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SEMIGROUPS, AMENABILITY AND FIXED POINT THEOREMS

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Abstract

This master thesis is focused on the study of amenability theory and its connection with fixed point theory for dynamical systems. A semigroup is said to be amenable if it admits a finitely additive probability measure defined in all of its subsets that is invariant by translations.

We aim to give a general overview on amenability theory and its relationship with the existence on invariant means, which deeply makes use of techniques from functional analysis. We show techniques to produce examples of amenable and non-amenable groups. During this approach we prove and apply extensively Markov-Kakutani and Day's fixed point theorems, which provide a characterization of amenability in terms of the existence of a common fixed point for a semigroup action. These results lie within the scope of the so-called "Kakutani-type" fixed point theorems in the context of topological dynamical systems. Last chapter will be devoted to the study of a wide class of this type of theorems for semigroup actions with special focus on the a Ryll-Narzdewski fixed point theorem. Some applications and further linear and non-linear extensions will be considered.

Contents

Introduction

This work is focused on the study of amenable semigroups and its properties with special emphasis on its connection with fixed point theory for dynamical systems. A semigroup is said to be amenable if it admits a finitely additive probability measure defined in all of its subsets that is invariant for the inner action of the semigroup.

Amenability theory started with the measure problem. That is, the existence of a finitely additive extension of the Lebesgue measure to all subsets in $\mathbb{R}^n$ that is invariant under rigid motions. This statement was proved to be true for $n = 1, 2$ in 1923 by Banach in [1]. Later, Banach and Tarski showed a paradoxical decomposition of the sphere in $\mathbb{R}^3$ [2]. That is, a partition of the sphere that can be rearranged, by rigid motions, to create the original sphere twice.

Amenability arises as the explanation of the difference between dimensions in the measure problem. The groups of isometries for one and two dimensions are amenable, that is, they each admit a finitely additive measure defined in all of their subsets that is invariant with respect to the group. On the other hand, the isometry group is not amenable for $n \geq 3$. The notion of amenability can also be considered for semigroups.

Amenability theory was connected to functional analysis when in 1932 Banach related finitely additive measures and a set of linear operators called means. This is considered the birth of modern amenability. Taking advantage of this, in 1961 Mahlon M. Day gave a major result regarding amenable semigroups [7], which we cover in this manuscript. This result is known as Day's fixed point Theorem and it characterizes amenable semigroups by a fixed point property, which shows an interplay among amenability theory, functional analysis and fixed point theory.

Day's fixed point Theorem is called a "Kakutani-type" fixed point result. That is, a statement of the form: Given a group or semigroup S of continuous affine transformations on a compact convex subset K of a locally convex vector space X

into itself. Then, under suitable conditions, S has a common fixed point in K.

The Kakutani-type name was coined by Namioka in [23] and still in recent day it has experienced new advances, including connections to amenability. We remark the works of various authors such us Wiśnicki [34] or Lau [19] and [20].

This work is structured as follows: In Chapter 1, we present some preliminary notions that will appear throughout the rest of this manuscript. We mainly include results on locally convex vector spaces and the Banach space $\ell^\infty(E)$ of bounded functions from an arbitrary set E into $\mathbb{R}$.

The second chapter covers the existence of an isometry between finitely additive measures of bounded variation on a set E and the dual space of $\ell^\infty(E)$. We make use of this isometry to introduce the concept of means and relate it to probability finitely additive measures. Then, in preparation of the next chapter, some properties of the set of means are given. During this chapter, various examples were added in order to illustrate each concept.

Making use of the previous sections, Chapter 3 is devoted to the notion of amenable semigroups, invariant means and their connection. The first Section is a summary on some algebraic notions regarding these structures. Another key concept on amenability are actions generated by semigroups. Specifically, we define a natural action of a semigroup S on the dual space of $\ell^\infty(S)$ to state an equivalent definition of amenability in terms of the existence of invariant means. In this regard, Section 3.4 is dedicated to two remarkable examples: the additive semigroup $\mathbb{N}$, which we relate to the study of Banach limits; and the main example of non-amenable groups, free groups.

In Chapter 4 we provide the Markov-Kakutani and Day's fixed point theorems. The Markov-Kakutani theorem requires commutative operators and will be applied to prove that abelian semigroups are amenable. On the other hand, Day's theorem is a proper characterization of amenability by fixed point properties, making it remarkably useful to prove that certain families of semigroups are amenable. This fact is mainly showcased in Section 5.1.

In Chapter 5, we first showcase different techniques to generate amenable semigroups from any given amenable semigroup. Among others, we show that these examples of semigroups are amenable: subgroups of amenable groups, quotient groups of amenable groups and homomorphic images of amenable semigroups. As a consequence every group containing a free subgroup is not amenable. This raised what is known as the von-Neumann conjecture which will be explained along Section 5.2.

Besides within this Chapter, we will display some other characterizations of amenability through the study of paradoxical decompositions obtained by Følner and Tarski: a group is amenable if and only if it does not admit a paradoxical decomposition if and only if it satisfies the so called Følner condition. This will drive us to connect amenability with the measure problem. The most famous paradoxical decomposition is the one given by Banach and Tarski of the unit sphere in $\mathbb{R}^3$ and the group of isometries.

Chapter 6, inspired by Namioka [23], is devoted to various Kakutani-type results in the setting of topological dynamical systems. Markov-Kakutani and Day fixed point Theorems studied in Chapter 4 connected to amenability are just simple examples. Through this chapter, we pursue two main objectives.

Firstly, we aim to prove the so called Ryll-Narzdewski fixed point Theorem and give some of its applications, as for instance, the existence of a Haar measure on every compact topological group. Lastly, several classical and recent common fixed point results under the actions of semigroups will be displayed, among which we find extensions and nonlinear counterparts of the Ryll-Narzdewski theorem.

Although they do not lie within the scope of this master thesis, amenability has found in the recent years many applications in multiple areas such as Ergodic Theory, Banach Algebra, Cellular Automata and Group Theory. For more proper references regarding these topics the reader can consult the monographs [4], [16], [25] or [28].

Chapter 1

Preliminaries

In this first section, we aim to provide the reader with some general context in regard to topological locally convex vector spaces. Also, we will give some results on which subsequent sections are based.

Since this section is designed as a simple refresh of a basic functional analysis course, some proofs will be excluded. Nevertheless, these results will each have a referenced proof. The main references for this Section are [9] and [27].

1.1 Topological locally convex vector spaces

We recall the following definition of topological vector spaces.

Definition 1.1.1. *A topological vector space (X, τ) is a vector space X over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ together with a topology τ such that (X, τ) is Hausdorff and the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous.*

It follows from the definition that the maps $y \to v + y$ for each fixed $v \in X$ and $y \to ay$ for each fixed scalar $a \neq 0$ are homeomorphisms from X onto itself. Hence, the neighborhood system of the origin determines the whole topology. The dual space X^* of a topological vector space X is the space of all linear and continuous maps $x^* : (X, \tau) \to \mathbb{R}$ $[\mathbb{C}]$. The evaluation $x^*(x)$ can be noted as x^*x or as $\langle x^*, x\rangle$.

In this work, we are interested in those topological vector spaces that have a base of convex neighborhoods of the origin. These spaces are said to be locally convex.

Proposition 1.1.2. *The dual space of a locally convex space X separates points of X. More formally, if X is a locally convex space, then for each distinct pair $x, y \in X$ there exists a map $x^* \in X^*$ such that $x^*x \neq x^*y$.*

Proof. Corollary 3.5 in [27]. $\square$

Given a locally convex space X, we focus our interests on two useful topologies, the weak and the weak* topologies given upon X and X^* respectively. They are usually noted as $\sigma(X, X^*)$ and $\sigma(X^*, X)$ or ω and ω^* respectively. Both topologies are defined as initial topologies although they can be understood in terms of convergence. The weak topology ω is the initial topology with respect to the family X^*. In other words, it is the coarsest topology on X such that each element in X^* remains a continuous function. In regards to net convergence: let $\{x_\alpha\}$ be a net X, then $x_\alpha \to x$ in the weak topology if and only if $x^*x_\alpha \to x^*x$ for every $x^* \in X^*$. This is also noted as $x_\alpha \xrightarrow{\omega} x$ or $\omega - \lim_\alpha x_\alpha = x$.

On the other hand, the weak* topology is the coarsest topology that makes the maps

$$\begin{aligned} x : X^* &\longrightarrow \mathbb{R}\ [\mathbb{C}] \\ x^* &\longmapsto x^*(x) \end{aligned}$$

continuous for each $x \in X$. It translates to convergence as: a net $\{x^*_\alpha\}$ in X^* is convergent to $x^* \in X^*$ with regards to the weak* topology if and only if $x^*_\alpha x \to x^*x$, for all $x \in X$. Such limit is also noted as $x^*_\alpha \xrightarrow{\omega^*} x^*$ or $\omega^* - \lim_\alpha x^*_\alpha = x^*$.

We shall consider as well the space of linear and continuous operators from a topological vector space X into itself. This space is usually noted as $\mathcal{L}(X)$. Given an operator $T \in \mathcal{L}(X)$, we consider the adjoint operator T^* defined on the dual space X^* as

$$\begin{aligned} T^* : X^* &\longrightarrow X^* \\ x^* &\longmapsto x^* \circ T. \end{aligned}$$

One can verify that an adjoint operator is linear and continuous. We provide now two properties that will be used in Section 3.2.

Proposition 1.1.3. *An adjoint operator is ω^*-to-ω^* continuous.*

Proof. Let $T \in \mathcal{L}(X)$ and let (x^*_α) be a net in X^* verifying $x^* = \omega^* - \lim x^*_\alpha$. Then, by definition of T^*,

$$\langle x, T^* x^*_\alpha \rangle = \langle Tx, x^*_\alpha \rangle,$$

for each $x \in X$.

Now, by weak* convergence of (x^*_α), we have that for every $y \in$

$$\langle y, x^*_\alpha \rangle \to \langle y, x^* \rangle.$$

Thus, writing $y = Tx$ we have

$$\langle x, T^* x^*_\alpha \rangle = \langle Tx, x^*_\alpha \rangle \to \langle Tx, x^* \rangle = \langle x, T^* x^* \rangle,$$

for every $x \in X$. Hence T^* is ω^*-to-ω^* continuous. $\square$

We conclude this section with two results related to the ω topology. Both proofs are based on the Hahn-Banach separation Theorem, which we now recall from [27, Theorem 3.4].

Theorem 1.1.4. *Let A and B be disjoint, nonempty, convex sets in a topological vector space X.*

(a) If A is open there exist $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\mathrm{Re}(\Lambda x) < \gamma \leq \mathrm{Re}(\Lambda y)$$

for every $x \in A$ and for every $y \in B$.

(b) If A is compact, B is closed, and X is locally convex, then there exist $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\mathrm{Re}(\Lambda x) < \gamma_1 < \gamma_2 < \mathrm{Re}(\Lambda y)$$

for every $x \in A$ and for every $y \in B$.

Let C be a subset of a vector space X. The closure with regards to a given topology σ is noted as $\overline{C}^\sigma$.

Proposition 1.1.5. *Let K be a convex subset of a locally convex vector space (X, τ). Then, $\overline{K}^\tau = \overline{K}^\omega$.*

Proof. Since the weak topology is weaker than the τ topology, $\overline{K}^{\tau} \subseteq \overline{K}^{\omega}$ is immediate. Now, let $x \notin \overline{K}^{\tau}$. Then, since $\overline{K}^{\tau}$ is a convex closed set and $\{x\}$ is a convex compact set, Theorem 1.1.4 yields the existence of an $x^* \in X^*$ and a scalar $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}(x^*x) < \alpha < \mathrm{Re}(x^*y),$$

for all $y \in \overline{K}^{\tau}$. Thus, since x^* is weakly continuous, $(\mathrm{Re}(x^*))^{-1}((-\infty, \alpha))$ is a weakly open set that does not intersect K. This implies that $x \notin \overline{K}^{\omega}$, concluding the proof. $\square$

Proposition 1.1.6. *Let X be a topological real vector space and let $T : X \to X$ be an affine τ-continuous operator. Then, T is ω-to-ω continuous.*

Proof. Let (x_α) be a weakly convergent net and let $x = \omega - \lim_\alpha x_\alpha$. Assume $\omega - \lim_\alpha Tx_\alpha \neq Tx$. Then there exist $\varepsilon > 0$ and a subnet denoted again by (x_α), such that either $|\mathrm{Re}(x^*(Tx_\alpha)) - \mathrm{Re}(x^*(Tx))| > \varepsilon$ or $|\mathrm{Im}(x^*(Tx_\alpha)) - \mathrm{Im}(x^*(Tx))| > \varepsilon$. As both cases are analogous, we assume that the first inequality holds. Then, we have that

$$\mathrm{Re}(x^*(Tx_\alpha)) \notin (\mathrm{Re}(x^*(Tx)) - \varepsilon, \mathrm{Re}(x^*(Tx)) + \varepsilon).$$

In particular, we can extract a further subnet, which we also denote (x_α), such that

$$\mathrm{Re}(x^*(Tx_\alpha)) < \mathrm{Re}(x^*(Tx)) - \varepsilon \text{ [or } \mathrm{Re}(x^*(Tx_\alpha)) > \mathrm{Re}(x^*(Tx)) + \varepsilon],$$

which implies

$$\mathrm{Re}(x^*(Ty)) < \mathrm{Re}(x^*(Tx)) - \varepsilon, \tag{1.1}$$

for all $y \in \mathrm{co}(\{x_\alpha\}_\alpha)$, since T is affine.

Now apply Proposition 1.1.5 to obtain $x \in \overline{\mathrm{co}}^{\omega}(x_\alpha) = \overline{\mathrm{co}}^{\tau}(x_\alpha)$. Then there is a net $(y_\beta) \subset \mathrm{co}(x_\alpha)$ such that $y_\beta \xrightarrow{\tau} x$ and since T is τ-to-τ continuous, $T(y_\beta) \to T(x)$ which contradicts (1.1). $\square$

1.2 The space $\ell^\infty(E)$

We give in this section some properties of what is the context space of well over half of this work. First, we recall the definition of a Banach space. Let X be a vector space. A map $\|\cdot\| : X \to \mathbb{R}$ is a norm if it verifies that $\|x - y\|$ defines a distance in

X and that $\|\lambda x\| = |\lambda|\, x$ for every $x \in X$ and for every $\lambda \in \mathbb{R}$ [$\mathbb{C}$]. A vector space X together with a norm $\|\cdot\|$ is called a normed space and it is an example of a locally convex vector space. A Banach space is a complete normed space.

Given a normed space X, the following norm is named the natural norm of the dual space X^*,

$$\|\Delta\| = \sup\left\{\frac{|\Delta(x)|}{\|x\|} : x \in X, x \neq 0\right\}.$$

Theorem 1.2.1. *The dual space of a Banach space with its natural norm defined above is a Banach space.*

Proof. See Theorem 4.1 in [27, p. 92]. □

The next result is an application of Tychonov's Theorem.

Theorem 1.2.2 (Banach-Alaoglu)**.** *Let X be a Banach space. Then closed unit ball in X^* is compact in the ω^*-topology.*

Proof. Theorem 3.15 in [27, p. 68]. □

Now we introduce the space $\ell^\infty(E)$ that plays an essential role in the upcoming chapters.

Definition 1.2.3. *Given a set E, we note by $\ell^\infty(E)$ the space of all bounded functions $f : E \to \mathbb{R}$. More formally,*

$$\ell^\infty(E) = \{f : E \to \mathbb{R} : \exists M > 0 \text{ such that } |f(x)| \leq M, \forall x \in E\}.$$

Proposition 1.2.4. *The space $\ell^\infty(E)$ together with the norm*

$$\|f\| = \sup\{|f(x)| : x \in E\}$$

is a Banach space.

Sketch of the proof. Let (f_n) be a Cauchy sequence in $\ell^\infty(E)$ and apply the completeness of $\mathbb{R}$ to prove that $(f_n(x))$ is a convergent sequence. Then, it is a simple verification that the function f defined as $f(x) = \lim f_n(x)$ is in $\ell^\infty(E)$.

Thus, we shall consider the dual space $\ell^\infty(E)^*$ of all linear and continuous maps from $\ell^\infty(E)^*$ to $\mathbb{R}$. This space, together with its natural norm

$$\|\Delta\| = \sup\left\{\frac{|\Delta(f)|}{\|f\|} : f \in \ell^\infty(E), f \neq 0\right\},$$

verifies all the properties discussed above.

Chapter 2

The dual of $\ell^{\infty}(E)$: Bounded finitely additive measures and means

This section is devoted to the properties of finitely additive measures and the introduction of the concept of *means*. Specifically, the majority of this segment is dedicated to a representation theorem analogous to the Riesz representation theorem. This representation will allow us to apply results from functional analysis and therefore make use of a wealthy collection of techniques. This approach will be present throughout Chapters $2, 3$ and 4.

2.1 Bounded finitely additive measures

This segment is dedicated to the properties of finitely additive measures of bounded variation on a set E. The main reference in this matter is [8, Chapter VII] .

Let $\mathcal{P}(E)$ denote the set of all subsets of E i.e.

$$\mathcal{P}(E) = \{A : A \subseteq E\}.$$

Contrarily to classical measure theory, in this work we are only interested in measures defined in all subsets, meaning there is no concept of measurability.

Definition 2.1.1. *A map* $\mathrm{m} : \mathcal{P}(E) \to \mathbb{R}$ *is called a finitely additive measure on* E *if it is finitely additive. That is,* $\mathrm{m}(A \cup B) = \mathrm{m}(A) + \mathrm{m}(B)$ *for all disjoint subsets* $A, B \in \mathcal{P}(E)$*.*

Definition 2.1.2. *Given a finitely additive measure* m *on* E*, we define its variation as*

$$|\mathrm{m}|(A) = \sup \left\{ \sum_{i=1}^{n} |\mathrm{m}(A_i)| : n \in \mathbb{N}, A = \bigcup_{i=1}^{n} A_i \text{ and } A_i \cap A_j = \emptyset, \forall i \neq j \right\},$$

for each $A \subseteq E$*.*

Definition 2.1.3. *A measure* m *is called of bounded variation if* $|\mathrm{m}|(E)$ *is finite. The set of all finitely additive measures of bounded variation on a set* E *is noted as* $\boldsymbol{ba}(E)$*.*

Proposition 2.1.4. *The variation of a measure in* $\boldsymbol{ba}(E)$ *is itself a measure in* $\boldsymbol{ba}(E)$*.*

Proof. Since the variation of a measure $\mathrm{m} \in \mathbf{ba}(E)$ is bounded, the mapping $|\mathrm{m}|$ is defined from $\mathcal{P}(E)$ to $[0, +\infty)$. Also, for every $A, B \subset E$ such that $A \cap B = \emptyset$, any finite partition $\mathcal{C} = \{C_i\}_{i=1}^{n}$ of $A \cup B$ defines two disjoint partitions of A and B respectively defined as:

$$\begin{aligned} \mathcal{A} &= \{C_i \in \mathcal{C} : C_i \subset A\}, \\ \mathcal{B} &= \{C_i \in \mathcal{C} : C_i \subset B\}. \end{aligned}$$

Hence,

$$\begin{aligned} |\mathrm{m}|(A \cup B) = & \sup \left\{ \sum_{i=1}^{n} |\mathrm{m}(C_i)| : A \cup B = \bigcup_{i=1}^{n} C_i \text{ and } C_i \cap C_j = \emptyset, \forall i \neq j \right\} \\ = & \sup \left\{ \sum_{i=1}^{n} |\mathrm{m}(A_i)| : A = \bigcup_{i=1}^{n} A_i \text{ and } A_i \cap A_j = \emptyset, \forall i \neq j \right\} \\ & + \sup \left\{ \sum_{i=1}^{n} |\mathrm{m}(B_i)| : B = \bigcup_{i=1}^{n} B_i \text{ and } B_i \cap B_j = \emptyset, \forall i \neq j \right\} \\ = & |\mathrm{m}|(A) + |\mathrm{m}|(B). \end{aligned}$$

Thus, $|\mathrm{m}| : \mathcal{P}(E) \to \mathbb{R}$ is finitely additive, and since the variation of $|\mathrm{m}|$ is $|\mathrm{m}|$ itself, then $|\mathrm{m}|$ is in $\mathbf{ba}(E)$. $\square$

For the following proposition, we need to give a meaning to two operations in the space $\mathbf{ba}(E)$. Let $\mathrm{m}_1, \mathrm{m}_2 \in \mathbf{ba}(E)$ and $\lambda \in \mathbb{R}$. Then, we have the following operations

- The addition $\mathrm{m}_1 + \mathrm{m}_2$ is defined as $(\mathrm{m}_1 + \mathrm{m}_2)(A) = \mathrm{m}_1(A) + \mathrm{m}_2(A)$ for each $A \subset E$.

- The scalar multiplication $\lambda \mathrm{m}_1$ is defined as $(\lambda \mathrm{m}_1)(A) = \lambda \mathrm{m}_1(A)$ for each $A \subset E$.

It is easy to check that, with these operations, $\mathbf{ba}(E)$ is a vector space.

Proposition 2.1.5. *The mapping that assigns each measure* $\mathrm{m} \in \boldsymbol{ba}(E)$ *its variation evaluated at* E, $|\mathrm{m}|(E) = \|\mathrm{m}\|$, *is a norm.*

Proof. It is easy to check that $|\mathrm{m}|(E) = 0$ if and only if $\mathrm{m}(A) = 0$ for all $A \subseteq E$, since $|\mathrm{m}(A)| + |\mathrm{m}(E \setminus A)| > 0$ for any subset A with non-zero measure.

