INSTITUT DE FRANCE
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Comptes Rendus
Mécanique

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Published online: 9 June 2023

https://doi.org/10.5802/crmeca.187

Part of Special Issue: The scientific legacy of Roland Glowinski
Guest editors: Gregoire Allaire (CMAP, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, France), Jean-Michel Coron (Laboratoire Jacques-Louis Lions, Sorbonne Université) and Vivette Girault (Laboratoire Jacques-Louis Lions, Sorbonne Université)

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Les Comptes Rendus. Mécanique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1873-7234

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The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

A posteriori Variational Multiscale Methods for the 1D convection-diffusion equations

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Abstract. The present work is a continuation of a paper presented by the two first authors in the proceedings of the "Computational Science for the 21st century" conference held in Tours in 1997 honouring the 60th birthday of Roland Glowinski. It is devoted to the solution of 1D convection-diffusion equations in dominant convection regime situations. In that paper, an "a posteriori" VMS filtering technique was introduced. We present an extension of this technique to nonlinear convection-diffusion equations (a traffic model), providing an efficient method for the resolution of shocks from just the Galerkin solution at targeted times. We also present a residual-based "a posteriori" VMS filtering, that provides quite accurate stable solutions, can be extended to multi-dimensional problems, and can be applied locally. We finally present some numerical tests exhibiting the high accuracy of the obtained solutions.

Keywords. VMS methods, dominant convection, a posteriori filtering, stabilisation, convection-diffusion equation, traffic flow equation.

Funding. The work of T. Chacón and M. Gómez has been partially supported by project PID2021-123153OB-C21 funded by MCIN /AEI /10.13039/501100011033 / FEDER, UE. The work of A. Domínguez has been partially funded by the Spanish Research Agency Plan Estatal 2017-2020 Retos - Proyectos I+D+i Referencia: PID2020-117722RB-I00 Grant.

Published online: 9 June 2023

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1. Introduction

This paper continues the “a posteriori Variational Multiscale” filtering procedure that the two first authors presented in the proceedings of the “Computational Science for the 21st century” conference held in Tours in 1997 celebrating the 60th birthday of Roland Glowinski [1].

We address the numerical treatment of the instabilities generated by the variational discretisation of convection-diffusion equations. These instabilities are due to the dominance of the convection terms on the diffusion ones, when the discretisation parameters are not small enough. This generates spurious oscillations in the discrete solutions that therefore are unreliable for practical applications. The Variational Multiscale (VMS) method, introduced by Hughes in [2], is a general technique to stabilise these unstable solutions (see Hughes (cf. [3, 4]).

The VMS methods are based upon the approximate solution of the small scale problem in terms of the resolved scales, and the substitution of the resolved approximate small scales in the resolved scales equation. This provides improved stable solutions, as under the VMS formulation the resolved scales appear as solutions of a problem with enhanced diffusion, stemming from the action of the sub-grid scales on the resolved ones (cf.[5–7]).

The purpose of the procedure introduced in [1] is to “cure” a solution of the convection-diffusion equations presenting spurious oscillations due to convection dominance, without need of solving again the equation on a finer grid or with stabilised methods. Only the information provided by the oscillating solution is used. The basic idea is to take advantage of the stabilising effect of the sub-grid scales. This is reached by means of a nodal-based “a posteriori” VMS filtering of the discrete solution, that provides a quite accurate numerical solution at the even nodes of the computational grid.

In this paper, on one hand we introduce a residual-based “a posteriori” VMS filtering that needs the residual minimisation of a convenient projection of the oscillating solution on the grid with double size. This procedure has the advantage of being able to be applied locally and admits a ready extension to multi-dimensional problems. It provides quite close filtered solutions to the exact one on the grid of double size, if the initialisation of the minimisation procedure for the least-squares problem is conveniently provided.

We also extend the nodal-based “a posteriori” VMS to evolution non-linear convection-diffusion problems that generate nearly discontinuous solutions in finite time. We actually address the one-lane traffic equation. We have developed an iterative filtering procedure of the Galerkin solution, using very small time steps to the solution at a targeted time. The filtering procedure does not need to be applied in preceding times. It provides quite accurate solutions, again at the nodes of a grid with double grid size.

The outline of the paper is the following. In Section 2 we describe the basics of the “a posteriori” VMS and the nodal version introduced in [1]. In Section 3 we address the residual-based “a posteriori” VMS, while in Section 4 we extend it to non-linear convection-diffusion problems. Finally, in Section 5 we present some numerical tests that exhibit the excellent accuracy of the filtered solutions for all three filtering procedures.

