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A value for games on colored communication structures.

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Keywords Colored graphs; Game theory; Myerson value

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A value for games on colored communication structures.

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Abstract Colored graphs have been used in many areas of technological research such as computer science, multiprocessor systems, network topology, etc .... Normally, the term colored graph is used in a graph where its nodes have been colored. However, we work with edge colored graphs, where the ones that have been colored are the edges.

The main aim of this paper is to determine a way based on cooperative game theory to measure the importance of the nodes in a network using edge colored graphs.

Keywords Edge colored graphs · Game theory · Myerson value.

1 Introduction

The theory of graphs had its beginning in *the problem of the bridges of Königsberg* which was solved by Leonard Euler in 1736. Later, Gustav Kirchhoff used it to analyze the electrical networks and that he published in his famous laws in 1874. Other applications have been found in later years in fields like topology, chemistry, etc.

Graph coloring are born as the tool used by F. Guthrie for the coloring of maps. From here it derives the conjecture of the four colors. There are other problems related to graph coloring as the problem of chromatic number or allocation of records in compilers. They are also used, among many other applications, to manage the resources used by computer programs, for the design and management of databases and for network topology.

We deal with edge colored graphs where each color expresses the nature of the relations. Our aim in this paper is to introduce a new problem that consists in measuring the importance of a node in an edge colored graph according to the position that he has on the graph. We apply the concepts of cooperative

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game theory to construct a value (function) to measure this importance. Then we apply this measure, as an application, to an example of multiprocessor tasks. However, there are other examples that could be studied as the following.

1. *Aircraft Scheduling.* Suppose that we have m planes, and we have to assign them to m flights, where the k-th flight is during the time interval I_k. Here two nodes will be connected if the time intervals overlap. The idea is to create an interval graph that is colored in a polynomial time and decide which flight is most important at that time. [**?**,**?**,**?**]
2. *Map Coloring and GSM Mobile Phone Networks.* Consider a mobile phone network that operates with p frequencies. This network is divided into hexagonal zones where each zone has a communications tower. In turn, mobile phones connect to the areas in its environment to communicate. The colors are assigned by zones, a color per emission frequency, where two zones are communicated if they emit in the same frequency. It is intended to know which area is more important, at the communications level, at any given time. [**?**,**?**,**?**,**?**,**?**]. Can we give an order of importance to these cellular regions?
3. *Art Gallery Problem.* Imagine that we went into an art gallery. If we observe it well we see that its walls, full of important pictures, are monitored by a camera systems. These systems may be parallel or alternate circuits. That is, several camera systems that operate simultaneously in case one or more of the circuits fails. Our goal is to try to give a measure of the importance to the positions where the cameras are placed depending on the ability to monitor the frames when any of the systems fail and their surveillance range. To achieve what was said before we will model the problem. We can see an art gallery as a closed polygon in two dimensions. The nodes of the polygon will be the points where the cameras will be placed that will also be the intersection of two walls of the gallery. We say that a camera covers a wall if all the points of the wall are reachable by the camera through straight segments. Next we will give colors to each type of camera. We will say that a wall has a color if there is a camera of that color that covers it. As the walls of the gallery are represented by the links of the polygon it is clear then, that the polygon contains colored graphs, as many as types of colors or there are cameras.

Section 2 contains necessary preliminaries about cooperative games and communication structures. In section 3 we present an example as motivation of the kind of problems to solve. Section 4 introduces the value which allow us to quantify the importance of the nodes, and it is axiomatized in last section.

2 Preliminaries

2.1 Cooperative games.

A *cooperative game with transferable utility*, game since now, is a pair (N, v) where N is a finite set of elements and $v : 2^N \to \mathbf{R}$ is an application such that

$v(\emptyset) = 0$. The elements of N are named players, the subsets of players are said coalitions and the application v the characteristic function of the game. If (N, v) is a game and $S \subseteq N$ then (S, v_S) is a new game where v_S is the restriction of the characteristic function to 2^S. We call Γ_N the set of all games on N.

