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Random attractors for a stochastic nonlocal delayed reaction–diffusion equation on a semi-infinite interval

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The aim of this paper is to prove the existence and qualitative property of random attractors for a stochastic non-local delayed reaction–diffusion equation (SNDRDE) on a semi-infinite interval with a Dirichlet boundary condition at the finite end. This equation models the spatial–temporal evolution of the mature individuals for a two-stage species whose juvenile and adults both diffuse that lives on a semi-infinite domain and subject to random perturbations. By transforming the SNDRDE into a random evolution equation with delay, by means of a stationary conjugate transformation, we first establish the global existence and uniqueness of solutions to the equation, after which we show the solutions generate a random dynamical system. Then, we deduce uniform a priori estimates of the solutions and show the existence of bounded random absorbing sets. Subsequently, we prove the pullback asymptotic compactness of the random dynamical system generated by the SNDRDE with respect to the compact open topology, and hence obtain the existence of random attractors. At last, it is proved that the random attractor is an exponentially attracting stationary solution under appropriate conditions. The theoretical results are illustrated by application to the stochastic non-local delayed Nicholson's blowfly equation.

Keywords: Random attractor; stochastic delayed reaction–diffusion equations; semi-infinite interval; non-local; age-structured population model.

1. Introduction

When modelling the growth of mature population of a two-stage species (juvenile and adult, with a fixed maturation time τ) whose mature individuals and immature individuals both diffuse, one faces the following delayed reaction–diffusion equation with spatial non-locality derived in So *et al.* (2001):

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \mu u(t,x) + \varepsilon \int_{\mathcal{O}} \Gamma(\alpha,x,y) f(u(t-\tau,y))\mathrm{d}y,\ (t,x) \in (0,\infty) \times D. \tag{1.1}$$

Here, $\mathcal{O} \subseteq \mathbb{R}^N$ is the spatial domain, and $u(t,x)$ stands for the total mature population at location x and time t. The positive constant μ represents the death rate of the mature population. ε and the immature mobility constant α are defined by $\varepsilon = e^{-\int_0^r d_I(a)\mathrm{d}a}$ and $\alpha = \int_0^r D_I(a)\mathrm{d}a$, where $d_I(a)$ and $D_I(a)$, $a \in [0,\tau]$, are the age-dependent death rate and diffusion rate of the immature population of the species, respectively. The diffuse kernel $\Gamma(\alpha,x,y)$ is obtained by integrating along the characteristic based on the general model in Metz & Diekmann (1986), representing the probability of the new-born individuals located at y that can survive to be matured and moved to location x. Generally speaking, explicit forms of $\Gamma(\alpha,x,y)$ can only be obtained for some special cases, see Liang *et al.* (2003). When the spatial domain $\mathcal{O}$ is bounded with a Dirichlet boundary value condition (DBVC), existence, uniqueness and attractiveness of the positive steady state and threshold dynamics are important and have been explored by Yi & Zou (2013). When a Neumann boundary value condition (NBVC) is imposed, Zhao (2009) established the global attractiveness of the positive steady state of (1.1) by adopting a fluctuation method.

In the real world, there are also species whose individuals live in the whole space, such as the fishes in the infinite ocean. In the situation $\mathcal{O} = \mathbb{R}$, the lack of compactness of the infinite domains and the complexity of non-local delayed term cause the global dynamics analysis of (1.1) becomes quite difficult and hence, the existing works mainly focus on the travelling wave solutions. See, for instance, So *et al.* (2001); Wu & Zou (2001); Yi & Zou (2015). To circumvent this difficulty, Yi *et al.* (2012) made a first attempt to describe the global dynamics of model (1.1) by adopting the compact open topology combined with delicate analysis of the asymptotic properties of the non-local term and the diffusion operator. Moreover, there are also species whose individuals live in a semi-infinite domain which is neither bounded nor is the whole space. For example, animals living in a big land that has the shore of an ocean or a lake at one side of the land provides such a scenario. For the species whose individuals live in a semi-infinite domain, the kernel function and the spatial domain are neither symmetric nor compact, implying the problem becomes more challenging. Recently, Yi & Zou (2016) derived the kernel

$$\Gamma(\alpha,x,y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}} - \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x+y)^2}{4\alpha}} \tag{1.2}$$

in the scenario $\mathcal{O} = \mathbb{R}_+ = [0,\infty)$ with the homogeneous DBVC at the finite end, and investigated global dynamics of Eq. (1.1). The results have been extended to the half plane case, which are more realistic in real world modelings by Hu & Duan (2018), Hu *et al.* (2018) and Wang (2014).

However, the evolution of the mature population is inevitably affected by random perturbations, including the noise generated by the internal self-excitation of the system and the random interference of the external environment. Consequently, for the species living on $\mathcal{O} = \mathbb{R}_+$ that are perturbed by some random effects, a more accurate mathematical model should be the following stochastic non-local delayed reaction–diffusion equation (SNDRDE):

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \varepsilon \int_{\mathcal{O}} \Gamma(\alpha,x,y) f(u(t-\tau,y))\mathrm{d}y + \sum_{j=1}^{m} g_j(x)\frac{dw_j}{dt},\ (t,x) \in (0,\infty) \times \mathbb{R}_+, \tag{1.3}$$

which is obtained by adding an additive noise $\sum_{j=1}^m g_j(x)\frac{\mathrm{d}w_j}{\mathrm{d}t}$ to (1.1). Here, $\Gamma(\alpha,x,y)$ is defined by (1.2), $\{g_j(x)\}_{j=1}^m$ are twice continuously differentiable on $(0,\infty)$, standing for the intensity and the shape of noise and $\{w_j\}_{j=1}^m$ are mutually independent two-sided real-valued Wiener process on an appropriate probability space to be specified later. For presentation simplicity, here we only consider $\mathbb{R}_+ = [0, \infty)$ in the one-dimensional space as earlier works (So *et al.*, 2001; Wu & Zou, 2001; Yi *et al.*, 2012; Yi & Zou, 2015) did. Indeed, the results can also be extended to half plane case or even three dimensions by similar techniques in Hu & Duan (2018), Hu *et al.* (2018) and Wang (2014).

In order to obtain the global complex dynamics and non-local analysis of the qualitative properties of random dynamical systems (RDSs), Crauel and Flandoli proposed the concept of random attractors for infinite dimensional random system in Crauel & Flandoli (1994); Flandoli & Schmalfuss (1996); Crauel (2002), by generalizing the theory of global attractors of infinite-dimensional dissipative systems. Since then, the existence, finite dimensionality and structure of random attractors for various stochastic nonlinear evolution equations or stochastic functional differential equations have been extensively and intensively investigated by adopting the framework in Crauel & Flandoli (1994); Crauel (2002). For example, for the stochastic reaction–diffusion equation without time delay, Caraballo *et al.* (2000), Gao *et al.* (2014) and Li & Guo (2008) explored the existence of global attractors on bounded domains. For the stochastic delayed reaction–diffusion equation on bounded domains, the existence of random attractors and their structure have been studied in Caraballo *et al.* (2007); Bessaih *et al.* (2014); Chueshov *et al.* (2014); Wang *et al.* (2015); Li & Guo (2020).

In our recent works Hu & Zhu (2021) and Hu & Zhu (2022), we have obtained the existence, uniqueness and stability of solutions to (1.1) as well as the existence of random attractors when the domain $\mathcal{O}$ is bounded with a DBVC. Therefore, similar questions arise naturally, i.e. under what conditions does (1.3) admit a unique global solution? Under what conditions does (1.3) generate an RDS ? Under what conditions does (1.3) possess random attractors? Moreover, under what conditions is the attractor of (1.3) a random fixed point? In the recent works (Bates *et al.*, 2009), Wang *et al.* (2018) and Zhou (2017), the authors obtained the existence of global attractors for stochastic reaction–diffusion equations on unbounded domains. The unboundedness of the domain causes the Sobolev embedding to no longer be compact and the asymptotic compactness of solutions cannot be obtained by a standard method. Therefore, in order to overcome the difficulty caused by the unboundedness of the domain, Bates *et al.* (2009) established uniform estimates on the far-field values of solutions. Nevertheless, it follows from (1.2) that the kernel $\Gamma(\alpha, x, y)$ is asymmetric and the domain is non-symmetric and non-compact, which together with the time delay imply that the analysis of the long time behaviour of solutions to (1.3) on the semi-infinite interval $\mathbb{R}_+ = [0, \infty)$ is more difficult. This motivates us to establish a new method to analyse the asymptotic behaviour of the following stochastic initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \varepsilon \int_{\mathbb{R}_+} \Gamma(\alpha,x,y) f(u(t-\tau,y))\mathrm{d}y + \sum_{j=1}^m g_j(x)\frac{dw_j}{dt}, \\ u(t,0) = 0, \quad t > 0, \\ u(t,x) = \phi(t,x), \quad (t,x) \in [-\tau, 0] \times \mathbb{R}_+. \end{cases} \tag{1.4}$$

In the case of deterministic equations, to obtain the global dynamics of (1.1), Yi & Zou (2016) established a priori estimates for non-trivial solutions by exploring the asymptotic properties of the non-local delayed effect and the diffusion operator. This method has also been adopted by Hu et al. (Hu & Duan, 2018; Hu *et al.*, 2018) to explore the global dynamics of some non-local delayed differential equations on different half spaces with various boundary conditions. Introducing random factors makes the analysis of the asymptotic behaviour of (1.4) quite different from the deterministic case, since the existence and uniqueness of global solution to SNDRDE (1.4) and whether it generates a random dynamic system are not so natural as in the deterministic case. Moreover, the framework to deal with random attractors is also quite different from that of the deterministic case. In this paper, we carry out a first attempt to extend the method of exploring the asymptotic properties of the deterministic non-local delayed effect and the diffusion operator to the random case, and prove the existence and qualitative property of random attractors for the SNDRDE (1.4) on the unbounded domain $\mathbb{R}_+$. Unlike the previous works (Caraballo *et al.*, 2007; Bessaih *et al.*, 2014; Chueshov *et al.*, 2014; Wang *et al.*, 2015; Li & Guo, 2020), where the phase space is a Hilbert space, we need to work here with a Banach space as natural phase space. Due to the lack of an inner product, we prove the existence of a global solution and obtain uniform a priori estimates of the solution by using the semigroup approach together with a careful analysis of the diffusion operator instead of taking inner product. Moreover, to overcome the difficulty caused by the non-compactness of the spatial domain, we also adopt the compact open topology to describe the asymptotic behaviour. It is clear that our method can be used for a variety of other equations on half spaces, as it was done for the deterministic case (Hu *et al.*, 2018).

