1}{2}}\Bigg)\\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_r^m\|^2_{C_\gamma(V)}\Big)+2c_1^2L^2_{\widetilde{g}_2}\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds.
\end{aligned}
\tag{3.24}
$$

It follows from (3.23)-(3.24) that, for all $t\in[\tau,\tau+T]$,

$$
\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_r^m\|^2_{C_\gamma(V)}\Big)\leq 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big)+4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t\|\widetilde{f}(s)\|^2ds\Big)+c_2\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds,\tag{3.25}
$$

where $c_2=2c_1(1+2c_1L^2_{\widetilde{g}_2})$. Set

$$
c_T:=4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big)+4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2ds\Big),\tag{3.26}
$$

which is finite due to $\phi\in L^4(\Omega,C_\gamma(V))$ and $\widetilde{f}\in I^4(\tau,\tau+T;V)$. Applying the Gronwall lemma to (3.25), we find, for all $t\in[\tau,\tau+T]$,

$$
\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_r^m\|^2_{C_\gamma(V)}\Big)\leq c_Te^{c_2T}=:R_{11}.\tag{3.27}
$$

Finally, we infer from (3.21) and (3.24) that, for all $t\in[\tau,\tau+T]$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\int_\tau^r\|u^m(s)\|^2_{D(A)}ds\Big)&\leq \mathbb{E}\Big(\|\phi(0)\|^2\Big)+2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2ds\Big)\\
&\quad+\frac{c_2}{2}\int_\tau^{\tau+T}\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds+\frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_r^m\|^2_{C_\gamma(V)}\Big),
\end{aligned}
\tag{3.28}
$$

which, together with (3.27), implies that there exists a positive constant R_{12},

$$
\mathbb{E}\Big(\int_\tau^t\|u^m(s)\|^2_{D(A)}ds\Big)\leq R_{12},\ \forall\ m.\tag{3.29}
$$

Combining (3.27) and (3.29), we obtain (3.17) for $R_1=R_{11}+R_{12}$.

We also need to give the following estimate:

$$
\mathbb{E}\Big(\sup_{\tau\leq r\leq\tau+T}\|u_r^m\|^4_{C_\gamma(V)}\Big)+\mathbb{E}\Big(\int_\tau^{\tau+T}\|u^m(s)\|^4ds\Big)\leq R,\tag{3.30}
$$

where R depends on T, $\mathbb{E}\big(\|\phi\|^4_{C_\gamma(V)}\big)$ and $\mathbb{E}\big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^4ds\big)$.

Indeed, by (3.18) and Ito's formula, we infer from (2.26) that

$$
\begin{aligned}
&\|u^m(t)\|^4+4\int_\tau^t\|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds\\
&\leq\|\phi(0)\|^4+4\int_\tau^t\|u^m(s)\|^2\Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m),u^m(s)\Big)\Big)ds+2\int_\tau^t\|u^m(s)\|^2\|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds\\
&\quad+4\int_\tau^t\|\widetilde{g}_2^*(s,u_s^m)u^m(s)\|^2_Kds+4\int_\tau^t\|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\\
&\leq\|\phi(0)\|^4+4\int_\tau^t\|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 6\int_\tau^t \|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}\|u^m(s)\|^2 ds + 4\int_\tau^t \|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\\
&\le \|\phi(0)\|^4 + 4\int_\tau^t \|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds\\
&+ 3L^4_{\widetilde{g}_2}\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds + 3\int_\tau^t \|u^m(s)\|^4 ds + 4\int_\tau^t \|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big), \qquad (3.31)
\end{aligned}
$$

where $\widetilde{g}_2^*$ is the adjoint operator of $\widetilde{g}_2$. By (2.13), the Young inequality and (2.26), we obtain

$$
\begin{aligned}
&4\int_\tau^t \|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds\\
&\le 4\widetilde{\lambda}_1^{-\frac{1}{2}}\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|_{D(A)}\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds\\
&\le 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|u^m(s)\|^2\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|^2 ds + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds\\
&\le 4\widetilde{\lambda}_1^{-1}\int_\tau^t \|u^m(s)\|^2\|\widetilde{f}(s)\|^2 ds + 4\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}\int_\tau^t \|u^m(s)\|^2\|u_s^m\|^2_{C_\gamma(V)}ds\\
&\quad + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds\\
&\le 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^4 ds + c_3\int_\tau^t \|u^m(s)\|^4 ds + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds\\
&\quad + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds, \qquad (3.32)
\end{aligned}
$$

where $c_3 = 2\widetilde{\lambda}_1^{-1}(1+L^2_{\widetilde{g}_1})$. Substituting (3.32) into (3.31),

$$
\begin{aligned}
&\|u^m(t)\|^4 + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds\\
&\le \|\phi(0)\|^4 + 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^4 ds + c_4\int_\tau^t \|u^m(s)\|^4 ds + c_5\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds\\
&\quad + 4\int_\tau^t \|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big), \qquad (3.33)
\end{aligned}
$$

where $c_4 = c_3+3, c_5 = 3L^4_{\widetilde{g}_2} + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}$. By (3.33), we find

$$
\begin{aligned}
\|u_t^m\|^4_{C_\gamma(V)} &\le \max\Big\{\sup_{\theta\le\tau-t} e^{4\gamma\theta}\|u^m(t+\theta)\|^4, \sup_{\tau-t\le\theta\le 0} e^{4\gamma\theta}\|u^m(t+\theta)\|^4\Big\}\\
&\le \max\Bigg\{\sup_{\theta\le\tau-t} e^{4\gamma\theta}\|\phi(t+\theta-\tau)\|^4, \sup_{\tau-t\le\theta\le 0} e^{4\gamma\theta}\Bigg(\|\phi(0)\|^4 + 2\widetilde{\lambda}_1^{-1}\int_\tau^{t+\theta}\|\widetilde{f}(s)\|^4 ds\\
&\quad + c_4\int_\tau^{t+\theta}\|u^m(s)\|^4 ds + c_5\int_\tau^{t+\theta}\|u_s^m\|^4_{C_\gamma(V)}ds\\
&\quad + 4\Big|\int_\tau^{t+\theta}\|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Bigg)\Bigg\}\\
&\le e^{-4\gamma(t-\tau)}\|\phi\|^4_{C_\gamma(V)} + \|\phi\|^4_{C_\gamma(V)} + 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^4 ds + c_4\int_\tau^t \|u^m(s)\|^4 ds\\
&\quad + c_5\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds + 4\sup_{\tau-t\le\theta\le 0} e^{4\gamma\theta}\Big|\int_\tau^{t+\theta}\|u^m(s)\|^2\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|. \qquad (3.34)
\end{aligned}
$$

Taking supremum and expectation of (3.34), we infer

$$
\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u_r^m\|^4_{C_\gamma(V)}\Big) \le 2\mathbb{E}\Big(\|\phi\|^4_{C_\gamma(V)}\Big) + 2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^4 ds\Big) + c_6\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u_r^m\|^4_{C_\gamma(V)}\Big)ds
$$

$$+ 4\mathbb{E}\Big(\sup_{\tau\le r\le t} \sup_{\tau-r\le\theta\le 0} e^{4\gamma\theta} \Big| \int_\tau^{r+\theta} \|u^m(s)\|^2 \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \Big| \Big), \tag{3.35}$$

where $c_6 = c_4 + c_5$. By the Burkholder-Davis-Gundy inequality and (2.26), the last term of (3.35) satisfies

$$\begin{aligned}
&4\mathbb{E}\Big(\sup_{\tau\le r\le t} \sup_{\tau-r\le\theta\le 0} e^{4\gamma\theta} \Big| \int_\tau^{r+\theta} \|u^m(s)\|^2 \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \Big| \Big) \\
&\le 4\mathbb{E}\Bigg(\sup_{\tau\le r+\theta\le t} \Big| \int_\tau^{r+\theta} \|u^m(s)\|^2 \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \Big| \Bigg) \\
&\le 4\mathbb{E}\Bigg(\sup_{\tau\le r\le t} \|u_r^m\|^2_{C_\gamma(V)} \sup_{\tau\le r+\theta\le t} \Big| \int_\tau^{r+\theta} \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \Big| \Bigg) \\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) + 8\mathbb{E}\Bigg(\sup_{\tau\le r+\theta\le t} \Big| \int_\tau^{r+\theta} \Big(\Big(u^m(s), \widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big) \Big|^2 \Bigg) \\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) + 8c_7\mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^2 \|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)} ds \Big) \\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) + 4c_7\mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^4 ds \Big) + 4c_7 L^4_{\widetilde{g}_2} \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u_r^m\|^4_{C_\gamma(V)} \Big) ds \\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) + c_8 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u_r^m\|^4_{C_\gamma(V)} \Big) ds,
\end{aligned} \tag{3.36}$$

where $c_8 = 4c_7(1 + L^4_{\widetilde{g}_2})$. Substituting (3.36) into (3.35), we obtain

$$\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) \le 4\mathbb{E}\Big(\|\phi\|^4_{C_\gamma(V)} \Big) + 4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^4 ds \Big) + c_9 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u_r^m\|^4_{C_\gamma(V)} \Big) ds, \tag{3.37}$$

where $c_9 = 2(c_6 + c_8)$. Setting

$$c^* := 4\mathbb{E}\Big(\|\phi\|^4_{C_\gamma(V)} \Big) + 4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T} \|\widetilde{f}(s)\|^4 ds \Big), \tag{3.38}$$

then applying the Gronwall lemma to (3.37), we deduce that

$$\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) \le c^* e^{c_9 T} =: R_{21},\ \forall\ t \in [\tau, \tau+T]. \tag{3.39}$$

It follows from (2.13), (3.33) and (3.36) that for all $t \in [\tau, \tau+T]$ such that

$$\begin{aligned}
2\widetilde{\lambda}_1\mathbb{E}\Big(\sup_{\tau\le r\le t} \int_\tau^r \|u^m(s)\|^4 ds \Big) &\le 2\mathbb{E}\Big(\sup_{\tau\le r\le t} \int_\tau^r \|u^m(s)\|^2 \|u^m(s)\|^2_{D(A)} ds \Big) \\
&\le \mathbb{E}\Big(\|\phi(0)\|^4 \Big) + 2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^4 ds \Big) \\
&\quad + \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{\mathbb{C}_\gamma(V)} \Big) + \frac{c_9}{2} \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u_r^m\|^4_{\mathbb{C}_\gamma(V)} \Big) ds,
\end{aligned} \tag{3.40}$$

which, together with (3.39), $\phi \in L^4(\Omega, C_\gamma(V))$ and $\widetilde{f} \in I^4(\tau, \tau+T; V)$, implies that there exists a positive constant R_{22} such that

$$\mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^4 ds \Big) \le R_{22},\ \forall\ t \in [\tau, \tau+T]. \tag{3.41}$$

Combining (3.39) and (3.41), we obtain

$$\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_r^m\|^4_{C_\gamma(V)} \Big) + \mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^4 ds \Big) \le R := R_{21} + R_{22},\ \forall\ t \in [\tau, \tau+T], \tag{3.42}$$

which implies (3.30) as desired.