A value on Γ_N is a function ψ which determines for each $(N, v) \in \Gamma_N$ a vector $\psi(N, v) \in \mathbf{R}^N$. The vector assigned to each game is usually interpreted as the payoff vector, that is, the allocation of the cooperation profit among all the players. The *Shapley value* is the most known value. It is defined for all game $(N, v) \in \Gamma_N$ and each player $i \in N$ as

$$\phi_i(N, v) = \sum_{\{S \subset N : i \in S\}} \frac{1}{s} \binom{n}{s}^{-1} [v(S) - v(S \setminus i)]. \tag{1}$$

2.2 Communication structures.

Let N be a finite set of players and $L_N = \{\{i, j\} \in N \times N : i \neq j\}$ the set of unordered pairs of elements in N. For simplicity, we will denote $\{i, j\}$ by ij. A *communication structure* on N is a graph (N, L) where N is the set of nodes and $L \subseteq L_N$ is the set of links. Hence we identify a communication structure, (N, L), for a fixed set N with the set of links L. If $L = \emptyset$ then we obtain the graph with only isolated nodes (without links).

A coalition $S \subseteq N$ which nodes are connected by the links of L is called *connected*. The maximal connected coalitions correspond to the sets of nodes of the connected components of the graph (N, L) and we denote them as N/L. This family N/L is actually a partition of N.

Let $(N, v) \in \Gamma_N$ a game on N. Let (N, L) a communication structure on N. If $S \subseteq N$ is a coalition then $L_S = \{ij \in L : i, j \in S\}$ is a new communication structure over N using only the links from L between players in S. Myerson [**?**] defined a game associate to the communication structure L as (N, v_L) where $v_L(S) = \sum_{K \in S/L_S} v(K)$ for all $S \subseteq N$. Particularly (S, v_{L_S}) represents the restriction to S of the game v and the communication structure taking also only the nodes in S. We will use $S/L = S/L_S$. This game is known as the *graph-game*. Note that this game incorporates the information of the communication structure.

A value for games with communication structures is a mapping over this family of games obtaining a payoff vector for each game with communication structure. The *Myerson value* is a value for games with communication structure based on the Shapley value. Given (N, v, L), Myerson [**?**] defines its value as

$$\mu(N, v, L) = \phi(N, v_L). \tag{2}$$

where $\phi(N, v_L)$ is the Shapley value for this game. The most interesting properties of the Myerson value, which axiomatize this value, are *efficiency by* *components*,

$$\sum_{i\in S}\mu_i(N,v,L)=v(S)\quad\forall S\in N/L, \tag{3}$$

and *fairness*, for each $ij\in L$ we have:

$$\mu_i(N,v,L)-\mu_i(N,v,L\setminus\{ij\})=\mu_j(N,v,L)-\mu_j(N,v,L\setminus\{ij\}). \tag{4}$$

Here, $L\setminus\{ij\}$ means that we delete the link ij from L.

The Myerson value is also *decomposable* in the sense that if $i\in S\in N/L$ then

$$\mu_i(N,v,L)=\mu_i(S,v,L_S). \tag{5}$$

3 An application to multiprocessor allocation systems.

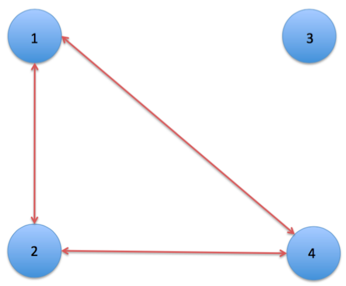
A distributed computer system consists of a set of processors that work together to perform a task and communicate through links present in the system. A distributed program assigns the work to processors using modules. The results are sent through the links between the processors in order to execute the program. Each processor can only perform one task at a time. There are usually several times of execution if a processor has to perform more than one task.

A failure in the system means that the program does not receive information from a processor in any of the times of execution. The program may work partially if one processor gives information about some tasks and fails on others. We will illustrate with this example the idea of giving different levels of importance to processors (nodes) depending on the level of connection of these graphs.

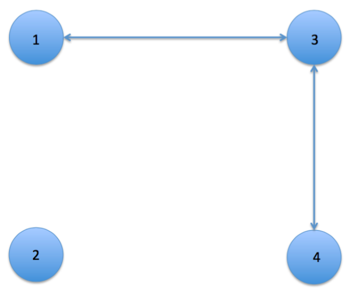
Let's suppose that the system has four processors and three execution times. In the first time the processors one, two and four communicate results. At time two processors one, three and four communicate results and the third time those who communicate are the processors two and three. Once all the execution times are finished the program checks all the processors that have worked to gather information. If any processor had not communicated in time the program does not work at least at full capacity, being able to fail. For example, if the one and two runtime communications failed, processors one and four would not have been able to give information to the program and it would not have been executed. We will name this color graph 1-connected. That is to say, any communication can be dropped since the other two allow the access of information by the program to all the processors. In this case the program would work although it would collect less information of the nodes but is functioning will be limited.