The remaining part of this paper is structured as follows. In Section 2, we recall some basic results from the theory of RDSs and random attractors as well as some notation and preliminary lemmas needed for the proof of our main results. In Section 3, by means of the Ornstein–Uhlenbeck (O-U) process, we first transform SNDRDE (1.4) into a random partial differential equation with delay, and we then show that SNDRDE (1.4) has global solutions by the Banach fixed point theorem together with the properties of the semigroup generated by the linear part of (1.4). Furthermore, we show that solutions to (1.4) generate RDSs. To prove the existence of random attractors for SNDRDE (1.4), we first establish uniform a priori estimates of the solutions in Section 4, and we then show the asymptotic compactness of RDSs generated by (1.2) with respect to the compact open topology, implying the existence of random attractors by the results in Crauel & Flandoli (1994); Crauel (2002). In Section 5, we derive sufficient conditions ensuring the random attractor becomes an exponentially attracting stationary solution. In Section 6, the theoretical results are applied to the stochastic non-local delayed Nicholson's blowfly equation. At last, we summarize the paper by pointing out some potential directions deserving further research.

2. Preliminaries

We first recall some notation to be used throughout this paper, and then we introduce the theory of RDSs as well as random attractors. We denote by $BUC\left(\mathbb{R}_+, \mathbb{R}\right)$ the set of all bounded and uniformly continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, and by $\mathcal{C} = C([-\tau, 0], X)$ the set of all continuously functions from $[-\tau, 0]$ to X equipped with the usual supremum norm $\|\varphi\|_{\mathcal{C}} = \sup\{\|\varphi(\xi)\|_X : \xi \in [-\tau, 0]\}$ for any $\varphi \in \mathcal{C}$. For any real interval $J \subseteq \mathbb{R}$, set $J + [-\tau, 0] = \{t + \xi : t \in I \text{ and } \xi \in [-\tau, 0]\}$. For any $u : (J + [-\tau, 0]) \to X$ and $t \in J$, we define $u_t(\cdot) \in \mathcal{C}$ by $u_t(\xi) = u(t + \xi)$ for all $\xi \in [-\tau, 0]$.

In the sequel, we introduce the concept of random attractor and RDS following Arnold (1998) and Crauel & Flandoli (1994); Flandoli & Schmalfuss (1996); Crauel (2002).

Definition 2.1. Let $\left\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\right\}$ be a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t\omega$ is measurable and $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t\theta_s$, for all $s, t \in \mathbb{R}$. The flow θ_t together with the probability space $\left(\Omega, \mathcal{F}, P, \left(\theta_t\right)_{t\in\mathbb{R}}\right)$ is called a metric dynamical system.

For a given complete separable metric space (X, d), denote by $\mathcal{B}(X)$ the Borel-algebra of open subsets in X.

DEFINITION 2.2. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ is said to be an RDS on a complete separable metric space (X, d) with Borel σ-algebra $\mathcal{B}(X)$ over the metric dynamical system $\left(\Omega, \mathcal{F}, P, \left(\theta_t\right)_{t\in\mathbb{R}}\right)$ if

(i) $\Phi(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \omega, \cdot)$ is the identity on X for P-a.e. $\omega \in \Omega$;

(iii) $\Phi(t+s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$, for all $t, s \in \mathbb{R}^+$ and P-a.e. $\omega \in \Omega$.

An RDS Φ is continuous or differentiable if $\Phi(t, \omega, \cdot) : X \to X$ is continuous or differentiable for all $t \in \mathbb{R}^+$ and P-a.e. $\omega \in \Omega$.

DEFINITION 2.3. A set-valued map $\Omega \ni \omega \mapsto D(\omega) \in 2^X$ is said to be a random set in X if the mapping $\omega \mapsto d(x, D(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for any $x \in X$, where $d(x, D(\omega)) \triangleq \inf_{y\in D(\omega)} \mathrm{d}(x, y)$ is the distance in X between the element x and the set $D(\omega) \subset X$.

DEFINITION 2.4. A random set $\{D(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d\left(D\left(\theta_{-t}\omega\right)\right) = 0, \qquad \text{for all } \beta > 0,$$

where $d(D) = \sup_{x\in D} \|x\|_X$.

DEFINITION 2.5. Let $\mathcal{D} = \{D(\omega) \subset X, \omega \in \Omega\}$ be a family of random set. A random set $K(\omega) \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Φ if for P-a.e. $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \omega) > 0$ such that

$$\Phi\left(t, \theta_{-t}\omega, B\left(\theta_{-t}\omega\right)\right) \subseteq K(\omega) \qquad \text{for all } t \geq T.$$

If, in addition, for all $\omega \in \Omega$, $K(\omega)$ is a closed non-empty subset of X and $K(\omega)$ is measurable in Ω with respect to $\mathcal{F}$, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

DEFINITION 2.6. An RDS Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\left\{\Phi\left(t_n, \theta_{-t_n}\omega, x_n\right)\right\}_{n\geq1}$ has a convergent subsequence in X whenever $t_n \to \infty$ and $x_n \in D\left(\theta_{-t_n}\omega\right)$ for any given $D \in \mathcal{D}$.

DEFINITION 2.7. A compact random set $\mathcal{A}(\omega)$ is said to be a $\mathcal{D}$-pullback random attractor associated with the RDS Φ if it satisfies the invariance property

$$\Phi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}\left(\theta_t\omega\right), \qquad \text{for all } t \geq 0,$$

and the pullback attracting property

$$\lim_{t\to\infty} \operatorname{dist}\left(\Phi\left(t, \theta_{-t}\omega\right) D\left(\theta_{-t}\omega\right), \mathcal{A}(\omega)\right) = 0, \qquad \text{for all } t \geq 0, D \in \mathcal{D}, P - a.e.\ \omega \in \Omega,$$

where $\operatorname{dist}(\cdot,\cdot)$ denotes the Hausdorff semidistance

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \mathrm{d}(x, y), \quad A, B \subset X.$$

LEMMA 2.1. Let (θ, Φ) be a continuous RDS. Suppose that Φ is $\mathcal{D}$-pullback asymptotically compact and has a closed pullback $\mathcal{D}$-absorbing set $K = \{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then it possesses a random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$, where

$$\mathcal{A}(\omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\Phi\left(t, \theta_{-t}\omega, K\left(\theta_{-t}\omega\right)\right)}.$$

For convenience, we introduce the following Gronwall inequality in Bessaih *et al.* (2014) that will be frequently used in our subsequent proofs.

LEMMA 2.2. Let $T > 0$ and u, α, f and g be non-negative continuous functions defined on $[0, T]$ such that

$$u(t) \leq \alpha(t) + f(t)\int_0^t g(r)u(r)dr, \qquad \text{for } t \in [0, T].$$

Then

$$u(t) \leq \alpha(t) + f(t)\int_0^t g(r)\alpha(r)e^{\int_r^t f(\tau)g(\tau)d\tau}dr, \qquad \text{for } t \in [0, T].$$

3. Global solutions and RDSs

In this section, we will prove the existence of global solutions to SNDRDE (1.4) under the given initial condition, and then show that the solutions generate an RDS. By the Fourier sine transform defined by Eq. (10.5.39) in Haberman (2004), we can obtain that the semigroup $S(t)$ generated by the linear system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \mu u, \quad t > 0 \\ u(t, 0) = 0, \quad t \geq 0 \\ u(0, x) = \phi(x), \quad x \in \mathbb{R}_+ \end{cases} \tag{3.1}$$

is

$$\begin{cases} S(0)[\phi](x) = \phi(x), \\ S(t)[\phi](x) = \frac{\exp(-\mu t)}{\sqrt{4\pi t}}\int_0^\infty \phi(y)\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right]\mathrm{dy}, \quad t > 0, \end{cases} \tag{3.2}$$

for $(x, \phi) \in \mathbb{R}_+ \times X$. Let $Z = BUC(\mathbb{R}, \mathbb{R})$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the usual supremum norm $\|\cdot\|_Z$. Then, the Fourier transformation method indicates that the semigroup $U(t) : Z \to Z$ generated by $\Delta - \mu I$ is defined as

$$\begin{cases} U(0)[\phi](x) = \phi(x), \\ U(t)[\phi](x) = \frac{\exp(-\mu t)}{\sqrt{4\pi t}}\int_{-\infty}^\infty \phi(y)\exp\left(-\frac{(x-y)^2}{4t}\right)\mathrm{dy} \text{ for all } t \in (0, \infty), \end{cases} \tag{3.3}$$

for $(x, \phi) \in \mathbb{R} \times Z$, which is analytic and strongly continuous on Z.

We introduce the following results concerning the properties of semigroup $S(t)$, which is frequently used thorough the whole paper. The details of the proof can be found in Yi & Zou (2016) Lemma 2.1.

LEMMA 3.1. Let $S(t)$ and $U(t)$ be defined in (3.2) and (3.3), respectively, then we have the following results.

(i) $S(t)[\phi](x) = e^{-\mu t}U(t)[\tilde{\phi}](x)$ for all $\phi \in X$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$, where $\tilde{\phi}$ represents the odd extension of ϕ.

(ii) $S(t)$ is an analytic and strongly continuous semigroup on X.

(iii) For all $t \in (0,\infty)$ and $(x,\phi) \in (0,\infty) \times X$, there hold

$$\|S(t)[\phi]\| \leq e^{-\mu t}\|\phi\|, \quad \left|\frac{\partial S(t)[\phi](x)}{\partial t}\right| \leq \frac{(1+\mu t)\exp(-\mu t)\|\phi\|}{t},$$
$$\left|\frac{\partial S(t)[\phi](x)}{\partial x}\right| \leq \frac{\exp(-\mu t)\|\phi\|}{\sqrt{\pi t}}, \quad \left|\frac{\partial^2 S(t)[\phi](x)}{\partial x^2}\right| \leq \frac{\exp(-\mu t)\|\phi\|}{t}.$$

(iv) For any $t_1, t_2 \in (0,\infty), x_1, x_2 \in \mathbb{R}_+$ and $\phi \in X$, there holds

$$\left|S\left(t_1\right)[\phi]\left(x_1\right) - S\left(t_2\right)[\phi]\left(x_2\right)\right| \leq \frac{\left(1+\mu\min\left\{t_1,t_2\right\}\right)\exp\left(-\mu\min\left\{t_1,t_2\right\}\right)||\phi||}{\min\left\{t_1,t_2\right\}}\left|t_2-t_1\right|$$
$$+\frac{\exp\left(-\mu\min\left\{t_1,t_2\right\}\right)||\phi||}{\sqrt{\pi\min\left\{t_1,t_2\right\}}}\left|x_2-x_1\right|.$$

For the purpose of later use, we prove the following property on the non-local diffusion operator of (1.4).

LEMMA 3.2. Define $K : X \to X$ by

$$K(\phi)(\cdot) = \int_{\mathbb{R}_+} \Gamma(\alpha,\cdot,y)\phi(y)\mathrm{d}y$$

for all $\phi \in X$. Then, $\|K\| \triangleq \sup\{\frac{\|K(\phi)\|}{\|\phi\|} : \|\phi\| \neq 0\} \leq 1$.