2. Nodal “a posteriori” VMS method

The “a posteriori” VMS method, introduced in [1], takes advantage of the VMS stabilising effect to generate a stable and accurate post-processing of the numerical solution, without needing to re-solve the targeted PDE (in our case, the convection-diffusion equations). To describe it, let us consider an elliptic variational problem:

$$\text{Find } x \in H \text{ such that } b(x, w) = l(w), \quad \forall\, w \in H, \tag{1}$$

where H is a Hilbert space, $b : H \times H \mapsto \mathbb{R}$ is bounded coercive bilinear form and $l \in H'$. We solve equation (1) by the Galerkin method, constructed with a family of finite element sub-spaces of H, $\{X_h\}_{h>0}$:

$$\text{Find } x_h \in X_h \text{ such that } b(x_h, w_h) = l(w_h), \quad \forall\, w_h \in X_h. \tag{2}$$

We decompose X_h into $X_h = Y_h \oplus Z_h$ (that is, $X_h = Y_h + Z_h$ and $Y_h \cap Z_h = \{0\}$), for some subspaces Y_h and Z_h of X_h. We consider that Y_h and Z_h respectively are a "large scales" and a "small scales" spaces. For $x_h \in X_h$, there is a unique decomposition $x_h = y_h + z_h$ such that $y_h \in Y_h$ and $z_h \in Z_h$. We may re-write problem (2) as an equivalent variational ("condensed") problem for the only unknown y_h. To build this condensed problem, we need the static condensation operator on Z_h, $\mathscr{R}_h : H' \mapsto Z_h$ defined for $\varphi \in H'$ by:

$$b\big(\mathscr{R}_h(\varphi),\, w_h\big) = \langle \varphi, w_h \rangle, \quad \forall\, w_h \in Z_h.$$

We may then state the condensed variational formulation to problem (2) as:

$$\text{Find } y_h \in X_h \text{ such that } b_c(y_h, v_h) = l_c(v_h), \quad \forall\, v_h \in X_h \tag{3}$$

with

$$b_c(y, v) = b(y, v) - b\big(\mathscr{R}_h\left(\mathscr{A}^* v\right), \mathscr{R}_h\left(\mathscr{A} y\right)\big), \quad l_c(v) = l(v) - b\big(\mathscr{R}_h\left(\mathscr{A}^* v\right), \mathscr{R}_h(f)\big), \quad \forall\, y, v \in H;$$

where $\mathscr{A}^*$ denotes the adjoint of the operator $\mathscr{A} : H \mapsto H$ defined for $v \in H$ by

$$\langle \mathscr{A} v,\, w \rangle = b(v, w), \quad \forall\, w \in H.$$

Then, in [1] is is proved that:

Theorem 1. *Assume that the spaces Y_h and Z_h satisfy $Y_h \cap Z_h = \emptyset$. Then:*

(1) *Let $x_h = y_h + z_h$ be the unique decomposition that x_h admits with $y_h \in Y_h$ and $z_h \in Z_h$. Then, x_h is the solution of the Galerkin method* (2) *if and only if y_h is the solution of the condensed variational formulation* (3)*, and $z_h = \mathscr{R}_h(l - \mathscr{A}(y_h))$.*

(2) *Assume, in addition, that the family of pairs of spaces $\{(Y_h, Z_h)\}_{h>0}$ satisfies the saturation property: there exists an angle $\sigma > 0$ such that*

$$\arccos\left(\sup_{y_h \in Y_h \setminus \{0\},\, z_h \in Z_h \setminus \{0\}} \frac{(y_h, z_h)_X}{\|y_h\|_H \|z_h\|_H} \right) > \sigma \quad \forall\, h > 0. \tag{4}$$

Then, there exists a constant $C > 0$ such that

$$\|y_h\|_H + \|z_h\|_H \le C\, \|l\|_{H'}, \quad \|c_h\|_H \le C\, \|l\|_{H'}, \tag{5}$$

where $c_h = \mathscr{R}_h(\mathscr{A}(y_h))$.

This results also holds if Z_h is replaced by some infinite-dimensional small scale sub-space Z of H such that $H = Y_h \oplus Z$ and the angle between Y_h and Z is not zero (that is, Y_h and Z are topological complements on H).

Observe that the estimate in (5) for c_h may be interpreted as a stabilisation effect. Indeed, the static condensation operator is a discrete Riesz operator that represents (the small scale components of the) elements of H' on Z_h. Then, c_h may be interpreted as a representation on Z_h of the small-scale components of the convection-diffusion operator $\mathscr{A}$ acting on the large-scale component y_h of the solution x_h. By (5), c_h is uniformly bounded in H norm.

Let us now apply this theory to the convection-diffusion equation

$$\left.\begin{aligned} (w\,u)' - \nu\, u'' &= f \qquad \text{in} \quad [0,1] \\ u(0) &= \alpha, \qquad u(1) = \beta \end{aligned}\right\} \tag{6}$$

where w is the fluid convection velocity, ν is the diffusion coefficient and α, β are given constants. We in principle assume that w and f are functions, although these will be considered as constants in some parts of the paper. We approximate this problem by the Galerkin method constructed on piecewise affine finite elements. Consider a partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the interval $[0,1]$ with an uniform spatial step $h = 1/N$.