Step 3: *We prove the existence of solutions to Eq.* (3.1) *by using the monotonicity method.* Indeed, by **Step 2**,

$$\begin{aligned}&u^m \text{ is bounded in } \ L^4(\Omega, L^\infty(\tau,\tau+T;V))\cap I^4(\tau,\tau+T;V)\cap I^2(\tau,\tau+T;D(A)),\\&u^m(\tau+T) \text{ is bounded in } \ L^2(\Omega;V).\end{aligned}$$

By (B2), (3.39) and (3.40), $\widetilde{B}(u^m)$ is bounded in $I^2(\tau,\tau+T;(D(A))^*)$. Moreover, by (2.26) and **Step 2**,

$$\begin{aligned}&\widetilde{g}_1(t,u^m_t) \text{ is bounded in } \ I^2(\tau,\tau+T;V),\\&\widetilde{g}_2(t,u^m_t) \text{ is bounded in } \ I^2(\tau,\tau+T;\mathcal{L}^2(K,V)).\end{aligned}$$

Thus, there exists a subsequence u^m (still denoted by itself) and five elements

$$u\in L^4(\Omega, L^\infty(\tau,\tau+T;V))\cap I^4(\tau,\tau+T;V)\cap I^2(\tau,\tau+T;D(A)),$$

$\mu\in L^2(\Omega;V)$, $\iota\in I^2(\tau,\tau+T;(D(A))^*)$, $\kappa_1\in I^2(\tau,\tau+T;V)$ and $\kappa_2\in I^2(\tau,\tau+T;\mathcal{L}^2(K;V))$ such that

$$\begin{aligned}&u^m \overset{*}{\rightharpoonup} u \ \text{ in } \ L^4(\Omega, L^\infty(\tau,\tau+T;V)),\\&u^m \rightharpoonup u \ \text{ in } \ I^4(\tau,\tau+T;V),\\&u^m \rightharpoonup u \ \text{ in } \ I^2(\tau,\tau+T;D(A)),\\&u^m(\tau+T) \rightharpoonup \mu \ \text{ in } \ L^2(\Omega;V),\\&-\widetilde{A}u^m-\widetilde{B}(u^m) \rightharpoonup \iota \ \text{ in } \ I^2(\tau,\tau+T;(D(A))^*),\\&\widetilde{g}_1(t,u^m_t) \rightharpoonup \kappa_1 \ \text{ in } \ I^2(\tau,\tau+T;V),\\&\widetilde{g}_2(t,u^m_t) \rightharpoonup \kappa_2 \ \text{ in } \ I^2(\tau,\tau+T;\mathcal{L}^2(K,V)).\end{aligned}$$

As in [19], we extend Eq. (3.16) to an open interval $(-\delta+\tau,\tau+T+\delta)$ for any $\delta>0$ such that all terms are equal to 0 outside of the interval $[\tau,\tau+T]$.

Let $\psi(t)$ be a function in $\mathrm{W}^{1,4/3}(-\delta+\tau,\tau+T+\delta)$ with $\psi(\tau)=1$. Put $w_j(t)=\psi(t)w_j$ for all integers $j\geq 1$, where we recall that $\{w_j\}$ is the Hilbert basis of V such that $\{w_j; j\geq 1\}\subset D(A)$. Applying the Ito formula to the function $(u^m(t),w_j(t))$, we obtain

$$\begin{aligned}(u^m(\tau+T),w_j(\tau+T))&=(u^m(\tau),w_j)+\int_\tau^{\tau+T}\Big(u^m(s),\frac{dw_j(s)}{ds}\Big)ds+\int_\tau^{\tau+T}\langle-\widetilde{A}u^m(s)-\widetilde{B}(u^m(s)),w_j(s)\rangle ds\\&\quad+\int_\tau^{\tau+T}\big((\widetilde{f}(s)+\widetilde{g}_1(s,u^m_s),w_j(s))\big)ds+\int_\tau^{\tau+T}\big((w_j(s),\widetilde{g}_2(s,u^m_s)dW(s))\big).\end{aligned}\tag{3.43}$$

Taking limit of (3.43) as $m\to\infty$, we refer to the similar calculation as in [39, Theorem 2.6], then

$$\begin{aligned}-\int_\tau^{\tau+T}\Big(u(s),\frac{dw_j(s)}{ds}\Big)ds&=(\phi(0),w_j)+\int_\tau^{\tau+T}\langle\iota,w_j\rangle\psi(s)ds+\int_\tau^{\tau+T}\big((\widetilde{f}(s)+\kappa_1(s),w_j)\big)\psi(s)ds\\&\quad+\int_\tau^{\tau+T}\psi(s)\big((w_j,\kappa_2(s)dW(s))\big)-(\mu,w_j)\psi(\tau+T).\end{aligned}\tag{3.44}$$

Consider a sequence of functions $\{\psi_k\}$ such that $\psi_k\to 1_{[\tau,\tau+T]}$ and the time derivative of ψ_k tends to υ_t weakly as $k\to\infty$. We use ψ_k in (3.44) to replace ψ and then let $k\to\infty$, we find that

$$(u(t),w_j)=(\phi(0),w_j)+\int_\tau^t\langle\iota(s),w_j\rangle ds+\int_\tau^t\big((\widetilde{f}(s)+\kappa_1(s),w_j)\big)ds+\int_\tau^t\big((w_j,\kappa_2(s)dW(s))\big)\tag{3.45}$$

for all $t<\tau+T$ with $(u(\tau+T),w_j)=(\mu,w_j)$ for all $j\geq 1$, then

$$u(t)=\phi(0)+\int_\tau^t(\iota(s)+\widetilde{f}(s)+\kappa_1(s))ds+\int_\tau^t\kappa_2(s)dW(s)\tag{3.46}$$

with $u(\tau+T)=\mu$.

Let $\varrho(t) = \int_\tau^t \|y(s)\|_{D(A)}^2 ds$, where $y \in I^2(\tau, \tau+T; D(A))$ with $y(\tau+s) = \phi(s),\ s \le 0$. Applying Ito's formula to the process $e^{-\sigma\varrho(t)}\|u(t)\|^2$ and $e^{-\sigma\varrho(t)}\|u^m(t)\|^2$ respectively, where σ is the same as in Lemma 3.2,

$$\begin{aligned}\mathbb{E}\Big(e^{-\sigma\varrho(t)}\|u(t)\|^2\Big) = {}& \mathbb{E}\Big(\|\phi(0)\|^2\Big) - \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u(s)\|^2 ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle\iota(s), u(s)\rangle ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{f}(s)+\kappa_1, u(s)\Big)\Big) ds\Big)\\ &+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\kappa_2\|_{\mathcal{L}^2(K,V)}^2 ds\Big), \end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\mathbb{E}\Big(e^{-\sigma\varrho(t)}\|u^m(t)\|^2\Big) = {}& \mathbb{E}\Big(\|u^m(\tau)\|^2\Big) - \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u^m(s)\|^2 ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s)\rangle ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m), u^m(s)\Big)\Big) ds\Big)\\ &+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s^m)\|_{\mathcal{L}^2(K,V)}^2 ds\Big). \end{aligned} \tag{3.48}$$

Define three elements X_m, Y_m and Z_m by

$$\begin{aligned} X_m = {}& -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u^m(s)-y(s)\|^2 ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s)-y(s)\rangle ds\Big)\\ &- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u^m(s)-y(s)\rangle ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,u_s^m)-\widetilde{g}_1(s,y_s), u^m(s)-y(s)\Big)\Big) ds\Big)\\ &+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s^m)-\widetilde{g}_2(s,y_s)\|_{\mathcal{L}^2(K,V)}^2 ds\Big).\\ Y_m = {}& -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u^m(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s)\rangle ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,u_s^m), u^m(s)\Big)\Big) ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s^m)\|_{\mathcal{L}^2(K,V)}^2 ds\Big).\\ Z_m = {}& -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\Big(\|y(s)\|^2 - 2(u^m(s), y(s))\Big) ds\Big)\\ &+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), -y(s)\rangle ds\Big)\\ &- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u^m(s)-y(s)\rangle ds\Big)\\ &- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,y_s), u^m(s)-y(s)\Big)\Big) ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,u_s^m), -y(s)\Big)\Big) ds\Big)\\ &+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_2(s,y_s) - 2\widetilde{g}_2(s,u_s^m), \widetilde{g}_2(s,y_s)\Big)\Big)_{\mathcal{L}^2(K,V)} ds\Big). \end{aligned}$$

We infer from the above equalities that $X_m = Y_m + Z_m$. By (3.3) in Lemma 3.2 and (3.9), we have $X_m \le 0$,

$$0 \ge \liminf_{m\to\infty} X_m$$

$$
\begin{aligned}
&\geq -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u(s)-y(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)-y(s)\rangle ds\Big)\\
&\quad - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s)-\widetilde{B}(y(s)), u(s)-y(s)\rangle ds\Big)\\
&\quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\kappa_1-\widetilde{g}_1(s,y_s), u(s)-y(s)\Big)\Big) ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\kappa_2-\widetilde{g}_2(s,y_s)\|_{\mathcal{L}^2(K,V)}^2 ds\Big). \quad (3.49)
\end{aligned}
$$