Proof. For any $x \in \mathbb{R}_+$, we have

$$\begin{aligned}|K(\phi)(x)| &= \left|\int_{\mathbb{R}_+}\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{(x-y)^2}{4\alpha}}\phi(y)\mathrm{d}y - \int_{\mathbb{R}_+}\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{(x+y)^2}{4\alpha}}\phi(y)\mathrm{d}y\right| \\ &= \left|\int_{-\infty}^{x}\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{u^2}{4\alpha}}\phi(x-u)\mathrm{d}u - \int_{x}^{\infty}\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{u^2}{4\alpha}}\phi(u-x)\mathrm{d}u\right| \\ &\leq \|\phi\|\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{u^2}{4\alpha}}\mathrm{d}u = \|\phi\|.\end{aligned} \tag{3.4}$$

Therefore, we have $\|K\| \triangleq \sup_{\|\phi\|\neq 0}\frac{\|K(\phi)\|}{\|\phi\|} \leq 1$. □

In the sequel, we always impose the following assumptions on the nonlinear drift term f.

(**H**) $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $f(0) = 0$, and $\left|\frac{df(s)}{ds}\right| \le L_f$.

Here, L_f is a positive constant, representing the bound of the derivation. Hence, it is clear that for all $s_1, s_2 \in \mathbb{R}$

$$\left|f(s_1) - f(s_2)\right| \le L_f \left|s_1 - s_2\right|, \tag{3.5}$$

In this paper, we consider the canonical probability space $(\Omega, \mathcal{F}, P)$ with

$$\Omega = \left\{\omega = \left(\omega_1, \omega_2, \dots, \omega_m\right) \in C\left(\mathbb{R}; \mathbb{R}^m\right) : \omega_i(0) = 0\right\}$$

and $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, while P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then, we identify W with

$$W(t, \omega) = \left(w_1(t), w_2(t), \dots, w_m(t)\right) \quad \text{for } t \in \mathbb{R}.$$

Moreover, we define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R}.$$

Then, $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}}\right)$ is a metric dynamical system.

In order to construct the conjugate transformation, we consider the stochastic stationary solution of the one dimensional Ornstein–Uhlenbeck equation

$$\mathrm{d}z_j + \mu z_j \mathrm{d}t = \mathrm{d}w_j(t), j = 1, \dots, m. \tag{3.6}$$

The solution to (3.6) is given by

$$z_j(t) \triangleq z_j\left(\theta_t \omega_j\right) = -\mu \int_{-\infty}^{0} e^{\mu s} \left(\theta_t \omega_j\right)(s)\mathrm{d}s, \quad t \in \mathbb{R}, \tag{3.7}$$

By Definition 2.4, one can see that the random variable $\left|z_j(\omega_j)\right|$ is tempered and $z_j\left(\theta_t \omega_j\right)$ is P-a.e. ω continuous. Therefore, Proposition 4.3.3 in Arnold (1998) implies that there exists a tempered function $0 < r(\omega) < \infty$ such that

$$\sum_{j=1}^{m} \left|z_j\left(\omega_j\right)\right|^2 \le r(\omega), \tag{3.8}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r\left(\theta_t \omega\right) \le e^{\frac{\mu}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \tag{3.9}$$

Combining (3.8) with (3.9), we obtain that for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} \left|z_j\left(\theta_t \omega_j\right)\right|^2 \le e^{\frac{\mu}{2}|t|} r(\omega), \quad t \in \mathbb{R} \tag{3.10}$$

Putting $z\left(\theta_t\omega\right)(x)=\sum_{j=1}^{m} g_j(x)z_j\left(\theta_t\omega_j\right)$, by (3.6), we have

$$\mathrm{d}z+\mu z\mathrm{d}t=\sum_{j=1}^{m} g_j(x)\mathrm{d}w_j.$$

To prove that (1.4) possesses a global solution that generates an RDS, we consider the transformation $v(t)=u(t)-z\left(\theta_t\omega\right)$, where u is a solution of (1.4), and show that v is a global solution of the transformed equation and generates an RDS. Then, we show that (1.4) also has a global solution and generates a conjugated RDS due to the inverse transformation. This method has also been adopted by Hu & Zhu (2022), Li & Guo (2020) and Wang *et al.* (2015) to deal with random attractors as well as Duan *et al.* (2003, 2004); Lu & Schmalfuß (2007) and Lu & Schmalfuß (2008) to tackle invariant manifolds of stochastic partial differential equations with or without delay. Apparently, v satisfies

$$\frac{\partial v(t,x)}{\partial t}=\Delta v(t,x)-\mu v(t,x)+F\big(\left(v_t+z\left(\theta_{t+\cdot}\omega\right)\right)\big)(x)+\Delta z\left(\theta_t\omega\right)(x), t>0, x\in(0,\infty) \tag{3.11}$$

with boundary condition

$$v(t,0)=0, \quad \text{for} \quad t\in(0,\infty), \tag{3.12}$$

and initial condition

$$v(\xi,x,\omega)=\psi(\xi,x,\omega)\triangleq\phi(x,\xi)-z\left(\theta_\xi\omega\right)(x) \quad \text{for} \quad (x,\xi)\in\mathbb{R}_+\times[-\tau,0]. \tag{3.13}$$

Here, $F:\mathcal{C}\to X$ is defined by

$$\begin{aligned}F(\varphi_t+z\left(\theta_{t+\cdot}\omega\right))(x)&=\varepsilon\int_{\mathbb{R}_+}\Gamma(\alpha,x,y)f\big(\varphi(t-\tau,y)+z\left(\theta_{t-\tau}\omega\right)\big)(y)\mathrm{d}y\\&=\varepsilon K[f\big(\varphi(t-\tau,\cdot)+z\left(\theta_{t-\tau}\omega,\cdot\right)\big)](x),\end{aligned}$$

for any $\varphi\in\mathcal{C}$.

We now show that the pathwise deterministic problem (3.11)–(3.13) has a global mild solution under assumption (**H**). We aim at solving the following integral equation:

$$v(t,\omega,\psi)=\begin{cases} S(t)\psi(0)+\int_0^t S(t-r)F\left(v_r+z\left(\theta_{r+\cdot}\omega\right)\right)dr+\int_0^t S(t-r)\Delta z\left(\theta_r\omega\right)dr,\\ \psi(t), t\in[-\tau,0],\end{cases} \tag{3.14}$$

for the initial data $\psi\in\mathcal{C}$. We have the following results.

THEOREM 3.1. Assume that f satisfies (**H**). Then, for any $\psi\in\mathcal{C}$ and for P-a.e. $\omega\in\Omega$, there exists a global mild solution to (3.11)–(3.13). Moreover, if $f:\mathcal{C}\to X$ is globally bounded, i.e. there exists $M>0$ such that $\|f(\varphi)\|\le M$ for all $\varphi\in\mathcal{C}$, then the solution is pullback bounded, i.e., there exists $C(\omega)>0$ such that $\|v(t,\theta_{-t}\omega,\psi)\|\le C(\omega)$ for P-a.e. $\omega\in\Omega$.

Proof. We first prove that (3.11)–(3.13) has a local mild solution and then show it can be extended to a global one by an argument of steps. For any $\psi\in\mathcal{C}$ and P-a.e. $\omega\in\Omega$, we show in the following that there exist $T(\omega) > 0$ and $v \in C([-\tau, T(\omega)]; X)$ satisfying (3.14) on $[-\tau, T(\omega)]$ due to the Banach fixed point theorem. For a fixed ω, we consider the complete metric subset X_ψ^T of $C([-\tau, T], X)$ defined by

$$X_\psi^T = \{v \in C([-\tau, T]; X) : u(s) = \psi(s), s \in [-\tau, 0]\}.$$

For such a $T > 0$ to be determined later, and $t \in [-\tau, T]$, we define the following operator $\Lambda : X_\psi \to X_\psi$ (where we omit T since no confusion is possible)

$$\Lambda(\zeta)(t) = \begin{cases} S(t)\psi(0) + \int_0^t S(t-r)F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr + \int_0^t S(t-r)\Delta z\left(\theta_r\omega\right) dr, t \in (0, T] \\ \psi(t), t \in [-\tau, 0]. \end{cases} \tag{3.15}$$

We show in the sequel that Λ is well defined, maps X_ψ into itself and is a contraction on $C([-\tau, T]; X)$, leading to the existence of a unique fixed point in X_ψ with T being determined according to the Banach fixed point theorem. It follows from Lemma 3.1 (ii) and (iii), $F : \mathcal{C} \to X$ and g_j is twice continuously differentiable that we have $\Lambda(\zeta)(t) \in X$ for any fixed $t \in [-\tau, T]$. Now we prove the continuity. If $t_1, t_2 \in [-\tau, 0]$, the result is obvious. Let us then pick $t_1, t_2 \in (0, T]$, and assume without loss of generality, that $t_1 < t_2$. Therefore, we have

$$\begin{aligned} \|\Lambda(\zeta)(t_1) - \Lambda(\zeta)(t_2)\| = \|[S(t_1) - S(t_2)]\psi(0)\| + \| \int_0^{t_1} S(t_1 - r)F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ - \int_0^{t_2} S(t_2 - r)F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr\| + \| \int_0^{t_1} S(t_1 - r)\Delta z\left(\theta_r\omega\right) dr \\ - \int_0^{t_2} S(t_2 - r)\Delta z\left(\theta_r\omega\right) \|dr \triangleq I_1 + I_2 + I_3. \end{aligned} \tag{3.16}$$

We estimate each term on the right-hand side of (3.16) due to Lemmas 3.1 and 3.2.

$$I_1 = \|S(t_1) - S(t_2)]\psi(0)\| \le \frac{\left(1 + t_1\right) \exp\left(-\mu t_1\right) ||\psi(0)||}{t_1} \left|t_2 - t_1\right|. \tag{3.17}$$

$$\begin{aligned} I_2 &= \| \int_0^{t_1} [S(t_1 - r) - S(t_2 - r)]F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr - \int_{t_1}^{t_2} S(t_2 - r)F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr\| \\ &\le \int_0^{t_1 - \sqrt{\delta}} [S(t_1 - r) - S(t_2 - r)]F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ &+ \int_{t_1 - \sqrt{\delta}}^{t_1} [S(t_1 - r) - S(t_2 - r)]F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ &+ \| \int_{t_1}^{t_2} S(t_2 - r)F\left(\zeta_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr\| \\ &\le \varepsilon \left|t_2 - t_1\right| \int_0^{t_1 - \sqrt{\delta}} \frac{\left(1 + \mu(t_1 - r)\right) \exp\left(-\mu(t_1 - r)\right) M}{t_1 - r} dr + 2\varepsilon M\sqrt{\delta} + \varepsilon M|t_2 - t_1| \\ &\le \varepsilon M \left|t_2 - t_1\right| \left(\frac{1}{\sqrt{\delta}} + \mu\right) + 2\varepsilon M\sqrt{\delta} + \varepsilon M|t_2 - t_1|. \end{aligned} \tag{3.18}$$

$$I_3 = \| \int_0^{t_1} [S(t_1 - r) - S(t_2 - r)] \Delta z \left(\theta_r \omega\right) dr - \int_{t_1}^{t_2} S(t_2 - r) \Delta z \left(\theta_r \omega\right) dr] \|$$
$$\leq \int_0^{t_1 - \sqrt{\delta}} [S(t_1 - r) - S(t_2 - r)] || \Delta z \left(\theta_r \omega\right) || dr + \int_{t_1 - \sqrt{\delta}}^{t_1} [S(t_1 - r) - S(t_2 - r)] || \Delta z \left(\theta_r \omega\right) || dr$$
$$+ \| \int_{t_1}^{t_2} S(t_2 - r) || \Delta z \left(\theta_r \omega\right) || dr \|$$
$$\leq \left| t_2 - t_1 \right| \int_0^{t_1 - \sqrt{\delta}} \frac{\left(1 + \mu(t_1 - r)\right) \exp\left(-\mu(t_1 - r)\right) || \Delta z \left(\theta_r \omega\right) ||}{t_1 - r} dr$$
$$+ \delta || \Delta z \left(\theta_r \omega\right) || + || \Delta z \left(\theta_r \omega\right) || | t_2 - t_1 |$$
$$\leq || \Delta z \left(\theta_r \omega\right) || [\left| t_2 - t_1 \right| \left(\frac{1}{\sqrt{\delta}} + 1 + \mu \right) + \delta]. \tag{3.19}$$