In general, we need information to reach all processors so that they can work. If a processor does not receive the information, the system does not work. The graph that joins the processors is actually the sum of a set of graphs that generate the color graph. This aggregated graph changes if we eliminate any color. The important goal in the example is to keep the connection between the four processors to be sure that all the information reaches them, no matter the channel (color). In this example, the connection is also maintained when any color drops. However it does not keep falling down two. That's why we say 1-connected. The probability distribution then acts on that no $\sigma(0)$ system falls or that 1, $\sigma(1)$ falls. These must add one because if the system works only these situations can appear. In order to explain our idea we will give the same color to the links that join the processors involved in each time. For example, red to those of time one, blue to those of time two and green to those of time three. This can be seen in the figures below.

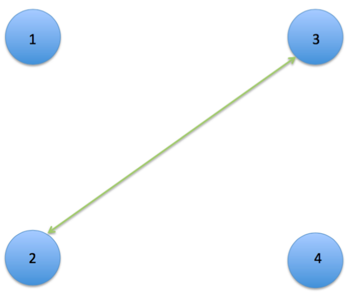
Execution time one



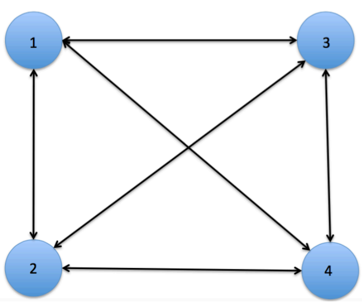
Execution time two



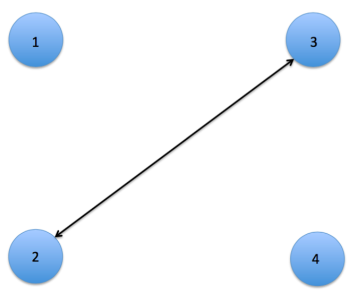
Execution time three



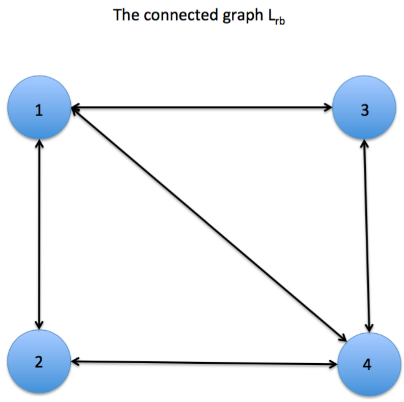
Connection lines for the program. Connected graph.

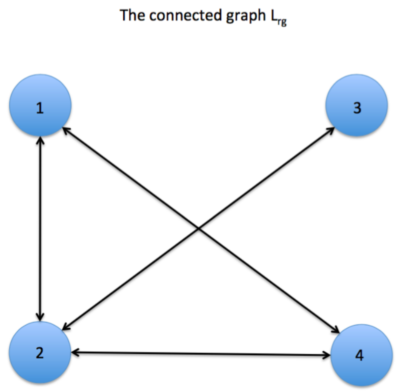


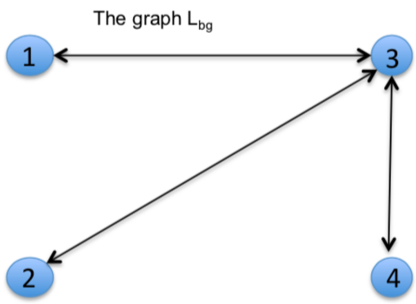
Communication failure. Disconnected graph.
Program Failure.



Now, we will explain the strategy to evaluate the importance of each position. Suppose that we have a probability distribution, $\{\sigma(0), \sigma(1)\}$ two no negative numbers with $\sigma(0)+\sigma(1)=1$, which indicates the probability that in any execution time there are no failures or that there are failures in any one runtime. We will also assume that we have a characteristic function $v : 2^N \to \mathbf{R}$ on the subsets of the positions of the nodes where $N=\{1,2,3,4\}$. The worth $v(S)$ means the quantification of the profit obtained with the information of the nodes in S without losses of the program. Let M be the set of colored graphs. In our case $M=\{r,b,g\}$ and we denote their corresponding graphs as $\mathcal{L}=\{L_r, L_b, L_g\}$. We will need to define a an operation of sum of graphs. If two graphs or more share in a determined link this link is reflected as a multi-link with as many levels of colors as graphs coincide in it. Once the operation is applied we get seven new graphs $\{L_r, L_b, L_g, L_{rb}, L_{rg}, L_{bg}, L_{rbg}\}$. In order to define our measure we consider these graphs as monochrome. The graphs $\{L_{rb}, L_{rg}, L_{bg}\}$ are shown in figure below.