Let $\mathrm{m}_1, \mathrm{m}_2 \in \mathbf{ba}(E)$ and $\lambda \in \mathbb{R}$. Let $\{A_i\}_{i=1}^n$ be a disjoint partition of E. Then,

$$\sum_{i=1}^{n} |(\mathrm{m}_1 + \mathrm{m}_2)(A_i)| = \sum_{i=1}^{n} |\mathrm{m}_1(A_i) + \mathrm{m}_2(A_i)| \leq \sum_{i=1}^{n} |\mathrm{m}_1(A_i)| + \sum_{i=1}^{n} |\mathrm{m}_2(A_i)|,$$

which implies $|\mathrm{m}_1 + \mathrm{m}_2|(E) \leq |\mathrm{m}_1|(E) + |\mathrm{m}_2|(E)$. Similarly, it holds that

$$\sum_{i=1}^{n} |\lambda \mathrm{m}_1(A_i)| = |\lambda| \sum_{i=1}^{n} |\mathrm{m}_1(A_i)|$$

and thus $|\lambda \mathrm{m}|(E) = |\lambda| \, |\mathrm{m}|(E)$. □

Proposition 2.1.6. *Let* $\mathrm{m} \in \boldsymbol{ba}(E)$ *be a bounded finitely additive measure on* E. *Then, for all* $A, B \in \mathcal{P}(E)$, *one has:*

1) $\mathrm{m}(\emptyset) = 0$;

2) $\mathrm{m}(A \cup B) = \mathrm{m}(A) + \mathrm{m}(B) - \mathrm{m}(A \cap B)$;

3) If $A \subset B$, *then* $\mathrm{m}(B \setminus A) = \mathrm{m}(B) - \mathrm{m}(A)$.

If $\mathrm{m} \in \boldsymbol{ba}(E)$ *is positive, the following also hold:*

4) $\mathrm{m}(A \cup B) \leq \mathrm{m}(A) + \mathrm{m}(B)$;

5) If $A \subset B$, *then* $\mathrm{m}(A) \leq \mathrm{m}(B)$;

Proof. First property derives from $\mathrm{m}(E) = \mathrm{m}(E \cup \emptyset) = \mathrm{m}(E) + \mathrm{m}(\emptyset)$. For all $A, B \in \mathcal{P}(E)$, we have $\mathrm{m}(A \cup B) = \mathrm{m}(A) + \mathrm{m}(B \setminus A)$ and $\mathrm{m}(B) = \mathrm{m}(A \cap B) + \mathrm{m}(B \setminus A)$, which gives 2) and, if m is non-negative, implies 4). Finally, since $B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \emptyset$; we have $\mathrm{m}(B) = \mathrm{m}(A) + \mathrm{m}(B \setminus A)$, yielding 5). Now applying the same reasoning as for 4), 5) holds. $\square$

We now define the concept of finitely additive probability measure. We will give this definition its full meaning applying the bijection in Theorem 2.2.1.

Definition 2.1.7. *A map* $\mathrm{m} : \mathcal{P}(E) \to [0, 1]$ *is called a finitely additive probability measure on* E *if it satisfies the following properties:*

1. $\mathrm{m}(E) = 1$,

2. $\mathrm{m}(A \cup B) = \mathrm{m}(A) + \mathrm{m}(B)$ *for all* $A, B \in \mathcal{P}(E)$ *such that* $A \cap B = \emptyset$.

The set of all probability measures on a set E is noted as $\mathcal{PM}(E)$. Trivially, the set of probability measures on E is a subset of $\mathbf{ba}(E)$.

Example. Let $x \in E$ be any element. The map defined as

$$\mathrm{m}_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A; \end{cases}$$

is a finitely additive probability measure. Clearly $\mathrm{m}_x(E) = 1$, since $x \in E$. Also, since any two disjoint subsets $A, B \subset E$ verify that x is in one and only one of them, the finitely additive property holds.

Note that, as probability measures are positive, they verify all properties in Proposition 2.1.6.

2.2 Representation Theorem

The upcoming results might be recognized by anyone already familiar with the Riesz representation theorem. We aim to prove that the Banach spaces $\ell^\infty(E)^*$ and $\mathbf{ba}(E)$ are isometrically identified.

For each subset $A \subset E$, we denote by χ_A the characteristic map of A, that is, the map $\chi_A : E \to \mathbb{R}$ defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Trivially

$\chi_A \in \ell^\infty(E)$ for all $A \subseteq E$. The bijection relies on the following statement. For each $\mu \in \ell^\infty(E)^*$, one can define a measure m as

$$\begin{aligned} \mathrm{m} : \mathcal{P}(E) &\longrightarrow \mathbb{R} \\ A &\longrightarrow \mathrm{m}(A) = \mu(\chi_A). \end{aligned}$$

Observe that by linearity of μ,

$$\mathrm{m}(A \cup B) = \mu(\chi_{A\cup B}) = \mu(\chi_A + \chi_B) = \mu(\chi_A) + \mu(\chi_B)$$

for all $A, B \subset E$ such that $A \cap B = \emptyset$. Moreover, for any disjoint partition $\{A_i\}_{i=1}^n$ of E,

$$\begin{aligned} \sum_{i=1}^n |\mathrm{m}(A_i)| &= \sum_{i=1}^n |\mu(\chi_{A_i})| \\ &= \sum_{i=1}^n \mu(\chi_{A_i})\mathrm{sgn}\ \mu(\chi_{A_i}) \\ &= \mu\left(\sum_{i=1}^n \chi_{A_i}\mathrm{sgn}\ \mu(\chi_{A_i})\right) \leq \|\mu\|, \end{aligned}$$

since $\|\chi_{A_i}\mathrm{sgn}\ \mu(\chi_{A_i})\| = 1$ for every A_i and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Thus, taking supremum on the disjoint partitions $\{A_i\}_{i=1}^n$ of E, we get

$$\|\mathrm{m}\| \leq \|\mu\| \tag{2.1}$$

Hence, m is a measure of bounded variation i.e. $\mathrm{m} \in \mathbf{ba}(E)$.

Theorem 2.2.1. *The map $\Phi : \ell^\infty(E)^* \to \boldsymbol{ba}(E)$ that assigns each mean $\mu \in \ell^\infty(E)^*$ the measure $\mathrm{m} \in \boldsymbol{ba}(E)$ defined as:*

$$\mathrm{m}(A) = \Phi\mu(A) = \mu(\chi_A),$$

for all $A \subset E$, is linear and bijective. Moreover, $\|\mu\| = \|\mathrm{m}\|$. Thus, $\boldsymbol{ba}(E)$ with its natural norm is a Banach space that is isometrically identified as the dual of $\ell^\infty(E)$.

The proof of this theorem is divided in several steps in the form of the following lemmas. The proof is strongly based on the density of the subspace of simple functions in $\ell^\infty(E)$. Although this fact can be known, we introduce all the details on the sake of completeness.

The objective is, given a measure, to define a bounded linear operator in a dense subspace of $\ell^\infty(E)$ and extend it by continuity to provide an operator on all $\ell^\infty(E)$. Specifically, we call $S(E)$ the set of all simple functions defined on E, that is, functions that take finite many values of $\mathbb{R}$. In fact, we will use the following representation of these functions: Let $x \in S(E)$ and $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ the values x takes, then

$$x = \sum_{i=1}^{n} \lambda_i \chi_{A_i},$$

where $A_i = x^{-1}(\lambda_i)$. When there is no worry of confusion, we also note the function with constant value 1 as $\chi_E = 1$ for simplicity of notation.

Lemma 2.2.2. *The vector subspace $S(E)$ is dense in $\ell^\infty(E)$.*

Proof. Let $x \in \ell^\infty(E)$. Now set $\alpha = \inf_E x$ and $\beta_E = \sup_E x$. Since x is bounded $|\alpha|, |\beta| < +\infty$, we can define, for every $n \in \mathbb{N}$, the finite set $\{\lambda_i\}_{i=1}^n$, where $\lambda_i = \alpha + i(\beta - \alpha)/n$ for $i = 1, \dots, n$. Now consider the map $y_n : E \longrightarrow \mathbb{R}$ defined by

$$y_n(a) = \min\{\lambda_i : x(a) \leq \lambda_i\},$$

for all $a \in E$. The map y_n is a simple function since it takes values in the set $\{\lambda_1, \dots, \lambda_n\}$. This function also satisfies by construction that

$$\|x - y_n\|_\infty \leq (\beta - \alpha)/n.$$

Consequently, $S(E)$ is dense in $\ell^\infty(E)$ in regards to $\|\cdot\|_\infty$, since we have defined a sequence of simple functions $\{y_n\}$ convergent to x for an arbitrary $x \in \ell^\infty(E)$. $\square$

Let m be a finitely additive measure on E. We define the map $\overline{\mu} : S(E) \to \mathbb{R}$ such as, for all $x \in S(E)$,

$$\overline{\mu}(x) = \overline{\mu}\left(\sum_{i=1}^{n} \lambda_i \chi_{A_i}\right) = \sum_{i=1}^{n} \lambda_i \mathrm{m}(A_i), \tag{2.2}$$

where $\{\lambda_1, \dots, \lambda_n\}$ are the finite values of x and $A_i = x^{-1}(\lambda_i)$.

Lemma 2.2.3. *For each measure* $\mathrm{m} \in \boldsymbol{ba}(E)$*, the map* $\overline{\mu} : S(E) \longrightarrow \mathbb{R}$ *defined above is linear and continuous with* $|\overline{\mu}(x)| \leq \|x\| \, \|\mathrm{m}\|$*, for all* $x \in S(E)$*.*

Proof. Let $x, y \in S(E)$ and $\psi, \eta \in \mathbb{R}$. Denote V (respectively W) the set of values taken by x (resp. y). Then the family of subsets $\{x^{-1}(\alpha) \cap y^{-1}(\beta)\}_{(\alpha,\beta)\in V\times W}$ is a partition of E. By definition of $\overline{\mu}$ we get:

$$\begin{aligned}\overline{\mu}(\psi x + \eta y) = & \sum_{(\alpha,\beta)\in V\times W} \mathrm{m}(x^{-1}(\alpha) \cap y^{-1}(\beta))(\psi\alpha + \eta\beta) = \\ & = \psi \sum_{(\alpha,\beta)\in V\times W} \mathrm{m}(x^{-1}(\alpha) \cap y^{-1}(\beta))\alpha + \\ & \quad + \eta \sum_{(\alpha,\beta)\in V\times W} \mathrm{m}(x^{-1}(\alpha) \cap y^{-1}(\beta))\beta = \\ & = \psi\overline{\mu}(x) + \eta\overline{\mu}(y).\end{aligned}$$

Consequently, $\overline{\mu}$ is linear.

Let $x \in S(E)$, $V = \{\lambda_1, \dots, \lambda_n\}$ be the set of values taken by x and $A_i = x^{-1}(\{\lambda_i\})$. Let $\lambda = \max\{|\lambda_i| : 1 \leq i \leq n\}$. Then, since all A_i are disjoint,

$$|\overline{\mu}(x)| = \left|\sum_{i=1}^{n} \lambda_i \mathrm{m}(A_i)\right| \leq \lambda \left(\sum_{i=1}^{n} |\mathrm{m}(A_i)|\right) \leq \lambda \|\mathrm{m}\| = \|x\| \, \|\mathrm{m}\| \, .$$

Hence, we have that $\|\overline{\mu}\| \leq |\mathrm{m}(E)| < +\infty$ i.e. $\overline{\mu}$ is continuous. □

Lemma 2.2.4. *Let* X *be a normed vector space and let* Y *be a dense vector subspace of* X*. Suppose that* $\varphi : Y \longrightarrow \mathbb{R}$ *is a continuous linear map. Then the exists a continuous linear map* $\tilde{\varphi} : X \longrightarrow \mathbb{R}$ *such that* $\tilde{\varphi}|_Y = \varphi$ *with* $\|\tilde{\varphi}\| = \|\varphi\|$*.*

Proof. Let $x \in X$. Since Y is dense, there exists a sequence $\{y_n\}_{n\geq 0}$ in Y such that $\lim y_n = x$. For all, $p, q \geq 0$, we have that

$$|\varphi(y_p) - \varphi(y_q)| = |\varphi(y_p - y_q)| \leq \|\varphi\| \, \|y_p - y_q\| \, .$$

Thus, $\{\varphi(y_n)\}$ is a Cauchy sequence and it converges as $\mathbb{R}$ is complete. Now set $\tilde{\varphi}(x) = \lim \varphi(y_n)$. Notice this definition does not depend on the sequence. Let $\{y_n'\}$ be another sequence on y which tends to x then it holds that

$$|\varphi(y_n) - \varphi(y_n')| \leq \|\varphi\| \, \|y_n - y_n'\| \, .$$

Linearity is inherited by the linearity of φ. Let $x, \hat{x} \in X$ and $\{y_n\}, \{\hat{y}_n\} \subset Y$ sequences convergent to x and $\hat{x}$ respectively. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\lim \varphi(\alpha y_n + \beta \hat{y}_n) = \lim(\alpha\varphi(y_n) + \beta\varphi(\hat{y}_n)) = \alpha \lim \varphi(y_n) + \beta \lim \varphi(\hat{y}_n) = \alpha\tilde{\varphi}(x) + \beta\tilde{\varphi}(\hat{x}).$$

It holds that $\|\varphi(y_n)\| \leq \|\varphi\| \, \|y_n\|$, for all $n \geq 0$. Thus

$$\|\tilde{\varphi}(x)\| \leq \|\varphi\| \, \|x\| \,, \tag{2.3}$$

for all $x \in X$. This shows that $\tilde{\varphi}$ is continuous. Also, if $x \in Y$ we can take $y_n = x$ for all $n \geq 0$, so $\tilde{\varphi}|_Y = \varphi$.

It remains to prove that $\|\tilde{\varphi}\| = \|\varphi\|$. From (2.3) we have $\|\tilde{\varphi}\| \leq \|\varphi\|$ and, since $\tilde{\varphi}(x) = \varphi(x)$ for all $x \in S(E)$, we have that:

$$\|\tilde{\varphi}\| = \sup\{|\tilde{\varphi}(x)| : x \in X, \|x\| = 1\} \geq \{|\tilde{\varphi}(x)| : x \in Y, \|x\| = 1\} = \|\varphi\| \,.$$

Thus $\|\tilde{\varphi}\| = \|\varphi\|$. □

Proof of Theorem 2.2.1. Let $\mathrm{m} \in \mathbf{ba}(E)$. Apply the extension in Lemma 2.2.4 with $Y = S(E)$, $X = \ell^\infty(E)$ and $\varphi = \overline{\mu}$ from (2.2). Call μ the extension of $\overline{\mu}$. By construction of μ, it holds that,

$$\mu(\chi_A) = \overline{\mu}(\chi_A) = \mathrm{m}(A),$$

for all $A \subset E$. Thus $\phi(\mu) = \mathrm{m}$ and ϕ is surjective.

It remains to show that ϕ is injective. Consider $\mu_1, \mu_2 \in \ell^\infty(E)^*$ such that $\phi(\mu_1) = \phi(\mu_2)$, which implies $\mu_1(\chi_A) = \mu_2(\chi_A)$, for every $A \subset E$. By linearity, μ_1 and μ_2 agree on every element in $S(E)$. Since $S(E)$ is dense in $\ell^\infty(E)$ and μ_1, μ_2 are continuous they agree in all of $\ell^\infty(E)$. Lastly, combine the norm inequalities in (2.1) and 2.2.3 to get

$$\|\mu\| = \|\overline{\mu}\| \leq \|\mathrm{m}\| \leq \|\mu\| \,,$$

i.e. $\|\mathrm{m}\| = \|\mu\|$. □

2.3 Means and probability measures

Now that we have our representation theorem identifying $\ell^\infty(E)^*$ with $\mathbf{ba}(E)$ assembled, we introduce the notion of means as operators in $\ell^\infty(E)^*$ that satisfy certain properties. The next objective is to prove the relationship between means and probability finitely additive measures.

Definition 2.3.1. *Let μ in $\ell^\infty(E)^*$. Then μ is called a mean on $\ell^\infty(E)$ if it satisfies:*

$$\inf\{f(x) : x \in E\} \leq \mu(f) \leq \sup\{f(x) : x \in E\}, \tag{2.4}$$

for all $f \in \ell^\infty(E)$.

The set of all means on the set E is noted by $\mathcal{M}(E)$. The following basic operators are examples of means.

Example. Let E be any set. For each $e \in E$, the functor:

$$\begin{aligned} \delta_e : \ell^\infty(E) &\longrightarrow \mathbb{R} \\ f &\longrightarrow f(e), \end{aligned}$$

is trivially a mean, since $\inf_E f \leq \delta_e(f) = f(e) \leq \sup_E f$. Thus, the set of means on E, $\mathcal{M}(E)$, is non-empty.

We now prove equivalent conditions for linear and continuous maps in $\ell^\infty(E)^*$ to be means. Recall that the notation $1 \in \ell^\infty(E)$ corresponds to the constant function of value 1, χ_E.

Proposition 2.3.2. *Let $\mu \in \mathcal{M}(E)$ be a mean. Then μ satisfies the following properties:*

$$\mu(f) \geq 0 \quad \text{if} \quad f \geq 0; \tag{I}$$

$$\mu(1) = 1; \tag{II}$$

$$\|\mu\| = 1. \tag{III}$$

Conversely, if $\mu \in \ell^\infty(E)^*$ satisfies any two of the conditions (I), (II) *and* (III), *then μ is a mean.*

Proof. We start assuming $\mu \in \ell^\infty(E)^*$ is a mean and proving the three conditions above. First and second properties are immediately derived from the inequality in the definition 2.3.1:

$$\mu(f) \geq \inf\{f(x) : x \in E\} \geq 0, \; \forall f \geq 0$$

and

$$1 = \inf\{1(x) : x \in E\} \leq \mu(1) \leq \sup\{1(x) : x \in E\} = 1.$$

Consequently, (II) and (2.3.1) implies (III) since, for any $f \in \ell^\infty(E)$,

$$\|\mu(f)\| = |\mu(f)| \leq \sup\{|f(x)| : x \in E\} = \|f\|$$

thus, $\|\mu\| \leq 1$ and,

$$1 = |\mu(1)| \leq \|\mu\| \, \|1\| = \|\mu\| \, .$$

We will now prove that any two conditions in the statement imply (2.4):

1. First assume (I) and (II). Then let $f \in \ell^\infty(E)$ and define

$$\begin{aligned} a &= \inf\{f(x) : x \in E\}, \\ b &= \sup\{f(x) : x \in E\}. \end{aligned}$$

   thus $f - a, b - f \geq 0$. Now, simply apply the assumptions:

$$\begin{aligned} 0 \leq \mu(f-a) = \mu(f) - \mu(a) = \mu(f) - a\mu(1) = \mu(f) - a \Longrightarrow \mu(f) \geq a, \\ 0 \leq \mu(b-f) = \mu(b) + \mu(-f) = b\mu(1) - \mu(f) = b - \mu(f) \Longrightarrow \mu(f) \leq b. \end{aligned}$$

2. Assuming (I) and (III) we will proof μ is a mean by proving (II) holds by our last reasoning. We will prove $\mu(1) = 1$ proving both inequalities.
   First, (I) and linearity yield $\mu(1) \geq 1$, since $\mu(0) \geq 0$ and $\mu(1) - 1 = \mu(0) \geq 0$.
   Now apply (III) and get

$$\mu(1) = |\mu(1)| \leq \|\mu\| \, \|1\| = 1.$$

3. Lastly, assuming (II) and (III) we will prove (I). Let $f \geq 0$ in $\ell^\infty(E)$ and define a and b like in 1). Then we have $b \geq a \geq 0$ and $b - f \geq 0$. We also have, by (II), that $\mu(c) = c\mu(1) = c$ for all constants c. Finally, also applying linearity and (III) we have

$$\begin{aligned} b - \mu(f) =& \mu(b-f) \leq \|\mu\| \, \|b-f\| = \|b-f\| = \\ =& \sup\{b - f(x) : x \in E\} = b + \sup\{-f(x) : x \in E\} = b - a. \end{aligned}$$

   Thus, $\mu(f) \geq a \geq 0$ for all $f \geq 0$.

□

Taking advantage of the results in Section 2.2, we can now easily prove the existence of a bijection between the set of probability measures $\mathcal{PM}(E)$ and the set of means $\mathcal{M}(E)$.

For each subset $A \subset E$, we denote by χ_A the characteristic map of A, that is, the map $\chi_A : E \to \mathbb{R}$ defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. The bijection relies on the following statement. Let $\mu \in \mathcal{M}(E)$ be a mean, then one can define the measure $\mathrm{m} \in \mathcal{PM}(E)$ such as

$$\begin{aligned} \mathrm{m} : \mathcal{P}(E) &\longrightarrow [0,1] \\ A &\longrightarrow \mathrm{m}(A) = \mu(\chi_A). \end{aligned}$$

Observe that $\mathrm{m}(E) = \mu(\chi_E) = \mu(1) = 1$ and, by linearity of μ,

$$\mathrm{m}(A \cup B) = \mu(\chi_{A\cup B}) = \mu(\chi_A + \chi_B) = \mu(\chi_A) + \mu(\chi_B)$$

for all $A, B \subset E$ such that $A \cap B = \emptyset$. Thus, m is indeed a measure in $\mathcal{PM}(E)$.

Theorem 2.3.3. *The map $\Phi : \ell^\infty(E)^* \to \boldsymbol{ba}(E)$ from Theorem 2.2.1 defines a bijection between $\mathcal{M}(E)$ and $\mathcal{PM}(E)$.*
More formally, the map $\Phi : \mathcal{M}(E) \to \mathcal{PM}(E)$ that assigns each mean $\mu \in \mathcal{M}(E)$ the measure $\mathrm{m} \in \boldsymbol{ba}(E)$ defined as:

$$\mathrm{m}(A) = \Phi\mu(A) = \mu(\chi_A),$$

for all $A \subset E$, is bijective.