In equations (3.18) and (3.19), δ satisfies $\delta \in (0,1)$ and $t_1 < t_2 < t_1 + \delta$ with $\delta \to 0$, then $I_1 \leq \varepsilon M(\sqrt{\delta} + \mu\delta) + 2\varepsilon M\sqrt{\delta} + \varepsilon M\delta$, $I_3 \leq || \Delta z \left(\theta_r \omega\right) || [\sqrt{\delta} + (2 + \mu)\delta]$. Hence, when $\delta \to 0$, it holds that $t_2 \to t_1$, $I_1 + I_2 + I_3 \to 0$, implying the continuity of $\Lambda(\zeta)(t) \in H$ with respect to $t \in [-\tau, T]$. Thus, we have obtained that Λ is well defined in X_ψ.

In the sequel, we show the contraction property of Λ on X_ψ. Let $\zeta^1, \zeta^2 \in X_\psi$, then for $t \in [-\tau, 0]$, it holds $\zeta^1(t) = \zeta^2(t)$. Owing to (**H**), for $t \in [0, T)$ we have

$$\begin{aligned} \left\| \left(\Lambda\left(\zeta^1\right)(t) - \Lambda\left(\zeta^2\right)\right)(t) \right\| &= \| \int_0^t S(t - r)[F\left(\zeta_r^1 + z\left(\theta_{r+\cdot}\omega\right)\right) - F\left(\zeta_r^2 + z\left(\theta_{r+\cdot}\omega\right)\right)] dr \|. \\ &\leq \varepsilon L_f \int_0^t e^{-\mu(t-r)} \left\| \zeta_r^1 - \zeta_r^2 \right\|_{\mathcal{C}} dr \\ &\leq \frac{\varepsilon L_f}{\mu} (1 - e^{-\mu t}) \left\| \zeta_r^1 - \zeta_r^2 \right\|_{\mathcal{C}}. \end{aligned} \tag{3.20}$$

Hence, if $\frac{\varepsilon L_f}{\mu} \leq 1$, then for any $t \geq 0$, Λ is a contraction on X_ψ. However, in the scenario $\frac{\varepsilon L_f}{\mu} > 1$, we can choose $T = \frac{1}{2\mu} \ln(\frac{\varepsilon L_f}{\varepsilon L_f - \mu})$, and therefore Λ is a contraction on X_ψ, which indicates the existence of a unique local mild solution to (3.14).

In the following text, we will derive the existence of a global mild solution by an argument of steps. Denote $T_1(\omega) = \frac{1}{2\mu} \ln(\frac{\varepsilon L_f}{\varepsilon L_f - \mu})$ and let us build the solution in the next time interval, say $\left[T_1(\omega), T_2(\omega)\right]$. It suffices to find $T_2(\omega)$ such that (3.14) also admits a local mild solution in the last interval. We only need to solve

$$v(t, \omega, \psi) = \begin{cases} S(t - T_1(\omega))\psi(0) + \int_0^{t - T_1(\omega)} S(t - T_1(\omega) - r) F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ \quad + \int_0^t S(t - T_1(\omega) - r) \Delta z\left(\theta_r \omega\right) dr, \\ v_1(t), t - T_1(\omega) \in [-\tau, 0], \end{cases} \tag{3.21}$$

where v_1 denotes the solution obtained on $\left[-\tau, T_1(\omega)\right]$. Taking $s := t - T_1$, the above system is equivalent to solve the problem for $y(s) = v\left(s + T_1(\omega)\right)$

$$y(s,\omega,\psi) = \begin{cases} S(s)\hat{\psi}(0) + \int_0^s S(s-r)F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right)dr + \int_0^s S(s-r)\Delta z\left(\theta_r\omega\right)dr, \\ \hat{\psi}(s) \triangleq u_1(s + T_1(\omega)), s \in [-\tau, 0], \end{cases} \tag{3.22}$$

which is the same as the previous step, but with initial condition $\hat{\psi}$. Taking the same steps as before, we can obtain a new piece given by a local solution defined now in the interval $\left[T_1 - \tau, T_2\right]$. Thus, by repeating the same procedure, one can obtain a sequence of time T_n. We prove in the sequel that $T_n \to \infty$. We only need to show that for any given $t > 0$, there exist $i \in \mathbb{N}$ such that $T_i > t$. If $T(\omega) := T_1(\omega) \geq t$ there is nothing to show. If this is not case, let s^* be the unique solution of the equation

$$\frac{\varepsilon L_f}{\mu}(1 - e^{-\mu t}) = 1/2,$$

which is trivially a positive lower bound of $T(\omega)$. If $T_2(\omega) \geq t$ we are done. Otherwise, $t > T_2(\omega) = T(\omega) + T\left(\theta_{T(\omega)}\omega\right)$, i.e. $T\left(\theta_{T(\omega)}\omega\right) < t - T(\omega)$, and therefore the previous inequality implies that $t^* \leq T\left(\theta_{T(\omega)}\omega\right)$, and in particular that $T_2(\omega) \geq 2t^*$. Repeating this method it turns out that there exists $i \in \mathbb{N}$ such that $T_i(\omega) \geq it^* > t$.

In what follows, we prove the pullback boundedness of the solution provided f is bounded. Since g_j is twice continuously differentiable, by (3.9) and (3.10), there must exist a constant $c > 0$ such that $\|\Delta z(\theta_{-t}\omega)\| \leq ce^{\frac{\mu t}{2}}r(\omega)$. It follows from (3.14) and the boundedness of f that, for P-a.e. $\omega \in \Omega$,

$$\begin{aligned} \left\|v(t,\theta_{-t}\omega,\psi)\right\| &= |S(t)|\|\psi(0)\| + M\int_0^t e^{-\mu(t-r)}dr + c\int_0^t e^{-\mu(t-r)}e^{\frac{\mu(t-r)}{2}}r(\omega)dr \\ &\leq e^{-\mu t}\|\psi(0)\| + M\frac{1}{\mu}(1 - e^{-\mu t}) + cr(\omega)\frac{2}{\mu}(1 - e^{-\mu t/2}) \\ &\leq \|\psi(0)\| + (M + cr(\omega))\frac{2}{\mu}. \end{aligned} \tag{3.23}$$

Therefore, the pullback boundedness of v is clear by taking $C(\omega) = \|\psi(0)\| + (M + cr(\omega))\frac{2}{\mu}$. □

REMARK 3.1. By Corollary 2.2.5 in Wu (1996) and the analyticity of the semigroup $S(t)$ given in Lemma 3.1 (ii), we know that a mild solution of problem (3.11)–(3.13) is also a classical solution of problem (3.11)–(3.13) for all $t > \tau$. Hence, $u(t,\omega,\phi) = v(t,\omega,\psi) + z\left(\theta_t\omega\right)$ is a global solution to (1.4).

In the sequel, we show that the solution of (3.14) generates an RDS. To this end we will prove that the cocycle property in Definition 2.2 holds.

THEOREM 3.2. The global mild solution v of (3.11)–(3.13) generates an RDS $\Phi : \mathbb{R}^+ \times \Omega \times \mathcal{C} \to \mathcal{C}$ defined by $\Phi(t,\omega,\psi)(\cdot) = v_t(\cdot)$, i.e.,

$$\Phi(t,\omega,\psi)(\cdot) = \begin{cases} S(t+\cdot)\psi(0) + \int_0^{t+\cdot} S(t+\cdot-r)F\left(v_r + z\left(\theta_{r-\tau+\cdot}\omega\right)\right)dr + \int_0^{t+\cdot} S(t-r+\cdot)\Delta z\left(\theta_r\omega\right)dr, \\ \psi(t+\cdot), t+\cdot \in [-\tau, 0]. \end{cases} \tag{3.24}$$

Proof. We prove the result in three cases. In the situation $t, \rho \geq \tau$ so that $t+s, \rho+s \geq 0$, for all $s \in [-\tau, 0]$, we have

$$\begin{aligned}
\Phi(t+\rho,\omega,\psi)(\zeta) &= S(t+\zeta+\rho)\psi(0) + \int_0^{t+\zeta+\rho} S(t+\zeta+\rho-r)[F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) + \Delta z\left(\theta_r\omega\right)]dr \\
&= S(t+\zeta)S(\rho)\psi(0) + S(t+\zeta)\int_0^{\rho} S(\rho-r)[F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) + \Delta z\left(\theta_r\omega\right)]dr \\
&\quad + \int_{\rho}^{t+\zeta+\rho} S(t+\zeta+\rho-r)[F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) + \Delta z\left(\theta_r\omega\right)]dr \\
&= S(t+\zeta)[S(\rho)\psi(0) + \int_0^{\rho} S(\rho-r)F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right)dr + \int_0^{\rho} S(\rho-r)\Delta z\left(\theta_r\omega\right)dr] \\
&\quad + \int_0^{t+\zeta} S(t+\zeta-r)F\left(v_{\rho+r} + z\left(\theta_{r+\cdot+\rho}\omega\right)\right)dr + \int_0^{t+\zeta} S(t+\zeta-r)\Delta z\left(\theta_{r+\rho}\omega\right)dr \\
&= S(t+\zeta)\Phi(\rho,\omega,\psi)(0) + \int_0^{t+\zeta} S(t+\zeta-r)F\left(v_{\zeta+r} + z\left(\theta_{r+\cdot}\theta_\rho\omega\right)\right)dr \\
&\quad + \int_0^{t+\zeta} S(t+\zeta-r)\Delta z\left(\theta_r\theta_\rho\omega\right)dr \\
&= \Phi(t,\theta_\rho\omega,\cdot)\Phi(\rho,\omega,\psi)(\zeta), \qquad (3.25)
\end{aligned}$$

which indicates the cocycle property in this situation.