Then, the space of approximation is

$$X_h = \left\{ x_h \in C^0[0,1] \,|\, x_h|_{]x_{j-1},x_j[} \in P_1, \quad 1 \le j \le N \right\}.$$

Let $\{\varphi_i\}_{i=0}^N$ be the Lagrange interpolation canonical base of X_h. The solution u of (6) is approximated by

$$u_h = \sum_{i=0}^{N} u_i \varphi_i \in X_h,$$

with $u_0 = \alpha$, $u_N = \beta$ and $u_i = u_h(x_i)$. Let

$$Pe_h = \frac{|w|\,h}{2\nu}$$

be the so-called Péclet mesh number associated with the discretization. It is well known that when w is constant, if $Pe_h < 1$, the discrete maximum principle is satisfied by the sequence $\{u_i\}_{i=0}^N$, while this fails to occur when $Pe_h > 1$. When this last happens, typically in situations of dominant convection, spurious oscillations appear in the discrete solution of (6).

The SUPG (Streamline Upwind Petrov–Galerkin) method provides a technique to eliminate these spurious oscillations (in one space dimension) or to reduce them (in higher space dimensions). We recall that the stabilisation proposed by the SUPG method (cf. [8]) is equivalent for 1D advection-diffusion problems and constant data w, ν and f to discretising by the Galerkin method in X_h the equation

$$w\,u' - (\nu + \nu_{num})\,u'' = f, \tag{7}$$

where $\nu_{num} = \nu\, Pe_h\, \xi(Pe_h)$, and

$$\xi(\alpha) = \coth(\alpha) - \frac{1}{\alpha}.$$

The above choice of ν_{num} guarantees that the approximate solution u_h provided by the Galerkin method applied to equation (7), coincides with the exact solution of equation (6) at all grid nodes, i.e., $u_h(x_i) = u(x_i), \quad 0 \le i \le N$.

In [9], it is shown that the numerical scheme provided by the SUPG method is equivalent to adding to the space X_h, an "optimal" bubble per element and to "condense" it afterwards. Here, "bubble" on, for example, the element $]x_{i-1}, x_i[$ means any function of $H_0^1(x_{i-1}, x_i)$, extended by zero to the whole interval $[0,1]$. Then, problem (7) is cast as the condensed problem (3) when

$$Z_h = Z = \bigoplus_{i=1}^{N} H_0^1(x_{i-1}, x_i).$$

A similar result is introduced in [1] in the fully discrete framework when w and f are constant: if the number of nodes of the grid is odd, there exists a choice of the sub-grid scales space Z_h such that the filtered solution y_h coincides with the exact solution u at the even nodes of the grid. The subspace Y_h is taken as the space of piecewise affine functions on the partitioning $0 = x_0 < x_2 < \ldots < x_{N-2} < x_N = 1$ of the interval $[0,1]$ with a uniform spatial step equal to $2h$, assuming $N = 2n$ for some integer $n \ge 1$. Let us denote by $\{\phi_i\}_{i=0}^n$ the Lagrange basis of the piecewise finite element space Y_h. We set the space $Z_h = \text{Span}\{\varphi_1, \varphi_3, \ldots, \varphi_{2n-1}\}$ for some suitable $\varphi_1, \varphi_3, \ldots, \varphi_{2n-1} \in X_h$ such that the decomposition

$$u_h = \sum_{i=1}^{n-1} y_i \phi_i + \sum_{i=1}^{n} z_{2i-1} \varphi_{2i-1}, \quad \text{with } y_i = u(x_{2i}) \text{ for } i = 1, \cdots, n-1, \tag{8}$$

is fulfilled.

For this, the minimum support that must have φ_{2i-1}, $2 \le i \le n$, is the interval $[x_{2i-3}, x_{2i}]$, and the minimum support of φ_1, must be $[x_0, x_2]$. In this case, (8) can be uniquely solved analytically.

Indeed, imposing that $\varphi_{2i-1}(x_{2i-1}) = 1$, then φ_{2i-1} is uniquely determined by its slope a_{2i-1} in the interval (x_{2i-2}, x_{2i-1}), as follows,

$$\varphi_{2i-1}(x) = \begin{cases} -\frac{1}{h}(x - x_{2i-1}) + 1, & \text{if } x \in [x_{2i-1}, x_{2i}], \\ a_{2i-1}(x - x_{2i-1}) + 1, & \text{if } x \in [x_{2i-2}, x_{2i-1}[, \\ \left(\frac{1}{h} - a_{2i-1}\right)(x - x_{2i-3}), & \text{if } x \in [x_{2i-3}, x_{2i-2}[\text{ and } i \geq 2, \\ 0 & \text{in other case.} \end{cases} \tag{9}$$

Imposing now that the discrete solution coincides at the even nodes with the exact solution, we recursively determine the coefficients z_{2i-1} and the slopes a_{2i-1} for $i = 1, 2, \ldots, n$ by

$$z_1 = u_h(x_1) - \frac{1}{2} u(x_2), \tag{10}$$

$$a_1 = \frac{1}{h}, \tag{11}$$

and for $i = 2, 3, \ldots, n$,

$$z_{2i-1} = u_h(x_{2i-1}) - \frac{1}{2}\big(u(x_{2i-2}) + u(x_{2i})\big), \tag{12}$$

$$a_{2i-1} = \frac{1}{h}\left[1 - \frac{u_h(x_{2i-2}) - u(x_{2i-2})}{z_{2i-1}}\right]. \tag{13}$$

The slopes a_{2i-1} in practice are positive, due to the oscillatory nature of u_h that lets z_{2i-1} have a different sign than $u_h(x_{2i-2}) - u(x_{2i-2})$. Actually, a_{2i-1} increases as i increases (see Figure 1).