We are now able to define the value that we will associate to each processor. This value can be interpreted as a measure of the importance of the node in the graph. We define the value as

$$\eta_i(N, v, \mathcal{L}, 1, \sigma) =$$

$$\frac{\sigma(0)\mu_i(N,v,L_{rbg})+\sigma(1)[\mu_i(N,v,L_{rb})+\mu_i(N,v,L_{rg})+\mu_i(N,v,L_{bg})]}{\sigma(0)\binom{3}{0}+\sigma(1)\binom{3}{1}}$$

where $\mu_i(N,v,L^*)$ is the Myerson value for i on the monochrome graph L^* and $\frac{1}{\sigma(0)\binom{3}{0}+\sigma(1)\binom{3}{1}}$ is the probability that deleting any set of colors the sum of the remaining colors is connected.

We define the value in the previous way because the value of Myerson is a fair and efficient value on graphs. Thus the processor is taken into account in all possible situations that can occur regarding the fail or not of a system weighted by the probability corresponding to that situation. The sum of the importance of a processor in all situations allows us to give a total measure of the importance of that processor to keep the program in good working order.

4 Games on colored communication structures

In this section we will approach our problem in a general way. To do this we will first introduce a series of tools that will be useful to solve it.

A *edge colored graph* (by links) over N is a finite set $\mathcal{L}=\{L_1,....,L_m\}$ of graphs over N. we identify each graph L_k with a color, the color k. We will call $M=\{1,...,m\}$ the *palette* of the edge colored graph.

If $A\subseteq M$ we define $\mathcal{L}_A$ as $\mathcal{L}_A=\{L_p\}_{p\in A}$, namely a new egde colored graph that is built by reducing the general palette M to set A. We denote as

$$L_A=\bigcup_{p\in A}L_p$$

the sum of all the graphs painted with colors in A. We can consider it as another graph with a new color A.

We say that a colored graph $\mathcal{L}$ is *k-connected* if deleting the links of any k colors of the palette, the sum of the remaining $n-k$ edge colored graphs is connected, and if we delete $k+1$ colors the sum of the remaining $n-k-1$ edge colored graphs is not connected. Hence, if $\mathcal{L}$ is k-connected then k is the maximum number of colors that we can delete of the palette so that the sum of all remaining edge colored graphs remains connected. Particularly $\mathcal{L}$ is 0-connected if L_M is connected but $L_{M\setminus\{p\}}$ is not connected for all $p\in M$. Number k is named *connection level* of $\mathcal{L}$.

In order to describe our problem we select a set of nodes N. We have a game over N with a characteristic function v establishing the worth of each group of agents. A colored graph $\mathcal{L}$ over N represents different systems of communication which sum guarantees the connection although certain number k (the connection level). Finally we have a decreasing probability distribution σ over $\{0,1,...,k\}$, i.e. with $\sigma(0)\geq\sigma(1)\geq\cdots\geq\sigma(k)$ shows the robustness of the system. Obviously we suppose $\sigma(r)=0$ or negligible for all $r>k$.

Definition 1 A game on a colored communication structure is a quintet $(N,v,\mathcal{L},k,\sigma)$ where (N,v) is a game, $\mathcal{L}=\{L_1,...,L_m\}$ is a colored graph such that there exists $k \in \{0, ..., m-1\}$ with $\mathcal{L}$ k-connected and σ is a decreasing probability distribution over $\{0, ..., k\}$. We denote as Υ the family of games on a colored communication structure.

We define a Myerson value over Υ, following our proposition for the multiprocessor allocation system problem. A value for games on a colored communication structure is a mapping over Υ obtaining a payoff vector for the agents in N for each $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$.

Definition 2 Let $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$ be a game on a colored communication structure. The colored Myerson value is defined for each player i as

$$\eta_i(N, v, \mathcal{L}, k, \sigma) = \frac{1}{C(m,k,\sigma)} \sum_{r=0}^{k} \sigma(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A}), \qquad (6)$$

where $C(m,k,\sigma) = \sum_{r=0}^{k} \sigma(r) \binom{m}{r}$.

Next, we compute the Myerson colored value of the multiprocessor allocation system problem exposed in section 3.