Proof. Since $\Phi : \ell^\infty(E)^* \to \mathbf{ba}(E)$ is a bijection, it suffices to prove that

$$\Phi(\mathcal{M}(E)) = \mathcal{PM}(E).$$

1. From Theorem 2.3.3, we have that Φ is an isometry. Hence, $\Phi_{|\mathcal{M}(E)}$ is injective and $\|\Phi(\mu)\| = \|\mu\| = 1$, for all $\mu \in \mathcal{M}(E)$. Moreover, since $\mu(f) \geq 0$ for any $f \geq 0$, we have that
$$\Phi(\mu)(A) = \mu(\chi_A) \geq 0.$$
Thus, $\Phi(\mu)$ is a finitely additive positive measure verifying $\|\Phi(\mu)\| = \Phi(\mu)(E) = 1$, i.e. $\Phi(\mu) \in \mathcal{PM}(E)$. Hence, $\Phi(\mathcal{M}(E)) \subseteq \mathcal{PM}(E)$.

2. On the other hand, from the definition of Φ we have that, given a measure $\mathrm{m} \in \mathcal{PM}(E)$, then the operator $\Phi^{-1}(\mathrm{m})$ verifies $\|\Phi^{-1}(\mathrm{m})\| = \|\mathrm{m}\| = 1$ and

$$\mu(1) = \mu(\chi_E) = \mathrm{m}(E) = 1.$$

Thus, by Proposition 2.3.2, $\Phi^{-1}(\mathrm{m})$ is a mean. Hence, $\Phi^{-1}(\mathcal{PM}(E)) \subseteq \mathcal{M}(E)$.

Combining both inclusions, we conclude that $\Phi : \mathcal{M}(E) \to \mathcal{PM}(E)$ is bijective. □

2.4 Properties of the set of means

Our next interest is to provide various properties of the set of means that will be used in plenty of the upcoming results. In this segment, various techniques from functional analysis, such us the Hahn-Banach separation Theorem 1.1.4, will be applied.

Proposition 2.4.1. *The subset $\mathcal{M}(E)$ of means on E is a convex and compact subset of $\ell^\infty(E)^*$ with respect to the ω^*-topology.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{M}(E)$ and $\alpha \in [0,1]$ arbitrary. Then, by Proposition 2.3.2, to show $(\alpha\mu_1 + (1-\alpha)\mu_2)$ is a mean it suffices to prove that $(\alpha\mu_1 + (1-\alpha)\mu_2)(1) = 1$ and $(\alpha\mu_1 + (1-\alpha)\mu_2)(f) \geq 0$, for all $f \geq 0$. The first holds, since $\mu_1(1) = 1 = \mu_2(1)$. Now let $f \geq 0$. Then $\mu_1(f), \mu_2(f) \geq 0$ and

$$\alpha\mu_1(f) + (1-\alpha)\mu_2(f) \geq 0$$

for every $\alpha \in [0,1]$. It remains to prove that $\mathcal{M}(E)$ is compact. Let $\{\mu_i\}_{i \in I}$ be a net in $\mathcal{M}(E)$ converging to $u \in \ell^\infty(E)^*$. Then for all $f \in \ell^\infty(E)$ we have:

$$\inf_E f \leq \mu_i(f) \leq \sup_E f,$$

and taking limits we get

$$\inf_E f \leq u(f) \leq \sup_E f.$$

Thus, every convergent net in $\mathcal{M}(E)$ converges to a mean i.e. $\mathcal{M}(E)$ is closed in $\ell^\infty(E)^*$ with respect to the ω^*-topology. Lastly, since $\mathcal{M}(E)$ is contained in

$$\{u \in \ell^\infty(E)^* : \|u\| \leq 1\},$$

which is compact by the Banach-Alaoglu Theorem 1.2.2, $\mathcal{M}(E)$ is compact.

□

To expand our understanding on means, recall that the space $\ell_1(E)$ is the set of all functions $\phi : E \to \mathbb{R}$ such that $\|\phi\|_1 = \sum_{e\in E} |\phi(e)| < +\infty$. It is well known that the dual space of $\ell_1(E)$ is the space $\ell^\infty(E)$ and that $\ell_1(E)$ can be embedded in $\ell^\infty(E)^*$. This mapping is defined as $Q : \ell_1(S) \to \ell^\infty(E)^*$ that assigns each function $\phi \in \ell_1(E)$ the operator $Q(\phi) \in \ell^\infty(E)^*$ defined as

$$Q(\phi)(f) = \sum_{e\in E} \phi(e)f(e),$$

for each $f \in \ell^\infty(E)$. This mapping is well defined since

$$\sum_{e\in E} |\phi(e)f(e)| \leq \|f\| \sum_{e\in E} |\phi(e)| = \|f\| \, \|\phi\|_1 < +\infty.$$

Now let $x \in E$ fixed and let $\phi_x \in \ell_1(E)$ be the function $\phi(e) = 1$ if $e = x$ and $\phi(e) = 0$ otherwise. Notice then, that for each $f \in \ell^\infty(E)$ we have that

$$Q(\phi_x)(f) = \sum_{e\in E} \phi(e)f(e) = f(x).$$

That is, $Q(\phi_x) = \delta_x$ from the example after Definition 2.3.1.

Moreover, let $\{c_i\}_{i=1}^n$ be a finite sequence of real numbers verifying $\sum c_i = 1$ and $c_i \geq 0$ for all $i = 1, \ldots, n$. Let $\{x_i\}_{i=1}^n$ be a finite sequence of elements in a set E. Then, the function defined as $\phi(e) = c_i$ if $e = x_i$ and $\phi(e) = 0$ otherwise is clearly in $\ell_1(E)$. Thus, we have

$$Q(\phi)(f) = \sum_{e\in E} \phi(e)f(e) = \sum_{i=1}^{n} c_i f(x_i) = \sum_{i=1}^{n} c_i \delta_{x_i} f.$$

Therefore, any convex combination of δ_x operators in $\ell^\infty(E)^*$ can be identified with a function in $\ell_1(E)$. Thus, the set in the following definition is in fact the image by the embedding map Q of the set of convex combinations of the functions $\phi_x = \chi_{\{x\}}$, for $x \in E$.

Definition 2.4.2. *The set of finite means $\mathcal{M}_1(E)$ is the convex hull of the set of this means, i.e.*

$$\mathcal{M}_1(E) = \operatorname{co}(\{\delta_e : e \in E\}).$$

A mean lying on the set $\mathcal{M}_1(E)$ is called a finite mean.

By Proposition 2.4.1, $\mathcal{M}(E)$ is convex and thus $\mathcal{M}_1(E) \subset \mathcal{M}(E)$. As a consequence of the above discussion, the norm closure of the set of finite means $\overline{\mathcal{M}_1(E)}^{\|\cdot\|}$ is exactly the image by the embeddiment Q of the set of functions in $\ell_1(E)$ such that $\|f\|_1 = 1$ and $f(e) \geq 0$ for all $e \in E$. That is,

$$\overline{\mathcal{M}_1(E)}^{\|\cdot\|} = \mathcal{M}(E) \cap \ell_1(E) = \{f \in \ell_1(E) : \|f\|_1 = 1, f(e) \geq 0, \forall e \in E\}.$$

The true impact of the set of finite means is seen in next result, as it is in fact dense in the set of all means. Its proof is based in the Hahn-Banach separation Theorem 1.1.4.

Theorem 2.4.3. *The set* $\mathcal{M}_1(E) \subset \mathcal{M}(E)$ *verifies*

$$\overline{\mathcal{M}_1(E)}^{\omega^*} = \mathcal{M}(E).$$

Proof. As seen in the previous proposition, $\mathcal{M}(E)$ is convex and compact with regards to the ω^*-topology, so $\mathcal{M}(E)$ must contain the set $\overline{\mathcal{M}_1(E)}^{\omega^*}$. Thus we only need to prove that there is no mean lying outside of this space.

Suppose there exists a mean μ that belongs outside of $\overline{\mathcal{M}_1(E)}^{\omega^*}$. Now we apply the Hahn-Banach separation Theorem (b) 1.1.4 with $A = \{\mu\}$ the compact convex space, $B = \overline{\mathcal{M}_1(E)}^{\omega^*}$ the convex closed set and $X = \ell^\infty(E)^*$ with the ω^*-topology the topological vector set. Then there exists $f \in \ell^\infty(E)$ and $\gamma \in \mathbb{R}$ such that

$$\mu(f) < \gamma < \lambda(f),$$

for all $\lambda \in B$. Finally, since $\{\delta_e : e \in E\} \subset B$, we have the following inequalities:

$$\mu(f) < \gamma < \inf\{\lambda(f) : \lambda \in B\} < \inf\{\delta_e(f) : e \in E\} = \inf\{f(e) : e \in E\}.$$

Thus μ does not verify the definition of a mean so there is no mean lying outside of $\overline{\mathcal{M}_1(E)}^{\omega^*}$. $\square$

Chapter 3

Amenability

In this chapter, we introduce the notion of amenability for groups and semigroups. The first two segments will be dedicated to preliminary definitions and properties of the underlying concepts: semigroups and their actions.

We will define amenable semigroups in terms of probability measures and relate them to means through the isometry given in the previous chapter, obtaining different characterizations of amenability. Some examples of amenable and non-amenable semigroups will be given to illustrate these concepts.

3.1 Algebraic prelude

In this section we give some algebraic notions related to semigroups and groups that will appear through the manuscript.

Definition 3.1.1. *A semigroup is a non-empty set S together with a binary associative operation on S (usually noted as multiplication) i.e. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in S$. Also, if the semigroup operation verifies that $x \cdot y = y \cdot x$ for all $x, y \in S$, then S is a commutative or abelian semigroup.*

Definition 3.1.2. *A group G is a semigroup that verifies the following properties:*

1) *There exists an element $e \in G$ verifying $e \cdot g = g = g \cdot e$ for all $g \in G$. This element is called the neutral element of G.*

2) For every $g \in G$ there exists an element $h \in G$ such that $g \cdot h = e = h \cdot g$. The element h is called inverse of g and is noted as g^{-1}.

To avoid confusion, when working with different groups the neutral element might be noted as e_G, thus specifying where it belongs to. A major property of groups that further on will allow us to prove various results is the cancellation property. That is, for every $x, y \in G$ if $gx = gy$ for some $g \in G$, then $x = y$.

Example.

- The set of positive integers $\mathbb{N}$ with the usual addition $+$ is a semigroup while $\mathbb{Z}$ with the same operation is a group.

- The set of polynomials with the composition operator is a semigroup. The same property holds for either the set of rational functions or the set of continuous functions.

- A semigroup of linear operators. That is, a set of linear operators over a vector space closed with respect to the composition. Likewise, any set of operators generate a semigroup through composition.

While working with abstract semigroups the operation will be omitted ($xy = x \cdot y$).

Definition 3.1.3. *A subset R of a semigroup S is a subsemigroup if R with the same operation as S is a semigroup. If R with the same operation is a group, then R is a subgroup of S.*

Definition 3.1.4. *Let S and R be two semigroups. A semigroup homomorphism is a map*

$$\begin{aligned} \phi : S &\longrightarrow R \\ s &\mapsto \phi(s), \end{aligned}$$

such that $\phi(s_1 s_2) = \phi(s_1)\phi(s_2)$ where each product represents the operation of S and R respectively.
When ϕ is bijective, it is called an isomorphism. Two semigroups S and R are isomorphic if there exists and isomorphism between them.

Now we focus our interests to groups. Specially we want to define the concept of quotient group. First we need the following definition.

Definition 3.1.5. *Let G be a group. A subgroup $H \subset G$ is said to be normal if*

$$g^{-1}Hg \subseteq H,$$

for all $g \in G$.

Let G be a group and H be a normal subgroup of G. Then, it is easy to check that the relation $\sim_H$ defined as $x \sim_H y$ if $xy^{-1} \in H$ is an equivalence relation. Given an element $x \in G$, we define the equivalence class of x as

$$[x] = \{y \in G : x \sim_h y\}.$$

Definition 3.1.6. *Let G be a group and H be a normal subgroup of G. Then, we define the equivalence relation $\sim_H$ as $x \sim_H y$ if $xy^{-1} \in H$. Then the quotient defined as*

$$G/H = G/\sim_H$$

is a group. Equivalently, G/H is the group of all equivalence classes of $\sim_H$ in G.

Given a normal subgroup H of a group G, it is an easy verification that the map that assigns each element of G its equivalence class in G/H is a surjective homomorphism. This map is called the canonical homomorphism from G to G/H. The above definition can only be considered when working with normal subgroups. For an in depth discussion on quotient groups see [26, Chapter 4].

We now introduce the definition of right coset and discuss its existence. This definition is used in a major result on amenability theory in Section 5.1.

Definition 3.1.7. *Let H be a subgroup on a group G. A set $R \subset G$ is a right coset for H in G if for every $g \in G$, there exists a unique pair (h, x) of elements in H and R respectively such that $hx = g$.*

The existence of a right coset is given by the following equivalence relation. Define $\sim$ in G as $x \sim y$ if there exists $h \in H$ such that $hx = y$ i.e. $yx^{-1} \in H$. It is clearly reflexive and, since H is a subgroup, it is symmetric. Transitivity holds from $x \sim y \sim z$ if $yx^{-1}, zy^{-1} \in H$, then $zy^{-1}yx^{-1} = zx^{-1} \in H$.

Now apply the Axiom of Choice to create a set R conformed by one and only one element from each equivalence class of the set $G/\sim$. Notice then that for each $g \in G$, there exists an element $x \in R$ such that $g \in Hx$ hence there exists an element $h \in H$ such that $g = hx$. Moreover, this pair is unique since x is clearly determined by g and for every pair $h_1, h_2 \in H$ such that $h_1 x = g = h_2 x$, the cancellation property of groups implies $h_1 = h_2$.

Definition 3.1.8. *A solvable group G is a group that admits a finite chain of normal subgroups*

$$\{e\} = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G,$$

such that each successive quotient is abelian. More formally, the group H_k/H_{k-1} is abelian for each $k = 1, \ldots, n$.

To conclude this section we introduce some relevant groups that will appear throughout the upcoming sections.

Example. For each $n \in \mathbb{N}$, let A represent a $n \times n$ matrix and let c represent an element in $\mathbb{R}^n$. We define the following groups:

- Translations, $\mathbf{T}_n$. This group is always abelian.

- Rotations,

$$\mathbf{SO}_n = \{\Delta : \Delta(x) = Ax,\ A^{-1} = A^{\mathrm{T}},\ \det(A) = 1\}.$$

  This group is non-abelian for every $n \geq 3$ and it is abelian for $n = 1, 2$.

- Rotations and translations,

$$\mathbf{SG}_n = \{\Delta : \Delta(x) = Ax + c,\ A^{-1} = A^{\mathrm{T}},\ \det(A) = 1\}.$$

  This group is not abelian for each $n \geq 2$.

- Rigid Motions on $\mathbb{R}^n$,

$$\mathbf{G}_n = \{\Delta : \Delta(x) = Ax + c,\ A^{-1} = A^{\mathrm{T}}\}.$$

  With this groups we can consider the following chain of normal subgroups,

$$\mathbf{T}_n \subset \mathbf{SG}_n \subset \mathbf{G}_n.$$

We conclude this section with the definition of free groups.

Definition 3.1.9. *Let M be a set. The free group F with generating set M is the group of all finite concatenations, or words, of letters from $\{\sigma, \sigma^{-1} : \sigma \in M\}$, where two words are equivalent if they differ in finite pairs of adjacent letters of the form $\sigma\sigma^{-1}$ or $\sigma^{-1}\sigma$, called trivial syllables.*

In a free group F, a word with no trivial syllables is called a reduced word and F may be taken to consist of all reduced words. A free group with n generators is noted as $\mathbb{F}_n$.

This definition can be extended to semigroups, with a free semigroup generated by M being the set of finite concatenations $\{\sigma : \sigma \in M\}$. Notice that, in free semigroups, trivial syllables might not exist.

3.2 Left and right translations. Semigroup actions

In this section we work with the definition of a semigroup acting on a set. Then, we will define the specific actions of semigroups in the spaces $\mathbf{ba}(S)$ and $\ell^\infty(S)^*$. Although the definitions are given separately, the isometry seen in last section indicates a relation between such actions.

Definition 3.2.1. *Let S be a semigroup. An action of S on a set E is defined as a mapping of the form*

$$\begin{aligned} S \times E &\longrightarrow E \\ (s, x) &\longmapsto s(x). \end{aligned}$$

that satisfies,

$$(st)x = s(tx),$$

for all $s, t \in S$ and for all $x \in E$.
When no confusion might arise, we will note actions as $s(x) = sx$.

Notice that a semigroup operation defines a natural action of the semigroup on itself. More formally, when $E = S$, $(s, t) \mapsto st$ is called the left action of the semigroup S on itself. Another relation we can consider is sometimes called the right anti-action of S on itself as it is the converse of the left action, that is $(s, t) \mapsto ts$.

Definition 3.2.2. *Let S be a semigroup. We define the left [right] mappings of S on the space $\boldsymbol{ba}(S)$,*

$$\begin{aligned} S \times \boldsymbol{ba}(S) &\longrightarrow \boldsymbol{ba}(S) \\ (s, \mathrm{m}) &\longmapsto s\mathrm{m}\ [\mathrm{m}s], \end{aligned}$$

as

$$s\mathrm{m}(A) = \mathrm{m}(s^{-1}A)\ [\mathrm{m}s(A) = \mathrm{m}(As^{-1})],$$

for each $A \subset S$ and $s \in S$, where $s^{-1}A = \{t \in S : st \in A\}$ $[As^{-1} = \{t \in S : ts \in A\}]$.

The definition of $s^{-1}A$ $[As^{-1}]$ is not required for groups, since s^{-1} exists for every element of the group and thus $s^{-1}A$ agrees with the usual concept $s^{-1}A = \{s^{-1}t : t \in A\}$.

It is worth remark, that the commutative property yields that left and right mappings are equivalent. More formally, if S is a commutative semigroup, then $st \in A$ if and only if $ts \in A$ yielding

$$s^{-1}A = \{t \in S : st \in A\} = \{t \in S : ts \in A\} = As^{-1}.$$

Example. Let $(\mathbb{Z}, +)$ be the additive group of integers. Let $\mathrm{m} \in \mathcal{PM}(\mathbb{Z})$ be a measure and let $k \in \mathbb{Z}$. Then the definition above translates to: for each $A \subset \mathbb{Z}$,

$$k\mathrm{m}(A) = \mathrm{m}(-k + A) = \mathrm{m}(A - k) = \mathrm{m}(\{n - k : n \in A\}).$$

Notice that $-k + A = A - k$ because the addition is commutative.

Now we translate this notion to the semigroup $(\mathbb{N}, +)$. In this case, we must use the formal definition of $s^{-1}A = \{t \in S : st \in A\}$. Let $k \in \mathbb{N}$ and let $A \subset \mathbb{N}$. Then, the definition remains as follows:

$$k\mathrm{m}(A) = \mathrm{m}(k^{-1}A) = \mathrm{m}(\{n \in \mathbb{N} : k + n \in A\}) = \mathrm{m}(\{n - k : n \in A\} \cap \mathbb{N}).$$

Notice how with this simple semigroup example, we can show that $s^{-1}A$ can be empty even if A is nonempty. For example, if $A = \{1, \dots, m\}$ then $(m + 1)^{-1}A = \{n \in \mathbb{N} : m + 1 + n \in A\} = \emptyset$.

Example. Another illustrative example is the unit circumference $\partial\mathbb{D}$ in $\mathbb{C}$ which, along with the usual product, yields a group. Each $z \in \partial\mathbb{D}$ represents a rotation. Thus, let $\mathrm{m} \in \mathcal{PM}(\partial\mathbb{D})$ and let $w \in \partial\mathbb{D}$, then $w^{-1} = \overline{w}$ and

$$w\mathrm{m}(A) = \mathrm{m}(w^{-1}A) = \mathrm{m}(\overline{w}A).$$

Proposition 3.2.3. *For any given measure* $\mathrm{m} \in \boldsymbol{ba}(S)$ *and for every* $s \in S$*, the mapping* $s\mathrm{m}$ $[\mathrm{m}s]$ *is a measure in* $\boldsymbol{ba}(S)$*.*

Proof. It suffices to prove that, for any given $s \in S$, every disjoint covering of S, $\{A_i\}_{i=1}^n$, verifies that $\{s^{-1}A_i\}_{i=1}^n$ is also a disjoint covering of S.

First, let $t \in s^{-1}A_i \cap s^{-1}A_j$. Then $st \in A_i \cap A_j$ which only makes sense if $i = j$. On the other hand, let $t \in S$. Then, $st \in S$ and hence st lies inside one and only one A_i and, by definition, $t \in s^{-1}A_i$. Thus $\{s^{-1}A_i\}_{i=1}^n$ is a disjoint covering of S. □

Recall that $|\mathrm{m}| = \mathrm{m}$ for any positive measure m. Thus, as a trivial case of last property, we have that any probability measure m in $\mathcal{PM}(S)$ verifies that $s\mathrm{m}, \mathrm{m}s \in \mathcal{PM}(S)$.