Taking $y(t)=u(t)$ in (3.49), since $e^{-\sigma\varrho(t)}$ is bounded with respect to $t\in[\tau,\tau+T]$, we find that $\kappa_2=\widetilde{g}_2(t,u_t),\ t\in[\tau,\tau+T]$. By (3.48), we derive

$$
Y_m = \mathbb{E}\Big(e^{-\sigma\varrho(s)}\|u^m(t)\|^2\Big) - \mathbb{E}\Big(\|u^m(\tau)\|^2\Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{f}(s), u^m(s)\Big)\Big) ds\Big), \quad (3.50)
$$

which, together with (3.47), implies

$$
\begin{aligned}
\liminf_{m\to\infty} Y_m &\geq \mathbb{E}\Big(e^{-\sigma\varrho(s)}\|u(s)\|^2\Big) - \mathbb{E}\Big(\|\phi(0)\|^2\Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{f}(s), u(s)\Big)\Big) ds\Big)\\
&= -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)\rangle ds\Big)\\
&\quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\kappa_1, u(s)\Big)\Big) ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s)\|_{\mathcal{L}^2(K,V)}^2 ds\Big). \quad (3.51)
\end{aligned}
$$

Besides,

$$
\begin{aligned}
\liminf_{m\to\infty} Z_m &\geq -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\Big(\|y(s)\|^2 - 2(u(s),y(s))\Big) ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, -y(s)\rangle ds\Big)\\
&\quad - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s)-\widetilde{B}(y(s)), u(s)-y(s)\rangle ds\Big)\\
&\quad - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,y_s), u(s)-y(s)\Big)\Big) ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\kappa_1, -y(s)\Big)\Big) ds\Big)\\
&\quad + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_2(s,y_s) - 2\widetilde{g}_2(s,u_s), \widetilde{g}_2(s,y_s)\Big)\Big)_{\mathcal{L}^2(K,V)} ds\Big). \quad (3.52)
\end{aligned}
$$

Therefore, by (3.49), (3.51) and (3.52), we have

$$
\begin{aligned}
0 &\geq \liminf_{m\to\infty} X_m = \liminf_{m\to\infty} Y_m + \liminf_{m\to\infty} Z_m\\
&\geq -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u(s)-y(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)-y(s)\rangle ds\Big)\\
&\quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\kappa_1-\widetilde{g}_1(s,y_s), u(s)-y(s)) ds\Big)\Big)\Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s)-\widetilde{B}(y(s)), u(s)-y(s)\rangle ds\Big)\\
&\quad + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,y_s)\|_{\mathcal{L}^2(K,V)}^2 ds\Big). \quad (3.53)
\end{aligned}
$$

We further obtain

$$
\begin{aligned}
0 &\leq \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,y_s)\|_{\mathcal{L}^2(K,V)}^2 ds\Big)\\
&\leq 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s)-\widetilde{B}(y(s)), u(s)-y(s)\rangle ds\Big)\\
&\quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\Big(\widetilde{g}_1(s,y_s)-\kappa_1, u(s)-y(s)\Big)\Big) ds\Big)\\
&\quad - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)-y(s)\rangle ds\Big) + \sigma\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|y(s)\|_{D(A)}^2\|u(s)-y(s)\|^2 ds\Big). \quad (3.54)
\end{aligned}
$$

Let $y(t) = u(t) - \vartheta z(t)$ with for any $z \in I^2(\tau, \tau+T; D(A)) \cap I^4(\tau, \tau+T; V)$ and $\vartheta \in [0,1]$, then

$$\begin{aligned} 0 \leq\ & 2\mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \langle -\widetilde{A}(u - \vartheta z) - \widetilde{B}(u - \vartheta z), \vartheta z \rangle ds \bigg) \\ & + 2\mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\Big(\widetilde{g}_1(s, u_s - \vartheta z) - \kappa_1, \vartheta z \Big) \Big) ds \bigg) \\ & - 2\mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota, \vartheta z \rangle ds \bigg) + \sigma\vartheta^2 \mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \|y(s)\|^2_{D(A)} \|z\|^2 ds \bigg). \end{aligned} \tag{3.55}$$

Dividing by ϑ on both sides of (3.55), and then letting $\vartheta \to 0$, we have

$$\mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota + \widetilde{A}u(s) + \widetilde{B}(u(s)), z \rangle ds \bigg) + \mathbb{E}\bigg(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\Big(\kappa_1 - \widetilde{g}_1(s, u_s), z \Big) \Big) ds \bigg) \leq 0. \tag{3.56}$$

Since $I^2(\tau, \tau+T; D(A)) \cap I^4(\tau, \tau+T; V)$ is dense in $I^2(\tau, \tau+T; V)$, we find

$$e^{-\sigma\varrho(s)} \Big(\iota + \widetilde{A}u(s) + \widetilde{B}(u(s)) + \kappa_1 - \widetilde{g}_1(s, u_s) \Big) = 0, \ \textit{a.e. } t \in [\tau, \tau+T], \ \omega \in \Omega. \tag{3.57}$$

Note that $\kappa_2 = \widetilde{g}_2(t, u_t), \ t \in [\tau, \tau+T]$, we can rewrite (3.46) as

$$\begin{aligned} & u(t) + \int_\tau^t \widetilde{A}u(s) ds + \int_\tau^t \widetilde{B}(u(s)) ds \\ & = \phi(0) + \int_\tau^t (\widetilde{f}(s) + \widetilde{g}_1(s, u_s)) ds + \int_\tau^t \widetilde{g}_2(s, u_s) dW(s), \ \textit{a.e. } t \in [\tau, \tau+T], \ \omega \in \Omega. \end{aligned} \tag{3.58}$$

Therefore, the existence of a weak solution has been proved.

Step 4: *We derive the estimate* (3.14). For each $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* τ_n^m as follows.

$$\tau_n^m = \inf \Big\{ t \leq \tau + T : \|u^m(t)\|^2 + \int_\tau^t \|u^m(s)\|^2_{D(A)} ds \geq n \Big\}. \tag{3.59}$$

For fixed m, the sequence $\{\tau_n^m; n \geq 1\}$ is increasing to $\tau + T$. By (3.42) and (3.59), we obtain

$$\mathbb{E}\bigg(\sup_{r \in [\tau, t \wedge \tau_n^m]} \|u_r^m\|^4_{C_\gamma(V)} \bigg) + \mathbb{E}\bigg(\int_\tau^{t \wedge \tau_n^m} \|u^m(s)\|^4 ds \bigg) \leq R, \ \forall\, t \in [\tau, \tau+T]. \tag{3.60}$$

Thanks to (3.60) and Fatou's lemma, we deduce that (3.14) holds for every $T > 0$.

Step 5: *We prove the uniqueness of solutions to Eq.* (3.1). Let u, v be two solutions of Eq. (3.1) with the same initial condition $u(s) = v(s) = \phi(s - \tau), \ s \leq \tau$, and let $\overline{w} := u - v$. For every $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* T_n by

$$T_n = \inf \Big\{ t \leq \tau + T : \int_\tau^t \|v(s)\|^2_{D(A)} ds \geq n \Big\}. \tag{3.61}$$

In addition, let $\varsigma(t) := e^{-\sigma \int_\tau^t \|v(s)\|^2_{D(A)} ds}$, where σ is given by (3.3) in Lemma 3.2 and $\mathbb{E}(\int_\tau^t \|v(s)\|^2_{D(A)} ds)$ is finite due to the steps 2-3. Applying Ito's formula to the process $\varsigma(t)\|\overline{w}(t)\|^2$, we infer from (3.3) in Lemma 3.2 that

$$\begin{aligned} \varsigma(t \wedge T_n) \|\overline{w}(t \wedge T_n)\|^2 = & -\sigma \int_\tau^{t \wedge T_n} \varsigma(s) \|v(s)\|^2_{D(A)} \|\overline{w}(s)\|^2 ds \\ & + 2 \int_\tau^{t \wedge T_n} \varsigma(s) \langle -\widetilde{A}\overline{w}(s) - \widetilde{B}(u) + \widetilde{B}(v), \overline{w}(s) \rangle ds \\ & + 2 \int_\tau^{t \wedge T_n} \varsigma(s) \Big(\Big(\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), \overline{w}(s) \Big) \Big) ds \\ & + 2 \int_\tau^{t \wedge T_n} \varsigma(s) \Big(\Big(\overline{w}(s), (\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)) dW(s) \Big) \Big) \end{aligned}$$

$$
\begin{aligned}
&+\int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\\
&\le -\int_\tau^{t\wedge T_n} \varsigma(s)\|\overline{w}(s)\|^2_{D(A)}ds + 2\int_\tau^{t\wedge T_n}\varsigma(s)\Big(\Big(\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s),\overline{w}(s)\Big)\Big)ds\\
&+2\int_\tau^{t\wedge T_n}\varsigma(s)\Big(\Big(\overline{w}(s),\big(\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\big)dW(s)\Big)\Big)\\
&+\int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds. \qquad (3.62)
\end{aligned}
$$

Taking supremum and expectation of (3.62), we find

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le r\le t}\varsigma(r\wedge T_n)\|\overline{w}(r\wedge T_n)\|^2\Big)+\mathbb{E}\Big(\int_\tau^t \varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2_{D(A)}ds\Big)\\
&\le 2\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\Big(\Big(\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s),\overline{w}(s)\Big)\Big)ds\Big|\Big)\\
&+2\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\Big(\Big(\overline{w}(s),\big(\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\big)dW(s)\Big)\Big)\Big|\Big)\\
&+\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big|\Big). \qquad (3.63)
\end{aligned}
$$

The Young inequality and (H5) imply

$$
\begin{aligned}
&2\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\Big(\Big(\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s),\overline{w}(s)\Big)\Big)ds\Big|\Big)\\
&\le \mathbb{E}\Big(\int_\tau^{t\wedge T_n}\varsigma(s)\|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2ds\Big)+\mathbb{E}\Big(\int_\tau^{t\wedge T_n}\varsigma(s)\|\overline{w}(s)\|^2ds\Big)\\
&\le (\widetilde{C}_{\widetilde{g}_1}+1)\mathbb{E}\Big(\int_\tau^t\varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2ds\Big)\\
&\le (\widetilde{C}_{\widetilde{g}_1}+1)\mathbb{E}\Big(\int_\tau^t\sup_{\tau\le\theta\le s}\varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2ds\Big). \qquad (3.64)
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality and (H5), we have

$$
\begin{aligned}
&2\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\Big(\Big(\overline{w}(s),\big(\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\big)dW(s)\Big)\Big)\Big|\Big)\\
&\le 2c_{10}\mathbb{E}\Big(\Big(\int_\tau^{t\wedge T_n}\Big(\varsigma^2(s)\|\overline{w}(s)\|^2\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}\Big)ds\Big)^{\frac{1}{2}}\Big)\\
&\le 2c_{10}\mathbb{E}\Big(\sup_{\tau\le s\le t}\varsigma^{1/2}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|\Big(\int_\tau^{t\wedge T_n}\varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac{1}{2}}\Big)\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le s\le t}\varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big)+c_{11}\mathbb{E}\Big(\int_\tau^{t\wedge T_n}\varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le s\le t}\varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big)+c_{11}\widetilde{C}_{\widetilde{g}_2}\int_\tau^t\mathbb{E}\Big(\sup_{\tau\le\theta\le s}\varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2\Big)ds, \qquad (3.65)
\end{aligned}
$$

where $c_{11}=2c_{10}^2$. By (H5), the last line of (3.63) is bounded by

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge T_n}\Big|\int_\tau^r\varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big|\Big)\\
&\le \mathbb{E}\Big(\int_\tau^{t\wedge T_n}\varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)
\end{aligned}
$$

$$\leq \widetilde{C}_{\widetilde{g}_2} \int_\tau^t \mathbb{E}\bigg(\sup_{\tau \leq \theta \leq s} \varsigma(\theta \wedge T_n) \|\overline{w}(\theta \wedge T_n)\|^2 \bigg) ds. \tag{3.66}$$