With this construction Z_h is a subspace of $H_0^1(]0,1[)$. In addition, it is straightforward that $X_h = Y_h + Z_h$. Indeed, any function $x_h \in X_h$ has the decomposition

$$x_h = y_h + z_h, \text{ with } y_h = \sum_{i=1}^{n} \widehat{y}_i \phi_i \in Y_h, \; z_h = \sum_{i=1}^{n} \widehat{z}_{2i-1} \varphi_{2i-1} \in Z_h, \tag{14}$$

where the coefficients $\widehat{y}_i$ and $\widehat{z}_i$, are recursively given by

$$\widehat{z}_{2i-1} = \frac{x_h(x_{2i-2}) - 2x_h(x_{2i-1}) + \widehat{y}_i}{-1 - h a_{2i-1}}, \quad \widehat{y}_{i-1} = 2\big(x_h(x_{2i-1}) - \widehat{z}_{2i-1}\big) - \widehat{y}_i, \tag{15}$$

for $i = n, n-1, \ldots, 1$. As the slopes a_{2i-1} are positive, the denominators $-1 - h a_{2i-1}$ are strictly negative and, therefore, the values of $\widehat{z}_{2i-1}$ are well defined for $1 \leq i \leq n$.

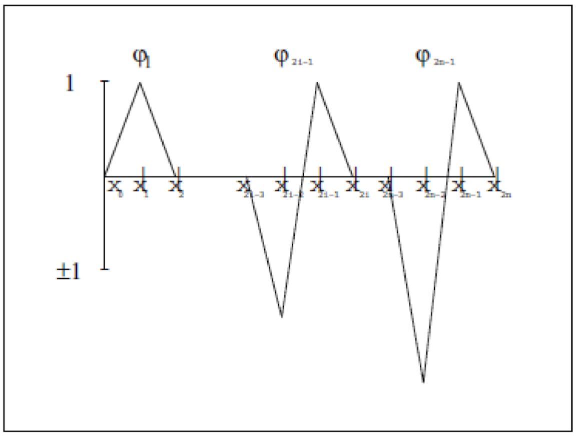


Figure 1. Optimal Z_h basis for the convection-diffusion equation (6), for $Pe_h = 10$.

Figure 1 shows the Z_h basis functions for $Pe_h = 10$. It can be observed how the slopes a_{2i-1} increase as the points x_{2i-1} approach the boundary layer. The projection z_h of u_h on the small-scales space Z_h determined by (14) "absorbs" the oscillations of u_h, letting its projection y_h on the large scales space Y_h coincide with the exact solution at the even nodes.

On the other hand, it is clear that Z_h depends on the exact solution u. Now, for constant w, ν and $f = 0$, it holds

$$u\left(x_{2j}\right) = \frac{\alpha\, e^{Pe} - \beta}{e^{Pe} - 1} + \frac{\beta - \alpha}{e^{Pe} - 1}\, e^{4\, j\, sign(w)\, Pe_h}, Pe = \frac{w}{\nu}. \tag{16}$$

Then, Z_h depends only on the global and the Péclet mesh numbers. This allows to generalise the definition of Z_h to problems with variable convection velocity by changing the Péclet mesh number that defines each slope a_{2i-1} (through identity (16)), by an average of the Péclet mesh number in a neighbourhood of x_{2i-1}, and computing Pe as an average Péclet number in the same neighbourhood.

The method presented above can be seen as an "a posteriori" Variational Multiscale method, in which for the case of constant coefficients, the exact solution is recovered at the nodes of the mesh of grid size $2h$. Our "resolved" scales are searched for in the space Y_h, while the "subgrid" scales (with respect to Y_h) belong to the space Z_h. Further, space Z_h is fully determined by the values of the solution u at the even nodes and the (oscillating) solution u_h on X_h.

3. Residual-based "a posteriori" VMS

An alternative strategy that does not need the knowledge of the exact solution is to minimise, among all possible choices of the small-scales space Z_h, the residual of the projection of x_h on the large scales space Y_h. This strategy has two additional advantages: it is readily extended to multi-dimensional problems and can be applied locally, in a region of the domain that concentrates the oscillations of the numerical solution.

To specify this procedure, we remark that the slopes a_{2i-1}, $i = 1, \cdots, n$ uniquely determine the functions φ_{2i-1}, $i = 1, \cdots, n$ by (9), and consequently the slopes a_{2i-1} determine a unique small-scale space Z_h, spanned by these piecewise affine functions φ_{2i-1}. Also, that once Z_h is determined in this way, there is a unique decomposition of any $x_h \in X_h$ given by (14)-(15), $x_h = y_h + z_h$ with $y_h \in Y_h$ and $z_h \in Z_h$ (excepting for very particular slopes, given by $a_{2i-1} = -1/h$).