Example. We take the game v as

$$v(S) = \begin{cases} |S| \ if \ |S| > 1 \\ 0 \ \ if \ |S| = 1 \end{cases}$$

where $|S|$ is the cardinality of set S. Suppose then that the interest of the processor only depends on the position. As distribution of probability we take $\sigma(0) = 0.9$ and $\sigma(1) = 0.1$. We will detail the value assigned to processor 1 for the graph L_{rbg}. The rest will be placed in the table 1.

$$\phi_1(N, v^{L_{rbg}}) = \binom{4}{1}^{-1} [v^{L_{rgb}}(1) - v^{L_{rgb}}(\emptyset)] +$$

$$\frac{1}{2}\binom{4}{2}^{-1} [v^{L_{rgb}}(12) - v^{L_{rgb}}(2) + v^{L_{rgb}}(13) - v^{L_{rgb}}(3) + v^{L_{rgb}}(14) - v^{L_{rgb}}(4)] +$$

$$\frac{1}{3}\binom{4}{3}^{-1} [v^{L_{rgb}}(123) - v^{L_{rgb}}(23) + v^{L_{rgb}}(134) - v^{L_{rgb}}(34) + v^{L_{rgb}}(124) - v^{L_{rgb}}(24)] +$$

$$\frac{1}{4}\binom{4}{4}^{-1} [v^{L_{rgb}}(1234) - v^{L_{rgb}}(234)] =$$

$$\frac{1}{12}6 + \frac{1}{12}3 + \frac{1}{4}1 = 1$$

Now, we can compute the colored Myerson value for all the processors. First, note that, in this case, $\sum_{r=0}^{k} \sigma(r) \binom{m}{r} = 1.2$. Then

$$\eta_1(N, v, \mathcal{L}, 1, \sigma) = (0.9 + 0.1[36/12])/1.2 = 1$$

$$\eta_2(N, v, \mathcal{L}, 1, \sigma) = (0.9 + 0.1[5/6 + 19/12 + 11/12])/1.2 = 37/36$$

$$\eta_3(N, v, \mathcal{L}, 1, \sigma) = (0.9 + 0.1[5/6 + 7/12 + 19/12])/1.2 = 1$$

$$\eta_4(N, v, \mathcal{L}, 1, \sigma) = (0.9 + 0.1[7/6 + 7/12 + 11/12])/1.2 = 35/36$$

A	$\phi(N, v^{L_A})$	A	$\phi(N, v^{L_A})$
$\{r, b, g\}$	$(1, 1, 1, 1)$	$\{r, b\}$	$(7/6, 5/6, 5/6, 7/6)$
$\{r, g\}$	$(11/12, 19/12, 7/12, 11/12)$	$\{b, g\}$	$(11/12, 11/12, 19/12, 7/12)$

Table 1 The Myerson value in the example

5 An axiomatization of the colored Myerson value

In this section we explain several interesting properties satisfied by our value. We will finish the section showing that these properties allow us to axiomatize the value. Let Ξ be any value over Υ in all the section.

At least our graph is 0-connected, thus it is connected. We propose then that the payoff vector obtained by Ξ be an allocation of the worth of the great coalition.

Efficiency. We say that Ξ is efficient if for all game on a colored communication structure $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$, it holds

$$\sum_{i \in N} \Xi_i(N, v, \mathcal{L}, k, \sigma) = v(N).$$

Fairness, see section 2, is one of the most important properties of the Myerson value. In the classical case we have a (N, v, L) game with communication structure L where L is connected. We can translate this situation to our language of k-colors in the following way, $\mathcal{L} = \{L\}$, $k = 0$ and $\sigma(0) = 1$. We denote the subfamily of this kind of problems as Υ_1.

1-Fairness. For all $(N, v, L) \in \Upsilon_1$ it holds that for all $ij \in L$,

$$\Xi_i(N, v, L) - \Xi_i(S_i, v, L_{S_i}) = \Xi_j(N, v, L) - \Xi_j(S_j, v, L_{S_j})$$

where S_i, S_j are the connected components in $L \setminus \{ij\}$ containing i, j respectively.

These properties are minor modifications that also satisfies the Myerson value.