It follows from (3.63)-(3.66) that

$$\mathbb{E}\bigg(\sup_{\tau \leq r \leq t} \varsigma(r \wedge T_n) \|\overline{w}(r \wedge T_n)\|^2 \bigg) \leq c_{12} \int_\tau^t \mathbb{E}\bigg(\sup_{\tau \leq \theta \leq s} \varsigma(\theta \wedge T_n) \|\overline{w}(\theta \wedge T_n)\|^2 \bigg) ds, \ \forall\, t \in [\tau, \tau + T],$$

where $c_{12} = 2\Big(\widetilde{C}_{\widetilde{g}_1} + 1 + c_{11}\widetilde{C}_{\widetilde{g}_2} + \widetilde{C}_{\widetilde{g}_2}\Big)$. The Gronwall Lemma, together with $0 < \varsigma \leq 1$, implies

$$\mathbb{E}\bigg(\sup_{\tau \leq r \leq t} \|\overline{w}(r \wedge T_n)\|^2 \bigg) = 0, \ \forall\, t \in [\tau, \tau + T],$$

and thus,

$$u(r \wedge T_n) = v(r \wedge T_n), \text{ a.e., } \omega \in \Omega.$$

Furthermore, by Markov's inequality,

$$P(T_n < \tau + T) = P\bigg(\int_\tau^t \|v(s)\|^2_{D(A)} ds \geq n \bigg) \leq \frac{\mathbb{E}\Big(\int_\tau^t \|v(s)\|^2_{D(A)} ds \Big)}{n}.$$

We infer from $\mathbb{E}\Big(\int_\tau^t \|v(s)\|^2_{D(A)} ds \Big) < \infty$ that $T_n \to \tau + T$ as $n \to \infty$. Therefore, $u(r) = v(r)$, a.e., $\omega \in \Omega$ for all $r \leq \tau + T$. The proof is concluded. □

4. Stationary solutions and their stability results

In this section, we are concerned with existence, uniqueness and stability properties of the stationary solutions to (1.1). For this end, we need to assume that $\widetilde{f}(t) \equiv \widetilde{f} \in (D(A))^*$ (i.e. $f(t) \equiv f \in \mathbb{H}^{-1}(\mathcal{O})$), which is independent of the time.

4.1. Existence and uniqueness of stationary solutions

We now consider the abstract equation associated to Eq. (1.1):

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f} + \widetilde{g}_1(t, u_t) + \widetilde{g}_2(t, u_t)\dfrac{dW}{dt}, \quad \forall\, t > 0, \\ u(t) = \phi(t), t \in (-\infty, 0]. \end{cases} \tag{4.1}$$

We denote by $u(t) := u(t; \phi)$ the solution of (1.1) with $\tau = 0$, where $\phi = u_0$.

By a stationary solution to (4.1), we mean a constant solution (in other words, an equilibrium point) of (4.1). Therefore, $u_\infty \in D(A)$ will be a stationary solution if formally

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_1(t, u_\infty) + \widetilde{g}_2(t, u_\infty)\frac{dW}{dt}, \quad \forall\, t \geq 0. \tag{4.2}$$

However, this equation depends on t and a noisy term. Therefore, we would need to assume that $\widetilde{g}_1$ and $\widetilde{g}_2$ would not depend on t, moreover, to get rid of the noise, we must assume that $\widetilde{g}_2(t, u_\infty) = 0$.

Consequently, we will focus on the existence of stationary solutions for the deterministic equation (i.e. $\widetilde{g}_2 = 0$ in (4.1)) which will be any $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_1(t, u_\infty), \quad \forall\, t \geq 0, \tag{4.3}$$

and then analyze the behavior of the solutions to (4.1) around these stationary solutions of (4.3).

Now, in order to study the existence of solutions to (4.3), we have to restrict ourselves to assume that for constant elements $\xi \in C_\gamma(V)$, $\widetilde{g}_i(t, \xi)$ $(i = 1, 2)$ can be rewritten as

$$\widetilde{g}_i(t, \xi) = \widetilde{\mathcal{G}}_i(\xi^*) \text{ if } \xi(s) = \xi^*, \forall s \leq 0, \tag{4.4}$$

where $\widetilde{\mathcal{G}}_1 : V \to V$ with $\widetilde{\mathcal{G}}_1(0) = 0$, $\widetilde{\mathcal{G}}_2 : V \to \mathcal{L}^2(K, V)$ with $\widetilde{\mathcal{G}}_2(0) = 0$, they are Lipschitz continuous, that is, there exist $L_{\widetilde{\mathcal{G}}_i} > 0$ $(i = 1, 2)$, for all $\eta, \zeta \in V$,

$$\|\widetilde{\mathcal{G}}_1(\eta) - \widetilde{\mathcal{G}}_1(\zeta)\| \le L_{\widetilde{\mathcal{G}}_1}\|\eta - \zeta\|, \tag{4.5}$$

$$\|\widetilde{\mathcal{G}}_2(\eta) - \widetilde{\mathcal{G}}_2(\zeta)\|_{\mathcal{L}^2(K,V)} \le L_{\widetilde{\mathcal{G}}_2}\|\eta - \zeta\|. \tag{4.6}$$

For example, if $\widetilde{g}_i$ $(i = 1, 2)$ are driven by unbounded variable delay, defined by

$$\widetilde{g}_i(t, \xi) = \widetilde{\mathcal{G}}_i(\xi(-h(t))),\ \ i = 1, 2, \tag{4.7}$$

with $\widetilde{\mathcal{G}}_i$ satisfying conditions (4.4)-(4.6), where $h \in C^1([0, +\infty)), h(t) \ge 0$ and $h^* = \sup_{t\ge0} h'(t) < 1$. In this case, the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ in our problem become $\widetilde{g}_i(t, u_t) = \widetilde{\mathcal{G}}_i(u(t - h(t)))$.

Another example is the case of infinite distributed delay, that is, the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ are defined by

$$\widetilde{g}_i(t, \xi) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}_i(s, \xi(s))ds, \tag{4.8}$$

where $\widetilde{\mathcal{H}}_1 : (-\infty, 0] \times V \to V$ with $\widetilde{\mathcal{H}}_1(s, 0) = 0$, and $\widetilde{\mathcal{H}}_2 : (-\infty, 0] \times V \to \mathcal{L}^2(K, V)$ with $\widetilde{\mathcal{H}}_2(s, 0) = 0$ are measurable, and they are Lipschitz continuous with respect to their second variable, that is, there exist $L_{\widetilde{\mathcal{H}}_i}(s) \in L^2(-\infty, 0)$ $(i = 1, 2)$ with $L_{\widetilde{\mathcal{H}}_i}(\cdot)e^{-(\gamma+\theta)\cdot} \in L^2(-\infty, 0)$, for certain $\theta > 0$, such that for all $s \in (-\infty, 0], \eta, \zeta \in V$,

$$\|\widetilde{\mathcal{H}}_1(s, \eta) - \widetilde{\mathcal{H}}_1(s, \zeta)\| \le L_{\widetilde{\mathcal{H}}_1}(s)\|\eta - \zeta\|, \tag{4.9}$$

$$\|\widetilde{\mathcal{H}}_2(s, \eta) - \widetilde{\mathcal{H}}_2(s, \zeta)\|_{\mathcal{L}^2(K,V)} \le L_{\widetilde{\mathcal{H}}_2}(s)\|\eta - \zeta\|. \tag{4.10}$$

In this case, we can rewrite the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ in our problem as $\widetilde{g}_i(t, u_t) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}_i(s, u(t+s))ds$ $(i = 1, 2)$.

The above both situations are within our framework, the conditions (H1)-(H5) are fulfilled for the infinite distributed delay in $C_\gamma(V)$ for $\gamma > 0$, but not necessarily for the unbounded variable delay. However, conditions (H1)-(H5) are satisfied for both delays in $C_{-\infty}(V)$.

Now, we are interested in studying the existence and uniqueness of a stationary solution to Eq. (4.3).

Theorem 4.1. *Assume that the above assumptions and notations hold. If $\widetilde{\lambda}_1 > L_{\widetilde{g}_1}$, then:*

(a) For all $\widetilde{f} \in (D(A))^*$, then there exists at least one stationary solution to (4.3)*, which belongs to $D(A)$ if $\widetilde{f} \in V$;*

(b) If $(1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$, then the stationary solution to (4.3) *is unique.*

Proof. We can prove this result by using the same method as in [14, Theorem 10] (or [13, Lemma 3.2]), which is based on the Lax-Milgram, the Schauder theorems. Therefore, we omit the details. □

4.2. Local stability of stationary solutions

In this subsection, we will prove the local stability of stationary solutions to (4.3) for general delay terms by using a direct method and then apply the abstract results to two specific situations.