We may then search for the filtered solution as the one that minimises the residual of the large-scales component y_h, with respect to the slopes a_{2i-1}, $i = 2, \cdots, n$. We thus define $K = (\mathbb{R} \setminus \{-1/h\})^{n-1}$ and consider the functional

$$J : K \to \mathbb{R} \quad \text{given by} \quad J(a_3, \cdots, a_{2n-1}) = \left\| R(y_h(a_3, \cdots, a_{2n-1})) \right\|^2_{H^{-1}(0,1)},$$

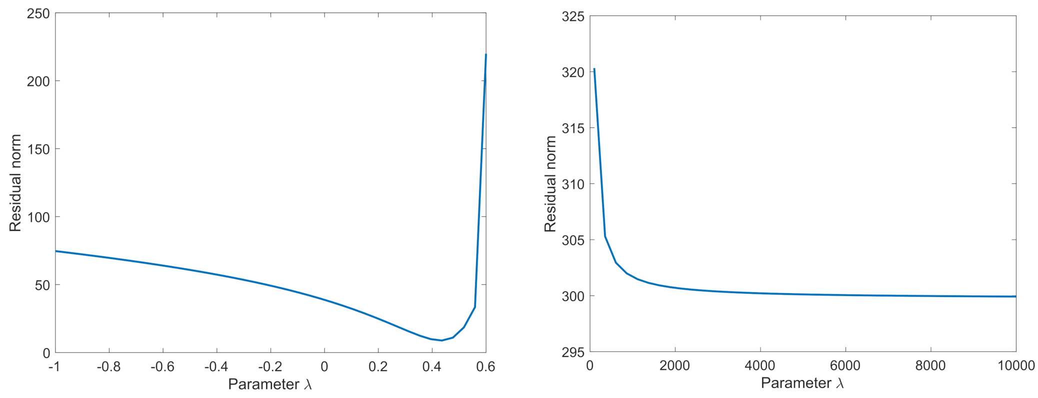
where $y_h(a_3, \cdots, a_{2n-1})$ is obtained by (14)-(15) with $x_h = u_h$ (that is, the Galerkin solution of the convection-diffusion problem (6) on X_h), and $R(y_h) \in H^{-1}(0,1)$ is the residual of y_h, that is $R(y_h) = l - \mathscr{A}(y_h)$. We define the residual minimisation-based "a posteriori" VMS filtered solution as

$$y_h\,(\tilde{a}_3, \cdots, \tilde{a}_{2n-1}) \quad \text{where} \quad (\tilde{a}_3, \cdots, \tilde{a}_{2n-1}) = argmin_K J.$$

To compare with the solution provided by the nodal-based procedure, it is preferable to replace the a_{2i-1}, $i = 2, \cdots, n$ as degrees of freedom, by the values α_{i-1}, $i = 2, \cdots, n$ reached by the φ_{2i-1}, $i = 2, \cdots, n$ at the nodes x_{2i-2}, that is

$$\alpha_{i-1} = \varphi_{2i-1}(x_{2i-2}) = 1 - h\, a_{2i-1}.$$

We then consider J as a functional depending on the values $\alpha_1, \cdots, \alpha_{n-1}$ defined on the domain $\Sigma = (\mathbb{R} \setminus \{2\})^{n-1}$, considering that $a_{2i-1} = -1/h \Leftrightarrow \alpha_{i-1} = 2$. We compute the norm of the residual



(a) **(b)**

Figure 2. *Residual-based a posteriori VMS.* Representation of the functional j given by (18) (squared dual norm of the residual) on two segments linking the parameters $\alpha_i = \alpha_i^{nb}$, corresponding to the optimal nodal solution, to the singular values $\alpha_i = 2$. The parameter range close to 2 is avoided as j reaches very large values in this range.

in $H^{-1}(0,1)$ as the norm in $H^1(0,1)$ of its Riesz representation. That is, if we denote $R_h = R(y_h(a_3,\cdots,a_{2n-1}))$, and define $r_h \in H_0^1(0,1)$ as the solution of

$$-r_h'' = R_h, \tag{17}$$

then

$$\|R_h\|_{H^{-1}(0,1)} = \|r_h\|_{H^1(0,1)}.$$

In practice we approximate r_h by its Galerkin finite element approximation on a grid of size smaller than h.

In Figure 2 we have represented the functional J when $Pe = 400$, $h = 1/22$ in a segment Γ linking the nodal-based optimal values $\alpha_i^{nb} = 1 - h\,a_{2i-1}^{nb}$ with a_{2i-1}^{nb} defined by (9) to (13), with the singular values $\alpha_i = 2$, that is, we represent the function

$$j : [a,b] \to \mathbb{R}; \quad j(\lambda) = J\Big((1-\lambda)\vec{\alpha}^{nb} + \lambda\vec{\beta}\Big), \tag{18}$$

where $\vec{\alpha}^{nb} = (\alpha_1^{nb},\cdots,\alpha_{n-1}^{nb})$ and $\vec{\beta} = (2,\cdots,2) \in \mathbb{R}^{n-1}$. Observe that j is a strictly convex function in a neigbourhood of its minimum. Moreover, $j(\lambda)$ tends to infinity as $\lambda \to 2$ and to constant values, larger than its minimum, as $\lambda \to -\infty$ and as $\lambda \to +\infty$. Note that the minimum of j in Γ is reached at a value $\gamma_{max} \in \Gamma$ somewhat larger than 0.4. Thus, it is close, but not equal, to the nodal-based optimal values (corresponding to $\lambda = 0$). Therefore, the nodal-based and the residual-based "a posteriori" VMS methods provide different filtered approximations.