Proposition 1 *The colored Myerson value satisfied efficiency and 1-fairness.*

Proof First we prove efficiency. Let $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$. Suppose $\mathcal{L} = \{L_1, ..., L_m\}$ with $M = \{1, ..., m\}$ and $0 \leq k \leq m - 1$. Observe that for each $A \subset M$ with $|A| \leq k$ we have that $L_{M \setminus A}$ is connected, and then, as the Myerson value is efficient by components,

$$\sum_{i \in N} \mu_i(N, v, L_{M \setminus A}) = v(N).$$

We get

$$\sum_{i \in N} \eta_i(N, v, \mathcal{L}, k, \sigma) = \sum_{i \in N} \frac{\sum_{r=0}^{k} \sigma(r) \left[\sum_{\{A \subset M, |A| = r\}} \mu_i(N, v, L_{M \setminus A}) \right]}{\sum_{r=0}^{k} \sigma(r) \binom{m}{r}}$$

$$= \frac{\sum_{r=0}^{k} \sigma(r) \sum_{\{A\subset M, |A|=r\}} \left[\sum_{i\in N} \mu_i(N, v, L_{M\setminus A})\right]}{\sum_{r=0}^{k} \sigma(r)\binom{m}{r}}$$

$$= v(N)\frac{\sum_{r=0}^{k} \sigma(r)\binom{m}{r}}{\sum_{r=0}^{k} \sigma(r)\binom{m}{r}} = v(N)$$

Note that the cardinality of $\{A \subset M, |A| = r\}$ is exactly the number $\binom{m}{r}$.

If $(N, v, L) \in \Upsilon_1$ then by definition $\Xi(N, v, L) = \mu(N, v, L)$. Graph L is connected but $L\setminus\{ij\}$ may not be. Suppose $S_i, S_j \in N/(L\setminus\{ij\})$ the connected components containing i, j respectively. If $L \setminus \{ij\}$ is connected then $S_i = S_j$. We use fairness and decomposability of the Myerson value, see section 2, to get

$$\begin{aligned}\eta_i(N, v, L) - \eta_j(N, v, L) &= \mu_i(N, v, L) - \mu_j(N, v, L)\\ &= \mu_i(N, v, L \setminus \{ij\}) - \mu_j(N, v, L \setminus \{ij\})\\ &= \mu_i(S_i, v, L_{S_i}) - \mu_j(S_j, v, L_{S_j})\\ &= \eta_i(S_i, v, L_{S_i}) - \eta_j(S_j, v, L_{S_j}).\end{aligned}$$

But we need more properties to characterize the colored Myerson value. Before stating them we introduce some concepts. Next property is only valid for a connection level less than $m-1$. It shows how to establish the importance of a position with respect to a color. To be able to enunciate it we define before two sets of graphs from a prefixed color.

Definition 3 Let $\mathcal{L} = \{L_1, ..., L_m\}$ be a colored graph and $M = \{1, ..., m\}$. If $q \in M$ then the contraction of $\mathcal{L}$ to q is the colored graph

$$\mathcal{L}^{+q} = \{L_{rq} : r \in M \setminus q\}.$$

The deletion of q in $\mathcal{L}$ is the colored graph

$$\mathcal{L}^{-q} = \mathcal{L}_{M\setminus q} = \mathcal{L} \setminus \{L_q\}.$$

Both, contraction and deletion have one color less than the original. Notice that the connection level of both, contraction and deletion, is one less than the original colored graph. Suppose now a game with a colored communication structure $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$ with palette $M = \{1, ..., m\}$ and $k < m - 1$. We modified the structure for the new palette. So, we consider

$$(N, v, \mathcal{L}^{+q}, k - 1, \sigma^{+q})$$

where $\sigma^{+q}(r) = \frac{\sigma(r)}{1-\sigma(k)}$ for all $r \in \{0, ..., k-1\}$. On the other hand the deletion has as connection level one less than the original. Associated to the deletion $\mathcal{L}^{-q}$, we take

$$(N, v, \mathcal{L}^{-q}, k - 1, \sigma^{-q})$$

where $\sigma^{-q}(r) = \frac{\sigma(r+1)}{1-\sigma(0)}$ for $r \in \{0, ..., k - 1\}$. Note that σ^{+q} and σ^{-q} are probability distributions.

Contraction-deletion. We say that Ξ satisfies contraction-deletion if for all $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$ with palette $M = \{1, ..., m\}$ and $k < m-1$ it holds for each color $q \in M$ that

$$\eta(N, v, \mathcal{L}, k, \sigma) = C^{+q}\eta(N, v, \mathcal{L}^{+q}, k-1, \sigma^{+q}) + C^{-q}\eta(N, v, \mathcal{L}^{-q}, k-1, \sigma^{-q}).$$

where $C^{+q} = (1-\sigma(k))\frac{C(m-1,k-1,\sigma^{+q})}{C(m,k,\sigma)}$ and $C^{-q} = (1-\sigma(0))\frac{C(m-1,k-1,\sigma^{-q})}{C(m,k,\sigma)}$