Theorem 4.2. *Suppose that the same hypotheses and notations in Theorem 3.4 and Theorem 4.1 hold. In addition, let*

$$2\widetilde{\lambda}_1 \ge \frac{2\widetilde{c}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2. \tag{4.11}$$

If $u(\cdot)$ is any solution of Eq. (4.1)*, u_∞ is the unique stationary solution of Eq.* (4.3) *and $w(t) = u(t) - u_\infty$, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \le \mathbb{E}\Big(\|w(0)\|^2\Big) + (C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2)\int_{-\infty}^{0} \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds. \tag{4.12}$$

Proof. Applying Ito's formula to $\|w(t)\|^2$, we obtain

$$\begin{aligned}\|w(t)\|^2 = \|w(0)\|^2 + 2\int_0^t \langle -\widetilde{A}w(s) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\ + 2\int_0^t \Big(\Big(\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty), w(s)\Big)\Big)ds + 2\int_0^t \Big(\Big(\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,u_\infty), w(s)\Big)\Big)dW(s) \\ + \int_0^t \|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,u_\infty)\|^2_{\mathcal{L}^2(K,V)}ds. \end{aligned} \tag{4.13}$$

Taking expectation of (4.13), thanks to Fubini's theorem,

$$\begin{aligned}\mathbb{E}\Big(\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds = \mathbb{E}\Big(\|w(0)\|^2\Big) - 2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \\ + 2\mathbb{E}\Big(\int_0^t \Big(\Big(\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty), w(s)\Big)\Big)ds\Big) \\ + \int_0^t \mathbb{E}\Big(\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,u_\infty)\|^2_{\mathcal{L}^2(K,V)}\Big)ds. \end{aligned} \tag{4.14}$$

By (2.13), (B2) and (3.7), we deduce

$$\begin{aligned}-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds &= -2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u_\infty, w(s)), w(s)\rangle\Big)ds \\ &\le 2\widetilde{c}\int_0^t \mathbb{E}(\|u_\infty\|\|w(s)\|^2_{D(A)})ds \\ &\le 2\widetilde{c}\widetilde{\lambda}_1^{-\frac{1}{2}}\int_0^t \mathbb{E}(\|u_\infty\|_{D(A)}\|w(s)\|^2_{D(A)})ds. \end{aligned} \tag{4.15}$$

By (2.13), (2.26) and (4.3), we find

$$\begin{aligned}\|u_\infty\|^2_{D(A)} &= \langle \widetilde{A}u_\infty, u_\infty\rangle \\ &= ((\widetilde{f}, u_\infty)) + ((\widetilde{g}_1(t,u_\infty), u_\infty)) \\ &\le \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u_\infty\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}\|u_\infty\|^2_{D(A)}, \end{aligned} \tag{4.16}$$

which, together with $\widetilde{\lambda}_1 > L_{\widetilde{g}_1}$, implies that

$$\|u_\infty\|_{D(A)} \le \frac{\widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}. \tag{4.17}$$

Thanks to (4.16)-(4.17), we can rewrite (4.15) as

$$-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \le \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}\int_0^t \mathbb{E}(\|w(s)\|^2_{D(A)})ds. \tag{4.18}$$

We now estimate the last two terms of (4.14) respectively. On the one hand, by (H4), (2.13) and the Young inequality, with $\epsilon_0 > 0$ to be specified later on, we deduce

$$\begin{aligned}&2\mathbb{E}\Big(\int_0^t \Big(\Big(\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty), w(s)\Big)\Big)ds\Big) \\ &\le 2\widetilde{\lambda}_1^{-\frac{1}{2}}\int_0^t \mathbb{E}\Big(\|\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty)\|\|w(s)\|_{D(A)}\Big)ds \\ &\le \frac{1}{\epsilon_0}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds + \epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}\int_{-\infty}^t \mathbb{E}\Big(\|w(s)\|^2\Big)ds \\ &\le \frac{1}{\epsilon_0}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds + \epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}\Big(\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds\end{aligned}$$

$$+\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big). \tag{4.19}$$

On the other hand, by (H4) and (2.13), we find

$$\begin{aligned}
&\int_0^t \mathbb{E}\Big(\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,u_\infty)\|^2_{\mathcal{L}^2(K,V)}\Big)ds\\
&\le C^2_{\widetilde{g}_2}\Big(\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds+\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big).
\end{aligned}\tag{4.20}$$

It follows from the above inequalities that

$$\begin{aligned}
\mathbb{E}\Big(\|w(t)\|^2\Big) &\le \mathbb{E}\Big(\|w(0)\|^2\Big)+\Big(\frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-2}C^2_{\widetilde{g}_1}+\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_2}-2\Big)\times\\
&\quad\Big(\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big)+(\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2})\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds\\
&\le \mathbb{E}\Big(\|w(0)\|^2\Big)+\widetilde{\lambda}_1^{-1}\Big(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{\widetilde{\lambda}_1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2}-2\widetilde{\lambda}_1\Big)\times\\
&\quad\Big(\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big)+(\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2})\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds.
\end{aligned}\tag{4.21}$$

In order to minimize the right-hand side of (4.21), we choose $\epsilon_0=\widetilde{\lambda}_1 C^{-1}_{\widetilde{g}_1}$ such that $\frac{\widetilde{\lambda}_1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}$ achieves its minimum value $2C_{\widetilde{g}_1}$. Then, by (4.11), we have (4.12) as desired. □

In what follows, we will discuss the local stability of stationary solutions to (4.3) when the delay terms have particular forms in $C_{-\infty}(V)$, and establish some sufficient conditions in the next corollaries. In this way, it is much easier for us to check the conditions than (4.11) in practical application.

Corollary 4.3. *Under the same hypotheses and notations in Theorem 3.4 and Theorem 4.1, let the delay terms* $\widetilde{g}_i(t,u_t)=\widetilde{\mathcal{G}}_i(u(t-h(t)))(i=1,2)$ *satisfy* (4.5)-(4.7)*, moreover,*

$$2\widetilde{\lambda}_1 \ge \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}+L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*} \tag{4.22}$$

is satisfied. If $u(\cdot)$ *is any solution of Eq.* (4.1)*,* u_∞ *is the unique stationary solution of Eq.* (4.3) *and* $w(t)=u(t)-u_\infty$*, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big)\le \mathbb{E}\Big(\|w(0)\|^2\Big)+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}+L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{4.23}$$

Proof. Taking $\widetilde{h}=s-h(s)$, we obtain $ds=1/(1-h'(s))d\widetilde{h}\le 1/(1-h^*)d\widetilde{h}$. Then, by (4.5), it follows

$$\begin{aligned}
\int_0^t \|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2ds &= \int_0^t \|\widetilde{\mathcal{G}}_1(u(s-h(s)))-\widetilde{\mathcal{G}}_1(v(s-h(s)))\|^2ds\\
&\le L^2_{\widetilde{\mathcal{G}}_1}\int_0^t \|u(s-h(s))-v(s-h(s))\|^2ds\\
&\le \frac{L^2_{\widetilde{\mathcal{G}}_1}}{1-h^*}\int_{-\infty}^t \|u(s)-v(s)\|^2ds,
\end{aligned}\tag{4.24}$$

Let $C_{\widetilde{g}_1}:=\frac{L_{\widetilde{\mathcal{G}}_1}}{\sqrt{1-h^*}}$, we deduce that, there exists $C_{\widetilde{g}_1}>0$ such that the first inequality in (H4) holds. Similarly, there exists $C_{\widetilde{g}_2}=\frac{L_{\widetilde{\mathcal{G}}_2}}{\sqrt{1-h^*}}>0$, such that the last one in (H4) holds. Thanks to (4.22), we find

$$2\widetilde{\lambda}_1 \ge \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}+L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}$$

$$\geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2, \tag{4.25}$$

which implies (4.11). Therefore, by Theorem 4.2, we obtain (4.12) as desired, and thus,

$$\begin{aligned}
\mathbb{E}\Big(\|w(t)\|^2\Big) &\leq \mathbb{E}\Big(\|w(0)\|^2\Big) + (C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2)\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds \\
&\leq \mathbb{E}\Big(\|w(0)\|^2\Big) + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L_{\widetilde{\mathcal{G}}_2}^2}{1-h^*}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \qquad (4.26)
\end{aligned}$$

The proof is concluded. □

Corollary 4.4. *Assume that the same hypotheses and notations in Theorem 3.4 and Theorem 4.1 hold. Let the delay terms* $\widetilde{g}_i(t,u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}_i(s,u(t+s))ds$ $(i=1,2)$ *satisfy* (4.8)-(4.10)*, moreover,*

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2\|L_{\widetilde{\mathcal{H}}_1}\|_{L^2(-\infty,0)} + \|L_{\widetilde{\mathcal{H}}_2}\|_{L^2(-\infty,0)}^2 \tag{4.27}$$

holds. If $u(\cdot)$ *is any solution of Eq.* (4.1)*,* u_∞ *is the unique stationary solution of Eq.* (4.3) *and* $w(t)=u(t)-u_\infty$*, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leq \mathbb{E}\Big(\|w(0)\|^2\Big) + \Big(\|L_{\widetilde{\mathcal{H}}_1}\|_{L^2(-\infty,0)} + \|L_{\widetilde{\mathcal{H}}_2}\|_{L^2(-\infty,0)}^2\Big)\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{4.28}$$

Proof. The proof is similar to the one of Corollary 4.3. It follows from (4.9) and (4.10) that there exist $C_{\widetilde{g}_i} = \|L_{\widetilde{\mathcal{H}}_i}\|_{L^2(-\infty,0)} > 0$ $(i=1,2)$ such that (H4) hold, and then by (4.27), we obtain (4.11). By Theorem 4.2, we deduce (4.28) as desired. □

Remark 4.5. In the case of infinite distributed delay, we can prove not only stability of stationary solutions in $C_{-\infty}(V)$ (see Corollary 4.4) even in $C_\gamma(V)$, but also their exponential asymptotic stability will be established as follows.

4.3. Exponential convergence of stationary solutions

Under suitable assumptions, we prove that the solution $u(t)$ to problem (4.1) with infinite distributed delay converges exponentially to the unique stationary solution u_∞ of Eq. (4.3) in $C_\gamma(V)$ for $\gamma > 0$.