4. Nodal-based "a posteriori" VMS for transient nonlinear convection-diffusion equations

In this section we apply the nodal-based "a posteriori" VMS procedure to a nonlinear transient diffusive convection problem, modelling the traffic flow in a one-lane road.

The traffic flow equation for a one-lane road can be written as

$$\begin{cases} u_{,t}(x,t) + F_{,x}(u(x,t)) - \nu\, u_{,xx}(x,t) = f(x,t), & x \in [0,1],\ t \in [0,T] \\ u(x,0) = u_0(x) & x \in [0,1], \\ u(0) = u_L, \quad u(1) = u_R, \end{cases} \tag{19}$$

where u represents the traffic density, f is a source term, u_0 is the given initial condition, u_L and u_R are given numbers, and the flux function F is twice differentiable and verifies

$$\begin{aligned} &F(0) = 0, \quad F(u_{max}) = 0, \\ &F(u) > 0 \quad \text{and} \quad F''(u) < 0 \quad \text{for} \quad 0 < u < u_{max}. \end{aligned} \tag{20}$$

The flux function can be modeled in different ways and is dependent on the road on which the traffic circulates; the model case we consider here is the given by

$$F(u) = u(1-u).$$

If we consider the Riemann problem

$$u_0(x) = \begin{cases} u_L \ \text{ if } x \in [0,1/2], \\ \\ u_R \ \text{ if } x \in [1/2,1], \end{cases} \tag{21}$$

with $u_L < u_R$, the solution is made of two steady states united by a diffusive shock which is shifted to the left.

To solve (19), we discretize in time by using a semi-implicit Euler scheme, obtaining an equation with the following structure:

$$\left.\begin{aligned} \frac{1}{\Delta t}u^{n+1} + w\left(u^n\right)\left(u^{n+1}\right)' - \nu\left(u^{n+1}\right)'' = f^{n+1} + \frac{1}{\Delta t}u^n \\ u^{n+1}(0) = u_L, \quad u^{n+1}(1) = u_R\,, \end{aligned}\right\} \tag{22}$$

where $w(v) = F'(v)$. The boundary conditions in (22) come from (21), and hold before the shock initially located at $x = 1/2$ reaches the boundary of the domain.

Now, we discretize (22) by the standard Galerkin method, using for this purpose the space of Finite Elements X_h. In this way, at each time step we solve the problem:

$$\left.\begin{aligned} &\text{Find } u_h^{n+1} \in X_h \quad \text{verifying that} \quad \forall\, v_h \in X_h, \\ &\frac{1}{\Delta t}\left(u_h^{n+1}, v_h\right) + \left(w\left(u_h^{n+1}\right)', v_h\right) - \nu\left(\left(u_h^{n+1}\right)', v_h'\right) = \langle f^{n+1}, v_h\rangle + \frac{1}{\Delta t}\left(u_h^n, v_h\right) \\ &u_h^{n+1}(0) = u_L, \quad u_h^{n+1}(1) = u_R\,. \end{aligned}\right\} \tag{23}$$

When this problem is solved by the Galerkin method using piecewise affine Finite Elements, again for high values of the Péclet mesh number the numerical solution is affected by strong oscillations, in this case around the shock (see Fig. 3).

To treat this instability, it is possible to apply to the solution of (23) the optimal filtering described in the previous section, at each time step. This eliminates the spurious oscillations.

It is also possible to apply this filtering only at the time instant at which one is interested to compute the solution accurately. Specifically, we consider the following algorithm to obtain a filtered solution of (23) at a pre-set time T:

Non-linear case Optimal Filtering Algorithm:

(0) **Initialization.** The values $N, \triangle t = T/N, u_h^0$ and $\triangle t^*$ are specified, where $\triangle t^*$ is a number close to the computer accuracy (but slightly higher).

(I) **Iteration in time without filtering.** The system (23) is solved, obtaining u_h^{n+1} from u_h^n for $n = 0, 1, \ldots, N-1$.

(II) **Filtering.**

(a) The system (23) is solved, obtaining $u_h^{N^*}$ from u_h^N using the time step $\triangle t^*$.

(b) The "optimal" filtering of $u_h^{N^*}$ is done, using the "optimal" basis of Z_h determined from the averages of $w_h^{N^*}$ over each interval $[x_{2i-2}; x_{2i}]$. That is, $u_h^{N^*}$ is computed by (14)-(15) using the values of z_{2i-1}, a_{2i-1} given by (12) and (13). In (13), the value of $u(x_{2i})$ is given by (16), computing Pe_h as an average value of the grid Péclet number in $[x_{2i-1}, x_{2i}]$.