In the scenario $t+\rho+\zeta \leq 0$, for $\zeta \in [-\tau, 0]$. Then, it is straightforward to see that

$$\Phi(t+\rho,\omega,\psi)(\zeta) = \psi(t+\rho+\zeta) = \Phi(\rho,\omega,\psi)(t+\zeta) = \Phi\left(t,\theta_\rho\omega,\cdot\right)\circ\Phi(\rho,\omega,\psi)(\zeta)$$

When $t+\rho+\zeta > 0$ but $\rho+\zeta \leq 0$ for $\zeta \in [-\tau, 0]$, we have

$$\psi(\rho+\zeta)(0) = v_{\rho+\zeta}(0) = \Phi(\zeta+\rho,\omega,\psi)(0) = \Phi\left(\rho,\omega,\psi\right)(\zeta)$$

Moreover, by (3.24), one can easily check that

$$\Phi(t+\rho,\omega,\psi)(\zeta) = \Phi\left(t,\theta_\rho\omega,\psi(\rho+\zeta)\right)(0).$$

Therefore, we have

$$\Phi(t+\rho,\omega,\psi)(\zeta) = \Phi(t,\theta_\rho\omega,\psi)(\rho+\zeta) = \Phi(t,\theta_\rho\omega,\psi(\rho+\zeta))(0) = \Phi\left(t,\theta_\rho\omega,\cdot\right)\circ\Phi(\rho,\omega,\psi)(\zeta).$$

□

By Remark 3.1, $u(t,\omega,\phi) = v(t,\omega,\psi) + z\left(\theta_t\omega\right)$ is the global solution to (1.4) with initial condition ϕ. We now define a mapping $\Psi : \mathbb{R}^+ \times \Omega \times \mathcal{C} \to \mathcal{C}$ by $\Psi(t,\omega,\phi) = u_t(\cdot,\omega,\phi) = v_t(\cdot,\omega,\psi) + z\left(\theta_{t+\cdot}\omega\right)$, where $u_t(\zeta,\omega,\phi) = u(t+\zeta,\omega,\phi)$ for $\zeta \in [-\tau, 0]$. By Theorem 3.2 and the cocycle property of z, Ψ is an RDS on $\mathcal{C}$ generated by (1.4).

4. Existence of random attractors

In this section, we are concerned with the existence of tempered pullback attractors for the SNDRDE (1.4) by first establishing a uniform estimation for the solution and then proving that Ψ is $\mathcal{D}$-pullback asymptotically compact. Nevertheless, due to the non-compactness of the spatial domain, it is quite difficult to prove the asymptotically compact of Ψ with respect to the usual supreme norm. Hence, similar to Yi *et al.* (2012), we introduce another more suitable topology called the compact open topology induced by the norms $\|\varphi\|_{co}^{X} = \sum_{n\geq 1} 2^{-n} \sup\{|\varphi(x)| : x \in [0,n],\ n \in \mathbb{N}\}$ for all $\varphi \in X$ and $\|\phi\|_{co}^{\mathcal{C}} = \sup\{\|\phi(\theta)\|_{co}^{X} : \theta \in [-\tau, 0]\}$ for all $\phi \in \mathcal{C}$, respectively, to describe the pullback asymptotic compactness of the RDS Ψ generated by (1.4). Moreover, we use X_{co} and C_{co} to denote the spaces $(X, \|\cdot\|_{co}^{X})$ and $(C, \|\cdot\|_{co}^{\mathcal{C}})$, respectively.

In order to adopt compact open topology to describe the global dynamics of (1.4), we first introduce without proof the following lemma, which gives sufficient and necessary condition for a sequence to be convergent and pre-compact with respect to the compact open topology. For details of the proof, the readers are referred to Lemma 2.1 in Wu (1996).

LEMMA 4.1. Given $r > 0$. Let $B_r = \{\phi \in * : \|\phi\|_* \leq r\}$ and $d_r(\phi, \psi) = \|\phi - \psi\|_{co}^{*}$, where $*$ stands for X or C. Then the following statements are true:
(i) For any $\phi_n, \phi \in B_r$ with $n \in \mathbb{N}$, $\lim_{n\to\infty} d_r(\phi_n, \phi) = 0$ if and only if

$$\lim_{n\to\infty} \sup\{|\phi_n(\zeta, x) - \phi(\zeta, x)| : \zeta \in [-\tau, 0], x \in I\} = 0$$

for any bounded domain $I = [0, i] \subset \mathbb{R}_+$ for all $i \in \mathbb{R}_+$.
(ii) Let $A \subseteq B_r$. Then A is pre-compact if and only if $A_I = \{\varphi|_I : \varphi \in A\}$ is a family of equicontinuous functions for any domain $I = [0, i] \subset \mathbb{R}_+$.

Throughout the rest of this paper, we always use $\mathcal{D}$ to denote the collection of all families of tempered non-empty subsets of C_{co}. The letters c and c_i, $(i = 1, 2, \cdots)$ are general positive constants whose values are not significant. Moreover, as for the asymptotic behaviour, we always assume that $t > \tau$ in the remaining part of this paper for convenience. The following lemma shows that the RDS Ψ has a random absorbing set respect to the compact open topology.

LEMMA 4.2. Assume that (**H**) is satisfied and $\varepsilon L_f e^{\mu\tau} - \mu < 0$, then there exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ satisfying that, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that

$$\Psi\left(t, \theta_{-t}\omega, B\left(\theta_{-t}\omega\right)\right) \subseteq K(\omega) \qquad \text{for all } t \geqslant T_B(\omega),$$

i.e. $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Ψ in $\mathcal{D}$.

Proof. We first derive uniform estimate on v by (3.14) and then obtain the existence of absorbing set of u by $u(t, \omega, \phi) = v(t, \omega, \psi) + z\left(\theta_t\omega\right)$. It follows from (3.14) and Lemma 3.1 (i) that for any $t > \tau$, we have

$$\begin{aligned} v(t, \omega, \psi) &= S(t)\psi(0) + \int_0^t S(t-s)F\left(v_s + z\left(\theta_{s+\cdot}\omega\right)\right) ds + \int_0^t S(t-s)\Delta z\left(\theta_s\omega\right) ds \\ &= e^{-\mu t}U(t)\tilde{\psi}(0) + \int_0^t e^{-\mu(t-s)}U(t-s)\tilde{F}\left(v_s + z\left(\theta_{s+\cdot}\omega\right)\right) ds + \int_0^t e^{-\mu(t-s)}U(t-s)\Delta\tilde{z}\left(\theta_s\omega\right) dr, \end{aligned} \tag{4.1}$$

where $\tilde{\psi}$, $\tilde{F}$ and $\tilde{z}$ represent the odd extension of ψ, F, z with respect to the spatial variable, respectively. Therefore, by Lemma 3.1(iii) and lemma 3.2, for any $\zeta \in [-\tau, 0]$, $n \in \mathbb{N}, x \in [0, n]$ and all $\psi \in \mathcal{C}$ we have

$$\begin{aligned}|v(t+\zeta,\omega,\psi)(x)| &\le e^{-\mu(t-\tau)}|\psi(0)(x)| + \varepsilon\int_0^{t+\zeta} e^{\mu(s-t-\zeta)}|K[v(s-\tau,\omega,\psi)+z(\theta_{s-\tau}\omega)](x)|ds\\ &+\int_0^{t+\zeta} e^{\mu(s-t-\zeta)}|\Delta z\left(\theta_s\omega\right)(x)|ds\\ &\le e^{-\mu(t-\tau)}|\psi(0)(x)| + \varepsilon L_f e^{\mu\tau}\int_0^t e^{\mu(s-t)}(|v(s-\tau,\omega,\psi)(x)| + |z\left(\theta_{s-\tau}\omega\right)(x)|)\mathrm{d}s\\ &+ e^{\mu\tau}\int_0^t e^{\mu(s-t)}|\Delta z\left(\theta_s\omega\right)(x)|\mathrm{d}s, \end{aligned} \tag{4.2}$$

which implies that

$$\begin{aligned}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|v(t+\zeta,\omega,\psi)(x)| &\le e^{-\mu(t-\tau)}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|\psi(0)(x)|\\ &+\varepsilon\int_0^{t+\zeta}e^{\mu(s-t-\zeta)}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|K[v(s-\tau,\omega,\psi)+z(\theta_{s-\tau}\omega)](x)|ds\\ &+\int_0^{t+\zeta}e^{\mu(s-t-\zeta)}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|\Delta z\left(\theta_s\omega\right)(x)|ds\\ &\le e^{-\mu(t-\tau)}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|\psi(0)(x)| + \varepsilon L_f e^{\mu\tau}\int_0^t e^{\mu(s-t)}\\ &\left(\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|v(s-\tau,\omega,\psi)(x)| + \sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|z\left(\theta_{s-\tau}\omega\right)(x)|\right)\mathrm{d}s\\ &+\int_0^t e^{\mu(s-t)}\sum_{n\ge1}2^{-n}\sup_{x\in[0,n]}|\Delta z\left(\theta_s\omega\right)(x)|\mathrm{d}s. \end{aligned} \tag{4.3}$$

Therefore, by the definition of compact open topology, we have

$$\begin{aligned}\|v(t+\zeta,\omega,\psi)\|_{co}^X &\le e^{-\mu(t-\tau)}\|\psi(0)\|_{co}^X + \varepsilon\int_0^{t+\zeta}e^{\mu(s-t-\zeta)}\|K[v(s-\tau,\omega,\psi)+z(\theta_{s-\tau}\omega)]\|_{co}^X ds\\ &+\int_0^{t+\zeta}e^{\mu(s-t-\zeta)}\|\Delta z\left(\theta_s\omega\right)\|_{co}^X ds\\ &\le e^{-\mu(t-\tau)}\|\psi(0)\|_{co}^X + \varepsilon L_f e^{\mu\tau}\int_0^t e^{\mu(s-t)}(\|v(s-\tau,\omega,\psi)\|_{co}^X + \|z\left(\theta_{s-\tau}\omega\right)\|_{co}^X)\mathrm{d}s\\ &+ e^{\mu\tau}\int_0^t e^{\mu(s-t)}\|\Delta z\left(\theta_s\omega\right)\|_{co}^X\mathrm{d}s, \end{aligned} \tag{4.4}$$

for P-a.e. $\omega \in \Omega$. Keep in mind that $\|v_t\|_{co}^{\mathcal{C}} = \sup\left\{\|v(t+\zeta)\|_{co}^{X} : \zeta \in [-\tau, 0]\right\}$. Hence, we can obtain

$$\begin{aligned}\|v_t(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} \le e^{\mu\tau}[e^{-\mu t}\|\psi\|_{co}^{\mathcal{C}} + \varepsilon L_f \int_0^t e^{\mu(s-t)} \|v_s(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} \mathrm{d}s \\ + \int_0^t e^{\mu(s-t)}(\|\Delta z\left(\theta_s\omega\right)\|_{co}^{X} + \varepsilon L_f\|z\left(\theta_{s-\tau}\omega\right)\|_{co}^{X})\mathrm{d}s.\end{aligned} \tag{4.5}$$