Theorem 4.6. *Assume that the same hypotheses and notations in Theorem 3.4 and Theorem 4.1 hold. Let the delay terms* $\widetilde{g}_i(t,u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}_i(s,u(t+s))ds$ $(i=1,2)$ *satisfy* (4.9)-(4.10)*, and moreover, there exists a constant* $0<\rho<2\gamma$ *such that for all* $t \geq 0$*,*

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2(2\rho)^{-\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} + \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2 + \rho \tag{4.29}$$

is satisfied. If $u(\cdot)$ *is any solution of Eq.* (4.1)*,* u_∞ *is the unique stationary solution of Eq.* (4.3) *and* $w(t) = u(t)-u_\infty$*, then*

$$\begin{aligned}
\mathbb{E}\Big(\|w(t)\|^2\Big) \leq e^{-\rho t}\Big(1 + &\frac{1}{2\rho(2\gamma-\rho)}\big((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\
&+ \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\big)\Big)\mathbb{E}\Big(\|\phi-u_\infty\|_{C_\gamma(V)}^2\Big), \qquad (4.30)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\Big(\|w_t\|_{C_\gamma(V)}^2\Big) \leq e^{-\rho t}\Big(2 + &\frac{1}{2\rho(2\gamma-\rho)}\big((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\
&+ \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\big)\Big)\mathbb{E}\Big(\|\phi-u_\infty\|_{C_\gamma(V)}^2\Big). \qquad (4.31)
\end{aligned}$$

Proof. Applying Ito's formula to $e^{\rho t}\|w(t)\|^2$ with $0<\rho<2\gamma$, we find, for all $t\geq 0$,

$$\begin{aligned}
e^{\rho t}\|w(t)\|^2 = \|w(0)\|^2 &+ \rho\int_0^t e^{\rho s}\|w(s)\|^2 ds \\
&+ 2\int_0^t e^{\rho s}\langle -\widetilde{A}(w(s)) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\
&+ 2\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}_1(r,u(s+r)) - \widetilde{\mathcal{H}}_1(r,u_\infty)\big)dr, w(s)\Big)\Big)ds \\
&+ \int_0^t e^{\rho s}\Big\|\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}_2(r,u(s+r)) - \widetilde{\mathcal{H}}_2(r,u_\infty)\big)dr\Big\|^2_{\mathcal{L}^2(K,V)} ds \\
&+ 2\int_0^t e^{\rho s}\Big(\Big(w(s), \Big(\int_{-\infty}^0 \widetilde{\mathcal{H}}_2(r,u(s+r)) - \widetilde{\mathcal{H}}_2(r,u_\infty)dr\Big)dW\Big)\Big). \qquad (4.32)
\end{aligned}$$

Taking expectation of (4.32), then using (2.13), we obtain

$$\begin{aligned}
&\mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds \\
&\quad\leq \mathbb{E}\Big(\|\phi - u_\infty\|^2_{C_\gamma(V)}\Big) + \rho\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds \\
&\quad\quad - 2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \\
&\quad\quad + 2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}_1(r,u(s+r)) - \widetilde{\mathcal{H}}_1(r,u_\infty)\big)dr, w(s)\Big)\Big)ds\Big) \\
&\quad\quad + \mathbb{E}\Big(\int_0^t e^{\rho s}\Big\|\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}_2(r,u(s+r)) - \widetilde{\mathcal{H}}_2(r,u_\infty)\big)dr\Big\|^2_{\mathcal{L}^2(K,V)} ds\Big). \qquad (4.33)
\end{aligned}$$

Thanks to (4.18), we deduce

$$-2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \leq \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds. \qquad (4.34)$$

By (2.13), (4.9) and the Young inequality with $\hat{\epsilon}>0$ to be specified later on, the fourth line of (4.33) is bounded by

$$\begin{aligned}
&2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}_1(r,u(s+r)) - \widetilde{\mathcal{H}}_1(r,u_\infty)\big)dr, w(s)\Big)\Big)ds\Big) \\
&\leq 2\widetilde{\lambda}_1^{-\frac{1}{2}}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)\cdot\|w(s)\|_{D(A)}ds\Big) \\
&\leq \hat{\epsilon}\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)^2 ds\Big) + \frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|^2_{D(A)}ds\Big) \\
&=: \hat{\epsilon}\widetilde{\lambda}_1^{-1}I + \frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|^2_{D(A)}ds\Big), \qquad (4.35)
\end{aligned}$$

where I is estimated as follows. By the Hölder inequality,

$$\begin{aligned}
I &= \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)^2 ds\Big) \\
&\leq \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)e^{-\gamma r}\|w_s\|_{C_\gamma(V)}dr\Big)^2 ds\Big) \\
&= \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)e^{-(\gamma+\rho)r}e^{\rho r}dr\Big)^2 ds\Big) \\
&\leq \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}\Big(\int_{-\infty}^0 L^2_{\widetilde{\mathcal{H}}_1}(r)e^{-2(\gamma+\rho)r}dr\int_{-\infty}^0 e^{2\rho r}dr\Big)ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}ds\Big)\\
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\times\\
&\quad \mathbb{E}\bigg(\int_0^t e^{\rho s}\max\Big\{\sup_{\theta\le -s}e^{2\gamma\theta}\|w(s+\theta)\|^2,\ \sup_{\theta\in[-s,0]}e^{2\gamma\theta}\|w(s+\theta)\|^2\Big\}ds\bigg)\\
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\bigg(\int_0^t\Big(e^{-(2\gamma-\rho)s}\|\phi-u_\infty\|^2_{C_\gamma(V)}\\
&\quad +\widetilde{\lambda}_1^{-1}\sup_{\theta\in[-s,0]}e^{(2\gamma-\rho)\theta}e^{\rho(s+\theta)}\|w(s+\theta)\|^2_{D(A)}\Big)ds\bigg).
\end{aligned}\tag{4.36}
$$

Thanks to (4.10), we obtain the following result by using the same method in (4.36):

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^t e^{\rho s}\Big\|\int_{-\infty}^0\big(\widetilde{\mathcal{H}}_2(r,u(s+r))-\widetilde{\mathcal{H}}_2(r,u_\infty)\big)dr\Big\|^2_{\mathcal{L}^2(K,V)}ds\Big)\\
&\quad\le \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_2}(r)\|w(s+r)\|dr\Big)^2ds\Big)\\
&\quad\le \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_2}(r)e^{-\gamma r}\|w_s\|_{C_\gamma(V)}dr\Big)^2ds\Big)\\
&\quad\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\bigg(\int_0^t\Big(e^{-(2\gamma-\rho)s}\|\phi-u_\infty\|^2_{C_\gamma(V)}\\
&\qquad +\widetilde{\lambda}_1^{-1}\sup_{\theta\in[-s,0]}e^{(2\gamma-\rho)\theta}e^{\rho(s+\theta)}\|w(s+\theta)\|^2_{D(A)}\Big)ds\bigg).
\end{aligned}\tag{4.37}
$$

Substituting (4.34)-(4.37) into (4.33), then by $0<\rho<2\gamma$, we have

$$
\begin{aligned}
\mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big)&\le \mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)+\widetilde{\lambda}_1^{-1}\bigg(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{\hat\epsilon}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\frac{\widetilde{\lambda}_1}{\hat\epsilon}\\
&\quad+\frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\rho-2\widetilde{\lambda}_1\bigg)\int_0^t\mathbb{E}\Big(\max_{r\in[0,s]}\{e^{\rho r}\|w(r)\|^2_{D(A)}\}\Big)ds\\
&\quad+\frac{1}{2\rho}\Big(\hat\epsilon\widetilde{\lambda}_1^{-1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\Big)\times\\
&\quad\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)\int_0^t e^{-(2\gamma-\rho)s}ds.
\end{aligned}\tag{4.38}
$$

Notice that

$$
\min_{\hat\epsilon>0}\Big\{\frac{\hat\epsilon}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\frac{\widetilde{\lambda}_1}{\hat\epsilon}\Big\}=2(2\rho)^{-\frac12}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)},\tag{4.39}
$$

which is achieved by $\hat\epsilon=(2\rho)^{\frac12}\widetilde{\lambda}_1\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^{-1}_{L^2(-\infty,0)}$. Then, we infer from (4.38), (4.39) and (4.29) that (4.30) holds.

By (4.30), and by $0<\rho<2\gamma$, and thus $e^{(2\gamma-\rho)\theta}\le 1$ when $\theta\le 0$, we find, for all $t\ge 0$,

$$
\begin{aligned}
\mathbb{E}\Big(\|w_t\|^2_{C_\gamma(V)}\Big)&=\mathbb{E}\Big(\sup_{\theta\le 0}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big)\\
&=\mathbb{E}\Big(\max\Big\{\sup_{\theta\in(-\infty,-t]}e^{2\gamma\theta}\|\phi(t+\theta)-u_\infty\|^2,\ \sup_{\theta\in[-t,0]}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)\\
&=\mathbb{E}\Big(\max\Big\{e^{-2\gamma t}\|\phi-u_\infty\|^2_{C_\gamma(V)},\ \sup_{\theta\in[-t,0]}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)\\
&\le \max\Big\{e^{-\rho t}\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big),e^{-\rho t}\Big(1+\frac{1}{2\rho(2\gamma-\rho)}\big((2\rho)^{\frac12}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}
\end{aligned}
$$

$$+\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\Big)\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)\Big\}$$

$$\le e^{-\rho t}\Big(2+\frac{1}{2\rho(2\gamma-\rho)}\big((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}$$

$$+\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\big)\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big). \tag{4.40}$$

Therefore, the proof is complete. □

Remark 4.7. In Section 4.2, we only analyzed the stability (rather than asymptotic stability) in the case of unbounded variable delay and proved in the current subsection the exponential stability in the particular case of distributed delay in $C_\gamma(V)$. Therefore, next, we are interested in studying the asymptotic stability for such variable delay in $C_{-\infty}(V)$. More precisely, on the one hand, we will prove the asymptotic stability by the method of Lyapunov functionals construction. On the other hand, we will prove the polynomial asymptotic stability with proportional delay, which is a particular case of unbounded variable delay.

4.4. Asymptotic stability: the Lyapunov functional method

In this subsection, we first investigate the asymptotic stability of the trivial solution of the following abstract nonlinear stochastic partial functional differential systems by constructing suitable Lyapunov functionals. In the last part of this subsection, we will apply the abstract results to Eq. (1.1).