We now introduce the definition of left and right translations of functions in $\ell^\infty(S)$. For each $s \in S$, the semigroup structure induces two linear operators ℓ_s and r_s from $\ell^\infty(S)$ to $\ell^\infty(S)$ for each $s \in S$, defined respectively as,

$$\ell_s f(t) = f(st) \text{ and } \mathrm{r}_s f(t) = f(ts),$$

for each $f \in \ell^\infty(S)$. Thus, we have the following left [right] translation.

$$\begin{aligned} S \times \ell^\infty(S) &\longrightarrow E \\ (s, f) &\longmapsto \ell_s f[\mathrm{r}_s f]. \end{aligned}$$

Since $|\ell_s f(t)|, |\mathrm{r}_s f(t)| \leq \|f\|$, for all $t \in S$; $\ell_s f$ and $\mathrm{r}_s f$ are in $\ell^\infty(S)$. Their adjoints, ℓ_s^* and r_s^*, are defined as

$$\ell_s^*\mu(f) = \mu(\ell_s f) \text{ and } \mathrm{r}_s^*\mu(f) = \mu(\mathrm{r}_s f),$$

for each $\mu \in \ell^\infty(S)^*$ and $f \in \ell^\infty(S)$. Analogously, we have the following maps,

$$\begin{aligned} S \times \ell^\infty(S)^* &\longrightarrow E \\ (s, \mu) &\longmapsto \ell_s^*\mu[\mathrm{r}_s^*\mu]. \end{aligned}$$

The definition of the left operator on $\ell^\infty(S)^*$ does hold the condition of action as $\ell_s^*(\ell_t^*\mu)(f) = \ell_t^*\mu(\ell_s f) = \mu(\ell_t\ell_s f) = \mu(\ell_{st} f) = \ell_{st}^*\mu(f)$. On the other hand, the right translation verifies $\mathrm{r}_s^*\mathrm{r}_t^*\mu = \mathrm{r}_{ts}^*\mu$.

These operators will be used to define amenability in terms of means. Next result will yield the continuity of left and right translations and their adjoints. The proof is only detailed for left maps as it is analogous for the right case, r_s and r_s^*.

Proposition 3.2.4. *The left [right] translations of S on $\ell^\infty(S)$ are linear and norm-continuous with $\|\ell_s\| = 1$ $[\|\mathrm{r}_s\| = 1]$. Consequently, its adjoint ℓ_s^* $[\mathrm{r}_s^*]$ is linear on $\ell^\infty(S)^*$ and $\|\ell_s^*\| = 1$ $[\|\mathrm{r}_s^*\| = 1]$.*

Proof. Linearity of both operators is easily checked by definition. Let $f_1, f_2 \in \ell^\infty(S)$, $\mu_1, \mu_2 \in \ell^\infty(S)^*$ and $\alpha, \beta \in \mathbb{R}$, then

$$\ell_s(\alpha f_1(t) + \beta f_2(t)) = \alpha f_1(st) + \beta f_2(st) = \alpha \ell_s f_1(t) + \beta \ell_s f_2(t),$$
$$\ell_s^*(\alpha\mu_1(f) + \beta\mu_2(f)) = \alpha\mu_1(\ell_s f) + \beta\mu_2(\ell_s f) = \alpha\ell_s^*\mu_1(f) + \beta\ell_s^*\mu_2(f).$$

As it was shown in the definition of ℓ_s, $|\ell_s(f)| \leq \|f\|$. Thus ℓ_s is continuous and $\|\ell_s(f)\| \leq 1$. In fact, $\|\ell_s\| = 1$ since $\ell_s(1) = 1$. Hence, $\|\ell_s^*\| = \|\ell_s\| = 1$, since taking adjoints preserves the norm. □

Corollary 3.2.5. *The operators ℓ_s and ℓ_s^* $[\mathrm{r}_s$ and $\mathrm{r}_s^*]$ are weak and weak* continuous respectively.*

Proof. Both statements are immediate, since strongly continuous and affine implies weakly continuous (Proposition 1.1.6) and Proposition 1.1.3 implies ℓ_s^* is weak* continuous. □

3.3 Amenability and means

We now introduce the definition of amenable semigroups.

Definition 3.3.1. *Let S be a semigroup. A measure $\mathrm{m} \in \mathcal{PM}(S)$ is called left [right] invariant if $s\mathrm{m}(A) = \mathrm{m}(A)$ $[\mathrm{m}s(A) = \mathrm{m}(A)]$, for each $s \in S$ and for each $A \subset S$. If m is both left and right invariant, then it is called a two-sided invariant measure or simply an invariant measure.*

Definition 3.3.2. *A semigroup S is called left amenable if it admits a measure $\mathrm{m} \in \mathcal{PM}(S)$ that is left invariant. Analogously, S is called right amenable if it admits a right invariant measure.*

If the semigroup S admits a measure $\mathrm{m} \in \mathcal{PM}(S)$ that is left and right invariant then S is said to be two-sided amenable.

Recall that left and right maps agree on commutative semigroups. Thus, $s\mathrm{m} = \mathrm{m}s$ for each $\mathrm{m} \in \mathcal{PM}(S)$ and for every $s \in S$. That is, any measure on a commutative semigroup that is left or right invariant is in fact two-sided invariant i.e. left, right and two-sided amenability are equivalent for commutative semigroups.

Example. To visualize the idea of an amenable semigroup we will show some initial examples.

- If G is a finite group, the it is amenable. We define the cardinal measure as the number of elements in a set C, $|C|$. Then, the following measure is left invariant,
$$\mathrm{m}(A) = \frac{|A|}{|G|}.$$
Since every element of G has an inverse, $g^{-1}A$ has the same number of elements as A and thus m is indeed a left invariant measure in G (in fact, it is a two-sided invariant measure).

- Let S be any semigroup with a null element 0, that is an element that verifies $s0 = 0s = 0$ for all $s \in S$. Then S is amenable with measure:
$$\mathrm{m}(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A. \end{cases}$$
It is indeed left and right invariant since $0 \in A$ if and only if $0 \in s^{-1}A[As^{-1}]$. We only detail left invariance since right invariance is analogous. First if $0 \in A$, then $s0[0s] \in A$ for all $s \in S$ and
$$0 \in s^{-1}A = \{t \in S : st \in A\}.$$
Also, if $0 \in s^{-1}A$ for any $s \in S$ then $s0 = 0 \in A$.

  In fact, This is the only left invariant measure on a semigroup with a 0 element. Notice that
$$0^{-1}A = \{t \in S : 0t \in A\} = \begin{cases} S & \text{if } 0 \in A, \\ \emptyset & \text{if } 0 \notin A. \end{cases}$$
Now, by definition of left invariant measure, m must verify that
$$\mathrm{m}(A) = \mathrm{m}(0^{-1}A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A. \end{cases}$$

Notice we have proved that $\mathbb{R}$ with the product is amenable. Another example of semigroup with a null element is the semigroup of linear operators on a vector space X with the composition. Since $T(0) = 0$ for every linear operator T, then the constant operator defined $N(x) = 0$ for all $x \in X$ is a null element for the composition $NT = N = TN$ for every linear operator T on X. Thus such semigroup is amenable.

Definition 3.3.3. *Let S be a semigroup.*

- *A mean μ on $\mathcal{M}(S)$ is called left [right] invariant if $\mu(\ell_s f) = \mu(f)$ $[\mu(\mathrm{r}_s f) = \mu(f)]$, for each $s \in S$ and for all $f \in \ell^\infty(S)$.*
- *A mean μ on $\mathcal{M}(S)$ is called a two-sided invariant mean if it is both left and right invariant.*

Notice that if a mean μ is left [right] invariant then it must verify $\ell_s^*\mu = \mu$ $[\mathrm{r}_s^*\mu = \mu]$ as elements of $\ell^\infty(S)^*$, for all $s \in S$.

Unsurprisingly, as it is shown in the following proposition, from the construction of the isometry defined in Theorem 2.3.3 naturally follows that invariant means are in correspondence with invariant probability measures. Hence, next result yields an equivalent definition of amenability.

Proposition 3.3.4. *Let $\mu \in \mathcal{M}(S)$ and Φ the isometry in Theorem 2.3.3. Then μ is left [right] invariant if and only if the measure $\mathrm{m} = \Phi(\mu)$ is left [right] invariant.*

Proof. As always the proof will only be detailed for left invariance. If μ is a left invariant mean, then $\mathrm{m} = \Phi(\mu)$ is clearly invariant since, for all $A \subset S$ and $s \in S$,

$$\mathrm{m}(A) = \mu(\chi_A) = \ell_s^*\mu(\chi_A) = \mu(\ell_s\chi_A) = \mu(\chi_{s^{-1}A}) = \mathrm{m}(s^{-1}A) = s\mathrm{m}(A).$$

Now let $\mathrm{m} \in \mathcal{PM}(S)$ be a left invariant probability measure. Then $\mu = \Phi^{-1}(\mathrm{m})$ is left invariant for simple functions. Indeed it holds that,

$$\ell_s^*\mu(\chi_A) = \mu(\chi_{s^{-1}A}) = s\mathrm{m}(A) = \mathrm{m}(A) = \mu(\chi_A),$$

for all $A \subset S$ and $s \in S$. Thus, by linearity of μ and ℓ_s^* for all $s \in S$, and continuity of ℓ_s^* (Proposition 3.2.4); μ is left invariant. $\square$

This discussion is summarized in the following corollary.

Corollary 3.3.5. *Let S be a semigroup. Then the following assertions are equivalent.*

a) S is left amenable.

b) There exists a probability measure $\mathrm{m} \in \mathcal{PM}(S)$ such that $\mathrm{m}(s^{-1}A) = \mathrm{m}(A)$, for all $s \in S$ and for all $A \subset S$.

c) There exists a mean $\mu \in \ell^\infty(S)^*$ such that $\mu(\ell_s f) = \mu(f)$ for all $s \in S$ and for all $f \in \ell^\infty(S)$. That is, $\ell_s^* \mu = \mu$ for all $s \in S$.

Analogous for right and two-sided amenability.

Notice that condition c) asserts that a left invariant mean is a fixed point of the adjoint operators ℓ_s^* for all $s \in S$.

To conclude this segment, we expose the main utility of the set of finite means $\mathcal{M}_1(S)$ and Theorem 2.4.3. Using the notation in Day's article [6], we have the following definition.

Definition 3.3.6. *Let S be a semigroup. A net $\{\mathrm{m}_\alpha\} \subset \mathcal{M}_1(S)$ of finite means is said to converge to left invariance if*

$$\omega^*\text{-}\lim \ell_s^* \mathrm{m}_\alpha - \mathrm{m}_\alpha = 0 \;\; i.e. \;\; \omega^*\text{-}\lim \ell_s \mathrm{m}_\alpha(f) - \mathrm{m}_\alpha(f) = 0,$$

for all $s \in S$ and for all $f \in \ell^\infty(S)$.

Proposition 3.3.7. *Let S be a semigroup. The following are equivalent:*

a) S is left amenable.

b) There exists a net (μ_α) of finite means that is convergent to left invariance.

Proof. By Corollary 3.3.5, S is left amenable if and only if there exists a left invariant mean in $\ell^\infty(S)^*$.

If $\mu \in \mathcal{M}(S)$ is a left invariant mean, the existence of a net (μ_α) of finite means converging to m is immediate by density of $\mathcal{M}_1(S)$ (Theorem 2.4.3). Thus, the net must verify

$$0 = \ell_s^* \mu - \mu = \omega^*\text{-}\lim \ell_s^* \mu_\alpha - \mu_\alpha,$$

for all $s \in S$, since the action ℓ_s^* is ω^*-to-ω^* continuous (Corollary 3.2.5).

On the other hand, let (μ_α) be a net of finite means on $\mathcal{M}_1(S)$ converging to left invariance. Then, taking a subnet if needed, by compactness of $\mathcal{M}(S)$ there exists a mean μ such that

$$\omega^*\text{-}\lim \mu_\alpha = \mu.$$

Now, since ℓ_s^* is continuous for every $s \in S$, we have that

$$0 = \omega^*\text{-}\lim \ell_s^* \mu_\alpha - \mu_\alpha = \ell_s^* \mu - \mu.$$

Thus, m is a left invariant mean on $\mathcal{M}(S)$. $\square$

As a consequence of the previous argument, we can remark that every cluster point of a net of finite means convergent to left invariance corresponds to a left invariant mean on $\mathcal{M}(S)$.

3.4 Examples

In this section we will study some examples of amenable and non-amenable semigroups. We will start with the semigroup of positive integers with the additive operation and the relation of left invariant means with the concept of Banach limits.

Proposition 3.4.1. *The additive semigroup of positive integers* $(\mathbb{N}, +)$ *is an amenable semigroup.*

Proof. Let δ_k be the mean defined in the example after Proposition 2.3.2 for each $k \in \mathbb{N}$. That is, $\delta_k(f) = f(k)$ for each $f \in \ell^\infty(\mathbb{N})$.

Consider the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ defined as

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_k.$$

Each μ_n is a finite mean as it is an affine combination of δ_k. We now prove that this net converges to left and right invariance. Since addition is commutative, left and right operators agree i.e. $\ell_s = \gamma_s$, for all $s \in \mathbb{N}$; so it suffices to show that

$$\lim_n \mu_n(\ell_s f) - \mu_n(f) = 0,$$

for all $f \in \ell^\infty(\mathbb{N})$. Note that, by definition and linearity of μ_n, $\mu_n(\ell_s f)$ is defined as

$$\mu_n(\ell_s f) = \frac{1}{n}\sum_{k=1}^{n} \delta_k(\ell_s f) = \frac{1}{n}\sum_{k=1}^{n} f(s+k).$$

Thus, for each fixed $s \in \mathbb{N}$ and $f \in \ell^\infty(\mathbb{N})$, we can assume that $n > s$, yielding the following

$$\begin{aligned}
|\mu_n(f) - \mu_n(\ell_s f)| &\leq \frac{1}{n}\sum_{k=1}^{n} |f(k) - f(s+k)| = \\
&= \frac{|f(1) + \cdots + f(s) - f(n+1) - \cdots - f(s+n)|}{n} \leq \\
&\leq \frac{2sC}{n} \xrightarrow{n\to\infty} 0,
\end{aligned}$$

where $C = \sup\{|f(k)| : k \in \mathbb{N}\}$. Thus, $\{\mu_n(f)\}$ is convergent to left and right invariance for all $f \in \ell^\infty(\mathbb{N})$ i.e. μ_n is convergent to both-side invariance. Thus, by Proposition 3.3.7, there exists a both-side invariant mean $\mu \in \ell^\infty(\mathbb{N})^*$. □

In the next note, we relate this proposition to Banach limits.

Definition 3.4.2. *A Banach limit is a continuous linear functional $\phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ that verifies the following properties for each $f \in \ell^\infty(\mathbb{N})$:*

1. *If $f \geq 0$, then $\phi(f) \geq 0$.*
2. *$\phi(f) = \phi(Sf)$, where $S : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ is the shift operator i.e. $Sf(n) = f(n+1)$ for all $n \geq 1$.*
3. *If $(f(n))_{n\in\mathbb{N}}$ is a convergent sequence, then $\phi(f) = \lim_n f(n)$.*

The next Proposition is an interesting relation between means, amenability and Banach limits. Notice that we specify the group $\mathbb{N}$ as the additive semigroup of positive integers, since as the multiplicative semigroup the statement would not hold.

Proposition 3.4.3. *Let $\mathbb{N}$ be the additive semigroup of positive integers. Then the set of all Banach limits on $\ell^\infty(\mathbb{N})$ is exactly the set of all invariant means in $\ell^\infty(\mathbb{N})^*$.*

Proof. Notice that a continuous linear functional $\phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ verifies 1. and 3. on the definition of a Banach limit then it is a mean. This holds from Proposition 2.3.2. Property 1. is in fact the property I in Proposition 2.3.2 and 3. implies $\phi(\mathbf{1}) = \lim_n 1 = 1$, where $\mathbf{1}$ is the constant function $f(n) = 1$, for all $n \geq 1$.

Now let μ be an invariant mean in $\ell^\infty(\mathbb{N})^*$. Then, μ is a continuous linear functional on $\ell^\infty(\mathbb{N})$ by definition and the shift operator S is the map ℓ_1. Thus, by invariance of μ, $\mu(Sf) = \mu(\ell_1 f) = \mu(f)$, verifying 2.. Lastly, 3. holds from 2. and the definition of mean since $S^k f(n) = f(n+k) = \ell_k f(n)$ i.e. $S^k = \ell_k$. Hence, if $f \in \ell^\infty(\mathbb{N})$ defines a convergent sequence $f(n) \to x$, then for every $\varepsilon > 0$ there exists $N > 0$ such that for every $n \geq N$,

$$x - \varepsilon < f(n) < x + \varepsilon.$$

That is $S^k f(n) \in (x - \varepsilon, x + \varepsilon)$ for each $k \geq N$ and for all $n \in \mathbb{N}$. Now, since by definition of mean

$$\inf_n f(n) \leq \mu(f) \leq \sup_n f(n)$$

for each $f \in \ell^\infty(\mathbb{N})$, we can write $\mu(f) = \mu(\ell_k f) = \mu(S^k f)$ to get

$$x - \varepsilon < \inf_n S^k f(n) \leq \mu(f) \leq \sup_n S^k f(n) < x + \varepsilon,$$

for all $k \geq N$. That is, for each $\varepsilon > 0$, $\mu(f) \in (x - \varepsilon, x + \varepsilon)$ yielding $\mu(f) = x$ i.e. 3. holds.

On the other hand, let ϕ be a Banach limit. Then, since we have proved that every $\varphi \in \ell^\infty(\mathbb{N})^*$ verifying 1. and 3. in the definition of Banach limit is a mean, ϕ is a mean. Now, since $S = \ell_1$ and $S^k = \ell_k$, we can apply induction on property 2. of a Banach limit to get

$$\phi(f) = \phi(Sf) = \mu(S^k f) = \mu(\ell_k f),$$

for every $k \in \mathbb{N}$. That is, ϕ is an invariant mean.

Thus, we have shown that the set of all Banach limits on $\ell^\infty(\mathbb{N})$ is exactly the set of all invariant means in $\ell^\infty(\mathbb{N})^*$. □

We now prove that free groups and semigroups of order two or greater defined in Section 3.1 are not amenable.

Proposition 3.4.4. *Free group of order two, or greater is neither left or right amenable.*

Proof. Let F be a free group or order at least two. Let $a, b \in F$ be two generators and define the sets A, A^{-1}, B, B^{-1} as the set of words starting on the left by a, a^{-1}, b, b^{-1} respectively. As non of the sets contain any trivial syllables (for example $a^{-1}a$), $a^{-1}A$ $[b^{-1}B]$ is the set of all words not starting at the left by a^{-1} $[b^{-1}]$. Thus, $a^{-1}A \cap A^{-1} = \emptyset = b^{-1}B \cap B^{-1}$. Then we have following equality ($\sqcup$ means disjoint union),

$$a^{-1}A \sqcup A^{-1} = F = b^{-1}B \sqcup B^{-1}.$$

Now assume that there exists a left invariant measure m defined on F. Notice that

$$F \setminus (A \cup A^{-1} \cup B \cup B^{-1}) = \{e\},$$

where $e \in F$ is the neutral element. Hence,

$$\begin{aligned}
\mathrm{m}(F) &= \mathrm{m}(a^{-1}A) + \mathrm{m}(A^{-1}) = \mathrm{m}(A) + \mathrm{m}(A^{-1}), \\
\mathrm{m}(F) &= \mathrm{m}(b^{-1}B) + \mathrm{m}(B^{-1}) = \mathrm{m}(B) + \mathrm{m}(B^{-1}), \\
\mathrm{m}(F) &= \mathrm{m}(A) + \mathrm{m}(A^{-1}) + \mathrm{m}(B) + \mathrm{m}(B^{-1}) + \mathrm{m}(\{e\}).
\end{aligned}$$

Thus, $\mathrm{m}(F) = 2\mathrm{m}(F) + \mathrm{m}(\{e\})$ which is a contradiction since $\mathrm{m}(F) = 1$ by definition of measure. The proof for right invariance is analogous defining the sets as the sets of words ending in a, a^{-1}, b, b^{-1} and applying right maps on m. $\square$

Proposition 3.4.5. *Free semigroup on two, or more generators, is neither left or right amenable.*

Proof. Let F be a free semigroup with two generators a, b such that $a^{-1} \notin F$. Notice that if $a^{-1} \in F$, for all generators of F we would be working with a free group, and thus the result is immediate by the previous proposition. Thus, as is a word starting with a on left for all $s \in F$.

Define $f \in \ell^\infty(F)$ as $f(s) = 1$ when s starts with a on the left, and $f(s) = 0$ otherwise. Then, the function

$$h = \ell_{ba} f - \ell_a f$$

is well defined and $h(s) = -1$ for all $s \in S$. Now assume that there exists $\mathrm{m} \in \mathcal{M}(F)$ be a left invariant mean. By definition of mean,

$$\mathrm{m}(h) \leq \sup\{h(s) : s \in S\} = -1,$$

while also, by linearity and left invariance of m,

$$\mathrm{m}(h) = \mathrm{m}(\ell_{ba} f) - \mathrm{m}(\ell_a f) = \mathrm{m}(f) - \mathrm{m}(f) = 0 > -1.$$

Hence, F is not left amenable. As usual, right amenability can be disproved by swapping the words left with right and ℓ with r. $\square$

The nonamenability of free groups will be used in Chapter 5 to generate further nonamenable groups and to raise the celebrated (and already solved) von-Neumann conjecture.

Chapter 4

Amenability Characterization Through Fixed Point Theorems

This chapter is dedicated to the first two "Kakutani-type" fixed point theorems in this master thesis: Markov-Kakutani and Day fixed point Theorems. We refer to "Kakutani-type" theorems as statements of the form:

Given a group or semigroup S acting as continuous affine transformations on a compact convex subset K of a locally convex vector space X into itself. Then, under suitable conditions, S has a common fixed point in K.

We will apply these two theorems to prove that abelian semigroups are amenable and to find a complete characterization of amenability in terms of fixed point properties.

First, we will prove the Markov-Kakutani fixed point Theorem that considers a family of commutative operators. We will dedicate the second section of this chapter to Day's fixed point Theorem. Day's Theorem is one of the most important results in this work and it will be strongly applied. Some other "Kakutani-type" theorems will be analyzed in Chapter 6.