Proposition 2 *The colored Myerson value satisfies contraction-deletion.*

Proof Let $(N, v, \mathcal{L}, k, \sigma)$ be a game on a colored communication structure with $\mathcal{L} = \{L_1, ..., L_m\}$ and $k < m-1$. Then, fixed a color $q \in M$,

$$\eta_i(N, v, \mathcal{L}, k, \sigma) = \frac{\sum_{r=1}^{k} \sigma(r) \left[\sum_{\{A \subset M, |A|=r, q \in A\}} \mu_i(N, v, L_{M \setminus A})\right]}{C(m, k, \sigma)} + \frac{\sum_{r=0}^{k-1} \sigma(r) \left[\sum_{\{A \subset M, |A|=r, q \notin A\}} \mu_i(N, v, L_{M \setminus A})\right]}{C(m, k, \sigma)}.$$

We multiply and divide by $(1-\sigma(k))$ the first term and by $(1-\sigma(0))$ the second one. Notice that we can identify each $A \subseteq M \setminus \{q\}$ with the same set of colors A but in the palette of $\mathcal{L}^{+q}$. Moreover $\sigma^{+q}(|A|) = \frac{\sigma(|A|)}{(1-\sigma(k))}$ for these sets. Each $A \subseteq M$ with $q \in A$ is identified to $A \setminus \{q\}$ subset in the palette of the contraction. Furthermore $\sigma^{-q}(|A|-1) = \frac{\sigma(|A|)}{(1-\sigma(0))}$. Then, we can rewrite the above expression as

$$\eta_i(N, v, \mathcal{L}, k, \sigma) = \sum_{r=0}^{k-1} \frac{\sigma^{+q}(r)(1-\sigma(k))}{C(m, k, \sigma)} \left[\sum_{\{A \subset M \setminus \{q\}, |A|=r\}} \mu_i(N, v, L_{M \setminus A} \cup L_q)\right] + \sum_{r=0}^{k-1} \frac{\sigma^{-q}(r)(1-\sigma(0))}{C(m, k, \sigma)} \left[\sum_{\{A \subset M, |A|=r, q \in A\}} \mu_i(N, v, L_{M \setminus A})\right] =$$

We multiply and divide the first term by $C(m-1, k-1, \sigma^{+q})$ and the second one by $C(m-1, k-1, \sigma^{-q})$

$$\frac{(1-\sigma(k))C(m-1, k-1, \sigma^{+q})}{C(m, k, \sigma)} \sum_{r=0}^{k-1} \frac{\sigma^{+q}(r)}{C(m-1, k-1, \sigma^{+q})} \left[\sum_{\{A \subset M \setminus \{q\}, |A|=r\}} \mu_i(N, v, L_{M \setminus A} \cup L_q)\right] +$$

$$\frac{(1-\sigma(0))C(m-1, k-1, \sigma^{-q})}{C(m, k, \sigma)} \sum_{r=0}^{k-1} \frac{\sigma^{-q}(r)}{C(m-1, k-1, \sigma^{-q})} \left[\sum_{\{A \subset M, |A|=r, q \in A\}} \mu_i(N, v, L_{M \setminus A})\right] =$$

$$C^{+q}\eta(N, v, \mathcal{L}^{+q}, k-1, \sigma^{+q}) + C^{-q}\eta(N, v, \mathcal{L}^{-q}, k-1, \sigma^{-q})$$

The following axiom is related to the case where the degree of connection is $k = m-1$. This case is not included in the previous axiom.

Reduction Let Ξ be a value on Υ Then for all $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$ it holds

$$C(m, m-1, \sigma)\Xi(N, v, \mathcal{L}, m-1, \sigma) =$$

$$= C(m, m-2, \tilde{\sigma})(1-\sigma(m-1))\Xi(N, v, \mathcal{L}, m-2, \tilde{\sigma}) + \sigma(m-1) \sum_{k=1,\ldots,m} \Xi(N, v, L_k)$$

where $\tilde{\sigma}_k = \frac{\sigma(k)}{1-\sigma(m-1)}$ and $(N, v, L_k) \in \Upsilon_1$. The reader can test that $\tilde{\sigma}$ is a probability distribution.