Now, we consider the following problem:

$$\begin{cases} du(t)=(\hat{A}(t,u(t))+\hat{F}_1(t,u_t))dt+\hat{F}_2(t,u_t)dW(t), \quad \forall\ t\in[0,T], \\ u(t)=\phi(t),\ t\in(-\infty,0], \end{cases} \tag{4.41}$$

where $\hat{A}(t,\cdot):D(A)\to(D(A))^*$ satisfies $\langle\hat{A}(t,u),u\rangle\le 0$, for all $u\in D(A)$, $\hat{F}_1(t,\cdot):C_{-\infty}(V)\to V$ and $\hat{F}_2(t,\cdot):C_{-\infty}(V)\to\mathcal{L}^2(K,V)$ satisfy the following conditions: $\hat{F}_1(t,0)=\hat{F}_2(t,0)=0$ and they are Lipschitz continuous, that is, there exist $L_{\hat{F}_i}>0$ $(i=1,2)$ such that for all $t\ge 0$ and $\eta,\zeta\in C_{-\infty}(V)$,

$$\|\hat{F}_1(t,\eta)-\hat{F}_1(t,\zeta)\|\le L_{\hat{F}_1}\|\eta-\zeta\|_{C_{-\infty}(V)},$$
$$\|\hat{F}_2(t,\eta)-\hat{F}_2(t,\zeta)\|_{\mathcal{L}^2(K,V)}\le L_{\hat{F}_2}\|\eta-\zeta\|_{C_{-\infty}(V)}. \tag{4.42}$$

By the similar estimates as in Section 3, the well-posedness of (4.41) can be proved. Fixed $T>0$ and given an initial value $\phi\in L^2(\Omega,C_{-\infty}(V))$, a solution to (4.41) is a stochastic process $u\in I^2(0,T;D(A))\cap L^2(\Omega,L^\infty(0,T;V))$ satisfying

$$\begin{cases} u(t)=\phi(0)+\displaystyle\int_0^t\hat{A}(s,u(s))ds+\int_0^t\hat{F}_1(s,u_s))ds \\ \qquad\quad +\displaystyle\int_0^t\hat{F}_2(s,u_s)dW(s),\ P-a.s.,\ \forall\ t\in[0,T], \\ u(t)=\phi(t),\ t\in(-\infty,0], \end{cases} \tag{4.43}$$

where the first equation is understood in $(D(A))^*$.

We denote by $u(\cdot;\phi)$ the solution of Eq. (4.41) corresponding to the initial condition ϕ.

Definition 4.8. The trivial solution of Eq. (4.41) is said to be p-stable, with $p>0$, if for any $\epsilon>0$, there exists $\delta>0$ such that $\mathbb{E}(\|u(t;\phi)\|^p)<\epsilon$, for all $t\ge 0$, provided that $\|\phi\|_1^p:=\sup_{\theta\le 0}\mathbb{E}(\|\phi(\theta)\|^p)<\delta$. If, besides, $\lim_{t\to+\infty}\mathbb{E}(\|u(t;\phi)\|^p)=0$ for every initial function ϕ, then the trivial solution of Eq. (4.41) is called asymptotically p-stable. In particular, if $p=2$, then the trivial solution of the system (4.41) is called asymptotically mean square stable.

Consider the stochastic differential of the process $\eta(t)=x(t,u(t))$, where $u(t)$ is a solution of the system (4.41) and the function $x:[0,\infty)\times D(A)\to\mathbb{R}_+$ has continuous partial derivatives:

$$x'_t=\frac{\partial x(t,u)}{\partial t},\ x'_u=\frac{\partial x(t,u)}{\partial u},\ x''_{uu}=\frac{\partial^2 x(t,u)}{\partial u^2}.$$

Applying Ito's formula to $\eta(t)$ we obtain

$$d\eta(t) = Lx(t, u(t))dt + \langle x'_u, \hat{F}_2(t, u_t)dW(t)\rangle, \tag{4.44}$$

where $\langle\cdot,\cdot\rangle$ denotes inner products in Hilbert spaces, and L is called the generator of Eq. (4.41), defined by

$$Lx(t, u_t) = x'_t(t, u(t)) + \langle x'_u(t, u(t)), \hat{A}(t, u(t)) + \hat{F}_1(t, u_t)\rangle + \frac{1}{2}\mathrm{tr}\Big(x''_{uu}(t, u(t))\hat{F}_2(t, u_t)Q\hat{F}_2^*(t, u_t)\Big).$$

We then apply the generator L to some functionals $U(t,\xi) : [0,\infty) \times L^2(\Omega, C_{-\infty}(V)) \to \mathbb{R}_+$. In addition, assume that $U(t,\xi) = U(t,\xi(0),\xi(\theta)), \theta < 0$, and for $\xi = u_t$, then set

$$\begin{aligned}&U_\xi(t, u) = U(t, \xi) = U(t, u_t) = U(t, u, u(t + \theta)),\ \theta < 0,\\ &u = \xi(0) = u(t).\end{aligned} \tag{4.45}$$

Let D be the universe of functionals which satisfy conditions (4.45). Any functional $U_\xi(t,u) \in D$ has a continuous derivative with respect to t and two continuous derivatives with regard to u. Then,

$$\begin{aligned}LU(t, u_t) = &\frac{\partial U_\xi(t, u(t))}{\partial t} + \Big\langle \frac{\partial U_\xi(t, u(t))}{\partial u}, \hat{A}(t, u(t)) + \hat{F}_1(t, u_t)\Big\rangle\\ &+ \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 U_\xi(t, u(t))}{\partial u^2}\hat{F}_2(t, u_t)Q\hat{F}_2^*(t, u_t)\Big).\end{aligned}$$

Thanks to Ito's formula, we obtain, for functionals from D,

$$\mathbb{E}\Big(U(t, u_t) - U(s, u_s)\Big) = \int_s^t \mathbb{E}\Big(LU(r, u_r)\Big)dr, \quad t \geq s. \tag{4.46}$$

In the next proposition, we generalize the idea of Shaikhet in [38, Theorem 2.1] to the infinite delay version of stochastic partial differential equations. Let us now prove the following result which plays an important role in our stability investigation.

Proposition 4.9. *Suppose that there exists a continuous functional $U(t,\xi) : [0,\infty) \times L^p(\Omega, C_{-\infty}(V)) \to \mathbb{R}_+$ such that for the solution $u(t)$ of problem* (4.41) *and $p \geq 2$, the following inequalities hold for some positive constants μ_1, μ_2 and μ_3,*

$$\begin{aligned}&\mathbb{E}(U(t, u_t)) \geq \mu_1\mathbb{E}(\|u(t)\|^p),\ \forall\ t \geq 0,\\ &\mathbb{E}(U(0, \phi)) \leq \mu_2\|\phi\|_1^p,\\ &\mathbb{E}(U(t, u_t) - U(0, \phi)) \leq -\mu_3\int_0^t \mathbb{E}(\|u(s)\|^p)ds,\ \forall\ t \geq 0.\end{aligned} \tag{4.47}$$

Then the trivial solution of equation (4.41) *is asymptotically p-stable, that is,*

$$\lim_{t\to+\infty} \mathbb{E}(\|u(t)\|^p) = 0. \tag{4.48}$$

Proof. We infer from (4.47) that

$$\mu_1\mathbb{E}(\|u(t)\|^p) \leq \mathbb{E}(U(t, u_t)) \leq \mathbb{E}(U(0, \phi)) \leq \mu_2\|\phi\|_1^p = \mu_2 \sup_{\theta\leq 0}\mathbb{E}(\|\phi(\theta)\|^p), \tag{4.49}$$

which proves the trivial solution of equation (4.41) is p-stable. Taking supremum of (4.49) with respect to t, we find

$$\sup_{t\geq 0}\mathbb{E}(\|u(t)\|^p) \leq \frac{\mu_2}{\mu_1}\|\phi\|_1^p. \tag{4.50}$$

Thanks to the last two lines of (4.47), we obtain

$$\int_0^\infty \mathbb{E}(\|u(s)\|^p)ds \leq \frac{1}{\mu_3}\mathbb{E}(U(0, \phi)) \leq \frac{\mu_2}{\mu_3}\|\phi\|_1^p < \infty. \tag{4.51}$$

Applying the generator L to the function $U(t, u_t) = \|u(t)\|^p$, by the Young inequality and (4.42), we have

$$\begin{aligned}
LU(t, u_t) &= L\|u(t)\|^p = p\|u\|^{p-2}\langle \hat{A}(u), u\rangle + p\|u\|^{p-2}((\hat{F}_1(t, u_t), u)) \\
&\quad + \frac{p}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t, u_t)\|^2_{\mathcal{L}^2(K,V)} + \frac{p(p-2)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t, u_t)\|^2_{\mathcal{L}^2(K,V)} \\
&\leq p\|u\|^{p-2}((\hat{F}_1(t, u_t), u)) + \frac{p(p-1)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t, u_t)\|^2_{\mathcal{L}^2(K,V)} \\
&\leq p\|u\|^{p-2}(\|\hat{F}_1(t, u_t)\| \cdot \|u\|) + \frac{p(p-1)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t, u_t)\|^2_{\mathcal{L}^2(K,V)} \\
&\leq \frac{p}{2}\|u(t)\|^p + \frac{p}{2}L^2_{\hat{F}_1}\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)} + \frac{p(p-1)}{2}L^2_{\hat{F}_2}\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)} \\
&= \frac{p}{2}\|u(t)\|^p + \hat{c_1}\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)} \\
&\leq \Big(\frac{p}{2} + \hat{c_2}\Big)\|u(t)\|^p + \hat{c_3}\|u_t\|^p_{C_{-\infty}(V)} \\
&\leq \Big(\frac{p}{2} + \hat{c_2} + \hat{c_3}\Big)\|u_t\|^p_{C_{-\infty}(V)} =: \hat{c_4}\|u_t\|^p_{C_{-\infty}(V)},
\end{aligned} \tag{4.52}$$

where $\hat{c_1} = \frac{p}{2}L^2_{\hat{F}_1} + \frac{p(p-1)}{2}L^2_{\hat{F}_2}$, $\hat{c_2} = \frac{\hat{c_1}(p-2)}{p}$ and $\hat{c_3} = \frac{2\hat{c_1}}{p}$. The above inequality implies

$$\mathbb{E}(LU(t, u_t)) \leq \hat{c_4}\mathbb{E}(\|u_t\|^p_{C_{-\infty}(V)}) =: \hat{c_5} < \infty. \tag{4.53}$$

Combining (4.46) and (4.53) , we obtain, for any $t \geq s \geq 0$,

$$\big|\mathbb{E}(\|u(t)\|^p) - \mathbb{E}(\|u(s)\|^p)\big| \leq \hat{c_5}(t - s), \tag{4.54}$$

which implies that $\mathbb{E}(|u(r)|^p)$ is Lipschitz continuous, together with (4.50) and (4.51), shows that $\mathbb{E}(\|u(t)\|^p) \to 0$ as $t \to +\infty$. This completes the proof. □

We state our asymptotic stability result by applying the previous abstract results to our model in the next theorem.