Through both sections, we will make use of the concept of representation. Given an action of a semigroup S on a set E as in Definition 3.2.1. Then, for each $s \in S$, one can define the following mappings

$$\begin{aligned} s : E &\longrightarrow E \\ x &\longmapsto sx, \end{aligned}$$

which, by definition of action, verify $(st)x = s(tx)$. We say that the map T that assigns each element $s \in S$ the mapping $s : E \to E$ is called a representation of S as actions on E.

We are interested in some specific representations of semigroups where the arbitrary set E is a compact convex subset of a locally convex space (X, τ). The results in this chapter will assume that each mapping T_s is a τ-continuous affine map from K into itself. A map $f : K \to K$ is affine if

$$f\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i f(x_i),$$

for every $\{x_i\}_{i=1}^n \subset K$ and for any $\{c_i\}_{i=1}^n$ verifying $c_i \geq 0$ and $\sum_{i=1}^n c_i = 1$.

Notice that the left actions defined on the set of means ℓ_s^* are a representation of S as continuous linear maps from $\mathcal{M}(S)$ into itself (see Section 3.2).

4.1 Markov-Kakutani Theorem and Abelian Semigroups

In this section, we will prove that all abelian semigroups are amenable. The proof makes use of the Markov-Kakutani Theorem on fixed point theory that will be proved beforehand following [4]. The Markov-Kakutani Theorem requires some previous results that we prove in the following lemmas.

Lemma 4.1.1. *Let K be a compact subset of a topological vector space X and let V be a neighborhood of 0 in X. Then there exists a real number $\alpha > 0$ such that $\lambda K \subset V$ for every scalar λ such that $|\lambda| < \alpha$.*

Proof. Since multiplication by a scalar is continuous, for each $x \in X$ there exists $\alpha_x \in \mathbb{R}$ and an open neighborhood $V_x \subset X$ such that

$$\lambda V_x \subset V, \tag{4.1}$$

for all $|\lambda| < \alpha_x$.

Since K is compact and

$$K \subset \bigcup_{x \in K} V_x,$$

there is a finite subcollection $V_1, \ldots, V_n \in \{V_x\}_{x \in K}$ covering K. Now, for each V_i, find α_i verifying (4.1) for all $|\lambda| < \alpha_i$. Finally, conclude the proof by choosing

$$\alpha = \min_{i=1,\ldots,n} \alpha_i,$$

which indeed verifies (4.1) for all $|\lambda| < \alpha$. □

Lemma 4.1.2. *Let K be a compact subset of a topological vector space X. Let $(x_i)_{i \in I}$ be a net of points in K and let $(\lambda_i)_{i \in I}$ be a net of real numbers converging to 0 in $\mathbb{R}$. Then the net $(\lambda_i x_i)_{i \in I}$ converges to 0 in X.*

Proof. Let V be a neighborhood of 0 in X. By the previous lemma, we can find $\alpha > 0$ such that $\lambda K \subset V$ for every λ such that $|\lambda| < \alpha$. As the net $(\lambda_i)_{i \in I}$ converges to 0, there exists $i_0 \in I$ such that $|\lambda_i| < \alpha$ for all $i \geq i_0$. Thus we have $\lambda_i x_i \in V$ for all $i \geq i_0$ and, by definition, the net $(\lambda_i x_i)$ converges to 0. □

Now we have everything we need to show the first result from fixed point theory. We start by proving the existence of a fixed point for a simple mapping under some initial conditions.

Lemma 4.1.3. *Let K be a nonempty convex compact subset of a topological vector space X and let $f : K \to K$ be an affine continuous map. Then f has a fixed point in K.*

Proof. Proving the statement is equal to proving that 0 belongs to the following set

$$C = \{x - f(x) : x \in K\}.$$

Choose an element $x \in K$ and define the sequence $(x_n)_{n \geq 1}$ as

$$x_n = \frac{1}{n} \sum_{k=0}^{n-1} \left(f^k(x) - f^{k+1}(x) \right).$$

By definition of C we have that $f^k(x) \in C$ for all $k \geq 0$ since $f^k(x) - f^{k+1}(x) = f^k(x) - f(f^k(x))$.

Since K is convex and f is affine, we can prove that C is also convex. That is, let $x, y \in C$ and $\alpha \in [0,1]$, then $x = x' - f(x')$ and $y = y' - f(y')$ for some $x', y' \in K$ and

$$\alpha x + (1-\alpha)y = \alpha x' + (1-\alpha)y' - f(\alpha x' + (1-\alpha)y') \in C,$$

since $\alpha x' + (1-\alpha)y' \in K$. Thus $x_n \in C$ for all $n \geq 1$.

Now, expanding the definition of x_n, we obtain

$$x_n = \frac{1}{n}x - \frac{1}{n}f^n(x).$$

Finally, since f is continuous, the map $x \mapsto x - f(x)$ is continuous and C inherits the compact property of K, as C is the image of a continuous map. Now applying Lemma 4.1.2 to the sequence $\{x - f^n(x)\}_{n\geq 1}$, $\{x_n\}$ converges to 0. As every compact set of a Hausdorff space is closed, C is closed and contains the limit of $\{x_n\}$ i.e. $0 \in C$. □

After these lemmas, the proof of the Markov-Kakutani theorem consists on applying the finite intersection property of compact sets. We now recall this property and its proof. A collection of sets $\mathcal{F} = \{C_i\}_{i\in I}$ has the finite intersection property if, for every finite set $A \subset I$, then

$$\bigcap_{i\in A} C_i \neq \emptyset.$$

Proposition 4.1.4. *Let X be a topological space X. Then X is compact if and only if for every collection of closed subsets of X, $\mathcal{F} = \{C_i\}_{i\in I}$, with the finite intersection property verifies*

$$\bigcap_{i\in I} C_i \neq \emptyset.$$

Proof. First we assume that a collection $\mathcal{F} = \{C_i\}_{i\in I}$ of closed subsets with the finite intersection property exists that verifies

$$\bigcap_{i\in I} C_i = \emptyset.$$

Then, by taking the complementary sets of C_i, it holds that

$$\bigcup_{i\in I} (C_i)^c = X.$$

Notice that, since all C_i are closed, all $(C_i)^c$ are open subsets. Thus $\mathcal{F}$ is a cover of X. Also, the fact that all finite subcollections of $\mathcal{F}$ have nonempty intersection yields that

$$\bigcup_{j=1}^{n}(C_j)^c \neq X.$$

That is, there is $\{(C_i)^c\}_{i\in I}$ is a covering of X that has no finite subcovering i.e. X is not compact.

Lastly, since this proof can be written backwards without any changes, the reciprocal statement is already proved. □

Theorem 4.1.5 (Markov-Kakutani)**.** *Let K be a nonempty convex compact subset of a topological vector space X. Let $\mathcal{F}$ be a family of continuous affine maps $f : K \to K$ such that all elements of $\mathcal{F}$ commute, that is, $f_1 \circ f_2 = f_2 \circ f_1$ for all $f_1, f_2 \in \mathcal{F}$. Then there exists a point in K which is fixed by all the elements of $\mathcal{F}$.*

Proof. First, for each $f \in \mathcal{F}$, define $\mathrm{Fix}(f)$ as the set of all fixed points of f on K. More formally,

$$\mathrm{Fix}(f) = \{x \in K : f(x) = x\}.$$

We want to show that

$$\bigcap_{f\in\mathcal{F}} \mathrm{Fix}(f) \neq \emptyset.$$

Every set $\mathrm{Fix}(f)$ is closed in X, since for every net $(x_i)_{i\in I}$ on $\mathrm{Fix}(f)$ converging to $x \in X$ we have that, by continuity of f,

$$f(x) = \lim f(x_i) = \lim x_i = x,$$

thus $x \in \mathrm{Fix}(f)$. Now, since K is compact, proving the theorem is equivalent to proving that the collection $\{\mathrm{Fix}(f)\}_{f\in\mathcal{F}}$ has the finite intersection property.

First, note that the set $\mathrm{Fix}(f)$ is nonempty for all $f \in \mathcal{F}$ by Lemma 4.1.3. It is compact since, as we proved above, it is a closed subset of the compact set K. Also, for all $x, y \in \mathrm{Fix}(f)$ and for all $\alpha \in [0, 1]$ it holds that, by linearity of f,

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y) = \alpha x + (1-\alpha)y.$$

Thus, $\mathrm{Fix}(f)$ is convex.

Now, let $g \in \mathcal{F}$ and $x \in \text{Fix}(f)$. Then the fact that f and g commute yields that

$$f(g(x)) = g(f(x)) = g(x),$$

that is, $g(x) \in \text{Fix}(f)$. Therefore,

$$g|_{\text{Fix}(f)} : \text{Fix}(f) \longrightarrow \text{Fix}(f)$$

is well defined and we can apply Lemma 4.1.3 to obtain a fixed point of f that is fixed as well by g i.e.

$$\text{Fix}(f) \cap \text{Fix}(g) \neq \emptyset.$$

By induction on the number of maps, we get that for any $f_1, \ldots, f_n \in \mathcal{F}$,

$$\bigcap_{i=1}^{n} \text{Fix}(f_i) \neq \emptyset.$$

To conclude the proof, apply the intersection property, since K is compact, which implies that

$$\bigcap_{f \in \mathcal{F}} \text{Fix}(f) \neq \emptyset.$$

That is, there exists a point in K that is fixed by all maps of $\mathcal{F}$.

□

We now proceed to prove that all abelian semigroups are two-sided amenable. We comment beforehand the fact that, if S is abelian $\ell_s = \mathrm{r}_s$ for all $s \in S$, since

$$\ell_s f(t) = f(st) = f(ts) = \mathrm{r}_s f(t),$$

for all $t \in S$ and $f \in \ell^\infty(S)$.

Corollary 4.1.6. *Let S be an abelian semigroup. Then S is two-sided amenable.*

Proof. By Corollary 3.3.5, it suffices to prove that there exists a mean in $\ell^\infty(S)^*$ which is left invariant. As a consequence, since S is abelian, S is two-sided amenable.

First, we recall from Proposition 2.4.1 that the set of means $\mathcal{M}(S)$ is a ω^*-compact and convex subset of $\ell^\infty(S)^*$. Also, we recall that, by Proposition 3.2.4, left actions on $\ell^\infty(S)^*$ are linear and continuous maps whose images of means are as well means. Then, if S is abelian, we have that

$$\ell_u \ell_v f(t) = \ell_u f(vt) = f(uvt) = f(vut) = \ell_v f(ut) = \ell_v \ell_u f(t),$$

for all $f \in \ell^\infty(S)$ and for all $t \in S$. This translates to the adjoint actions as the equality

$$\ell_u^* \ell_v^* \mu(f) = \ell_u^* \mu(\ell_v f) = \mu(\ell_u \ell_v f) = \mu(\ell_v \ell_u f) = \ell_v^* \mu(\ell_u f) = \ell_v^* \ell_u^* \mu(f),$$

for all $\mu \in \ell^\infty(S)^*$ and for all $f \in \ell^\infty(S)$.

We need to prove that the set of left actions has a common fixed point on $\mathcal{M}(S)$ i.e. there exists a mean μ such that $\ell_s^* \mu = \mu$ for all $s \in S$.

Apply the Markov-Kakutani theorem with $X = \ell^\infty(S)^*$, $K = \mathcal{M}(S)$ and $\mathcal{F} = \{\ell_s^* : s \in S\}$. Thus we obtain a mean that is a fixed point for all ℓ_s^* and r_s^*. That is, the mean μ satisfies

$$\ell_s^* \mu = \mathrm{r}_s^* \mu = \mu,$$

for all $s \in S$. That is,

$$\ell_s^* \mu(f) = \mu(\ell_s f) = \mu(f),$$

for all $f \in \ell^\infty(S)$ and for all $s \in S$ i.e. μ is left invariant. Finally, as discussed above, if S is abelian $\ell_s = \mathrm{r}_s$ and thus μ is two-sided invariant. □

4.2 Day's Fixed Point Theorem

As it could be checked in the previous section, fixed point theorems can provide the existence of left invariant means and, in particular, can determine when a semigroup is amenable. In this section, we will prove Day's Fixed Point Theorem that characterizes amenability through a common fixed point theorem. The proof was taken from Day's article [7].

Theorem 4.2.1. *Let S be a left amenable semigroup. Then every representation of S as continuous affine maps from a compact convex set K in a locally convex vector space (X, τ) into it-self has a common fixed point.*

Proof. Let S be a semigroup acting on a compact convex set K in a locally convex vector space X. As the action of S on K is continuous and affine by hypothesis, the mappings

$$\begin{aligned} s : (K,\tau) &\longrightarrow (K,\tau) \\ x &\longmapsto sx \end{aligned}$$

are continuous for the weak topology as well, by Proposition 1.1.6. Fix a point $p \in K$ and, for each $x^* \in X^*$, define

$$\begin{aligned} f_{x^*} : S &\longrightarrow \mathbb{R} \\ s &\longmapsto \langle x^*, sp\rangle, \end{aligned}$$

Note that $f_{x^*} \in \ell^\infty(S)$, since $sp \in K$ for every $s \in S$ and x^* is bounded on C.

We recall that the means δ_s are defined as $\delta_s(f) = f(s)$, for each $f \in \ell^\infty(S)$ (see Example after definition 2.3.1). Let $s_1, \dots, s_n \in S$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. Define the mean $\mu \in \mathcal{M}_1(S)$ as

$$\mu = \sum_{i=1}^n \lambda_i \delta_{s_i}.$$

and define x_μ as

$$x_\mu = \sum_{i=1}^n \lambda_i s_i p, \tag{4.2}$$

which, by convexity of K, is a point in K.

Notice that

$$\begin{aligned} \mu\left(f_{x^*}\right) &= \sum_{i=1}^n \lambda_i \delta_{s_i}\left(f_{x^*}\right) = \sum_{i=1}^n \lambda_i f_{x^*}(s_i) = \\ &= \sum_{i=1}^n \lambda_i \langle x^*, s_i p\rangle = \langle x^*, \sum_{i=1}^n \lambda_i s_i p\rangle = \langle x^*, x_\mu\rangle. \end{aligned}$$

Also, for every $t \in S$ we have that the action of t on K is affine and thus

$$\begin{aligned} \mu\left(\ell_t f_{x^*}\right) &= \sum_{i=1}^n \lambda_i \delta_{s_i}\left(\ell_t f_{x^*}\right) = \sum_{i=1}^n \lambda_i f_{x^*}(t s_i) = \\ &= \sum_{i=1}^n \lambda_i \langle x^*, t s_i p\rangle = \langle x^*, \sum_{i=1}^n \lambda_i t s_i p\rangle = \langle x^*, t\left(\sum_{i=1}^n \lambda_i s_i p\right)\rangle = \langle x^*, t x_\mu\rangle. \end{aligned}$$

Since S is left amenable, we can choose a left invariant mean μ_0. Now, by Proposition 3.3.7, there exists a net $\{\mu_\alpha\} \subset \mathcal{M}_1(S)$ of finite means verifying that $\mu_\alpha \xrightarrow{\omega^*} \mu_0$ and that $\{\mu_\alpha\}$ is convergent to left invariance, yielding

$$\lim_\alpha \mu_\alpha(\ell_t f) - \mu_\alpha(f) = 0, \tag{4.3}$$

for all $f \in \ell^\infty(S)$ and for all $t \in S$.

Finally, let x_{μ_α} defined as in (4.2). Then, since K is τ compact, there is a convergent subnet $(\mu_\beta) \subset (\mu_\alpha)$. Let $x_0 \in K$ be the limit of (x_{μ_β}). Let $t \in S$, it only remains to prove that $tx_0 = x_0$. Note that, since X is locally convex, it suffices to show that

$$\langle x^*, tx_0 \rangle = \langle x^*, x_0 \rangle,$$

for every $x^* \in X^*$.

Fix $x^* \in X^*$ and consider the corresponding function f_{x^*}. We showed at the beginning of the proof that $\mu_\beta(f_{x^*}) = \langle x^*, x_{\mu_\beta} \rangle$ and $\mu_\beta(\ell_t f_{x^*}) = \langle x^*, tx_{\mu_\beta} \rangle$. Thus, since t is ω-continuous,

$$\begin{aligned} \mu_\beta(f_{x^*}) &= \langle x^*, x_{\mu_\beta} \rangle \to \langle x^*, x_0 \rangle, \\ \mu_\beta(\ell_t f_{x^*}) &= \langle x^*, tx_{\mu_\beta} \rangle \to \langle x^*, tx_0 \rangle. \end{aligned}$$

Finally, apply (4.3) to f_{x^*} to yield the desired result

$$0 = \lim_\alpha \mu_\alpha(\ell_t f_{x^*}) - \mu_\alpha(f_{x^*}) = \lim_\beta \mu_\beta(\ell_t f_{x^*}) - \mu_\beta(f_{x^*}) = \langle x^*, tx_0 \rangle - \langle x^*, x_0 \rangle.$$

□

This result is indeed a generalization of the Markov-Kakutani Theorem 4.1.5 swapping the commutativity with the condition that the family of maps forms (or is contained) in a left amenable semigroup. In fact, it immediately provides a fixed point characterization of amenability.

Corollary 4.2.2. *Let S be a semigroup. The following assertions are equivalent:*

a) S is left amenable.

b) Every representation of S as continuous affine maps from a compact convex set K in a locally convex space (X, τ) into it-self has a common fixed point.

Proof. First implication comes from last Theorem. Thus assume that every representation of S as continuous affine maps from a compact convex set in a locally convex space into it-self has a common fixed point. Now let $\mathcal{M}(S)$ be the compact convex set on the Banach space $\ell^\infty(S)^*$ with the ω^*-topology. Naturally, $\{\ell_s^* : s \in S\}$ is a representation of S acting on the set of means $\mathcal{M}(S)$ which, by Proposition 3.2.4, are affine and continuous. Then, there exists a common fixed point of the actions of S on $\mathcal{M}(S)$ i.e. there is a mean $\mu \in \mathcal{M}(S)$ verifying $\ell_s^*\mu = \mu$ for all $s \in S$. Hence, by definition, S is left amenable. $\square$

Chapter 5

Amenability by Examples. Paradoxical Decompositions

In this chapter we will show different examples of amenable and nonamenable semigroups. Moreover, we will showcase a collection of processes through which one can get new amenable semigroups from any given amenable one. In order to do so, we will make use of all the results provided in previous chapters.

Afterwards, we will relate amenability theory with paradoxical decompositions. We will discuss the main statement that connects amenable groups with paradoxical results called the Tarski Theorem, which establishes that a group is paradoxical if and only if it is not amenable.

5.1 Amenability Properties

Throughout this segment we will provide several examples of amenable groups and semigroups. The main references for this section are: [4, Sections 4.5-4.6], [5, Section II.4], [33, Section 12] or [16, Chapter 3].

We first discuss a notable difference between amenability in groups and semigroups. For groups, left and right amenability coincide with two-sided amenability, as shown in next proposition. The proof was taken from [33, p. 221] and afterwards, in the particular case of groups, both-sided amenability and left [right] amenability will be treated indistinctly, referring to them simply as amenable groups.

Proposition 5.1.1. *Let G be a left amenable group, then G is two-sided amenable.*

Proof. Let m_l be a left invariant measure on $\mathcal{PM}(G)$ and define the measure $\mathrm{m}_r \in \mathcal{PM}(G)$ as $\mathrm{m}_r(A) = \mathrm{m}_l(A^{-1})$. Notice that m_r is right invariant, since

$$\mathrm{m}_r g(A) = \mathrm{m}_r(Ag^{-1}) = \mathrm{m}_l(gA^{-1}) = g^{-1}\mathrm{m}_l(A^{-1}) = \mathrm{m}_l(A^{-1}) = \mathrm{m}_r(A).$$

Now define the right invariant mean associated to m_r by $\mu = \Phi(\mathrm{m}_r)$, where Φ is the isometry in Theorem 2.3.3.

For each $A \subseteq G$, define f_A by $f_A(g) = \mathrm{m}_l(Ag^{-1})$. Notice that f_A is bounded by 1 and thus $f_A \in \ell^\infty(G)$, for each $A \subseteq G$. Now define the measure m_0 by $\mathrm{m}_0(A) = \mu(f_A)$. It is easy to check that $\mathrm{m}_0(G) = 1$ and, since m_l is finitely additive and μ is linear,

$$f_{A\cup B} = f_A + f_B \text{ if } A \cap B = \emptyset.$$

Hence m_0 is finitely additive meaning $\mathrm{m}_0 \in \mathcal{PM}(G)$. Now we only need to prove that m_0 is both-sided invariant.

Let $A \subseteq G$ and $h \in G$. Then, apply left invariance of m_l to get

$$\begin{aligned} f_{h^{-1}A}(g) &= \mathrm{m}_l(h^{-1}Ag^{-1}) = h\mathrm{m}_l(Ag^{-1}) = \mathrm{m}_l(Ag^{-1}) = f_A(g), \\ f_{Ah^{-1}}(g) &= \mathrm{m}_l(Ah^{-1}g^{-1}) = f_A(gh) = \mathrm{r}_h f_A(g). \end{aligned}$$

Now, by right invariance of μ,

$$\begin{aligned} h\mathrm{m}_0(A) &= \mathrm{m}_0(h^{-1}A) = \mu(f_{h^{-1}A}) = \mu(f_A) = \mathrm{m}_0(A), \\ \mathrm{m}_0 h(A) &= \mathrm{m}_0(Ah^{-1}) = \mu(f_{Ah^{-1}}) = \mu(\mathrm{r}_h f_A) = \mu(f_A) = \mathrm{m}_0(A). \end{aligned}$$

Hence, m_0 is two-sided invariant and G is amenable. □

On the other hand, semigroups do not check the statement above, although semigroups do verify that left and right amenable semigroups are two-sided amenable. This result is not covered in this manuscript and it requires the definition of a product operation on the space of means. For a complete view on this result see [5, Theorem II.2.12].