By replacing ω by $\theta_{-t}\omega$, we derive from (4.5) that, for all $t \ge \tau$,

$$\begin{aligned}\left\|v_t\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \le e^{\mu\tau}[e^{-\mu t}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} \\ + \int_0^t e^{\mu(s-t)}(\|\Delta z\left(\theta_{s-t}\omega\right)\|_{co}^{X} + \varepsilon L_f\|z\left(\theta_{s-t-\tau}\omega\right)\|_{co}^{X})\mathrm{d}s \\ + \varepsilon L_f \int_0^t e^{\mu(s-t)}\left\|v_s\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \mathrm{d}s].\end{aligned} \tag{4.6}$$

Since g_j are twice continuously differentiable and $z(\omega)(x) = \sum_{j=1}^{m} g_j(x)z_j(\omega_j)$, there exists constant c such that $p_1(\omega) \triangleq \|\Delta z(\omega)\|_{co}^{X} + \varepsilon L_f\|z\left(\theta_{-\tau}\omega\right)\|_{co}^{X} \le c\sum_{j=1}^{m}\left|z_j(\omega_j)\right|^2$. Therefore, it follows from (3.9) and (3.10) that

$$\int_0^t e^{\mu(s-t)} p_1\left(\theta_{s-t}\omega\right)\mathrm{d}s \le c\int_0^t e^{\frac{\mu}{2}(s-t)}r(\omega)\mathrm{d}s \le cr(\omega). \tag{4.7}$$

Incorporating (4.7) into (4.6) gives rise to

$$\begin{aligned}\left\|v_t\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \le e^{\mu\tau}[e^{-\mu t}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} \\ + \varepsilon L_f \int_0^t e^{\mu(s-t)}\left\|v_s\left(\cdot,\theta_{-s}\omega,\psi\left(\theta_{-s}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \mathrm{d}s + cr(\omega)].\end{aligned} \tag{4.8}$$

Multiply the both sides of (4.8) by $e^{\mu t}$ and adopt the Grönwall inequality, we have

$$\begin{aligned}e^{\mu t}\left\|v_t\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \le e^{\mu\tau}(\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} + ce^{\mu t}r(\omega)) + \varepsilon e^{\mu\tau}L_f\int_0^t (\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} \\ + ce^{\mu s}r(\omega))e^{\varepsilon L_f e^{\mu\tau}(t-s)}\mathrm{d}s \\ \le ce^{\mu\tau}e^{\mu t}r(\omega) + e^{\mu\tau}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} + \varepsilon e^{\mu\tau}L_f\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}\int_0^t e^{\varepsilon L_f e^{\mu\tau}(t-s)}\mathrm{d}s \\ + c\varepsilon e^{\mu\tau}L_f r(\omega)\int_0^t e^{\varepsilon L_f e^{\mu\tau}(t-s)}e^{\mu s}\mathrm{d}s.\end{aligned} \tag{4.9}$$

Therefore, we have

$$\left\|v_t\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} \le ce^{\mu\tau}r(\omega) + e^{-\mu t}e^{\mu\tau}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}} + \left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}(e^{(\varepsilon L_f e^{\mu\tau}-\mu)t}-1)$$
$$+\frac{c\varepsilon e^{\mu\tau}L_f}{\mu-\varepsilon e^{\mu\tau}L_f}[e^{-\varepsilon e^{\mu\tau}L_f}-e^{-\mu t}]r(\omega). \tag{4.10}$$

Note that $\psi(\omega)=\phi-z(\theta_{t+\cdot}\omega)$. The above estimate (5.1) implies that, for all $t\ge\tau$,

$$\begin{aligned}\left\|u_t\left(\cdot,\theta_{-t}\omega,\phi\right)\right\|_{co}^{\mathcal{C}} &\le \left\|v_t\left(\cdot,\theta_{-t}\omega,\psi\left(\theta_{-t}\omega\right)\right)\right\|_{co}^{\mathcal{C}} + \|z(\theta_{-t}\theta_{t+\cdot}\omega)\|_{co}^{\mathcal{C}}\\ &\le ce^{\mu\tau}r(\omega)+e^{-\mu t}e^{\mu\tau}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}+\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}(e^{(\varepsilon L_f e^{\mu\tau}-\mu)t}-1)\\ &+\frac{c\varepsilon e^{\mu\tau}L_f}{\mu-\varepsilon e^{\mu\tau}L_f}[e^{-\varepsilon e^{\mu\tau}L_f}-e^{-\mu t}]r(\omega)+ce^{\frac{\mu\tau}{2}}r(\omega).\end{aligned} \tag{4.11}$$

Therefore, if $\phi\in\mathcal{D}\left(\theta_{-t}\omega\right)$ and $\varepsilon L_f e^{\mu\tau}-\mu<0$, then there exists a $T_{\mathcal{D}}>\tau$ such that, for all $t\ge T_D(\omega)$,

$$e^{-\mu t}e^{\mu\tau}\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}+\left\|\psi\left(\theta_{-t}\omega\right)\right\|_{co}^{\mathcal{C}}(e^{(\varepsilon L_f e^{\mu\tau}-\mu)t}-1)-\frac{c\varepsilon e^{\mu\tau}L_f}{\mu-\varepsilon e^{\mu\tau}L_f}e^{-\mu t}r(\omega)\le c_1(\omega), \tag{4.12}$$

which, along with (4.11), shows that, for all $t\ge T_{\mathcal{D}}(\omega)$

$$\left\|u_t\left(\cdot,\theta_{-t}\omega,\phi\right)\right\|_{co}^{\mathcal{C}}\le 2ce^{\mu\tau}r(\omega)+\frac{c\varepsilon e^{\mu\tau}L_f}{\mu-\varepsilon e^{\mu\tau}L_f}e^{-\varepsilon e^{\mu\tau}L_f}r(\omega)+c_1(\omega). \tag{4.13}$$

Given $\omega\in\Omega$, define

$$K(\omega)=\{\varphi\in\mathcal{C}:\|\varphi\|_{co}^{\mathcal{C}}\le 2ce^{\mu\tau}r(\omega)+\frac{c\varepsilon e^{\mu\tau}L_f}{\mu-\varepsilon e^{\mu\tau}L_f}e^{-\varepsilon e^{\mu\tau}L_f}r(\omega)+c_1(\omega)\}. \tag{4.14}$$

Then, $K=\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Furthermore, (4.13) implies that K is a random absorbing set for the RDS Φ in $\mathcal{D}$. □

In the sequel, we first show that Ψ is a continuous random semiflow with respect to $\|\cdot\|_{co}^{\mathcal{C}}$.

LEMMA 4.3. Assume (**H**) is satisfied and f is globally bounded, then $\Phi\left(t,\theta_{-t}\omega,\phi\right)$ is a continuous random semiflow with respect to the compact open topology induced by the norm $\|\cdot\|_{co}^{\mathcal{C}}$.

Proof. It suffices to prove that for any given $\phi_n\in D\left(\theta_{-t_n}\omega\right)$ and $\phi\in D\left(\theta_{-t}\omega\right)$ such that $t_n\to t,\phi_n\to\phi$ then the sequence $\Psi\left(t_n,\theta_{-t_n}\omega,\phi_n\right)$ convergent to $\Psi\left(t,\theta_{-t}\omega,\phi\right)$ with respect to $\|\cdot\|_{co}^{\mathcal{C}}$. Here, for convenience, we assume that $t_n\ge t$ since the case $t_n<t$ can be proved similarly.

To prove the continuity, define $P:\mathbb{R}_+\times\Omega\times\mathcal{C}\to X_{co}$ by $P(t,\theta_{-t}\omega,\phi)(x)=\Psi(t,\theta_{-t}\omega,\phi)(0)(x)$ for all $(t,\phi)\in\mathbb{R}_+\times\mathcal{C}$. By Theorem 3.2 and the cocycle property of z, we only need to prove that $P(t,\theta_{-t}\omega,\phi)(x)$ is a random semiflow. Take $\left\{\left(t_n,\phi_n\right)\right\}_{n\in\mathbb{N}}\subset\mathbb{R}_+\times D\left(\theta_{-t_n}\omega\right)$ such that $\lim_{n\to\infty}\left|t_n - t\right| = 0$ and $\lim_{n\to\infty} d_r\left(\phi_n,\phi\right) = 0$. Denote by $G_n \triangleq |P(t_n,\theta_{-t_n}\omega,\phi_n)(x) - P(t,\theta_{-t}\omega,\phi)(x)|$. For any given a bounded closed interval I and any $x \in I$, we have

$$
\begin{aligned}
G_n &= \left|\Psi(t_n,\theta_{-t_n}\omega,\phi_n)(0)(x) - \Psi\left(t,\theta_{-t}\omega,\phi\right)(0)(x)\right| \\
&\le \left|\Phi(t_n,\theta_{-t_n}\omega,\psi_n)(0)(x) - \Phi\left(t,\theta_{-t}\omega,\psi\right)(0)(x)\right| + |z(\theta_t\omega)(x) - z(\theta_{t_n}\omega)(x)| \\
&\le |S(t_n)\psi_n(0,x) - S(t)\psi(0,x)| + |\int_0^{t_n} S(t_n - r)F\left(v_r^n + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x) \\
&\quad - \int_0^t S(t-r)F\left(v_r + z\left(\theta_{-t}\theta_{r+\cdot}\omega\right)\right)(x)dr| + |\int_0^{t_n} S(t_n - r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x)dr \\
&\quad - \int_0^t S(t-r)\Delta z\left(\theta_{-t}\theta_r\omega\right)(x)dr| + |z(\theta_t\omega)(x) - z(\theta_{t_n}\omega)(x)| \\
&\triangleq I_1 + I_2 + I_3 + I_4,
\end{aligned}
\tag{4.15}
$$

for P-a.e. $\omega \in \Omega$. Now, we estimate each term on the right-hand side of (4.15).

$$
\begin{aligned}
I_1 &\leqq |[S(t_n) - S(t)]\psi_n(0,x)| + |S(t)[\psi_n(0,x) - \psi(0,x)]| \\
&\le \frac{\left(1+t_n\right)\exp\left(-\mu t_n\right)|\psi_n(0,x)|}{t_n}\left|t - t_n\right| + e^{-\mu t}|\psi_n(0,x) - \psi(0,x)| \\
&\triangleq I_{11} + I_{12}.
\end{aligned}
\tag{4.16}
$$

It follows from $\lim_{n\to\infty} d_r\left(\phi_n,\phi\right) = 0$, the continuity of $z(\theta_t\omega)$ with respect to t and $\lim_{n\to\infty}\left|t_n - t\right| = 0$ that $\lim_{n\to\infty} I_{12} = 0$. Clearly, $I_{11} \to 0$ because of $t_n \to t$.