Theorem 4.10. *Assume that the same hypotheses and notations in Theorem 3.4 and Theorem 4.1 hold. In addition, let the delay terms $\widetilde{g}_i(t, u_t) = \widetilde{\mathcal{G}}_i(u(t - h(t)))(i = 1, 2)$ satisfy (4.5)-(4.7), $\widetilde{f} = 0$ and*

$$2\widetilde{\lambda}_1 \geq \frac{2(1 - h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1 - h^*}. \tag{4.55}$$

Then $u_\infty = 0$ is the unique stationary solution to problem (4.3). Moreover, the trivial solution of (4.1) is asymptotically mean square stable.

Proof. We first infer from the assumption $\widetilde{f} = 0$ and Theorem 4.1 that $u_\infty = 0$ is the unique stationary solution to Eq. (4.3). We then let

$$U(t, \xi) = \|\xi(0)\|^2 + \frac{(1 - h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1 - h^*}\int_{-h(t)}^0 \|\xi(s)\|^2 ds, \tag{4.56}$$

if ξ is replaced by u_t, then

$$U(t, u_t) = \|u(t)\|^2 + \frac{(1 - h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1 - h^*}\int_{t-h(t)}^t \|u(s)\|^2 ds, \tag{4.57}$$

and then let $\hat{A}(t, u) = -\widetilde{A}u(t) - \widetilde{B}(u(t))$, $\hat{F}_1(t, u_t) = \widetilde{g}_1(t, u_t) = \widetilde{\mathcal{G}}_1(u(t - h(t)))$, $\hat{F}_2(t, u_t) = \widetilde{g}_2(t, u_t) = \widetilde{\mathcal{G}}_2(u(t - h(t)))$ in (4.41), by (2.11), (2.13), (4.5) and (4.6), we obtain

$$\begin{aligned}
L\|u(t)\|^2 &= 2\langle -\widetilde{A}(u) - \widetilde{B}(u), u\rangle + 2((\widetilde{\mathcal{G}}_1(u(t - h(t))), u)) + \|\widetilde{\mathcal{G}}_2(u(t - h(t)))\|^2_{\mathcal{L}^2(K,V)} \\
&\leq -2\|u\|^2_{D(A)} + 2\|\widetilde{\mathcal{G}}_1(u(t - h(t)))\|\|u\| + L^2_{\widetilde{\mathcal{G}}_2}\|u(t - h(t))\|^2
\end{aligned}$$

$$\leq -2\widetilde{\lambda}_1\|u\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}}{(1-h^*)}\|u\|^2 + (1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}\|u(t-h(t))\|^2 + L^2_{\widetilde{\mathcal{G}}_2}\|u(t-h(t))\|^2$$

$$= \Big(-2\widetilde{\lambda}_1 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}}{(1-h^*)}\Big)\|u\|^2 + \Big((1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}\Big)\|u(t-h(t))\|^2, \tag{4.58}$$

then

$$LU(t,u_t) = L\bigg(\|u(t)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{t-h(t)}^t \|u(s)\|^2 ds\bigg)$$

$$\leq L\|u(t)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\|u(t)\|^2 - \Big((1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}\Big)\|u(t-h(t))\|^2$$

$$\leq \bigg(-2\widetilde{\lambda}_1 + \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\bigg)\|u(t)\|^2, \tag{4.59}$$

which, on account of (4.55), implies $LU(t,u_t) \leq 0$. Moreover, the functional $U(t,u_t)$ defined in (4.57) satisfies the conditions in Proposition 4.9, and thus the trivial solution of (4.1) is asymptotically mean square stable in the sense of Definition 4.8. □

Remark 4.11. By using the method of Lyapunov functionals construction, we obtain the asymptotic stability of the trivial solution to (4.1) with unbounded variable delay. Notice that condition (4.55) becomes exactly condition (4.22) when $\tilde{f} = 0$. Therefore, Theorem 4.10 ensures asymptotic stability under the same sufficient conditions which ensures only stability in Corollary 4.3, which means that the construction of Lyapunov functionals may provide better stability results. Furthermore, our analysis is also valid to study the asymptotic stability for the general case, that is, if the stationary solution is not the origin, in this case, we can shift it to the origin by a coordinate transformation.

4.5. Polynomial asymptotic stability for a particular case of unbounded variable delay

In this subsection, we study the polynomial asymptotic behaviour of solutions to deterministic pantograph equations. In the particular case of proportional delay, we not only prove asymptotic stability but we can determine that the rate of convergence is at least polynomial. Now, let us consider the following deterministic pantograph equation:

$$\begin{cases} X'(t) = a_1X(t) + a_2X(\theta t), \ \forall\, t \geq 0, \\ X(0) = X_0, \end{cases} \tag{4.60}$$

where $a_1, a_2 \in \mathbb{R}$, and $\theta \in (0,1)$.

Recall that the Dini derivative D^+F, where F is a continuous real-valued function of a real variable defined by

$$D^+F = \limsup_{\delta \downarrow 0} \frac{F(t+\delta) - F(t)}{\delta}.$$

Thanks to [2, Lemma 3.4], we present the following result which is useful to obtain the polynomial asymptotic stability of stationary solutions to (4.60).

Lemma 4.12. *Let $a_1 \in \mathbb{R}, a_2 > 0$ and $\theta \in (0,1)$. Assume that X satisfies* (4.60) *with $X_0 > 0$. If there exists a continuous non-negative function $t \mapsto Y(t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$,*

$$D^+Y(t) \leq a_1Y(t) + a_2Y(\theta t), \quad t \geq 0 \tag{4.61}$$

with $0 < Y(0) < X_0$, then $Y(t) \leq X(t)$ for all $t \geq 0$.

Lemma 4.13. *Assume that X is the solution of* (4.60)*. If $a_1 < 0$ and $a_2 \in \mathbb{R}$, there exists a constant $M_0 = M_0(a_1,a_2,\theta) > 0$,*

$$\limsup_{t\to+\infty} \frac{|X(t)|}{t^\beta} = M_0|X_0|, \tag{4.62}$$

where $\beta \in \mathbb{R}$ satisfies

$$a_1 + |a_2|\theta^{\beta} = 0. \tag{4.63}$$

Then, for some $M = M(a_1, a_2, \theta) > 0$,

$$|X(t)| \le M|X_0|(1+t)^{\beta}, \quad t \ge 0. \tag{4.64}$$

Proof. The proof is similar to [2, Lemma 3.5], thus the details are omitted here. □

Note that the polynomial asymptotic stability of the trivial solution to (4.60) is presented in the above Lemma when $\beta < 0$. In the following, we apply the idea to derive the polynomial asymptotic stability of stationary solution to (4.1).

Theorem 4.14. *Assume that the same hypotheses and notations in Theorem 3.4 and Theorem 4.1 hold. In addition, let the system (4.1) satisfy $\widetilde{f} = 0$, the delay terms $\widetilde{g}_i(t, u_t) = L_{\widetilde{g}_i} u(\theta t)$ $(i = 1, 2)$ with $\theta \in (0, 1)$ and $2\widetilde{\lambda}_1 > 2|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2}$, then the origin is the unique stationary solution of Eq. (4.3), moreover, any solution $u(t)$ of Eq. (4.1) converges to zero polynomially, that is, there exist $\widetilde{M} = \widetilde{M}(L_{\widetilde{g}_1}, L_{\widetilde{g}_2}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta < 0$,*

$$\mathbb{E}(\|u(t;\phi)\|^2) \le \widetilde{M}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^{\beta}, \quad t \ge 0, \tag{4.65}$$

where β satisfies $-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}| + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\theta^{\beta} = 0$.

Proof. The conclusion that the origin is the unique stationary solution of Eq. (4.3) follows from $\widetilde{f} = 0$ and Theorem 4.1. Applying Ito's formula to $\|u(t)\|^2$, then taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}(\|u(t)\|^2) - \mathbb{E}(\|u(0)\|^2) \\
&\le -2\mathbb{E}\Big(\int_0^t \|u(s)\|^2_{D(A)} ds\Big) + |L_{\widetilde{g}_1}|\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big) \\
&\le (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}|)\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big), \ \forall\, t > 0,
\end{aligned}$$

where we used (2.13). Let $v(t) = \mathbb{E}(\|u(t)\|^2)$, then

$$v'(t) \le (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}|)v(t) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})v(\theta t). \tag{4.66}$$

By Lemmas 4.12-4.13, we obtain that there exist $\widetilde{M} = \widetilde{M}(L_{\widetilde{g}_1}, L_{\widetilde{g}_2}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta \in \mathbb{R}$,

$$v(t) \le \widetilde{M}v(0)(1+t)^{\beta}. \tag{4.67}$$

Since $-2\widetilde{\lambda}_1 + 2|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2} < 0$, we deduce $\beta < 0$ and

$$\mathbb{E}(\|u(t)\|^2) \le \widetilde{M}\mathbb{E}(\|\phi\|^2)(1+t)^{\beta} \le \widetilde{M}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^{\beta}.$$

The proof is complete. □

Remark 4.15. As a matter of fact, we can take into account a more general case in the form of $\widetilde{g}_i(t, \xi) = \widetilde{G}_i(\xi(-(1-\theta)t))$, where $\widetilde{G}_i(\cdot)$ is Lipschitz continuous.

5. Conclusion

On the one hand, we proved some results on the existence and uniqueness of the solutions for a stochastic three-dimensional Lagrangian-averaged Navier-Stokes model with infinite delay. On the other hand, the stability and asymptotic stability of stationary solutions are established. We first proved the local stability of stationary solutions for general delay terms by using a direct method. It is worth mentioning that all conditions are general enough to include several kinds of delays, where we mainly consider unbounded variable delays and infinite distributed delays. As we know, it is still an open and challenging problem to obtain sufficient conditions ensuring the exponential stability of solutions in case of unbounded variable delay. Fortunately, we obtained the exponential stability of stationary solutions in the case of infinite distributed delay. However, we are able to further investigate the asymptotic ability of stationary solutions in the case of unbounded variable delay by constructing suitable Lyapunov functionals. Besides, we proved the polynomial asymptotic stability of stationary solutions for the particular case of proportional delay.

Acknowledgments

The research of the first author has been partially supported by CSC (China Scholarship Council, No. 202006990054) for 1 year's study abroad at the University of Sevilla. The research of the first three authors has been partially supported by the National Natural Science Foundation of China (No. 11571283). The research of the first and fourth authors has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00, and by Junta de Andalucía (Consejería de Economía y Conocimiento) under project US-1254251.

Data availability

We confirm that all the data necessary for the research carried out in this paper are included here.

Conflict of interest

We confirm that we do not have any conflict of interest.

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