For an example on a semigroup that is left amenable but not right amenable consider $S = \{s, t\}$ such that $st = t = tt$ and $ts = s = ss$. Then, left maps are neutral, meaning $\ell_s f = \ell_t f = f$ for all $f \in \ell^\infty(S)$, hence every mean in $\ell^\infty(S)^*$ is left invariant. Now define the function f as $f(s) = 1$ and $f(t) = 0$. Then, $\mathrm{r}_t f = f(t) = 0$ and $\mathrm{r}_s f = f(s) = 1$. Hence, any invariant mean $\mu \in \ell^\infty(S)^*$ must verify

$$\mu(f) = \mu(\mathrm{r}_t f) = 0,$$
$$\mu(f) = \mu(\mathrm{r}_s f) = 1,$$

yielding a contradiction. Then, the semigroup S does not admit a right invariant mean.

At this time, we recall that a first positive result on amenability for abelian semigroups was given as a consequence of the Markov-Kakutani Theorem in Corollary 4.1.6. Likewise, as an application of the characterization of left amenability for semigroups by using Day's Fixed Point Theorem we derive the following:

Corollary 5.1.2. *An homomorphic image of a left amenable semigroup is a left amenable semigroup.*

Proof. Let $h : S \to T$ be a semigroup homomorphism where S is left amenable. Consider the surjective homomorphism $h : S \to h(S)$. It is trivial that $h(S)$ is a subsemigroup of T. Let $h(S)$ be represented as continuous affine maps from a compact convex set $C \subset X$ onto itself where X is a locally convex space. Then $h(s) \in \mathrm{Im}(h)$ is a representation of $s \in S$ as continuous affine map from C onto itself. Hence, there is a common fixed point $x_0 \in C$ such that $h(s)x_0 = x_0$ for all $s \in S$. Thus, the semigroup $h(S)$ fixes x_0 which by Theorem 4.2.2 is equivalent to left amenability. □

Recall the definition of a quotient group from Section 3.1. Given a normal subgroup H of a group G, define G/H as the quotient $G/\sim_H$ where $\sim_H$ is the equivalence relation defined as $x \sim_H y$ if $xy^{-1} \in H$. As a consequence of last Corollary, the next result is immediate.

Corollary 5.1.3. *Every quotient group of an amenable group is amenable.*

Proof. Let H be a normal subgroup of an amenable group G. Applying last Corollary to the surjective canonical homomorphism

$$\begin{aligned} Q : G &\longrightarrow {}^{G}\!/_{H} \\ g &\longmapsto [g], \end{aligned}$$

we deduce that G/H is amenable. □

Next result does not require any use of means as the whole proof consists on an explicit construction of an invariant measure. To avoid overload, we discussed in Section 3.1 the definition and existence of a right coset.

Proposition 5.1.4. *Every subgroup H of an amenable group G is amenable.*

Proof. Let m be an invariant probability measure on G and let R be a right coset for H in G, that is a subset of G such that for every $g \in G$ there exist a unique pair (h, r) verifying $h \in H$, $r \in R$ and $hr = g$. We prove that the following map verifies the three properties of an invariant measure in $\mathcal{PM}(H)$. For each $A \subseteq H$ define

$$\tilde{\mathrm{m}}(A) = \mathrm{m}\left(\bigcup_{x \in R} Ax\right).$$

The first property is checked immediately by definition of right coset,

$$\tilde{\mathrm{m}}(H) = \mathrm{m}\left(\bigcup_{x \in R} Hx\right) = \mathrm{m}(G) = 1.$$

To prove finite additivity we first prove that, if $A, B \subseteq H$ are disjoint, then $Ax \cap By = \emptyset$ for all $x, y \in R$. Assume there exist an element $g \in Ax \cap By$ for some $x, y \in R$. Then, there exist an element $h_1 \in A$ and an element $h_2 \in H$ such that

$$h_1 x = g = h_2 y,$$

but, since R is a right coset, the pair (h, r) such that $hr = g$ is unique i.e. $h_1 = h_2$ and $x = y$, contradicting $A \cap B = \emptyset$. Hence,

$$\begin{aligned}\tilde{\mathrm{m}}(A \cup B) &= \mathrm{m}\left(\bigcup_{x \in R} (A \cup B)x\right) = \\ &= \mathrm{m}\left(\bigcup_{x \in R} Ax\right) + \mathrm{m}\left(\bigcup_{x \in R} Bx\right) = \tilde{\mathrm{m}}(A) + \tilde{\mathrm{m}}(B).\end{aligned}$$

Finally, since m is left invariant, for all $h \in H$

$$\begin{aligned}h\tilde{\mathrm{m}}(A) = \tilde{\mathrm{m}}(h^{-1}A) &= \mathrm{m}\left(\bigcup_{x \in R} h^{-1}Ax\right) = \\ &= \mathrm{m}\left(h^{-1}\bigcup_{x \in R} Ax\right) = \mathrm{m}\left(\bigcup_{x \in R} Ax\right) = \tilde{\mathrm{m}}(A).\end{aligned}$$

Thus, $\tilde{m}$ is a left invariant measure which implies that H is left amenable and, by Proposition 5.1.1, H is amenable. $\square$

On the other hand, semigroups do not check the statement above as even subsemigroups of a group can be non-amenable. An example can be found in the article [14] by M. Hochster. Another example will be shown in example after Proposition 3.4.5 using the group of affine transformations $(ax + b)$ on $\mathbb{R}$. Even though, we do provide a sufficient condition for a subsemigroup of a left amenable semigroup to be left amenable.

Proposition 5.1.5. *Let H be a subsemigroup of a left amenable semigroup S. If μ is a left invariant mean in $\ell^\infty(S)^*$ such that $\mu(\chi_H) > 0$, then H is left amenable.*

Proof. For each $f \in \ell^\infty(H)$, define an operator $T : \ell^\infty(H) \to \ell^\infty(S)$ as $Tf(s) = f(s)$ if $s \in S$ and $Tf(s) = 0$ otherwise. The linearity of T is immediate by definition and $\|Tf\| = \|f\|$ for all $f \in \ell^\infty(H)$ where each norm is taken in $\ell^\infty(S)$ and $\ell^\infty(H)$ respectively.

Now let $\mu \in \ell^\infty(S)^*$ be the left invariant mean on S such that $\mu(\chi_H) > 0$ given by hypothesis. Define $\mu_0 \in \ell^\infty(H)^*$ as

$$\mu_0(f) = \mu(Tf)/\mu(\chi_H),$$

for each $f \in \ell^\infty(H)$. Note that $\mu_0(\chi_H) = 1$. Also, since $Tf \geq 0$ for all $f \geq 0$, it holds that $\mu_0(f) = \mu(Tf)/\mu(\chi_H) \geq 0$ for all $f \geq 0$. Thus, by Proposition 2.3.2 μ_0 is a mean on $\ell^\infty(H)^*$.

The proof of left invariance of μ_0 requires some previous work that we divide in the following steps:

- By linearity and left invariance of μ, it suffices to prove that

  $$\mu\left(\ell_s(Tf) - T(\ell_s f)\right) = 0,$$

  for each $s \in H$ and for each $f \in \ell^\infty(H)$, which implies $\mu_0(\ell_s f) = \mu_0(f)$. To simplify notation, let $g = \ell_s(Tf) - T(\ell_s f)$.

- Define the product of two functions $g_1, g_2 \in \ell^\infty(S)$ as $g_1 g_2(t) = g_1(t) g_2(t)$. It is an easy check that $\|g_1 g_2\| \leq \|g_1\| \, \|g_2\|$. With this definitions, $T(\ell_s f)$ and

$\ell_s(Tf)$ expressions can be expanded as,

$$T(\ell_s f)(t) = f(st)\chi_H(t),$$
$$\ell_s(Tf)(t) = Tf(st) = f(st)\chi_H(st) = f(st)\chi_{s^{-1}H}(t).$$

Thus, since $H \subset s^{-1}H$ for all $s \in H$,

$$g = \ell_s(Tf) - T(\ell_s f) = (\ell_s f)\chi_{s^{-1}H} - (\ell_s f)\chi_H$$
$$= (\ell_s f)(\chi_{s^{-1}H} - \chi_H) = (\ell_s f)\chi_{s^{-1}H\cap(S\setminus H)}.$$

To simplify the notation, let $E = s^{-1}H \cap (S \setminus H)$.

- For every $f \in \ell^\infty(H)$ we can decompose $f = f_1 - f_2$, where $f_1, f_2 \geq 0$. Thus, since μ, T and ℓ_s are linear, we can assume f to be non-negative. Afterwards, we can generalize as follows

$$\mu\left(\ell_s(Tf) - T(\ell_s f)\right) = \mu\left(\ell_s(Tf_1) - T(\ell_s f_1)\right) - \left(\mu\left(\ell_s(Tf_2) - T(\ell_s f_2)\right)\right)$$

- Combining the previous statements, $g = (\ell_s f)\chi_E$ with $f \geq 0$. Then, $|g(t)| \leq \|g\|\, \chi_E(t)$ for all $t \in S$ and, by the definition of mean,

$$0 \leq \mu(g) \leq \|g\|\, \mu(\chi_E).$$

Thus, it suffices to prove that $\mu(\chi_E) = \mu(\chi_{s^{-1}H\cap(S\setminus H)}) = 0$.

For each $t \in S$ and for each $s \in H$, there is at most one element in $\{s^i t\}_{i=1}^\infty$ that belongs to E, since otherwise, let i be the first positive integer such that $s^i t \in E$, then $s^{i+1}t \in H$ and, since H is a subsemigroup, $s^{i+k}t = s^k s^i t \in H$ for each $k \geq 1$ and therefore $s^{i+k}t \notin E \subset S \setminus H$ for all $k \geq 1$.

Then, for every integer $n > 0$ and for every $t \in S$ we have that

$$\sum_{i=1}^{n} \ell_{s^i}\chi_E(t) = \sum_{i=1}^{n} \chi_E(s^i t) \leq 1.$$

Thus, applying left invariance of μ

$$n\mu(\chi_E) = \sum_{i=1}^{n} \mu\left(\chi_E(t)\right) = \sum_{i=1}^{n} \mu\left(\ell_{s^i}\chi_E(t)\right) = \mu\left(\sum_{i=1}^{n} \ell_{s^i}\chi_E(t)\right) \leq 1,$$

for all $n > 0$. Hence, $\mu(\chi_E) = 0$, concluding the proof.

$\square$

Proposition 5.1.6. *Let H be a normal subgroup of a group G. Then, G is amenable if and only if H and G/H are amenable.*

Proof. The amenability of G implies H and G/H are amenable immediately after Proposition 5.1.4 and Corollary 5.1.3.

Conversely, suppose H and G/H are amenable. Let G be represented as continuous affine maps from a compact convex set K in a locally convex space X into K. Since H is amenable, Theorem 4.2.2 yields that the set K_0 of all fixed points of H in K is not empty. More formally, the set

$$K_0 = \{k \in K : h(k) = k, \forall h \in H\}$$

is not empty. Also, K_0 is closed in K and, since $h \in H$ is represented as an affine map, K_0 is also convex. That is, K_0 is a compact convex set in X.

Now, for each $g \in G$, let $[g]$ represent the equivalence class of g in G/H. Since any two elements $g_1, g_2 \in G$ with the same equivalence class verify $g_1 = g_2 h$ for some $h \in H$, then

$$g_1(k) = g_2 h(k) = g_2(h(k)) = g_2(k),$$

for every $k \in K_0$. Moreover, since H is a normal subgroup, for every $g \in G$ and for every $h \in H$, there exist a $s \in H$ such that $hg = gs$. Then, for every $g \in G$ and for every $k \in K_0$,

$$h(g(k)) = hg(k) = gs(k) = g(k),$$

for each $h \in H$ and for some $s \in H$. Hence, these representation maps of G, map K_0 into itself.

Finally, we have proven that this representation of G as continuous affine maps on K_0 verify that all elements in the same equivalence class in G/H have the same representation in K_0. Thus, this representation of G gives a representation of G/H as continuous affine maps from K_0 into itself when we restrict each map to K_0. An explicit definition of this representation of G/H is constructed as follows: For each element $[g] \in G/H$, pick any representative g of the equivalence class $[g]$ and assign to it the map $g : K_0 \to K_0$ which we have proven to be well defined and independent of the chosen representative. Notice this definition directly involves the axiom of choice. Then, by amenability of G/H and Theorem 4.2.2, this representation has a fixed point $\overline{k} \in K_0$. That is, for each $g \in G$,

$$g(\overline{k}) = \overline{k},$$

and therefore G has a common fixed point. By Theorem 4.2.2, G is amenable. $\square$

As an immediate consequence we have the following result.

Corollary 5.1.7. *Let G_1 and G_2 be amenable groups. Then, the group $G = G_1 \times G_2$ is amenable.*

Proof. The set $H = \{(g_1, e_{G_2}) : g_1 \in G_1\}$ is a normal subgroup of G isomorphic to G_1 with quotient G/H isomorphic to G_2. Thus, Corollary 5.1.3 yields that G is amenable. $\square$

To conclude this section, we will showcase an important application of the previous results. Summarizing, we now know different families of amenable semigroups and groups: abelian semigroups, subgroups of amenable groups, quotient groups on amenable groups and homomorphic images of amenable semigroups. We also provided a tool to extend amenable groups, which is key in next corollary. First, we recall the definition of a solvable group from section 3.1. A group G is solvable if it admits a finite chain of normal subgroups $\{H_k\}_{k=0}^n$ such that $H_k \subset H_{k+1}$, $H_n = G$ and $H_0 = \{e\}$ as well as every quotient group H_k/H_{k-1} are abelian.

Corollary 5.1.8. *Every solvable group is amenable.*

Proof. Let G be a solvable group with $\{H_i\}_{i=0}^n$ normal subgroups verifying the definition. Since

$$H_0, {}^{H_1}\!/_{H_0}, \dots, {}^{H_n}\!/_{H_{n-1}}$$

are abelian, by Proposition 4.1.6, all of these groups are amenable. Now, by Proposition 5.1.6, H_1 is also amenable. Applying induction we get that H_i and H_{i+1}/H_i are amenable and thus H_{i+1} is amenable as well. Hence, $H_n = G$ is amenable. $\square$

5.2 Free groups and the von-Neumann conjecture

In Proposition 3.4.4, we proved that free groups of order two are not amenable by showing that any left invariant measure defined on such a group cannot verify the finitely additive property. This relates to the Banach-Tarski Paradox which will be discussed in Section 5.4.

Applying Proposition 5.1.4 we give an immediate generalization of Proposition 3.4.4.

Corollary 5.2.1. *Any group containing a subgroup isomorphic to a free group of order two is nonamenable.*

When considering semigroups this statements does not hold as even amenable groups may contain free subsemigroups. For an example, we reference the group of affine transformations $(ax + b)$ on $\mathbb{R}$ of positive real coefficients, which is amenable since it is solvable even though it contains a free subsemigroup on three generators, as Kolpakov and Talambutsa prove in [18, Example 5].

The reciprocal of Corollary 5.2.1: a group is amenable if and only if it does not contain a free subgroup of order two, is the so called *von-Neumann conjecture* stated by von-Neumann in 1929. Whether or not this was a characterization of group amenability was solved definitely in 1980 by Ol'shanskii [24], as he proved a counter example. The Tarski monster groups, as they were called, are a collection of non-amenable groups without any free subgroup of order two. An infinite group G is called a Tarski monster group for p if every nontrivial subgroup (i.e. every subgroup other than $\{e\}$ and G itself) has p elements. Ol'shanskii proved that for each prime number $p > 10^{75}$ there is a Tarski monster group. Since then, other more manageable examples have been found such as the one provided by Nicolas Monod in 2012 [21], in which the counterexample is built from piecewise projective homomorphisms, meaning homomorphisms defined piecewise as Möbius transformations.

5.3 Concrete examples of amenable and nonamenable groups

We give in this section some specific examples of amenable and nonamenable groups. Starting with a nonamenable group, we now prove the result that explains the existence of the Hausdorff paradox. For a complete study on the Hausdorff and Banach-Tarski Paradoxes see [33, Chapters 2 and 3]. Notice that in the next statement it is required that $n \geq 3$ as we will later prove that for $n = 1, 2$ the groups $\mathbf{SO}_n$ of all rotations in $\mathbb{R}^n$ with axis through the origin are in fact amenable.

Lemma 5.3.1. *The group* $\mathbf{SO}_n$ *contains a free subgroup of order two for all* $n \geq 3$. *Thus,* $\mathbf{SO}_n$ *is not amenable for all* $n \geq 3$.

Proof. Consider ϕ and ρ as the rotations around the exes x and z with angle $\arccos \frac{1}{3}$ on $\mathbb{R}^3$. Notice that these rotations can be seen in higher dimensions, thus it suffices to proof that $\langle \phi, \rho \rangle$ is a free group of order two. The rotations are represented as the following matrices

$$\phi = \begin{pmatrix} 1/3 & -2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \phi^{-1} = \begin{pmatrix} 1/3 & 2\sqrt{2}/3 & 0 \\ -2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \qquad \rho^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 2\sqrt{2}/3 \\ 0 & -2\sqrt{2}/3 & 1/3 \end{pmatrix}.$$

We need to prove that no reduced word with these elements is equal to the identity. Assume there exists a finite non trivial composition ω of $\phi^{\pm 1}$ y $\rho^{\pm 1}$ such that $\omega = e$, where e is the neutral element of $\mathbf{SO}_n$. To provide a contradiction, we will proof that $\omega(1, 0, 0)^{\mathrm{T}} = (a, b\sqrt{2}, c)/3^k$ for some integers a, b, c where $3 \nmid b$, and thus $\omega(1, 0, 0)^{\mathrm{T}} \neq (1, 0, 0)$.

We can assume ω starts by $\phi^{\pm 1}$ on the right, since we can always consider $\omega\phi\phi^{-1}$. Notice that if $\omega = \phi^{\pm 1}$, then

$$\omega(1, 0, 0)^{\mathrm{T}} = \frac{1}{3}(1, \pm 2\sqrt{2}, 0),$$

and apply induction on the length of ω.

We distinguish four cases depending on the first element on the left of ω. Either $\omega = \phi^{\pm 1}\omega'$ or $\omega = \rho^{\pm 1}\omega'$ where, by the induction hypothesis,

$$\omega'(1, 0, 0)^{\mathrm{T}} = (a', b'\sqrt{2}, c')/3^{k-1}$$

with $a, b, c \in \mathbb{Z}$ and $3 \nmid b$. Then, $\omega(1, 0, 0)^{\mathrm{T}}$ is of the form

$$\phi^{\pm 1}\omega'(1, 0, 0)^{\mathrm{T}} = \phi^{\pm 1}(a', b'\sqrt{2}, c')^{\mathrm{T}}/3^{k-1} = \frac{1}{3^k}(a' \mp 4b', (b' \pm 2a')\sqrt{2}, 3c')^{\mathrm{T}}, \tag{5.1}$$

$$\rho^{\pm 1}\omega'(1, 0, 0)^{\mathrm{T}} = \frac{1}{3^k}(3a', (b' \pm 2c')\sqrt{2}, c' \pm 4b')^{\mathrm{T}}. \tag{5.2}$$

To show $3 \nmid b$, decompose ω once more in the following possibilities

$$\omega = \begin{cases} \phi^{\pm 1}\rho^{\pm 1}v \\ \rho^{\pm 1}\phi^{\pm 1}v \\ \phi^{\pm 1}\phi^{\pm 1}v \\ \rho^{\pm 1}\rho^{\pm 1}v \end{cases}.$$

This time distinguish when the first two elements are the same (not regarding inverses) or not and discard the cases were the syllable is trivial. If the elements are different, we have $b = b' \mp 2a'$ or $b = b' \pm 2c'$ where either $3 \mid a'$ or $3 \mid c'$, since it comes from the first component of (5.2). In both cases, $3 \nmid b'$ implies $3 \nmid b$.

For the two remaining cases, applying the induction hypothesis, we now there are some integers a'', b'' y c'' such that

$$v(1,0,0)^{\mathrm{T}} = (a'', b''\sqrt{2}, c'')^{\mathrm{T}}.$$

where $3 \nmid b$. Respectively in each case, substitute $a' = a'' \mp 4b''$ and $\pm 2a'' = b' - b''$, or $c' = c'' \mp 4b''$ and $\pm 2c'' = b' - b''$, to obtain

$$b = b' \pm 2(a'' \mp 4b'') = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''.$$

Hence, $3 \nmid b'$ implies $3 \nmid b = 2b' - 9b''$.

Finally, we have shown that $\omega(1,0,0)^{\mathrm{T}} = (a, b\sqrt{2}, c)^{\mathrm{T}}/3^k$ with $a, b, c \in \mathbb{Z}$ and $b \not\equiv 0 \bmod 3$ which implies that ϕ and ρ are independent. $\square$

Example. We reference from [4, Example 4.5.3 and Lemma 2.3.2] that the matrix group generated by the matrices,

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is isomorphic to $\mathbb{F}_2$. Thus the group $\mathbf{SL}(2,\mathbb{Z})$ of 2×2 matrices of determinant 1 with coefficients in $\mathbb{Z}$ is not amenable by Proposition 5.2.1 as it contains a free subgroup of two generators.

As a consequence of Corollary 5.1.8 we can prove that $\mathbf{G}_1$ and $\mathbf{G}_2$ are amenable as they are solvable, since it is easy to check that the consecutive quotients in $\mathbf{T}_1 \subset \mathbf{G}_1$ and $\mathbf{T}_2 \subset \mathbf{SG}_2 \subset \mathbf{G}_2$ are all abelian.