Proposition 3 *Our value satisfies the reduction axiom.*

Proof Notice that, fixed $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$ we can decompose our value as:

$$\frac{1}{C(m, m-1, \sigma)} \sum_{r=0}^{m-1} \sigma(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A}) =$$

$$\frac{1}{C(m, m-1, \sigma)} \sum_{r=0}^{m-2} \sigma(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A}) +$$

$$+ \frac{\sigma(m-1)}{C(m, m-1, \sigma)} \sum_{\{A \subset M, |A|=m-1\}} \mu_i(N, v, , L_{M \setminus A}) =$$

$$= \frac{1}{C(m, m-1, \sigma)} \sum_{r=0}^{m-2} p(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A}) +$$

$$+ \frac{\sigma(m-1)}{C(m, m-1, \sigma)} \sum_{k=1}^{m} \eta_i(N, v, L_k, 0, 1).$$

because we have m choices of $m-1$ colored graphs and $L_{M \setminus A}$ reduces to only one color. We define $\tilde{\sigma}(k) = \frac{\sigma(k)}{1-\sigma(m-1)}$ as a new probability distribution when the level of connection is $m-2$. For this we can write

$$\frac{1}{C(m, m-1, \sigma)} \sum_{r=0}^{m-1} \sigma(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A}) =$$

$$\frac{(1-\sigma(m-1))C(m, m-2, \tilde{\sigma})}{C(m, m-1, \sigma)} \frac{\sum_{r=0}^{m-2} \tilde{\sigma}(r) \sum_{\{A \subset M, |A|=r\}} \mu_i(N, v, L_{M \setminus A})}{C(m, m-2, \tilde{\sigma})} =$$

$$\frac{(1-\sigma(m-1))C(m, m-2, \tilde{\sigma})}{C(m, m-1, \sigma)} \eta_i(N, v, m-2, \mathcal{L}, \tilde{\sigma}).$$

Then, by adding the second term we obtain the result

$$C(m, m-1, \sigma)\eta(N, v, \mathcal{L}, m-1, \sigma) =$$

$$= C(m, m-2, \tilde{\sigma})(1-\sigma(m-1))\eta(N, v, \mathcal{L}, m-2, \tilde{\sigma}) + \sigma(m-1) \sum_{k=1,\ldots,m} \eta(N, v, \mathcal{L}_k, 0, 1).$$

This proves the proposition.

Now, we are ready to prove that our value is the only one satisfying the above axioms.

Theorem 1 *There exists a unique value η on Υ such that satisfies efficiency, 1-fairness, contraction-delection and reduction axioms.*

Proof It remains to prove the uniqueness of the value. Let Ξ be another value on Υ), that satisfies the four axioms. First notice that as both values satisfies efficiency and 1-fairness then for all $(N, v, \mathcal{L}, 0, 1) \in \Upsilon$, where $\mathcal{L}$ is a single color, it holds

$$\Xi(N, v, \mathcal{L}, 0, 1) = \mu(N, v, L) = \eta(N, v, \mathcal{L}, 0, 1)$$

because the unique value that satisfies these axioms is the Myerson value. Suppose the uniqueness on Υ for all $k \leq m' - 1$ and $m' \leq m - 1$. Let $m' = m$, $k = m - 1$ and $(N, v, \mathcal{L}, k, \sigma) \in \Upsilon$. By the axiom of reduction we can express the value as

$$\begin{gathered} C(m, m-1, \sigma)\Xi(N, v, \mathcal{L}, m-1, \sigma) = \\ = C(m, m-2, \tilde{\sigma})(1 - \sigma(m-1))\Xi(N, v, \mathcal{L}, m-2, \tilde{\sigma}) + \\ + \sigma(m-1) \sum_{k=1,\ldots,m} \Xi(N, v, \mathcal{L}_k, 0, 1) = \\ = C(m, m-2, \tilde{\sigma})(1 - \sigma(m-1))\Xi(N, v, \mathcal{L}, m-2, \tilde{\sigma}) + \\ + \sigma(m-1) \sum_{k=1,\ldots,m} \eta(N, v, \mathcal{L}_k, 0, 1) \end{gathered}$$

by the induction hypothesis.

Now we apply the contraction-delection axiom to

$$\Xi(N, v, \mathcal{L}, m-2, \tilde{\sigma})$$

because $k = m - 2$ obtaining

$$\begin{gathered} C(m, m-2, \tilde{\sigma})(1 - \sigma(m-1))\Xi(N, v, \mathcal{L}, m-2, \tilde{\sigma}) = \\ = C(m, m-2, \tilde{\sigma})(1 - \sigma(m-1))[C^{+q}\Xi(N, v, \mathcal{L}^{+q}, k-1, \sigma^{+q}) + \\ C^{-q}\Xi(N, v, \mathcal{L}^{-q}, k-1, \sigma^{-q})] \end{gathered}$$

Applying successively the contraction-delection axiom we come to a situation where $m = 1$ and k is 0. As both values satisfiy 1-fairness and efficiency we obtain the result.

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6 References

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