$$
\begin{aligned}
I_2 &\leqq |\int_0^{t_n} S(t_n - r)F\left(v_r^n + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr - \int_0^t S(t_n - r)F\left(v_r^n + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr| \\
&\quad + |\int_0^t S(t_n - r)F\left(v_r^n + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr - \int_0^t S(t_n - r)F\left(v_r + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr| \\
&\quad + |\int_0^t S(t_n - r)F\left(v_r + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr - \int_0^t S(t - r)F\left(v_r + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr| \\
&\quad + |\int_0^t S(t_n - r)F\left(v_r + z\left(\theta_{-t_n}\theta_{r+\cdot}\omega\right)\right)(x)dr - \int_0^t S(t - r)F\left(v_r + z\left(\theta_{-t}\theta_{r+\cdot}\omega\right)\right)(x)dr| \\
&\triangleq I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
\tag{4.17}
$$

By Lemmas 3.1 and 3.2 and the boundedness of f, we have for any $x \in I$ and P-a.e. $\omega \in \Omega$

$$
I_{21} \le \varepsilon\int_t^{t_n} e^{-\mu(t_n - r)}f\left(v_r^n + z\left(\theta_{r+\cdot}\omega\right)\right)(x)dr \le \varepsilon M\int_t^{t_n} e^{-\mu(t_n-r)}dr. \tag{4.18}
$$

Since $t_n \to t$, we can see $I_{21} \to 0$.

$$I_{22} \leq \varepsilon L_f \int_0^t e^{-\mu(t_n-r)} \|v_r^n - v_r\|_{\mathcal{C}_{co}} dr \leq \varepsilon L_f |v_r(\cdot, \theta_{-t_n}\omega, \psi_n)(x) - v_r(\cdot, \theta_{-t_n}\omega, \psi)(x)| \int_t^{t_n} e^{-\mu(t_n-r)} dr. \tag{4.19}$$

By similar procedure as the proof of Theorem 2.8-(i) in Yi *et al.* (2012), we have $I_{22} \to 0$ provided $\psi_n \to \psi$.

$$\begin{aligned} I_{23} &\leq \int_0^t [S(t_n - r) - S(t-r)] F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ &= \int_0^{t_n - \sqrt{\delta}} [S(t_n - r) - S(t-r)] F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ &\quad + \int_{t_n - \sqrt{\delta}}^t [S(t_n - r) - S(t-r)] F\left(v_r + z\left(\theta_{r+\cdot}\omega\right)\right) dr \\ &\leq \varepsilon M \int_0^{t_n - \sqrt{\delta}} \frac{\left(1 + \mu(t_n - r)\right) \exp\left(-\mu(t_n - r)\right)}{t_n - r} (t_n - t) dr + 2M|t - t_n + \sqrt{\delta}| \\ &\leq \varepsilon M \left|t - t_n\right| \left(\frac{1}{\sqrt{\delta}} + \mu\right) + 2M|t - t_n + \sqrt{\delta}|, \end{aligned} \tag{4.20}$$

where $\delta \in (0, 1)$ and $t < t_n < t + \delta$ with $\delta \to 0$. Hence, we have $I_{23} \to 0$ when $t_n \to t$. Therefore, $\lim_{n\to\infty} I_2 = 0$, as $t_n \to t, \phi_n \to \phi$. In the following, we estimate I_3.

$$\begin{aligned} I_3 &\leqq |\int_0^{t_n} S(t_n - r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) - \int_0^t S(t-r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr| \\ &\leqq |\int_0^{t_n} S(t_n - r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr - \int_0^t S(t_n - r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr| \\ &\quad + |\int_0^t S(t_n - r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr - \int_0^t S(t-r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr| \\ &\quad + |\int_0^t S(t-r)\Delta z\left(\theta_{-t_n}\theta_r\omega\right)(x) dr - \int_0^t S(t-r)\Delta z\left(\theta_{-t}\theta_r\omega\right)(x) dr| \triangleq I_{31} + I_{32} + I_{33}. \end{aligned} \tag{4.21}$$

$$I_{31} \leq c|\Delta z\left(\theta_r\omega\right)(x)| \int_t^{t_n} e^{-\mu(t_n-r)} e^{\mu(t_n-r)/2} r(\omega) dr. \tag{4.22}$$

Since $\{g_j(x)\}_{j=1}^m$ are twice continuously differentiable, there exists $M > 0$ such that for any $x \in I$ and P-a.e. ω, $|\Delta z\left(\theta_r\omega\right)(x)| \leqslant M$. Thus, $\lim_{n\to\infty} I_{31} = 0$ and $\lim_{n\to\infty} I_{31} = 0$ in the case $t_n \to t$. It follows from $\Delta z\left(\theta_t\omega\right)$ is continuous with respect to t that $\lim_{n\to\infty} I_{33} = 0$. By the same arguments as the estimation of I_{23} in (4.20), we have $\lim_{n\to\infty} I_{32} = 0$, indicating that $\lim_{n\to\infty} I_3 = 0$. Moreover, since $z_j\left(\theta_t\omega_j\right)$ is P-a.e. ω continuous, we have that $\lim_{n\to\infty} I_4 = 0$. Summing up the above computation together with Lemma 4.1, we can see that $\Phi\left(t, \theta_{-t}\omega, \phi\right)$ is a continuous random semiflow with respect to the compact open topology induced by the norm $\|\cdot\|_{co}$. □

Now, we are in the position to prove the $\mathcal{D}$-pullback asymptotically compact in $\mathcal{D}$ with respect to $\|\cdot\|_{C}^{co}$.

Lemma 4.4. Assume that (**H**) holds and f is bounded. Then, the RDS Ψ generated by SNDRDE (1.4) is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{C}_{co}$ for $t > \tau$, i.e., for P-a.e. $\omega \in \Omega$, the sequence $\{\Psi(t_n, \theta_{-t_n}\omega, \phi_n\left(\theta_{-t_n}\omega\right))\}$ has a convergent subsequence in $\mathcal{C}_{co}$ provided $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\phi_n\left(\theta_{-t_n}\omega\right) \in B\left(\theta_{-t_n}\omega\right)$.

Proof. Take an arbitrary random set $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, a sequence $t_n \to +\infty$ and $\phi_n \in B\left(\theta_{-t_n}\omega\right)$. We have to prove that $\left\{\Psi\left(t_n, \theta_{-t_n}\omega, \phi_n\right)\right\}$ is precompact. Since $\{K(\omega)\}$ is a random absorbing for Ψ, then there exists $T > 0$ such that, for all $\omega \in \Omega$,

$$\Psi\left(t, \theta_{-t}\omega\right) B\left(\theta_{-t}\omega\right) \subset K(\omega) \tag{4.23}$$

for all $t \geq T$. Because $t_n \to +\infty$, we can choose $n_1 \geq 1$ such that $t_{n_1} - 1 \geq T$. Applying (4.23) for $t = t_{n_1} - 1$ and $\omega = \theta_{-1}\omega$, we find that

$$\eta_1 \triangleq \Psi\left(t_{n_1} - 1, \theta_{-t_{n_1}}\omega, \phi_{n_1}\right) \in K\left(\theta_{-1}\omega\right). \tag{4.24}$$

Similarly, we can choose a subsequence $\left\{n_k\right\}$ of $\{n\}$ such that $n_1 < n_2 < \cdots < n_k \to +\infty$ such that

$$\eta_k \triangleq \Psi\left(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right) \in K\left(\theta_{-k}\omega\right). \tag{4.25}$$

Hence, by the assumption we conclude that the sequence

$$\left\{\Psi\left(k, \theta_{-k}\omega, \eta_k\right)\right\} \tag{4.26}$$

is precompact. On the other hand by (4.25), we have

$$\Psi(k, \theta_{-k}\omega, \eta_k) = \Psi(k, \theta_{-k}\omega, \Psi(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k}) = \Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right), \tag{4.27}$$

for all $k \geq 1$. Combining (4.26) and (4.27), we obtain that the sequence $\left\{\Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right)\right\}$ is precompact. Therefore, $\left\{\Psi\left(t_n, \theta_{t_n}\omega, \phi_{n_k}\right)\right\}$ is precompact, which completes the proof. □

Lemma (4.2) says that the continuous RDS Ψ has a random absorbing set while Lemma (4.1) tells us that (θ, Ψ) is pullback asymptotically compact in C_{co}. Thus, it follows from Lemma 2.1 that the continuous RDS (θ, Ψ) possesses a random attractor. Namely, we obtain the following result.

Theorem 4.1. Assume that (**H**) holds, $\varepsilon L_f e^{\mu\tau} - \mu < 0$ and f is bounded, then the continuous RDS Ψ generated by (1.4) admits a unique $\mathcal{D}$-pullback attractor in $\mathcal{C}_{co}$ belonging to the class $\mathcal{D}$.

5. Existence of exponentially attracting stationary solutions

In this section, we are devoted to deriving sufficient conditions that guarantee the random attractor being an exponentially attracting random fixed point ξ^* by adopting the general Banach fixed point theorem. We first introduce the general Banach fixed point theorem, which was established in Schmalfus (1998) and extended in Duan *et al.* (2003) to infinite case in the following.

LEMMA 5.1. Let (Y, d_Y) be a complete metric space with bounded metric. Suppose that

$$\Phi(t, \omega, Y) \subset Y$$

for $\omega \in \Omega, t \geq 0$, and that $x \to \Phi(t, \omega, x)$ is continuous. In addition, we assume the contraction condition: There exists a constant $k < 0$ such that, for $\omega \in \Omega$,

$$\sup_{x \neq y \in Y} \log \frac{d_Y(\Phi(1, \omega, x), \Phi(1, \omega, y))}{d_Y(x, y)} \leq k$$

Then Φ has a unique generalized fixed point γ^* in Y. Moreover, the following convergence property holds:

$$\lim_{t \to \infty} \Phi\left(t, \theta_{-t}\omega, x\right) = \gamma^*(\omega)$$

for any $\omega \in \Omega$ and $x \in Y$.

THEOREM 5.1. Assume that f is bounded and satisfies (**H**). Moreover, assume that $0 < \tau < 1$ and $\mu > \max\{\frac{\varepsilon L_f}{1-\tau}, \varepsilon L_f e^{\mu\tau}\}$. Then the RDS Ψ generated by SNDRDE (1.4) possess a tempered random fixed point ξ^*, which is unique under all tempered random variables in $\mathcal{C}_{co}$ and attracts exponentially fast every random variable in $\mathcal{C}_{co}$.

Proof. If $\mu > \varepsilon L_f e^{\mu\tau}$, then the conditions of Theorem 4.1 hold and hence (1.4) possess random attractors in $\mathcal{C}_{co}$. We will prove that (1.4) admits a unique globally exponentially attracting random stationary solution in $\mathcal{C}_{co}$, which immediately implies the random attractor in $\mathcal{C}_{co}$ obtained in Theorem 4.1 is the random fixed point. If suffices to prove that the RDS Φ generated by (3.11) has a unique exponentially attracting generalized fixed point χ^*. Since the transformation $v(t) = u(t) - z\left(\theta_t\omega\right)$ and $u(t) = v(t) + z\left(\theta_t\omega\right)$ are conjugation, one can see that $\xi^* = \chi^* + z\left(\theta_t\omega\right)$ is a unique exponentially attracting generalized fixed point of (1.4) by conjugation technique.