For the last examples on amenable groups we need the definition of alternating groups. We use the notation in [4, Example 4.6.4].

Definition 5.3.2. *An alternating group is the group of even permutations of a finite set. The alternating group on a set of n elements is called the alternating group of degree n and it is denoted by* Sym_n^+.

Thus, the alternating group Sym_5^+ is finite and hence amenable. This is an example of a non solvable group that is amenable (see [4, Example 4.6.2 e)]. For an example of an infinite amenable group which is not solvable, apply Corollary 5.1.7 to $\mathbb{Z} \times \mathrm{Sym}_5^+$ (recall that $\mathbb{Z}$ is amenable since it is abelian).

It is worth remark that nilpotent groups are solvable (see [4, Proposition 4.6.6] and hence nilpotent groups are amenable by Proposition 5.1.8. A nilpotent group G is a group that has an upper central series that terminates with G i.e. it admits a finite chain of normal subgroups starting on $\{e\}$ and terminating in G. An example of a nilpotent group is the Heisenberg group over a ring R. This group is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in R$. Thus, the Heisenberg group is amenable.

5.4 Amenability, paradoxical decompositions and extension of measures

Along the manuscript we have already studied different characterizations of amenability in terms of the existence of left invariant means and in terms of a common fixed point theorem. This section is devoted to a new and surprising characterization of amenability in groups that is related to the notion paradoxical decompositions. Proofs in this section are just referenced or sketched since they deviate from the initial scope of this master thesis. We will finish with some comments related to the measure problem.

We start by defining paradoxical decompositions.

Definition 5.4.1. *Let G be a group acting on a set E. A G-paradoxical decomposition of E is a triplet $(K, (A_k)_{k\in K}, (B_k)_{k\in K})$ where K is a finite subset of G and, $(A_k)_{k\in K}, (B_k)_{k\in K}$ are disjoint families of subsets of E indexed by K such that*

$$E = \left(\bigsqcup_{k\in K} kA_k\right) \sqcup \left(\bigsqcup_{k\in K} kB_k\right) = \left(\bigsqcup_{k\in K} A_k\right) = \left(\bigsqcup_{k\in K} B_k\right)$$

Each A_k and B_k is called a piece of the decomposition. Note that each A_k or B_k may be empty sets. This definition can naturally be considered for a group acting on itself. That is, if the set E is also the group G. We give the following definition.

Definition 5.4.2. *A group G is paradoxical if it admits a paradoxical decomposition.*

This concept plays an essential role is the Banach-Tarski Paradox, where the paradoxical set is the unit sphere in $\mathbb{R}^3$ and the action group G is the group of rotations $\mathbf{G}_3$. Banach and Tarski showed that any subset of $\mathbb{R}^3$ with non-empty interior is paradoxical (it can be split into disjoint pieces that can be rearranged by rigid motions to form the original set twice), thus showing that there is no finitely additive extension to all of $\mathcal{P}(\mathbb{R}^3)$ of the Lebesgue measure. This is known as the measure problem and we will discuss it further at the end of this section for cases $n = 1, 2$.

As seen in last section (Lemma 5.3.1), the group of isometries is not amenable for $n \geq 3$. This idea is key in the understanding of the construction of the Banach-Tarski paradox (see [33, Theorem 3.10]).

We now relate amenability with paradoxical decompositions. We first define the so called Følner's condition.

Definition 5.4.3. *A group G is said to verify the Følner condition if for every finite subset $K \subset G$ and every real number $\varepsilon > 0$, there exists a nonempty finite subset $F \subset G$ such that*

$$\frac{|F \setminus kF|}{|F|} < \varepsilon,$$

for all $k \in K$, where $|\cdot|$ represents the cardinal measure.

For more details on Følner's condition see [4, Section 4.7] and [33, Section 12.4]. The main purpose of this definition is that it provides a new characterization on groups and is strongly used when proving Tarski's theorem.

Theorem 5.4.4 (Tarski-Følner)**.** *Let G be a group. The following properties are equivalent:*

a) G is amenable.

b) G satisfies Følner's condition.

c) G does not admit a paradoxical decomposition.

Proof. See [4, Theorem 4.9.2]. □

The Følner condition is also a necessity for semigroups to be amenable. More formally there is the following theorem.

Theorem 5.4.5. *Let S be a left amenable semigroup. Then, S satisfies the Følner's condition.*

Proof. See [22, Theorem 3.5]. □

As a remark, the Følner condition is not sufficient for a semigroup to be left amenable. One can prove that finite semigroups verify Følner's condition although the semigroup $S = \{s, t\}$ with $st = s = ss$ and $ts = t = tt$ is not amenable. To prove it, define the function $f(s) = 0$ and $f(t) = 1$. Then, consider $h = (f - \ell_t f) - (f - \ell_s f) = -1$ but then any left invariant mean on $\ell^\infty(S)^*$ must verify $\mu(h) = -1$ by definition and, by linearity and left invariance,

$$\mu(h) = \mu((f - \ell_t f) - (f - \ell_s f)) = \mu(f) - \mu(\ell_t f) - \mu(f) + \mu(\ell_s f) = 2\mu(f) - 2\mu(f) = 0.$$

To finish this chapter, we take one last look at the measure problem. That is, the existence of an extension of the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$. We proved that the isometry groups $\mathbf{G}_1$ and $\mathbf{G}_2$ on $\mathbb{R}$ and $\mathbb{R}^2$ respectively are amenable since they are solvable. This property allows the construction of a finitely additive extension of the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$. This is known as the Banach Theorem.

The proof of Banach's Theorem we showcase here is an application of [33, Theorem 12.11]. The notation in [33] was given in terms of measures and integrals. For a detailed view on this result see [33, Another proof of Corollary 12.9 p.235] or [30, Theorem 3.3.4]. Here we exhibit a sketch of the proof where means play the essential role and the Hahn-Banach extension Theorem [27, Theorem 3.2] is used.

Theorem 5.4.6 (Hahn-Banach)**.** *Let X be a real vector space and $p : X \to \mathbb{R}$ a functional verifying:*

$$p(x + y) \leq p(x) + p(y),$$
$$p(\alpha x) = \alpha p(x),$$

for every $x, y \in X$ and for ever $\alpha \geq 0$.
Let Y be a subspace of X. Then for every linear functional $\psi : Y \to \mathbb{R}$ such that $\psi \leq p$, there exists a linear functional $\overline{\psi} : X \to \mathbb{R}$ that extends ψ i.e. $\overline{\psi}_{|_Y} = \psi$ and verifies $\overline{\psi} \leq p$ on X.

Theorem 5.4.7 (Banach)**.** *There exists a finitely additive extension of the Lebesgue measure on $\mathbb{R}$ that is invariant with regards to isometries. This statement also holds true for $\mathbb{R}^2$.*

Sketch of the Proof: Let λ denote the Lebesgue integral on $\mathbb{R}$ (analogous for $\mathbb{R}^2$). Let $G = \mathbf{G}_1$. Let V_0 be the set of all Lebesgue integrable functions in $\ell^\infty(\mathbb{R})$. Define the set $V \subset \ell^\infty(\mathbb{R})$ as

$$V = \{f \in \ell^\infty(\mathbb{R}) : \exists f_0 \in V_0 \text{ such that } f_0 \geq f\}.$$

It is a simple verification that $V_0 \subset V \subset \ell^\infty(\mathbb{R})$ is a chain of vectorial subspaces.

Define the operator $p : V \to \mathbb{R}$ as,

$$p(f) = \inf \left\{ \int_{\mathbb{R}} f_0 : f_0 \in V_0, f_0 \geq f \right\},$$

for each $f \in \ell^\infty(R)$. This functional is well defined on V as for every $f \in V$ there is at least one function in V_0 that majores f. It is an easy check that this functional is well defined and that verifies

$$p(f_1 + f_2) \leq p(f_1) + p(f_2),$$
$$p(\alpha f_1) = \alpha p(f_1),$$

for all $f_1, f_2 \in V$ and for all $\alpha \geq 0$. Then, apply Hahn-Banach's extension Theorem to obtain a linear functional λ_0 that extends λ to V and is bounded by the functional p. Now, since $\mathbf{G}_1$ is amenable, there exist an invariant mean μ on $\ell^\infty(\mathbf{G}_1)$.

For each function, $f \in V$ define the function $h \in \ell^\infty(\mathbf{G}_1)$ as

$$h_f(g) = \lambda_0(gf), \tag{5.3}$$

where $gf(x) = f(gx)$ for each $x \in \mathbb{R}$. Hence, we can define a functional $\overline{\lambda}$ as

$$\overline{\lambda}(f) = \mu(h_f).$$

It follows from definition (5.3), that for each $s \in G$,

$$h_{sf}(g) = \lambda_0(gsf) = h_f(gs) = \mathrm{r}_s h_f(g).$$

Hence, by invariance of μ,

$$\overline{\lambda}(sf) = \mu(h_{sf}) = \mu(\mathrm{r}_s h_f) = \mu(h_f) = \overline{\lambda}(f).$$

That is, $\overline{\lambda}$ is G invariant.

Now define the measure $\mathrm{m}(A) = \overline{\lambda}(\chi_A)$ for each $A \subset \mathbb{R}$ if $\chi_A \in V$ and $\mathrm{m}(A) = +\infty$ otherwise. Then, this measure is $\mathbf{G}_1$-invariant, since $g\chi_E = \chi_{g^{-1}E}$, and it agrees with the Lebesgue measure on measurable sets and that is well defined for all $A \in \mathcal{P}(\mathbb{R})$. Thus, we have found an isometry invariant and finitely additive extension of the Lebesgue measure.

Chapter 6

Some "Kakutani-type" fixed point theorems for semigroups

In this chapter we introduce the notion of a "Kakutani-type" fixed point theorem, term coined by Namioka in [23] strongly inspired by Kakutani theorem for compact flows, and that gathers the flavour of different common fixed point theorems under the action of semigroups and strongly inspired by Kakutani fixed point theorem for compact flows.

So far, we have studied two results that lie within the above landscape: Markov-Kakutani's theorem and Day's theorem (see Chapter 4).

We will state some other "Kakutani-type" fixed point theorems, among which we will exhibit a detailed proof of Ryll-Narzdewski's theorem for noncontracting semigroups, that covers the previous cases. As a conclusion, some applications will be shown and further extensions to the linear and nonlinear case will be displayed.

6.1 Introduction and Preliminaries

It is an easy consequence of Bolzano Theorem that every continuous function $f : [a, b] \to [a, b]$ has a fixed point. Brouwer fixed point theorem asserts that every continuous selfmapping defined on a closed convex bounded subset of $\mathbb{R}^n$ has a fixed point and this result was extended to the setting of locally convex topological spaces by Schauder and Tychonoff: Every continuous mapping defined from a convex compact subset of a locally convex topological space into itself has a fixed point [32].

The situation is completely different if we try to seek for a common fixed point for more than one continuous function defined on the same convex compact space, even if these mappings commute. For instance, in 1969, Boyce [3] and Huneke [15] independently published examples of two commuting continuous maps f, g of $[0,1]$ into itself without a common fixed point.

Note that every collection of self-mappings defined over the same domain generates a semigroup acting by composition and that a point is fixed for every element of the family if and only if it is fixed by all the mappings belonging to the generated semigroup. Thus, in search of a common fixed point (f.p. in short), we can always assume that we are within the framework of the action of a semigroup over the shared domain.

In Chapter 4 we already proved Markov-Kakutani's theorem that claims that in case that S is a commutative semigroup and the mappings are continuous and affine, then there is a common fixed point whenever the domain is a convex compact subset of a locally convex space.

Afterwards, Day's fixed point Theorem (that characterizes amenability) shows that the previous conclusion can be extended replacing commutativity by amenability. During this last chapter of the manuscript, we will study sufficient conditions upon a semigroup of mappings acting over a convex compact domain so that there exists at least a common fixed point. Following [23] we introduce the following notation:

A statement is said to be a Kakutani-type fixed point theorem if it is of the form: Given a group or semigroup S acting on a compact convex subset K of a locally convex vector space X into itself. Assume that each action $s : K \to K$ is a continuous affine transformation. Then, under suitable conditions, S has a common fixed point in K.

These theorems are called "Kakutani-type" in reference to the fixed point theorems proved in Kakutani's paper [17]. For the sake of context, we introduce some notions on topological dynamic systems.

Definition 6.1.1. *A flow or dynamical system is a pair (S, C) where C is a nonempty subset of a topological space (X, τ) and S is a semigroup acting on C from the left where each action $s : C \to C$ is a continuous map.*
Additionally, a flow (S, C) is said to be:

- *a compact flow if C is compact,*

- *an affine flow if C is convex and $s : C \to C$ is an affine map, for each $s \in S$; or*

- *an equicontinuous flow if, for each neighborhood U of 0, there exists a neighborhood V of 0 such that $x, y \in C$ and $x - y \in V$ imply $sx - sy \in U$ for each $s \in S$.*

- *a distal flow if, it is a compact flow and, for each $x, y \in C$ such that $\lim_\alpha s_\alpha x = \lim_\alpha s_\alpha y$ for some net $(s_\alpha) \subset S$, then $x = y$.*

With this definitions, the Markov-Kakutani Theorem 4.1.5 can be enunciated as follows:
Let (S, K) be a compact affine flow. If S is commutative, then S admits a common f.p. in K.
Now we state two more Kakutani-type theorems that we will prove to be weaker versions of the upcoming Ryll-Narzdewski's Theorem 6.1.4, whose proof will be exhibited in detail in Section 6.3.

Theorem 6.1.2 (Kakutani's f.p. Theorem [17])**.** *Let (S, K) be a compact affine flow. If S is a group and equicontinuous on K, then S admits a common f.p. in K.*

Theorem 6.1.3 (Hahn's f.p. Theorem [13])**.** *Let (S, K) be a distal affine flow. Then S admits a common f.p. in K.*

It is in this context that Ryll-Narzdewski's theorem appears, which is as well a Kakutani-type f.p. theorem. We now give the statement of this theorem, although its proof is given in Section 6.3 as we will prepare some tools in advance.

Theorem 6.1.4 (Ryll-Nardzewski)**.** *Let K be a nonempty weakly compact convex subset of a locally convex space (X, τ) and let S be a semigroup of weakly continuous affine maps $s : K \to K$. If S is noncontracting on K, i.e., $0 \notin \overline{\{sx - sy : s \in S\}}^\tau$ for distinct $x, y \in K$, then S has a common fixed point in K.*

Notice that, since for every locally convex space (X, τ) the topology is generated by a set of seminorms $\mathcal{N}$, the assumption of noncontractiveness is equivalent to: for each $x \neq y$ there exists a seminorm $\rho \in \mathcal{N}$ such that

$$\inf_{s\in S} \rho(sx, sy) > 0.$$

In the next results we prove that Ryll-Narzdewski's result is indeed an improvement of Theorems 6.1.2 and 6.1.3. First we prove that, when S is a group, equicontinuous implies distal.

Proposition 6.1.5. *An affine compact flow (S, K) is distal if S is a group and (S, K) is an equicontinuous flow.*

Proof. Denote as X the topological space containing K. Let $x, y \in K$ be two distinct points. Then, there exists a neighborhood U of 0 in X such that $x - y \notin U$. By equicontinuity of S on K, there exists a neighborhood V of 0 in X such that $su - sv \in U$ for each $s \in S$ whenever $u, v \in K$ and $u - v \in V$. Then for each $s \in S$, $sx - sy \notin V$ as otherwise, $sx - sy \in V$ for some $s \in S$ and then $x - y = s^{-1}sx - s^{-1}sy = s^{-1}(sx) - s^{-1}(sy) \in U$, contradicting our choice of U. Consequently, there is no net $(s_\alpha) \subset S$, such that $\lim_\alpha s_\alpha x = \lim_\alpha s_\alpha y$. Hence (S, Q) is distal. □

We reference from [11] that there exist distal flows that are not equicontinuous.

Proposition 6.1.6. *Any distal affine flow (S, K) satisfies Ryll-Narzdewski's Theorem 6.1.4 hypotheses.*

Proof. Any distal affine flow (S, K) verifies K is weakly compact since it is τ-compact by definition. Also by τ-compactness, the assumption

$$0 \notin \overline{\{sx - sy : s \in S\}}^{\tau}$$

is equivalent to: no two τ-convergent nets $s_\alpha x = s_\alpha y$ verifying $x \neq y$. That is, a distal flow is noncontracting.

Also, by Proposition 1.1.6, every τ-continuous affine mapping is weakly continuous. Hence, the mappings of S on K are weakly continuous, concluding the proof. □

Ryll-Narzdewski Theorem was originally proved in [29]. The proof we show here is taken from Namioka's article [23] that uses extreme points and its properties for convex compact sets. These tools will be detailed in the next section.

Another essential tool we will use is Zorn's Lemma. We recall that the Zorn Lemma establishes that for any nonempty partially ordered set S in which every chain has a lower bound, then S has a minimal element.

Let K be a compact convex subset of a topological vector space X. Consider any finite number of properties that are inherited by intersection, such as compactness, convexity, invariance (with respect to a set of maps), etc. Then the family Ω of closed subsets of K verifying such properties is a partially ordered set when considering inclusion. Thus, for any chain of subsets, meaning a subfamily $\mathcal{C} \subset \Omega$ such that for any $A, B \subset \mathcal{C}$, then $A \subset B$ or $B \subset A$, the intersection is a lower bound which nonempty by compactness. Therefore by Zorn Lemma there is a minimal element in the family.

Such minimal set is tremendously convenient in some of the upcoming proofs. For instance, consider the family Ω of all compact and convex subsets of K, then any minimal set $M \in \Omega$ has the following property. Given any subset $C \subset M$, then $\overline{\text{co}}(C) = M$ since otherwise $\overline{\text{co}}(C)$ would be a compact and convex set that is a strict subset of M, contradicting its minimality.

Another tool used in Namioka's proof is given as Lemma 6.1.8 which uses the following result on topological vector regarding balanced sets. A subset B of a vector space X is said to be balanced if for every $|\alpha| \leq 1$, $\alpha B \subset B$ where $\alpha B = \{\alpha b : b \in B\}$.

Theorem 6.1.7. *In a topological vector space X,*

1. *every neighborhood of 0 contains a balanced neighborhood of 0, and*

2. *every convex neighborhood of 0 contains a balanced convex neighborhood of 0.*

Proof. See [27, Theorem 1.14]. □

As an immediate consequence of this theorem we prove the following statement.

Lemma 6.1.8. *Let X be a topological vector space and let U be a convex neighborhood of 0. Then there exists a closed convex subset $V \subset U$ such that V is a neighborhood of 0 and $V - V \subset U$.*

Proof. It follows from statement 2 in Theorem 6.1.7 that there exists a balanced convex neighborhood of 0 U' such that $U' \subset U$. Then, take $V = (1/2)U'$. This U' is convex and a neighborhood of 0 as well; and for every $x, y \in V$, there exists $u, v \in U'$

such that $x = (1/2)u$ and $y = (1/2)v$. Hence, since U' is balanced, $-y \in U' \subset U$ and, since U is convex, $x - y = (1/2)u + (1/2)(-v) \in U$ for every $x, y \in V$. That is, $V - V \subset U$. Finally if V is not closed, we can choose $V' = (1/2)\overline{V}$ which is convex closed and a neighborhood of 0 verifying $V' - V' \subset V - V \subset U$, thus concluding the proof. □

The next tool that will appear in the proof of Theorem 6.1.4 is a well known topology theorem, which proof we omit.

Theorem 6.1.9 (Baire). *If E is either*

1. *a complete metric space, or*
2. *a locally compact Hausdorff space,*

then the intersection of every countable collection of dense open subsets of E is dense in E.

Proof. See [27, Theorem 2.2]. □

6.2 Extremal Point Theorems

One of the main tools in the proof of Ryll-Narzdewski's Theorem is the description of a convex compact set in a locally convex topological space in terms of its extreme points. This section contains the results required for such description, the Krein and Milman Theorems for extreme points. Each proof was taken from [27, Theorems 3.23 and 3.25]. We first give the definition of extreme point.

Definition 6.2.1. *Let K be a convex set of a vector space X. A subset $S \subset K$ is a extreme subset of K if*

$$tx + (1-t)y \in S,$$

for any points $x, y \in K$ and $t \in (0,1)$ implies that $x, y \in S$.

Note that, when $S = \{p\}$, this definition is written as follows. A point $p \in K$ is called an extreme point if for every $x, y \in K$ such that there exists $t \in (0,1)$ verifying

$$tx + (1-t)y = p,$$

then $x = p = y$. The set of all extreme points of K is denoted by $E(K)$.

Theorem 6.2.2 (Krein-Milman)**.** *Suppose X is a locally convex vector space. If K is a nonempty compact convex set in X, then K is the closed convex hull of the set of its extreme points. In symbols, $K = \overline{\text{co}}(E(K))$.*

Proof. Consider the family $\mathcal{P}$ of compact extreme subsets of K, which is nonempty since $K \in \mathcal{P}$. It is immediate that any non-empty intersection $S = \cap_{i\in I} S_i$ of elements in $\mathcal{P}$ is compact. Moreover, for every $x, y \in S$ and $t \in (0,1)$ such that

$$tx + (1-t)y \in S,$$

then $x, y \in S_i$, for every $i \in I$; meaning $x, y \in S$. Hence, S is extreme on K and $S \in \mathcal{P}$.

We now aim to apply the Hanh-Banach separation Theorem 1.1.4. We shall before hand prove the following property: If $S \in \mathcal{P}$, $x^* \in X^*$, $\mu = \max\{\text{Re}\langle x^*, x\rangle : x \in S\}$, and

$$S_{x^*} = \{x \in S : \text{Re}\langle x^*, x\rangle = \mu\}$$

then $S_{x^*} \in \mathcal{P}$.