(c) If the filtered solution exhibits spurious oscillations, take $u_h^N = u_h^{N^*}$ and return to (IIa).

Let us note again that the filtering is not performed until it is reached just the time T at which we want to obtain the solution.

5. Numerical Tests

In this section we present a test for each of the three a-posteriori Variational Multi-Scale methods introduced in the previous sections.

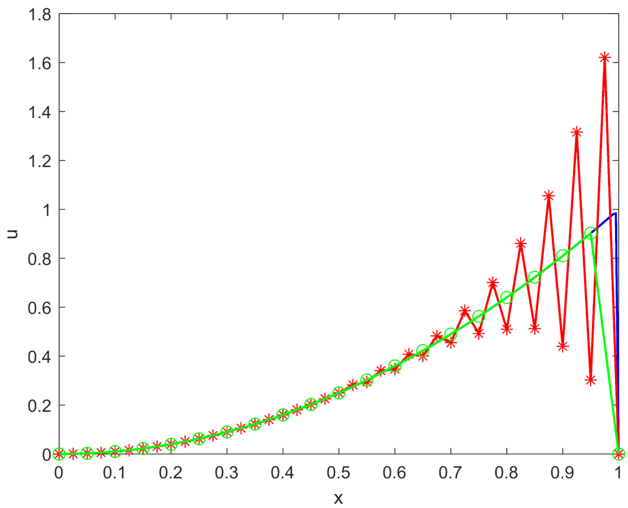


Figure 3. *Test 1.* Exact solution (line marked by +) of the convection-diffusion equation with linear source term and homogeneous boundary conditions, approximation by FEM Galerkin $\mathbb{P}1$ (line marked by $*$) and nodal "a posteriori VMS" filtering (line marked by $\circ$), for $Pe_h = 10$.

Test 1: Nodal filtering. Linear convection-diffusion equation

At first we reproduce the results of a test for the linear convection-diffusion equation (6), with linear source term $f(x) = x$, constant velocity $w = 400$ and diffusion $\nu = 1$, homogeneous boundary conditions $\alpha = \beta = 0$, and discretisation parameters $n = 40$ and $Pe_h = 10$, presented in the Computational Science for the 21st Century congress celebrating the 60th anniversary of R. Glowinski.

The result obtained is shown in Figure 3. As can be seen in this figure, the nodal “a posteriori VMS” filtering technique indeed allows to recover the exact solution at the nodes of the mesh with grid size $2h$.

Test 2: Residual-based filtering. Linear convection-diffusion equation

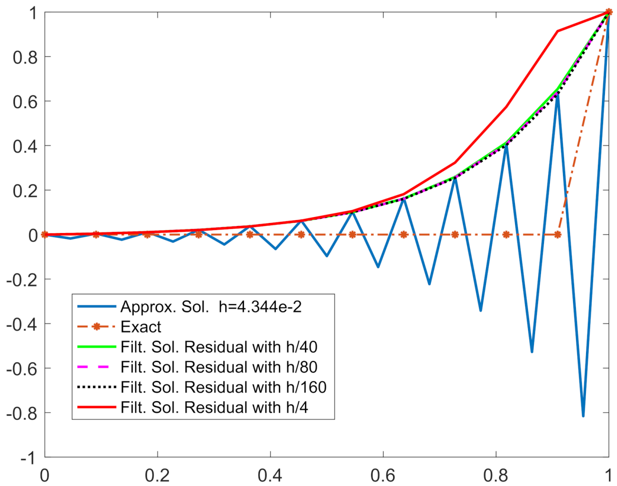


Figure 4. *Residual-based a posteriori VMS.* Filtered solutions obtained with initial values for the parameters α_i for the gradient method. The filtered solution approaches values very close to those of the oscillating solution at even grid nodes, as the residual is computed with successively refined grids.

In this section we test the residual-based “a posteriori VMS” filtering procedure. We have minimised the functional J by a gradient method with fixed step, that provides good results if the step is appropriately tuned. We present in Figure 4 the filtered solution when we use constant initial values $\alpha_i = -0.1$ for the gradient method. We use progressively refined grid sizes, smaller than h, to compute the residual for the convection-diffusion equation with constant velocity and $Pe_h = 10$. We observe that the filtered solution values at even grid nodes is quite close to those of the oscillating solution u_h. This filtered solution provides a local minimum of the residual norm, although it is rather far from the actual exact solution.

In Figure 5 we also show the filtered solutions when we use the nodal-based solution as initial condition for the gradient method, that is, the initial values are given by $\alpha_i = \alpha_i^{nb}$. We observe that the filtered solution is quite close to the exact one at the even grid nodes (although, as we have commented before, it does not coincide with the exact solution). We thus observe that the functional J has several local minima, and that these seem not to present spurious solutions. Also that some of them provide filtered solutions quite close to the exact one at even grid nodes. The choice of the initialization of the gradient method appears as a crucial choice to fit an appropriate residual-based “a posteriori” VMS solution.

Let us remark that this procedure can be applied locally, only in some sub-domain of the domain Ω where the solutions present rather large oscillations, to reduce the computational cost of the filtering procedure. This is quite important for the post-treatment of oscillating solutions of multi-dimensional flows.