By (4.5) and the Grönwall inequality, one can see that for any $\psi \in \mathcal{C}_{co}$

$$\begin{aligned} \left\|v_t\left(\cdot, \omega, \psi\left(\omega\right)\right)\right\|_{co}^{\mathcal{C}} \leq & c e^{\mu\tau} r(\omega) + e^{-\mu t} e^{\mu\tau} \left\|\psi\left(\omega\right)\right\|_{co}^{\mathcal{C}} + \left\|\psi\left(\omega\right)\right\|_{co}^{\mathcal{C}} \left(e^{(\varepsilon L_f e^{\mu\tau} - \mu)t} - 1\right) \\ & + \frac{c\varepsilon e^{\mu\tau} L_f}{\mu - \varepsilon e^{\mu\tau} L_f} [e^{-\varepsilon e^{\mu\tau} L_f} - e^{-\mu t}] r(\omega), \end{aligned} \tag{5.1}$$

which implies that for any $\psi \in \mathcal{C}_{co}$, $\Phi(t, \omega, \psi) \in \mathcal{C}_{co}$, i.e. $\mathcal{C}_{co}$ is invariant under the random semiflow Φ. Moreover, it follows from Lemma 4.3 that Φ is continuous in $\mathcal{C}$. Therefore, we only need to prove the contraction property. That is, there exists $k < 0$ such that

$$\sup_{\varphi \neq \psi \in \mathcal{C}_{co}} \|\Phi(1,\omega,\varphi) - \Phi(1,\omega,\psi)\|_{co}^{\mathcal{C}} \leq e^{k} \|\varphi - \psi\|_{co}^{\mathcal{C}}. \tag{5.2}$$

Hence, it suffices to prove that for any $\varphi, \psi \in \mathcal{C}$

$$\|\Phi(1,\omega,\varphi) - \Phi(1,\omega,\psi)\|_{co}^{\mathcal{C}} = \|v_1(\cdot,\omega,\varphi) - v_1(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} \leq e^{k} \|\varphi(\zeta,x) - \psi(\zeta,x)\|_{co}^{\mathcal{C}}. \tag{5.3}$$

By Eq. (3.14), we have for any $\varphi, \psi \in \mathcal{C}$

$$\begin{aligned} \|v_1(\cdot,\omega,\varphi) - v_1(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} &\leq \|S(1)[\varphi(0) - \psi(0)]\| \\ &+ \sup_{\zeta\in[-\tau,0]} \int_0^{1+\zeta} S(1+\zeta-r)\left[F\left(v_r^{\varphi} + z\left(\theta_{r+\cdot}\omega\right)\right) - F\left(v_r^{\phi} + z\left(\theta_{r+\cdot}\omega\right)\right)\right] dr \\ &\leq e^{\mu\tau}\left[e^{-\mu}\|\phi - \psi\|_{co}^{\mathcal{C}} + \varepsilon L_f \int_0^1 e^{-\mu(1-r)} \left\|v_r(\cdot,\omega,\phi) - v_r(\cdot,\omega,\psi)\right\|_{co}^{\mathcal{C}} \mathrm{d}r\right]. \end{aligned} \tag{5.4}$$

Multiply both sides of (6.3) by e^{μ} leads to

$$e^{u}\|v_1(\cdot,\omega,\varphi) - v_1(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} \leq e^{\mu\tau}[\|\varphi - \psi\|_{co}^{\mathcal{C}} + \varepsilon L_f \int_0^1 e^{\mu r} \left\|v_r(\cdot,\omega,\varphi) - v_r(\cdot,\omega,\psi)\right\|_{co}^{\mathcal{C}} \mathrm{d}r]. \tag{5.5}$$

Again, the Grönwall inequality gives rise to

$$e^{u}\|v_1(\cdot,\omega,\varphi) - v_1(\cdot,\omega,\psi)\|_{co}^{\mathcal{C}} \leq e^{\mu\tau+\varepsilon L_f}\|\varphi - \psi\|_{co}^{\mathcal{C}}, \tag{5.6}$$

implying that $\|v_{\varphi} - v_{\psi}\|_{co}^{\mathcal{C}} \leq e^{\mu(\tau-1)+\varepsilon L_f}\|\phi - \psi\|_{co}^{\mathcal{C}}$. Thus, $\mu > \frac{\varepsilon L_f}{1-\tau}$ means that $\mu(\tau-1) + \varepsilon L_f < 0$, indicating that (6.1) satisfies. Therefore, $\Phi(t,\omega,\cdot)$ admits a random exponentially attracting generalized fixed point χ^* and $\xi^* = \chi^* + z\left(\theta_t\omega\right)$ is a unique exponentially attracting generalized fixed point of (1.4). This completes the proof. □

6. Applications

In this section, we will apply our main results to the stochastic non-local delayed Nicholson's blowfly equation. As pointed before, we consider the ideal case, i.e. the one-dimensional domain. The results can be extended to half planes similarly to the deterministic case Wang (2014); Hu & Duan (2018); Hu *et al.* (2018).

We consider equation (1.4) with the birth function being Ricker's function $b(u) = pue^{-qu}$, which was first employed by Gurney *et al.* (1980) to fit the Nicholson's blowfly experiment data and has been widely used as a prototype of birth function for many species such as blowfly and fish. Here, p is the maximum per capita egg production, τ is the maturation time rate and $1/q$ is the size at which the population reproduces at its maximum rate. Thus, (1.4) can be written as

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \varepsilon \int_{\mathbb{R}_+} \Gamma(\alpha,x,y)pu(t-\tau,y)e^{-qu(t-\tau,y)}\mathrm{d}y + \sum_{j=1}^{m} g_j(x)\frac{dw_j}{dt}, \\ u(t,0) = 0, \quad t > 0, \\ u(t,x) = \phi(t,x), \quad (t,x) \in [-\tau,0] \times \mathbb{R}_+. \end{cases} \tag{6.1}$$

THEOREM 6.1. For (6.1), the following statements are true:
(i) If $q > e^2$, and $\mu > \varepsilon pqe^{-2}e^{\mu\tau}$, then (6.1) admits a random attractor. Furthermore, if the delay τ satisfies $0 < \tau < 1$ and $\mu > \max\{\frac{\varepsilon pqe^{-2}}{1-\tau}, \varepsilon pqe^{-2}e^{\mu\tau}\}$, then the random attractor is a random exponentially attracting generalized fixed point.
(ii) If $0 < q \le e^2$ and $\mu > \varepsilon pe^{\mu\tau}$, then (6.1) admits a random attractor. Furthermore, if the delay τ satisfies $0 < \tau < 1$ and $\mu > \max\{\frac{\varepsilon p}{1-\tau}, \varepsilon pe^{\mu\tau}\}$, then the random attractor is a random exponentially attracting generalized fixed point.

Proof. Differentiating f twice gives

$$f'(u) = p(1-qu)e^{-qu}, \tag{6.2}$$

$$f''(u) = -qp(2-qu)e^{-qu}. \tag{6.3}$$

Hence, $f(u) \le f(\frac{1}{q}) = \frac{p}{q}e^{-1}$, i.e. f is bounded by $\frac{p}{q}e^{-1}$. Moreover, by (6.3), we can see $f'(u)$ is decreasing on $[0, \frac{2}{q})$ and increasing on $(\frac{2}{q}, +\infty)$. Moreover, $f'(u) < 0$ on $(\frac{1}{q}, +\infty)$ and $f'(u) > 0$ on $[0, \frac{1}{q})$. Therefore, we have $|f'(u)| \le \max\{f'(0), |f'(\frac{2}{q})|\} = \max\{p, pqe^{-2}\}$. In the case $q > e^2$, we have

$$|f(u) - f(v)| \le pqe^{-2}|u - v|, \tag{6.4}$$

i.e. we can take $L_f = pqe^{-2}$. Thus, by Theorem 4.1, we have if

$$\mu > \varepsilon pqe^{-2}e^{\mu\tau}, \tag{6.5}$$

then (6.1) admits a random attractor. Furthermore, if the delay τ satisfies $0 < \tau < 1$ and $\mu > \max\{\frac{\varepsilon pqe^{-2}}{1-\tau}, pq\varepsilon e^{-2}e^{\mu\tau}\}$, then the random attractor is a random exponentially attracting generalized fixed point.

In the scenario $q \le e^2$, we can take $L_f = p$. Thus, by Theorem 4.1, we have if

$$\mu > \varepsilon pe^{\mu\tau}, \tag{6.6}$$

then (6.1) admits a random attractor. Furthermore, if the delay τ satisfies $0 < \tau < 1$ and $\mu > \max\{\frac{\varepsilon p}{1-\tau}, \varepsilon pe^{\mu\tau}\}$, then the random attractor is a random exponentially attracting generalized fixed point. □

In the following text, we give some comments of Theorem 6.1 from the biological point of view.

REMARK 6.1. Since $1/q$ is the size at which the population reproduces at its maximum rate, Theorem 6.1 says that if the mature population arrives its maximum reproduction rate at a small size, say $0 < u < \frac{1}{e^2}$, the parameter q affects the existence and the structure of random attractors, while in the case the mature population arrives its maximum reproduction rate at a large size, say more than $\frac{1}{e^2}$, then the existence and the structure of random attractors do not depend on the parameter q.

REMARK 6.2. For simplicity, we use Theorem 6.1 (ii) to explain the influence of other parameters on the existence and the structure of random attractors since it is the same for (i). If we fix the maximum birth rate p, the death rate μ of mature population and ε of immature population such that $\mu > \max\{\frac{\varepsilon p}{1-\tau}, \varepsilon p e^{\mu\tau}\}$, then the increase of maturation time τ may destroy the existence of attractors and the attractors being a random exponentially attracting fixed point, i.e. the long maturation time may make the population unstable. It is clear that if we fix other parameters such that the inequalities hold, then the increase of p, ε and τ will all destroy the random attractor. At last, if p, ε and τ are fixed, the left of $\mu > \varepsilon p e^{\mu\tau}$ increases at a linear rate, while the right increases at an exponential rate and hence with the increase of μ, the right will at last be larger than the left, which destroys the condition. In summary, when we fix other parameters, each single parameter in (6.1) may destabilize the system when it becomes large enough.

7. Summary

In this paper, we have obtained the existence and qualitative property of random attractors for (1.4) on a semi-infinite interval $\mathbb{R}_+$. We show that under certain conditions, the random attractor is a globally exponentially attracting random stationary solution. From dynamical system theory, the conditions for the attractors being fixed point are so strong that they could be hardly met in the real world applications. Indeed, from the dissipative system theory, if some estimates on the dimension of random attractors can be given, it will benefit the researchers a lot in studying the structure of the random attractor. Nevertheless, the lack of inner product of the phase space and the asymmetry as well as non-compactness of spatial domain made this problem quite challenging, requiring further studies. Furthermore, in order to obtain the global complex dynamics and non-local analysis of the qualitative properties of the system, existence and structure of the associated invariant manifolds of the stationary solutions, and the existence of connecting orbits (including the heteroclinic orbits or homoclinic orbits) are all of great significance and deserve much attention.

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