Assume $tx + (1-t)y = z \in S_{x^*}$ where $x, y \in K$ and $t \in (0,1)$. Then, since S is extreme in K and $z \in S$, $x, y \in S$. By definition of S_{x^*}, $\text{Re}\langle x^*, x\rangle \leq \mu$, $\text{Re}\langle x^*, y\rangle \leq \mu$ and $\text{Re}\langle x^*, z\rangle = \mu$. But, since $\text{Re}x^*$ is linear,

$$\text{Re}\langle x^*, x\rangle \leq \mu = \mu = \text{Re}\langle x^*, y\rangle.$$

Thus, $x, y \in S_{x^*}$, proving that S_{x^*} is extreme in K. Finally, compactness of S_{x^*} is immediate by definition yielding $S_{x^*} \in \mathcal{P}$.

We will use this fact to prove that the set of extreme points of K is nonempty and that, in fact, for all $S \in \mathcal{P}$, $S \cap E(K) \neq \emptyset$. We first make the following observation:

Let $S \in \mathcal{P}$ and $\mathcal{P}'$ be the subfamily of subsets of S that belong to $\mathcal{P}$. Notice that $\mathcal{P}'$ is non-empty since $S \in \mathcal{P}'$ and that $\mathcal{P}'$ is a partially ordered set considering the inclusion. Now consider a totally ordered subcollection $\Omega \subset \mathcal{P}'$. Then, any two sets $S_1, S_2 \in \Omega$ verify that either $S_1 \subset S_2$ or vice-versa. Hence, Ω trivially verifies the finite intersection property and since it is a collection of compact sets, $M = \cap_{S\in\Omega} S$ is non-emtpy, yielding $M \in \mathcal{P}$. Moreover, since M is the intersection of subsets of S, $M \subset S$ and thus $M \in \mathcal{P}'$.

Now we will apply the separating property of X^* from the hypothesis. Recall that the set $S_{x^*} \in \mathcal{P}$ for every $x^* \in X^*$. By definition, $S_{x^*} \in \mathcal{P}'$ and $M \subset S_{x^*}$ for every $x^* \in X^*$ by minimality of M. Thus every x^* is constant on M and since X^* separates points, M is singleton i.e. M is an extreme point of K. Therefore we have proven that

$$E(K) \cap S \neq \emptyset, \tag{6.1}$$

for every $S \in \mathcal{P}$.

Now we can prove the actual statement. Since K is compact and convex,

$$\overline{\text{co}}(E(K)) \subset K,$$

showing that $\overline{\text{co}}(E(K))$ is compact.

Assume then, to achieve contradiction, that there exists some $x_0 \in K$ such that $x_0 \notin \overline{\text{co}}(E(K))$. Then, by Theorem 1.1.4 *b*), there exists $x^* \in X$ such that

$$\text{Re}\langle x^*, x\rangle < \text{Re}\langle x^*, x_0\rangle$$

for every $x \in \overline{\text{co}}(E(K))$. Finally, consider K_{x^*} which is an element of $\mathcal{P}$. By definition of K_{x^*},

$$\text{Re}\langle x^*, x\rangle = \max\{\text{Re}\langle x^*, y\rangle : y \in K\},$$

for all $x \in K_{x^*}$; and thus

$$\text{Re}\langle x^*, x\rangle < \text{Re}\langle x^*, x_0\rangle \leq \max\{\text{Re}\langle x^*, y\rangle : y \in K\}.$$

Hence $K_{x^*} \cap \overline{\text{co}}(E(K)) = \emptyset$ which contradicts (6.1).

□

Theorem 6.2.3 (Milman)**.** *If K is a compact set in a locally convex space X, and if $\overline{\text{co}}(K)$ is also compact, then every extreme point of $\overline{\text{co}}(K)$ lies in K.*

Proof. Assume that some extreme point p of $\overline{\text{co}}(K)$ is not in K. Then there exists a convex balanced neighborhood V of 0 such that

$$(p + \overline{V}) \cap K = 0. \tag{6.2}$$

which implies $p \notin K + \overline{V}$. By compactness of K, let $x_1, \ldots, x_n$ be a finite subcollection of the indexes such that $K \subset \cup_{i=1}^{n}(x_i + V)$. Define the sets

$$A_i = \overline{\text{co}}(K \cap (x_i + V)),$$

for each $1 \leq i \leq n$. Notice that, since V is convex, $A_i \subset \overline{\text{co}}(x_i + V) = \text{co}(x_i + V)$. Also, each A_i is convex and, since $A_i \subset \overline{\text{co}}(K)$, compact. Moreover, $K \subset A_1 \cup \cdots \cup A_n$. Now, apply that the convex hull of the union of compact convex sets is compact (see [27] Theorem 3.20) to achieve

$$\overline{\text{co}}(K) \subset \overline{\text{co}}(A_1 \cup \cdots \cup A_n) = \text{co}(A_1 \cup \cdots \cup A_n).$$

The other inclusion also holds since $A_i \subset \text{co}(K)$ for each i, yielding

$$\overline{\text{co}}(K) = \text{co}(A_1 \cup \cdots \cup A_n)$$

Thus, every element $x \in \overline{\text{co}}(K)$ can be expressed as an affine combination of elements in A_i. More specifically, p can be written as

$$p = t_1 y_1 + \cdots + t_n y_n = t_1 y_1 + (1 - t_1)\frac{t_2 y_2 + \cdots + t_n y_n}{t_2 + \cdots + t_n},$$

for some $y_i \in A_i$ and $t_i \geq 0$ such that $\sum t_i = 1$, with $1 \leq i \leq n$.

Therefore, we have written p as an affine combination of two points in $\overline{\text{co}}(K)$, hence $y_1 = p$. Then, for some i

$$p \in A_i \subset x_i + \overline{V} \subset K + \overline{V},$$

contradicting (6.2). □

6.3 The Ryll-Narzdewski Fixed Point Theorem

This section is exclusively dedicated to the proof of Theorem 6.1.4. Recall that the Zorn Lemma will be applied to obtain a minimal set (see discussion in Section 6.1).

We introduce some notation that will appear in the theorem. Let S be a semigroup acting on a set E. Then, we say that a subset $C \subset E$ is S-stable if $s(C) \subset C$ for all $s \in S$, where $s(C) = \{sc : c \in C\}$. Similarly to this notation, the set $Sc = \{sc : s \in S\}$ is called the orbit of c by S.

Theorem 6.1.4 (Ryll-Nardzewski). Let K be a nonempty weakly compact convex subset of a locally convex space (X, τ) and let S be a semigroup of weakly continuous affine maps $s : K \to K$. If S is noncontracting on K, i.e., $0 \notin \overline{\{sx - sy : s \in S\}}^{\tau}$ for distinct $x, y \in K$, then S has a common fixed point in K.

Proof. Following the steps of the proof in [23], we start by justifying two assumptions that simplify the proof:

1. Applying Zorn Lemma, we obtain K_0 a minimal nonempty weakly compact convex and S-stable subset of K. Since the action of S on K_0 is noncontracting as well we may assume K to be minimal. Thus $\overline{co}(Sx) = \overline{co}(\{sx : s \in S\}) = K$ for each $x \in K$. We recall from Proposition 1.1.5 that the τ-closure and the weak closure agree on convex sets.

2. It suffices to prove that each finite subset of S has a common f.p. in K since for any finite set $\{s_1, \ldots, s_n\} \subset S$ the set $\{x \in K : s_j x = x, j = 1, \ldots, n\}$ is a closed subset of K. Thus, proving this set is nonempty for any finite set of elements in S, by the finite intersection property and the compactness of K, the set of f.p. in K shall be nonempty as well. Now, since we are interested in working with the semigroup structure, we instead assume that S is finitely generated $S = \langle s_1, \ldots, s_n \rangle$, which is countable.

Let $x \in K$. Then, as discussed in 1., $\overline{co}(Sx) = K$, where Sx is countable by simplification 2. It follows that K is τ-separable, rendering the assumptions above useful.

Applying Zorn's Lemma, let M be a minimal weakly compact nonempty S-stable subset of K. With an analogous reasoning, $K = \overline{co}(M)$. Using that the mappings are affine and weakly continuous, it remains to prove that M is singleton. Searching the contradiction, assume that there exists two distinct points $x, y \in M$. Then, by hypothesis, $0 \notin \overline{\{sx - sy : s \in S\}}^{\tau}$ so there is a convex τ-neighbourhood U of 0 in X such that $sx - sy \notin U$ for all $s \in S$. Apply Lemma 6.1.8 to obtain V be a convex τ-closed subset of U verifying $V - V \subset U$. Now, apply that K is τ-separable to obtain a countable dense subset $T \subset K$. It follows that $K \subset \cup_{t \in T}(V + t)$. Then, as M is weakly compact, (M, ω) is a Baire space. Hence, applying Theorem 6.1.9 we have that for some weakly open subset $W \subset X$, it holds that $\emptyset \neq M \cap W \subset V + t$ for some $t \in T$, since $\{V + t\}_{t \in T}$ forms a countable closed covering of K and consequently of M.

On the other hand, by the Krein-Milman Theorem, there exists and extreme point $u \in K$ and by Milman's Theorem, $u \in M$. Also, since Su is an S-stable non-empty subset of M, by minimality of M it must verify $\overline{Su}^{\omega} = M$ i.e. Su is weakly dense in

M. Hence for some $s_0 \in S$, $s_0 u \in M \cap W$. Now let $z = (x+y)/2$. Once again, apply Milman's theorem to $K = \overline{co}(Sz) = \overline{co}(\overline{Sz}^{\omega})$ and thus $u \in \overline{Sz}^{\omega}$. Then there is a net $\{s_\alpha\}$ in S such that

$$s_\alpha z = (1/2)(s_\alpha x + s_\alpha y) \xrightarrow{\omega} u$$

(recall that all $s_\alpha \in S$ are affine actions on K). Since K is weakly compact, taking subnets if necessary, there exists $a, b \in K$ such that $s_\alpha x \to a$ and $s_\alpha y \to b$ weakly. Therefore $u = (a+b)/2$ and thus $a = u = b$ since u is extreme in K.

Since s_0 is weakly continuous, $s_0 s_\alpha x \overset{\omega}{\to} s_0 a = s_0 u \in M \cap W$. Now recall that M is S-stable and $x \in M$, thus $s_0 s_\alpha x \in M$ for all α. Since $M \cap W$ is a weak-neighborhood of u relative to M, $s_0 s_\alpha x \in M \cap W \subset V + t$ eventually. Similarly, $s_0 s_\alpha y \in M \cap W \subset V + t$ eventually. Hence for some β,

$$s_0 s_\beta x - s_0 s_\beta y \in (V+t) - (V+t) = V - V \subset U,$$

contradicting our choice of U. This proves that M must be a singleton, and the theorem is proved. $\square$

6.4 Some applications

Recall from the discussion in 6.1 that the Ryll-Nardzewski theorem is a generalization of Kakutani and Hahn fixed point theorems (6.1.2, 6.1.3) from two different approaches.

Firstly, every group of equicontinuous mappings is distal (Proposition 6.1.5). Secondly, Ryll-Nardzewski theorem involves an interplay between the natural topology of a locally convex space (which is usually called the strong topology for emphasis) and its weak topology. If X is in particular a normed infinite vector space, then its closed unit ball is never strong compact (or compact for the norm topology). As an application we deduce that if K is a convex weakly compact subset of a normed space, then every group (or semigroup) of affine isometries (not necessarily onto) has at least one fixed point.

We comment in this Section some further results that base on Ryll-Narzdewski's work. Before displaying some applications of this theorem, we need to define a topological group.

Definition 6.4.1. *Let G be a group and τ a topology on G. The pair (G, τ) (or simply G) is a topological group if the group transformations on G, $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$, are τ-continuous mappings.*

Definition 6.4.2. *The set $C(G)$ is the space of all bounded and continuous real valued functions on G.*

As in the discrete case, we can consider the left transformations $\ell_g : C(G) \to C(G)$ for each $g \in G$, defined as $\ell_g f(t) = f(gt)$.

The first implication worth noting is the existence of an invariant measure on a compact group called Haar's measure, where a compact group G is a (G, τ) topological group that is also a τ-compact space.

We first give the definition of a mean for a given topological group G. Let $X = C(G)$, an element $\mu \in C(G)^*$ is a mean on $C(G)$ if $\mu(1) = 1$ and $\mu(f) \geq 0$, for all $f \geq 0$. Recall that we also note the constant unit function on G as 1.

This set is clearly nonempty since the operators $\delta_g f = f(g)$ are means for every $g \in G$ (see example after 2.3.1). Then, one can easily replicate Proposition 2.4.1 to prove that the set of means on a topological group G, $\mathcal{M}(G)$, is a ω^*-compact convex subset of $C(G)^*$.

Similarly to means in discrete semigroups, we can define the mappings $\ell_s : C(G) \to C(G)$ as $\ell_s f(t) = f(st)$. Then, the adjoint mappings give a left action on means, $\ell_s^* \mu(f) = \mu(\ell_s f)$, for all $f \in C(G)$. Hence, we can consider the compact affine flow $(G, \mathcal{M}(G))$ where G is represented as the mappings ℓ_s^*. This flow was proved to be equicontinuous by Kakutani in [17, Lemma 3.3].

Now, apply Ryll-Narzdewski's Theorem (or Kakutani's Theorem 6.1.2) to obtain an element $\mu \in \mathcal{M}(G)$ that is a fixed point for every mapping ℓ_s^* i.e. $\mu(\ell_s f) = \mu(f)$. Thus, by the Riesz Representation Theorem there exists a measure λ that verifies

$$\mu(f) = \int_G f(t)\, d\lambda(t)$$

for every $f \in C(G)$. Finally, since μ is left invariant, we have that

$$\int_G f(t)\, d\lambda(t) = \mu(f) = \mu(\ell_s f) = \int_G \ell_s f(t)\, d\lambda(t) = \int_G f(st)\, d\lambda(t),$$

for all $f \in C(G)$. That is, we have proved that G admits an invariant measure.

Note that these arguments are also used in [27, Theorem 5.14] where the Haar measure is constructed using Kakutani's Theorem.

For a second application, we introduce the concept of almost periodic functions.

Definition 6.4.3. *Let G be a topological group. A function $f \in C(G)$ is called a weakly [strong] almost periodic function if the set*

$$\overline{\text{co}}(\{\ell_g f : g \in G\})$$

is weakly [norm] compact.

Notice that, since G is a group, the sets $\{\ell_g f : g \in G\}$ and $\{\ell_g(\ell_h f) : g \in G\}$ are equal for every $h \in G$. This implies that the set of weakly [strong] almost periodic functions is closed under the left transformations of G. In [10, Theorem 4.2] Eberlein proved that the set of weakly almost periodic functions is a closed subspace of $C(G)$.

Let $W(G)$ denote the space of weakly almost periodic functions (w.a.p. in short). We give now an application of Ryll-Narzdewski's Theorem to find a left invariant mean in $W(G)$ that is, a left invariant functional $\mu \in W(G)^*$ that satisfies $\mu(1) = 1$ and $\mu(f) \geq 0$, for all $f \geq 0$ for all $f \in W(G)$. In the particular case that G is amenable, this was proved using Day's fixed point Theorem by Eberlein in [10] and it was raised the question whether or not $W(\mathbb{F}_2)$ admits a left invariant mean. This problem was completely solved for every topological group by Ryll-Narzdewski using his fixed point theorem. In order to finish this section, we give here a sketch of the proof.

Theorem 6.4.4. *Let G be a topological group and $f \in C(G)$ be w.a.p. function. Then, there is a constant function, which we denote by $M(f)$, in $\overline{\text{co}}(\{\ell_g f : g \in G\})$.*

Proof. In Ryll-Narzdewski's Theorem 6.1.4, let $X = C(G)$, $K = \overline{\text{co}}(\{\ell_g f : g \in G\})$, which is weakly compact since f is w.a.p., and G represented as $sf(t) = f(s^{-1}t)$. This defines an equicontinuous flow (as $\|\ell_s f\| = \|f\|$ for all $g \in G$) on a weakly compact convex subset of $C(G)$. Thus, we can apply Ryll-Narzdewski's to obtain a common fixed point in $\overline{\text{co}}(\{\ell_g f : g \in G\})$ i.e. a G-invariant function $M(f)$ in $\overline{\text{co}}(\{\ell_g f : g \in G\})$. That is, $gM(f)(t) = M(f)(g^{-1}t) = M(f)(t)$, for all $g, t \in G$. This, since G is a group, implies that $M(f)(t) = M(f)(s)$ for all $t, s \in G$ i.e. $M(f)$ is constant. □

Note that neither Kakutani's Theorem 6.1.2 nor Hahn's Theorem 6.1.3 can be applied above since the domain is not strong compact. The same happens in the next theorem.

Theorem 6.4.5. *Let (G, X) be an equicontinuous linear flow where G is a group. We define*

$$O_G(x) = \overline{\text{co}}(\{gx : g \in G\}),$$

called the convex G-orbit of an element $x \in X$. Then,

1. *for each $x \in X$ such that $O_G(x)$ is weakly compact, there exists a unique G-invariant element in $O_G(x)$ denoted as Mx.*

Moreover, if we define X_0 as the elements $x \in X$ such that $O_G(x)$ is weakly compact and M is the mapping defined on X_0 as $x \to Mx$, then

2. *the set X_0 is a closed subspace of X, and*

3. *the operator M is linear and $g(Mx) = M(gx) = M(Mx) = Mx$, for all $g \in G$ and for all $x \in X_0$.*

Proof. We proof each statement separately:

1. Let $x \in X_0$. Since, (G, X) is equicontinuous, $(G, O_G(x))$ is an affine equicontinuous flow with $O_G(x)$ weakly compact. Thus, we can apply Theorem 6.1.4 to yield the existence of a common fixed point in $O_G(x)$ for the action of G.

   To prove uniqueness, for each $x^* \in X^*$ and for each $y \in O_G(x)$ we define the function

   $$\begin{aligned} F_y : G &\longrightarrow \mathbb{R} \\ g &\longmapsto x^*(gy). \end{aligned}$$

   Then, in [29, Theorem 5] it is stated that F is w.a.p. and thus, by Theorem 6.4.4, we can obtain an invariant mean value of F, which we called $M(F_y)$. That is, for all $s \in G$

   $$M(F_y) = M(\ell_s F_y).$$

   Hence, for every pair $y_1, y_2 \in O_G(x)$, $y_1 = gy_2$ for some $g \in G$ and hence $M(F_{y_1}) = M(F_{y_2})$.

On the other hand, for every $x_0 \in O_G(x)$ G-invariant element, we have that $F_{x_0}(g) = x^*(gx_0) = x^*(x_0)$ for all $g \in G$, that is F_{x_0} is constant and thus $M(F) = x^*(x_0)$. Then we have that, for any pair $x_0, y_0 \in O_G(x)$ of G-invariant elements

$$x^*(x_0) = M(F_{x_0}) = M(F_{y_0}) = x^*(y_0).$$

Since this equality holds for every $x^* \in X^*$ and X is locally convex, $x_0 = y_0$.

2. By linearity of the action of G,

$$O_G(\alpha x + y) = \alpha O_G(x) + O_G(y),$$

for all $x, y \in X_0$ and for every scalar α.

To prove that X_0 is closed see [29, Proposition 1 and Theorem 5].

3. The fact that $M(Mx) = Mx$ is immediate since Mx is the unique fixed point in $O_G(x)$. Also, since Mx is G-invariant for every $x \in X_0$, $g(Mx) = Mx$ is obvious. Lastly, for every $O_G(x) = O_G(gx)$, for every $g \in G$, hence $M(gx) = M(x)$ for every $x \in X_0$.

□

6.5 Notes on further results

In this section we aim to display some other classical and recent common fixed point results in the setting of Kakutani-type theorems. Firstly, we remark that some extensions of Ryll-Narzdewski's Theorem are given in [12, Theorem 1.6].

Furstenberg fixed point theorem [23, Theorem 4.1] is another example of Kakutani-type fixed point theorem:

Theorem 6.5.1. *Let (S, Q) be a compact affine flow. Suppose there exists a nonempty compact S-stable subset K (i.e., $sK \subset K$ for each $s \in S$) of Q such that (S, K) is distal. Then there is a common f.p. of S in Q.*

In this result, the distal hypothesis is restricted to some (not necessarily convex) closed S-stable subset.

Some non-linear counterparts of the Ryll-Nardzewski Theorem have been recently obtained by Wiśnicki in [34]. The assumption of linearity is changed by considering nonexpansive mappings. In the case of a normed space a mapping $s : C \to C$ is nonexpansive if

$$\|sx - sy\| \leq \|x - y\|,$$

for all $x, y \in C$. For the case of locally convex vector spaces, as every locally convex space is determined by a family of seminorms $\mathcal{N}$, we have the following definition: the semigroup S acting on C is non-expansive if

$$p(sx - sy) \leq p(x - y),$$

for all $x, y \in C$ and for every $p \in \mathcal{N}$.

The theorem proved by Wiśnicki for locally convex spaces [34, Theorem B] states as follows.

Theorem 6.5.2. *Let K be a nonempty weakly compact convex subset of a locally convex space (X, τ) and let (S, K) be a nonexpansive and $\tau-$distal flow. Then, there is a common fixed point of S in K. Moreover, the set of fixed points is a nonexpansive retract of K.*

We conclude by stating a theorem that connects once more amenability with Kakutani-type theorems this time for nonexpansive semigroups, given by Takahashi in [31].

Theorem 6.5.3. *Let S be a left amenable semigroup. Then, every representation of S as non-expansive mappings from a non-empty compact convex subset K of a Banach space into itself has a common fixed point for S in K.*

Generalizations of this theorem in the context of convex weakly compact sets of Banach spaces for different types of nonexpansive semigroups and strongly connected to the geometry of the space can be found in [19] and [20].

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