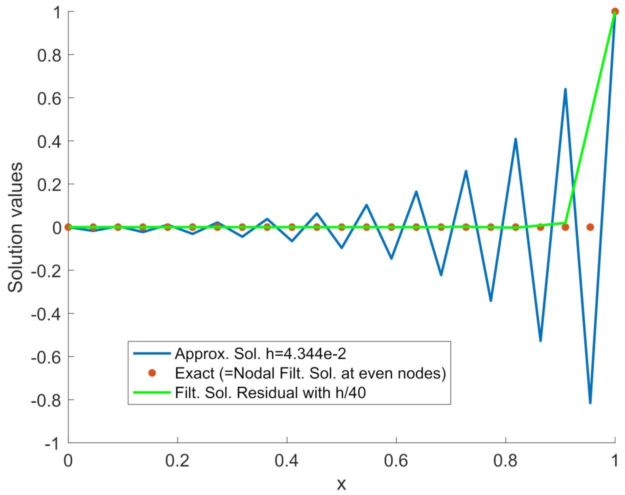


Figure 5. *Residual-based "a posteriori VMS".* Filtered solution obtained whit initial values for the parameters yielding the optimal nodal values α_i^{nb} for the gradient method. The filtered solution approaches values very close to those of the exact solution at even grid nodes, when the residual is computed with a fine enough grid.

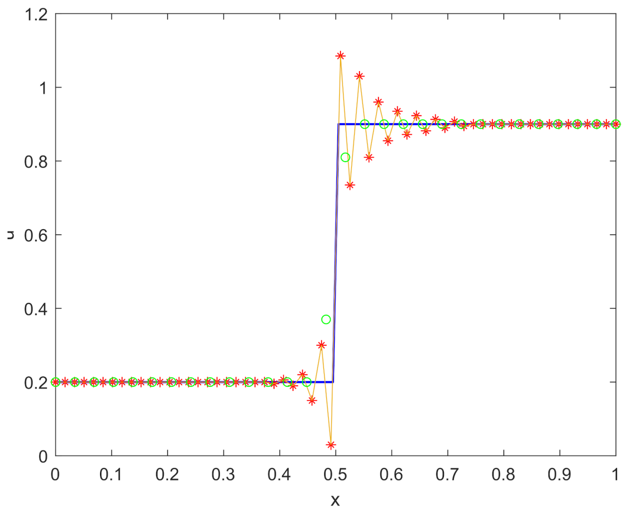


Figure 6. *Test 2.* Galerkin approximation (oscillating line), filtered approximation (line marked by ∘) and exact solution (solid line), for $N = 60, \nu = 0.001$ and $T = 0.1$.

Test 3: Evolution traffic flow equation

In the third test the post-processing technique has been applied to the nonlinear traffic problem given by (19) with the initial conditions (21). The test data are: $N = 60, \nu = 0.001, T = 0.1, \triangle t = 1/200, u_L = 0.2; u_R = 0.9$.

The grid Péclet number corresponding to these data is $Pe_h = 16.66$. Once the final time is reached (through 20 time steps), a total of 6 of filtering steps are performed (step II.3). The results obtained are presented in Figure 6. We observe that all spurious oscillations have disappeared, no overshoot or undershoot are presented in the filtered solution that, in addition, provides a rather accurate approximation to the shock solution at the targeted final time.

6. Conclusions

In this paper we have introduced two extensions of the "a posteriori VMS" filtering procedure that we presented in the proceedings of the "Computational Science for the 21$^{\text{st}}$ century" conference held in Tours in 1997 celebrating the 60$^{\text{th}}$ birthday of Roland Glowinski. The purpose of this procedure is to "cure" a solution of the convection-diffusion equations presenting spurious oscillations due to convection dominance, without need of solving again the equation on a finer grid or with stabilised methods. Only the information provided by the oscillating solution is used. The basic idea is to take advantage of the stabilising effect of the sub-grid scales.

On one hand we have introduced a residual-based "a posteriori" VMS that needs the solution of a least-squares problem (residual minimisation of a convenient projection of the oscillating solution on the grid with double size). Quite close filtered solutions to the exact one on the grid of double size are obtained, if the initialisation of the minimisation procedure for the least-squares problem is conveniently provided.

On another hand, we have applied the nodal-based "a posteriori" VMS to evolution non-linear convection-diffusion problems generating shocks in finite time (the one-lane traffic equation). We have developed an iterative filtering procedure using very small time steps to the solution at a targeted time. The filtering procedure does not need to be applied in preceding times. It provides quite accurate solutions, again at the nodes of a grid with double grid size.

Presently we are developing the extension of the residual-based "a posteriori" VMS procedure to multi-dimensional problems to be applied locally. The purpose is to obtain cures of the oscillating solutions only in sub-domains where it is needed.

Conflicts of interest

The authors have no conflict of interest to declare.

Acknowledgements

The authors are deeply indebted to the scientific teaching of Roland Glowinski, who directly or indirectly helped them to develop their careers in Computational Science.

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